1 Bessel process

The Bessel process is one of the most important one-dimensional diffusion processes. There are many ways that it arises. We will start by viewing the Bessel process as a Brownian motion “tilted” by a function of the current value.

To make this precise we use the Girsanov theorem. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which is defined a standard one-dimensional Brownian motion $X_t$ with filtration $\{\mathcal{F}_t\}$ with $X_0 > 0$. Let $T_x = \inf\{t : X_t = x\}$. Let $a \in \mathbb{R}$, and let

$$Z_t = X_t^a.$$  

The Bessel process with parameter $a$ will be the Brownian motion “weighted locally by $X_t^a$”. If $a > 0$, then the Bessel process will favor larger values while for $a < 0$ it will favor smaller values. The value $a = 0$ will correspond to the usual Brownian motion. For the moment, we will stop the process at time $T_0$ when it reaches the origin.

Itô’s formula shows that if $f(x) = x^a$, then

$$dZ_t = f'(X_t) dX_t + \frac{1}{2} f''(X_t) dt = Z_t \left[ \frac{a}{X_t} dX_t + \frac{a(a-1)}{2 X_t^2} dt \right], \quad t < T_0.$$  

For the moment, we will not consider $t \geq T_0$. Let

$$N_t = N_{t,a} = \left[ \frac{X_t}{X_0} \right]^a \exp \left\{ - \frac{(a-1)a}{2} \int_0^t \frac{ds}{X_s^2} \right\}, \quad t < T_0.$$  

Then the product rule combined with the Itô calculation above shows that $N_t$ is a local martingale for $t < T_0$ satisfying

$$dN_t = \frac{a}{X_t} N_t dX_t, \quad N_0 = 1.$$  

Let $\tau = \tau_{\epsilon,R} = T_\epsilon \wedge T_R$. For each fixed $\epsilon, R$ with $0 < \epsilon < X_0 < R$ and each $t < \infty$, we can see that $N_{t \wedge \tau}$ is uniformly bounded for $s \leq t$ satisfying

$$dN_{t \wedge \tau} = \frac{a}{X_{t \wedge \tau}} 1\{t < \tau\} N_{t \wedge \tau} dX_t.$$  

In particular, $N_{t \wedge \tau}$ is a positive continuous martingale.

We will use Girsanov’s theorem and we assume the reader is familiar with this. This is a theorem about positive martingales that can also be used to study positive local martingales. We discuss now in some detail how to do this, but later on we will not say as much. For each $t, \epsilon, R$, we define the probability measure $\tilde{\mathbb{P}}_{a,\epsilon,R,t}$ on $\mathcal{F}_t$ by

$$\frac{d\tilde{\mathbb{P}}_{a,\epsilon,R,t}}{d\mathbb{P}} = N_{t \wedge \tau}, \quad \tau = \tau_{\epsilon,R}.$$  

Since (for fixed $a, \epsilon, R$), $N_{t \wedge \tau}$ is a martingale, we can see (as in the usual set up for Girsanov’s theorem) that if $s < t$, then $\tilde{\mathbb{P}}_{a,\epsilon,R,t}$ restricted to $\mathcal{F}_{s \wedge \tau}$ is $\tilde{\mathbb{P}}_{a,\epsilon,R,s}$. For this reason, we write just $\tilde{\mathbb{P}}_{a,\epsilon,R}$. Girsanov’s theorem states that if

$$B_t = B_{t,a} = X_t - \int_0^t \frac{a ds}{X_s}, \quad t < \tau,$$  

$$1$$
then $B_t$ is a standard Brownian motion with respect to $\hat{P}_{a,\epsilon,R}$, stopped at time $\tau$. In other words,

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad t \leq \tau.$$ 

Note that as $X_t$ gets large, the drift term $a/X_t$ gets smaller. By comparison with a Brownian motion with drift, we can therefore see that as $R \to \infty$,

$$\lim_{R \to \infty} \hat{P}_{a,\epsilon,R}\{t \wedge \tau = T_R\} \to 0,$$

uniformly in $\epsilon$. Hence we can write $\hat{P}_a$, and see that

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad t \leq T_0,$$

Note that this equation does not depend on $\epsilon$ except in the specification of the allowed values of $t$. For this reason, we can write $\hat{P}_\epsilon$, and let $\epsilon \downarrow 0$ and state that

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad t < T_0,$$

where in this case we define $T_0 = \lim_{\epsilon \downarrow 0} T_\epsilon$. This leads to the definition.

**Definition** $X_t$ is a Bessel process starting at $x_0 > 0$ with parameter $a$ stopped when it reaches the origin, if it satisfies

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad t < T_0, \quad X_0 = x_0, \quad (2)$$

where $B_t$ is a standard Brownian motion.

We have written the parameter $a$ to make the equation as simple as possible. However, there are two more common ways to parametrize the family of Bessel process.

