Exercise 1  Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $H_\mathcal{F}$ denote the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Suppose $\mathcal{G} \subset \mathcal{F}$ is a sub $\sigma$-algebra, and let $H_\mathcal{G}$ denote the corresponding Hilbert space of $\mathcal{G}$-measurable, square-integrable random variables.

1. Show that $H_\mathcal{G}$ is a closed subspace of $H_\mathcal{F}$.

2. If $X \in H_\mathcal{F}$ denote by $\mathcal{E}(X \mid \mathcal{G})$ the projection of $X$ onto $H_\mathcal{G}$. Show that for all events $A \in \mathcal{G}$,
   \[ \mathbf{E}[X 1_A] = \mathbf{E}[\mathcal{E}(X \mid \mathcal{G}) 1_A] \] (1)

3. Suppose $X$ is a random variable in $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}[|X|] < \infty$ (note that we do not assume that $\mathbf{E}[X^2] < \infty$). Show that there is an integrable random variable $\mathcal{E}(X \mid \mathcal{G})$ that is $\mathcal{G}$-measurable and such that (1) holds for all $A \in \mathcal{G}$. (You may wish to consider $X \geq 0$ first. You may not use the Radon-Nikodym theorem.)

4. Suppose that $X, Y$ are square-integrable random variables and $Y$ is $\mathcal{G}$-measurable. Show that
   \[ \mathcal{E}(X Y \mid \mathcal{G}) = Y \mathcal{E}(X \mid \mathcal{G}). \]

5. Suppose $\tilde{\mathcal{G}}$ is independent of $\mathcal{G}$ and $X$ is $\tilde{\mathcal{G}}$-measurable. What is $\mathcal{E}(X \mid \mathcal{G})$?

6. Show that
   \[ \mathbf{E}[\mathcal{E}(X \mid \mathcal{G})^2] \leq \mathbf{E}[X^2], \]
   where we allow infinity as a possible value for the expectation. Give an example to show that the left-hand side can be finite and the right-hand side infinite.