- The dimension $d$,

$$a = \frac{d - 1}{2}, \quad d_a = 2a + 1.$$ 

This terminology comes from the fact that if $W_t$ is a standard $d$-dimensional Brownian motion, then the process $X_t = |W_t|$ satisfies the Bessel process with $a = (d - 1)/2$. We sketch the argument here. First note that

$$dX_t^2 = \sum_{j=1}^{d} d[(W_t^j)^2] = 2 \sum_{j=1}^{d} W_t^j dW_t^j + d dt,$$

which we can write as

$$dX_t^2 = d dt + 2 X_t dZ_t,$$

for the standard Brownian motion

$$Z_t = \sum_{j=1}^{d} \int_0^t \frac{|W_s^j|}{X_s} dW_s^j.$$
(To see that $Z_t$ is a standard Brownian motion, check that $Z_t$ is a continuous martingale with $(Z)_t = t$.) We can also use Itô’s formula to write
\[ dX_t^2 = 2X_t \, dX_t + d\langle X \rangle_t. \]

Equating the two expressions for $dX_t^2$, we see that
\[ dX_t = \frac{d-1}{X_t} \, dt + dZ_t. \]

The process $X_t^2$ is called the $d$-dimensional squared-Bessel process. This can be defined even if $d$ is not an integer.

- The index $\nu$,
  \[ a = \frac{2\nu + 1}{2}, \quad \nu_a = \frac{2a - 1}{2}. \]

  Note that $\nu_{1-a} = -\nu_a$. We will see below that there is a strong relationship between Bessel processes of parameter $a$ and parameter $1 - a$.

As we have seen, in order to construct a Bessel process, we can start with a standard Brownian motion $X_t$ on $(\Omega, \mathbb{P}, \mathcal{F})$, and then consider the probability measure $\hat{\mathbb{P}}_a$. (There is a minor technical issue that the next proposition handles. The measure $\hat{\mathbb{P}}_a$ is defined only up to time $T_{0+}$. The next proposition shows that we can replace this with $T_0$ and get continuity at time $T_0$.) Equivalently, we can start with a Brownian motion $B_t$ on $(\Omega, \mathbb{P}, \mathcal{F})$ and define $X_t$ to be the solution of the equation (2).

**Proposition 1.1.** Let $T_{0+} = \lim_{r \downarrow 0} T_r$.

1. If $a \geq 1/2$, then for each $x > 0, t > 0$,
   \[ \hat{\mathbb{P}}_a^x \{ T_{0+} \leq t \} = 0. \]

2. If $a < 1/2$, then for each $t > 0$,
   \[ \hat{\mathbb{P}}_a^x \{ T_{0+} \leq t \} > 0. \]

   Moreover, on the event $\{ T_{0+} \leq t \}$, except on an event of $\hat{\mathbb{P}}_a$-probability zero,
   \[ \lim_{t \uparrow T_0} X_t = 0. \]

**Proof.** If $0 < r < x < R < \infty$, let $\tau = T_r \lor T_R$. Define $\phi(x) = \phi(x; r, R, a)$ by
\[ \phi(x) = \begin{cases} x^{1-2a} - r^{1-2a} & a \neq 1/2, \\ \log x - \log r & a = 1/2. \end{cases} \]

This is the unique function on $[r, R]$ satisfying the boundary value problem
\[ x \phi''(x) = 2a \phi'(x), \quad \phi(r) = 0, \quad \phi(R) = 1. \quad (3) \]
Using Itô’s formula and (3), we can see that \( \phi(X_{t \wedge \tau}) \) is a bounded continuous \( \hat{P}_a \)-martingale, and hence by the optional sampling theorem

\[
\phi(x) = \hat{E}_a [\phi(X_{\tau})] = \hat{P}_a \{ T_R < T \}. 
\]

Therefore,

\[
\hat{P}_a \{ T_R < T \} = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq 1/2, 
\]

(4)

\[
\hat{P}_a \{ T_R < T \} = \frac{\log x - \log r}{\log R - \log r}, \quad a = 1/2. 
\]

In particular, if \( x > r > 0 \),

\[
\hat{P}_a \{ T < \infty \} = \lim_{R \to \infty} \hat{P}_a \{ T < T_R \} = \left\{ \begin{array}{ll} (r/x)^{2a-1}, & a > 1/2 \\ 1, & a \leq 1/2, \end{array} \right. 
\]

(5)

and if \( x < R \),

\[
\hat{P}_a \{ T_R < T_{0+} \} = \lim_{R \downarrow 0} \hat{P}_a \{ T_R < T \} = \left\{ \begin{array}{ll} (x/R)^{1-2a}, & a < 1/2 \\ 1, & a \geq 1/2. \end{array} \right. 
\]

(6)

If \( a > 1/2 \), then (5) implies that for each \( t, R < \infty \),

\[
\hat{P}_a \{ T_{0+} \leq t \} \leq \hat{P}_a \{ T_{0+} < T_R \} + \hat{P}_a \{ T_R < T_{0+}; T_R < t \}. 
\]

\[
\hat{P}_a \{ T_{0+} < \infty \} \leq \lim_{R \to \infty} \hat{P}_a \{ T_{0+} < T_R \} = 0. 
\]

Letting \( R \to \infty \) (and doing an easy comparison with Brownian motion for the second term on the right), shows that for all \( t \),

\[
\hat{P}_a \{ T_{0+} \leq t \} = 0, 
\]

and hence, \( \hat{P}_a \{ T_{0+} < \infty \} = 0 \).

If \( a < 1/2 \), let \( \tau_n = T_{2^{-2n}} \) and \( \sigma_n = \inf \{ t > \tau_n : X_t = 2^{-n} \} \). Then if \( x > 2^{-2n} \), (6) implies that

\[
\hat{P}_a \{ \sigma_n < T_{0+} \} = 2^{n(2a-1)}. 
\]

In particular,

\[
\sum_{n=0}^{\infty} \hat{P}_a \{ \sigma_n < T_{0+} \} < \infty, 
\]

and by the Borel-Cantelli lemma, with \( P_a \)-probability one, \( T_{0+} < \sigma_n \) for all \( n \) sufficiently large. On the event that this happens, we see that

\[
\lim_{t \uparrow T_{0+}} X_t = 0, 
\]

and hence \( T_0 = T_{0+} \). On this event, we have \( \max_{0 \leq t \leq T_0} X_t < \infty \), and hence

\[
\hat{P}_a \{ T_{0+} = \infty \} \leq \hat{P}_a \left\{ \sup_{0 \leq t < T_{0+}} X_t = \infty \right\} \leq \lim_{R \to \infty} \hat{P}_a \{ T_{0+} < T_R \} = 0. 
\]
With this proposition, we can view $\hat{P}_a^x$ for each $t$ as a probability measure on continuous paths $X_s, 0 \leq s \leq t \wedge T_0$.

**Proposition 1.2.** For each $x, t > 0$ and $a \in \mathbb{R}$, the measures $\mathbb{P}^x$ and $\hat{P}_a^x$, considered as measures on paths $X_s, 0 \leq s \leq t$, restricted to the event $\{T_0 > t\}$ are mutually absolutely continuous with Radon-Nikodym derivative

$$
\frac{d\hat{P}_a^x}{d\mathbb{P}^x} = \left[ \frac{X_t}{x} \right]^a \exp \left\{ -\frac{(a-1)a}{2} \int_0^t ds \frac{ds}{X_s^2} \right\}.
$$

In particular,

$$
\frac{d\hat{P}_a^x}{d\mathbb{P}^{1-a}} = \left[ \frac{X_t}{x} \right]^{2a-1}.
$$

(7)

**Proof.** The first equality is a restatement of what we have already done, and the second follows by noting that the exponential term is not changed if we replace $a$ with $1-a$. \qed

**Corollary 1.3.** Suppose $x < y$ and $a \geq 1/2$. Consider the measure $\hat{P}_a^y$ conditioned on the event $\{T_x < T_y\}$, considered as a measure on paths $X_t, 0 \leq t \leq T_x$. Then $\hat{P}_a^y$ is the same as $\hat{P}_{1-a}^y$, again considered as a measure on paths $X_t, 0 \leq t \leq T_x$.

We also see from (7), a rederivation of the fact that $\hat{P}_a^y\{T_x < \infty\} = (x/y)^{2a-1}$.

If $a < b$, then $\hat{P}_a^x$ and $\hat{P}_b^x$, considered as measures on paths up to time $t$, are mutually absolutely continuous restricted to the event $\{T_0 > t\}$. However, they are singular measures viewed on the paths $X_s, 0 \leq s \leq T_0$.

For fixed $t$, on the event $\{T_0 > t\}$, the measures $\mathbb{P}^x$ and $\hat{P}_a^x$ are mutually absolutely continuous. Indeed, if $0 < r < x < R$, and $\tau = T_r \wedge T_R$, then $\mathbb{P}^x$ and $\hat{P}^x$ are mutually absolutely continuous on $\mathcal{F}_\tau$ with

$$
\frac{d\hat{P}_a^x}{d\mathbb{P}^x} = N_{\tau,a} \in (0, \infty).
$$

However, if $a < b < 1/2$, the measures $\hat{P}_a$ and $\hat{P}_b$ viewed as measure on curves $X_t, 0 \leq t \leq T_0$, can be shown to be singular with respect to each other.

**Proposition 1.4** (Brownian scaling). Suppose $X_t$ is a Bessel process satisfying (2), $r > 0$, and

$$
Y_t = r^{-1} X_{r^2 t}, \quad t \leq r^{-2} T_0.
$$

Then $Y_t$ is a Bessel process with parameter $a$ stopped at the origin.

**Proof.** This follows from the fact that $Y_t$ satisfies

$$
dY_t = \frac{a}{Y_t} \, dt + dW_t,
$$

where $W_t = r^{-1} B_{r^2 t}$ is a standard Brownian motion. \qed
1.1 Logarithm

When one takes the logarithm of the Bessel process, the parameter $\nu = a - \frac{1}{2}$ becomes the natural one to use. Suppose $X_t$ satisfies

$$dX_t = \frac{\nu + \frac{1}{2}}{X_t} \, dt + dB_t, \quad X_0 = x_0 > 0.$$ 

If $L_t = \log X_t$, then Itô’s formula shows that

$$dL_t = \frac{1}{X_t} \, dX_t - \frac{1}{2X_t^2} \, dt + \frac{\nu}{X_t} \, dt + \frac{1}{X_t} \, dB_t.$$ 

Let

$$\sigma(s) = \inf \left\{ t : \int_0^t \frac{dr}{X_r^2} = s \right\},$$

$$\hat{L}_s = L_{\sigma(s)}.$$ 

Then $\hat{L}_s$ satisfies

$$d\hat{L}_s = \nu \, ds + dW_s,$$

where $W_s$ is a standard Brownian motion. In other words the logarithm of the Bessel process is a time change of a Brownian motion with (constant) drift. Note that

$$\sigma(\infty) = \int_0^\infty e^{-2\hat{L}_s} \, ds,$$

which is not hard to show is finite (with probability one) if and only if $\nu < 0$. Although it takes an infinite amount of time for the logarithm to reach $-\infty$ in the new parametrization, it only takes a finite amount of time in the original parametrization.

1.2 Density

We will now drop the bulky notation $\hat{P}_a^x$ and use $P$ or $P^x$. We assume that $B_t$ is a standard Brownian motion and $X_t$ satisfies

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \quad X_0 = x > 0.$$ 

This is valid until time $T = T_0 = \inf \{ t : X_t = 0 \}$. When $a$ is fixed, we will write $L, L^*$ for the generator and its adjoint, that is, the operators

$$Lf(x) = \frac{a}{x} f'(x) + \frac{1}{2} f''(x),$$

$$L^* f(x) = - \left[ \frac{a f(x)}{x} \right]' + \frac{1}{2} f''(x) = \frac{a}{x^2} f(x) - \frac{a}{x} f'(x) + \frac{1}{2} f''(x).$$

For $x, y > 0$, let $q_t(x, y; a)$ denote the density for the Bessel process stopped when it reaches the origin. In other words, if $I \subset (0, \infty)$ is an interval,

$$\mathbb{P}^x \{ T > t ; X_t \in I \} = \int_I q_t(x, y; a) \, dy.$$
In particular,
\[
\int_0^\infty q_t(x, y; a) \, dy = \mathbb{P}_x(T > t) \begin{cases} = 1, & a \geq 1/2, \\ < 1, & a < 1/2. \end{cases}
\]

If \( a = 0 \), this is the density of Brownian motion killed at the origin for which we know that \( q_t(x, y; 0) = q_t(y, x; 0) \). We can give an explicit form of the density by solving either of the “heat equations”
\[
\partial_t q_t(x, y; a) = L_x q_t(x, y; a), \quad \partial_t q_t(x, y; a) = L_y^* q_t(x, y; a),
\]
with appropriate initial and boundary conditions. The subscripts on \( L, L^* \) denote which variable is being differentiated with the other variables remaining fixed. The solution is given in terms of a special functions which arises as a solution to a second order linear ODE. We will leave some of the computations to an appendix, but we include the important facts in the next proposition. We will use the following elementary fact: if a nonnegative random variable \( T \) has density \( f(t) \) and \( r > 0 \), then the density of \( rT \) is \( r^{-1} f(t/r) \).

**Proposition 1.5.** Let \( q_t(x, y; a) \) denote the density of the Bessel process with parameter \( a \) stopped when it reaches the origin. Then for all \( x, y, t, r > 0 \),
\[
q_t(x, y; 1 - a) = (y/x)^{1-2a} q_t(x, y; a),
\]
\[
q_t(x, y; a) = q_t(y, x; a) \, (y/x)^{2a},
\]
\[
q_{t^2}(rx, ry; a) = r^{-1} q_t(x, y; a).
\]

Moreover, if \( a \geq 1/2 \),
\[
q_1(x, y; a) = y^{2a} \exp \left\{ -\frac{x^2 + y^2}{2} \right\} h_a(xy),
\]
where \( h_a \) is the entire function
\[
h_a(z) = \sum_{k=0}^\infty \frac{z^{2k}}{2^{a+2k-\frac{1}{2}} \Gamma(k+a+\frac{1}{2})}.
\]

In particular,
\[
q_1(0, y; a) := q_1(0+, y, a) = \frac{2^{1-a}}{\Gamma(a+\frac{1}{2})} y^{2a} e^{-y^2/2}, \quad a \geq 1/2.
\]

In the proposition we defined \( h_a \) by its the series expansion (13), but it can also be defined as the solution of a particular boundary value problem.

**Lemma 1.6.** \( h_a \) is the unique solution of the second order linear differential equation
\[
z h''(z) + 2a h'(z) - z h(z) = 0.
\]
with \( h(0) = 2^{1-a}/\Gamma(a+\frac{1}{2}), h'(0) = 0 \).

**Proof.** See Proposition 5.1.

**Remarks.**
• By combining (11) and (12) we get for \( a \geq 1/2 \),
\[
q_t(x, y; a) = \frac{1}{\sqrt{t}} q_1 \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a \right) = \frac{y^{2a}}{t^{a+\frac{1}{2}}} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} h_a \left( \frac{xy}{t} \right).
\] (16)

• The density is often written in terms of the modified Bessel function. If \( \nu = a - \frac{1}{2} \), then
\[
I_{\nu}(x) := x^\nu h_{\nu+\frac{1}{2}}(x)
\]
is the modified Bessel function of the first kind of index \( \nu \). This function satisfies the modified Bessel equation
\[
x^2 I''(x) + x I'(x) - [\nu^2 + x^2] I(x) = 0.
\]

• The expressions (12) and (16) hold only for \( a \geq 1/2 \). For \( a < 1/2 \), we can use (9) to get
\[
q_t(x, y; a) = \frac{1}{\sqrt{t}} q_1 \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a \right)
= \frac{1}{x^{2a-1} t^{\frac{1}{2}-a}} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} h_{1-a} \left( \frac{xy}{t} \right).
\]

• In the case \( a = 1/2 \), we can use the fact that the radial part of a two-dimensional Brownian motion is a Bessel process and write
\[
q_1(x, y; 1/2) = \frac{y}{2\pi} \int_0^{2\pi} \exp \left\{ \frac{(x - y \cos \theta)^2 + (y \sin \theta)^2}{2} \right\} d\theta = y e^{-(x^2+y^2)/2} h_{1/2}(xy),
\]
where
\[
h_{1/2}(r) = \frac{1}{2\pi} \int_0^{2\pi} e^r \cos \theta \, d\theta = I_0(r).
\]
The last equality is found by consulting an integral table. Note that \( h_{1/2}(0) = 1, h'_{1/2}(0) = 0 \).

• We can write (14) as
\[
c_d y^{d-1} e^{-y^2/2} \quad d = 2a + 1,
\]
which for positive integer \( d \) can readily be seen to be the density of the radial part of a random vector in \( \mathbb{R}^d \) with density proportional to \( \exp\{-(|x|^2)/2\} \). This makes this the natural guess for all \( d \) for which \( c_d \) can be chosen to make this a probability density, that is, \( a > -1/2 \). For \(-1/2 < a < 1/2 \), we will see that this is the density of the Bessel process reflected (rather than killed) at the origin.

• Given the theorem, we can determine the asymptotics of \( h_a(x) \) as \( x \to \infty \). Note that
\[
\lim_{x \to \infty} q_1(x, x; a) = \frac{1}{\sqrt{2\pi}},
\]
since for large \( x \) and small time, the Bessel process looks like a standard Brownian motion. Therefore,
\[
\lim_{x \to \infty} x^{2a} e^{-x^2} h_a(x^2) = \frac{1}{\sqrt{2\pi}},
\]
and hence
\[ h_a(x) \sim \frac{1}{\sqrt{2\pi}} x^{-a} e^x, \quad x \to \infty. \]

In fact, there is an entire function \( u_a \) with \( u_a(0) = 1/\sqrt{2\pi} \) such that for all \( x > 0 \),
\[ h_a(x) = u(1/x) x^{-a} e^x. \]

See Proposition 5.3 for details. This is equivalent to a well known asymptotic behavior for \( I_\nu \),
\[ I_\nu(x) = x^\nu h_{\nu+\frac{1}{2}}(x) \sim \frac{e^x}{\sqrt{2\pi x}}. \]

- The asymptotics implies that for each \( a \geq 1/2 \), there exist \( 0 < c_1 < c_2 < \infty \) such that for all \( 0 < x, y, t < \infty \),
\[ c_1 \left[ \frac{y}{x} \right]^{2a} \frac{1}{\sqrt{t}} e^{-(x-y)^2/2t} \leq q_t(x, y; a) \leq c_2 \left[ \frac{y}{x} \right]^{2a} \frac{1}{\sqrt{t}} e^{-(x-y)^2/2t}, \quad t \leq xy, \quad (17) \]
\[ c_1 \left[ \frac{y}{\sqrt{t}} \right]^{2a} \frac{1}{\sqrt{t}} e^{-(x^2+y^2)/2t} \leq q_t(x, y; a) \leq c_2 \left[ \frac{y}{\sqrt{t}} \right]^{2a} \frac{1}{\sqrt{t}} e^{-(x^2+y^2)/2t}, \quad t \geq xy, \quad (18) \]

**Proof.** The relation (9) follows from (7). The relation (10) follows from the fact that the exponential term in the compensator \( N_{t,a} \) for a reversed path is the same as for the path. The scaling rule (11) follows from Proposition 1.4 and (8).

For the remainder, we fix \( a \geq 1/2 \) and let \( q_t(x, y) = q_t(x, y; a) \). The heat equations give the equations
\[ \partial_t q_t(x, y; a) = \frac{a}{x} \partial_x q_t(x, y; a) + \frac{1}{2} \partial_{xx} q_t(x, y; a), \quad (19) \]
\[ \partial_t q_t(x, y; a) = \frac{a}{y^2} q_t(x, y; a) - \frac{a}{y} \partial_y q_t(x, y; a) + \frac{1}{2} \partial_{yy} q_t(x, y; a), \quad (20) \]

In our case, direct computation (See Proposition 5.2) shows that if \( h_a \) satisfies (15) and \( q_t \) is defined as in (14), then \( q_t \) satisfies (19) and (20). We need to establish the boundary conditions on \( h_a \). Let
\[ q_t(0, y; a) = \lim_{x \downarrow 0} q_t(x, y; a) \]

Since
\[ 1 = \int_0^\infty q_1(0+, y; a) \, dy = \int_0^\infty h_a(0) y^{2a} e^{-y^2/2} \, dy, \]
we see that
\[ \frac{1}{h_a(0)} = \int_0^\infty y^{2a} e^{-y^2/2} \, dy = \int_0^\infty (2u)^a e^{-u} \, du \frac{du}{\sqrt{2u}} = 2^{a-\frac{1}{2}} \Gamma \left( a + \frac{1}{2} \right). \]

Note that
\[ q_t(x, 1; a) = q_1(0, 1; a) + x h_a'(0) e^{-1/2} + O(x^2), \quad x \downarrow 0. \]
Proposition 1.8. Let $h'_a(0) = 0$, it suffices to show that $q_1(x, 1; 0) = q_1(0, 1; a) + o(x)$. Suppose $0 < r < x$, and we start the Bessel process at $r$. Let $\tau_x$ be the first time that the process reaches $x$. Then by the strong Markov property we have
\[ q_1(r, 1; a) = \int_0^1 q_{1-s}(x; 1; 0) dF(s), \]
where $F$ is the distribution function for $\tau_x$. Using (19), we see that $q_{1-s}(x; 1; 0) = q_1(x, 1; 0) + O(s)$. Therefore,
\[ |q_1(x, 1; a) - q_1(r, 1; a)| \leq c \int_0^1 s \, dF(s) \leq c \mathbb{E}^r[\tau_x]. \]
Using the scaling rule, we can see that $\mathbb{E}^r[\tau_x] = O(x^2).$ 
\[ \square \]

We note that the standard way to solve a heat equation 
\[ \partial_t f(t, x) = L_x f(t, x), \]
with zero boundary conditions and given initial conditions, is to find a complete set of eigenfunctions for $L$ 
\[ L \phi_j(x) = -\lambda_j \phi_j(x), \]
and to write the general solution as 
\[ \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} \phi_j(x). \]
The coefficients $c_j$ are found using the initial condition. This gives a series solution. This could be done for the Bessel process, but we had the advantage of the scaling rule (11), which allows the solution to be given in terms of a single special function $h_a$.

There are other interesting examples that will not have this exact scaling. Some of these, such as the radial Bessel process below, look like the Bessel process near the endpoints. We will use facts about the density of the Bessel process to conclude facts about the density of these other processes. The next proposition gives a useful estimate — it shows that the density of a Bessel process at $y \in (0, \pi/2]$ is comparable to that one would get by killing the process when it reaches level $7\pi/8$. (The numbers $3\pi/4 < 7\pi/8$ can be replaced with other values, of course, but the constants depend on the values chosen. These values will be used in discussion of the radial Bessel process.)

**Proposition 1.7.** Let $q_t(x, y; a)$ be the density of the Bessel process stopped at time $T = T_0 \land T_{7\pi/8}$. If $a \geq 1/2$, then for every $0 < t_1 < t_2 < \infty$, there exist $0 < c_1 < c_2 < \infty$ such that if $t_1 \leq t \leq t_2$ and $0 < x, y \leq 3\pi/4$, then
\[ c_1 y^{2a} \leq q_t(x, y; a) \leq q_t(x, y; a) \leq c_2 y^{2a}. \]

This is an immediate corollary of the following.

**Proposition 1.8.** Let $q_t(x, y; a)$ be the density of the Bessel process stopped at time $T = T_0 \land T_{7\pi/8}$. For every $a \geq 1/2$ and $t_0 > 0$, there exists $0 < c_1 < c_2 < \infty$ such that for all $0 \leq x, y \leq 3\pi/4$ and $t \geq t_0$,
\[ \hat{q}_t(x, y; a) \geq c e^{-\beta t} q_t(x, y; a), \]
\[ c_1 \mathbb{P}^x \{ T > t - t_0; X_t \leq 3\pi/4 \} \leq y^{-2a} \hat{q}_t(x, y; a) \leq c_2 \mathbb{P}^x \{ T > t - t_0 \} . \]
Proof. It suffices to prove this for $t_0$ sufficiently small. Note that the difference $q_t(x, y; a) - \hat{q}_t(x, y; a)$ represents the contribution to $q_t(x, y; a)$ by paths that visit $7\pi/8$ some time before $t$. Therefore, using the strong Markov property, we can see that

$$q_t(x, y; a) - \hat{q}_t(x, y; a) \leq \sup_{0 \leq s \leq t} [q_s(x, y; a) - \hat{q}_s(x, y; a)] \leq \sup_{0 \leq s \leq t} q_s(3\pi/4, y; a).$$

Using the explicit form of $q_t(x, y; a)$ (actually it suffices to use the up-to-constants bounds (17) and (18)), we can find $t' > 0$ such that

$$c_1 y^{2a} \leq \hat{q}_t(x, y; a) \leq c_2 y^{2a}, \quad t' \leq t \leq 2t', \quad 0 < x, y \leq \pi/2.$$

If $s \geq 0$, and $t' \leq t \leq 2t'$,

$$\hat{q}_{s+t}(x, y; a) = \int_0^{7\pi/8} \hat{q}_s(x, z; a) \hat{q}_t(z, y; a) dy \leq c \mathbb{P}^x\{T > s\} \sup_{0 \leq z \leq 7\pi/8} q_t(z, y; a) \leq c \mathbb{P}^x\{T > s\} y^{2a},$$

$$\hat{q}_{s+t}(x, y; a) \geq \int_0^{3\pi/4} \hat{q}_s(x, z; a) \hat{q}_t(z, y; a) dy \geq c \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\} \inf_{0 \leq z \leq 1} \hat{q}_t(z, y; a) \geq c \mathbb{P}^x\{T > s, X_s \leq 3\pi/4\} y^{2a}.$$

\(\square\)

Proposition 1.9. Suppose $X_t$ is a Bessel process with parameter $a < 1/2$ with $X_0 = x$, then the density of $T_0$ is

$$\frac{2^{a-\frac{1}{2}}}{\Gamma\left(\frac{3}{2} - a\right)} x^{1-2a} t^{\frac{3}{2}-\frac{3}{2}} \exp\{-x^2/2t\}, \quad (21)$$

Proof. The distribution of $T_0$ given that $X_0 = x$ is the same as the distribution of $x^2 T_0$ given that $X_0 = 1$. Hence by (8), we may assume $x = 1$. We will consider the density $q_t(1, y; a)$ as $y \downarrow 0$. We write $y_t = y/\sqrt{t}$ Using Proposition 1.5, we see that

$$q_t(1, y; a) = t^{-1/2} q_1(t^{-1/2}, y_t; a) = t^{-1/2} y^{2a} q_1(y_t, t^{-1/2}; a) = t^{-1/2} y q_1(y_t, t^{-1/2}; 1 - a).$$

Therefore,

$$\lim_{y \downarrow 0} y^{-1} q_t(x, y; a) = t^{-1/2} \lim_{z \downarrow 0} q_1(z, t^{-1/2}; 1 - a) = c t^{-1/2} (t^{-1/2})^{2(1-a)} e^{-1/2t} = c t^{a-\frac{3}{2}} e^{-1/2t}.$$

\(\square\)
To find the constant $c$ note that
\[
\int_0^\infty t^{a-\frac{3}{2}} e^{-t/2} dt = \int_0^\infty (2u)^{\frac{3}{2}-a} e^u (u^{-2} du/2) = 2^{\frac{1}{2}-a} \int_0^\infty u^{\frac{1}{2}-a} e^{-u} du = 2^{\frac{1}{2}-a} \Gamma \left( \frac{1}{2} - a \right).
\]

1.3 Geometric time scale

It is often instructive to consider the scaled Bessel process at geometric times (this is sometimes called the Ornstein-Uhlenbeck scaling). For this section we will assume $a \geq 1/2$ although much of what we say is valid for $a < 1/2$ up to the time that the process reaches the origin and for $-1/2 < a < 1/2$ for the reflected process.

Suppose $X_t$ satisfies
\[
 dX_t = \frac{a}{X_t} dt + dB_t,
\]
and let
\[
 Y_t = e^{-t/2} X_t, \quad W_t = \int_0^t e^{-s/2} dB_s.
\]

Note that
\[
 dX_t = \frac{a e^{t} dt}{X_t} + e^{t/2} dW_t = \frac{a}{Y_t} dt + dW_t,
\]
or
\[
 dY_t = \left[ \frac{a}{Y_t} - \frac{Y_t}{2} \right] dt + dW_t. \tag{22}
\]

This process looks like the usual Bessel process near the origin, and it is not hard to see that processes satisfying (22) with $a \geq 1/2$, never reaches the origin. Of course, we knew this fact from the definition of $Y_t$ and the fact that $X_t$ does not reach the origin.

Not as obvious is the fact that $Y_t$ is a recurrent process, in fact, a positive recurrent process with an invariant density. Let us show this in two ways. First let us note that the process $Y_t$ is the same as a process obtained by starting with a Brownian motion $Y_t$ and weighting locally by the function
\[
 \phi(x) = x^a e^{-x^2/4}.
\]

Indeed, using Itô’s formula and the calculations,
\[
 \phi'(x) = \left[ \frac{a}{x} - \frac{x}{2} \right] \phi(x),
\]
\[
 \phi''(x) = \left( \left[ \frac{a}{x} - \frac{x}{2} \right]^2 - \frac{a}{x^2} - \frac{1}{2} \right) \phi(x) = \left[ \frac{a^2 - a}{x^2} - (a + \frac{1}{2}) + \frac{x^2}{4} \right] \phi(x),
\]
we see that if
\[
 M_t = \phi(X_t) \exp \left\{ \int_0^t \left( \frac{a - a^2}{Y_s^2} + (a + \frac{1}{2}) - \frac{Y_s^2}{4} \right) ds \right\},
\]
then $M_t$ is a local martingale satisfying
\[
 dM_t = \left[ \frac{a}{Y_t} - \frac{Y_t}{2} \right] M_t dY_t.
\]
The invariant probability density for this process is given by

\[ f(x) = c \phi(x)^2 = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} x^{2a} e^{-x^2/2}. \]  

(23)

where the constant is chosen to make this a probability density. The equation for an invariant density is \( L^*f(x) = 0 \), where \( L^* \) is the adjoint of the generator

\[ L^*f(x) = -\left( \left( a - \frac{x}{2} \right) f(x) \right)' + \frac{1}{2} f''(x) \]

\[ = \left( \frac{a}{x^2} + \frac{1}{2} \right) f(x) + \left( \frac{x}{2} - \frac{a}{x} \right) f'(x) + \frac{1}{2} f''(x). \]

Direct differentiation of (23) gives

\[ f'(x) = \left( \frac{2a}{x} - x \right) f(x), \]

\[ f''(x) = \left( \left( \frac{2a}{x} - x \right)^2 - \frac{2a}{x^2} - 1 \right) f(x) = \left( \frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right) f(x), \]

so the equation \( L^*f(x) = 0 \) comes down to

\[ \frac{a}{x^2} + \frac{1}{2} + \left( \frac{x}{2} - \frac{a}{x} \right) \left( \frac{2a}{x} - x \right) + \frac{1}{2} \left( \frac{4a^2 - 2a}{x^2} - 4a - 1 + x^2 \right) = 0, \]

which is readily checked.

Let \( \phi_t(x,y) \) denote the density of a process satisfying (22). Then

\[ \phi_t(x,y) = e^{t/2} q_{et}(x, e^{t/2} y; a) = y^{2a} \exp \left\{ -\frac{e^{-t} x^2 + y^2}{2} \right\} h_a \left( e^{-t/2} xy \right). \]

In particular,

\[ \lim_{t \to \infty} \phi_t(x,y) = q_1(0,y;a) = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} y^{2a} e^{-y^2/2} = f(y). \]

### 1.4 Green’s function

We define the Green’s function (with Dirichlet boundary conditions) for the Bessel process by

\[ G(x,y,a) = \int_0^\infty q_t(x,y;a) \, dt. \]

If \( a \geq 1/2 \), then

\[ G(x,y,a) = \int_0^\infty q_t(x,y;a) \, dt = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} \int_0^\infty \frac{y^{2a}}{t^{a+\frac{1}{2}}} \exp \left\{ -\frac{x^2 + y^2}{2t} \right\} h_a \left( \frac{xy}{t} \right), \]

and

\[ G_{1-a}(x,y,a) = \int_0^\infty q_t(x,y;1-a) \, dt = (y/x)^{1-2a} \int_0^\infty q_t(x,y;a) \, dt = G(x,y;a). \]
In particular,
\[ G(1, 1; a) = G(1, 1; 1-a) = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} \int_0^\infty \frac{1}{t^{a+\frac{1}{2}}} e^{-1/t} h_a(1/t) \, dt = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} \int_0^\infty \frac{1}{u^{\frac{1}{2}-a}} e^{-u} h_a(u) \, du. \]

**Proposition 1.10.** For all \( a, x, y \),
\[ G(x, y; a) = G(x/y)^{1-2a} G(x, y; 1-a). \]
\[ G(x, y; a) = (y/x)^{2a} G(y, x; a), \]

If \( a = 1/2 \), \( G(x, y; a) = \infty \) for all \( x, y \). If \( a > 1/2 \), then
\[ G(r, ry; a) = C_a r [1 \wedge y^{1-2a}], \]

where
\[ C_a = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} \int_0^\infty \frac{1}{u^{\frac{1}{2}-a}} e^{-u} h_a(u) \, du < \infty. \]

**Proof.**
\[ G(x, y; 1-a) = \int_0^\infty q_t(x, y; 1-a) \, dt \]
\[ = (y/x)^{1-2a} \int_0^\infty q_t(x, y; a) \, dt = (y/x)^{1-2a} G(x, y; a). \]
\[ G(x, y; a) = \int_0^\infty q_t(x, y; a) \, dt = (y/x)^{2a} \int_0^\infty q_t(y, x; a) \, dt = (y/x)^{2a} G(y, x; a). \]
\[ G(rx, ry; a) = \int_0^\infty q_t(rx, ry; a) \, dt \]
\[ = \int_0^\infty \frac{1}{r} q_{t/r^2}(x, y; a) \, dt \]
\[ = \int_0^\infty r q_s(x, y; a) \, ds = r G(x, y; a). \]

We assume that \( a \geq 1/2 \) and note that the strong Markov property implies that
\[ G(x, 1; a) = \mathbb{P}_x \{ T_1 < \infty \} G(1, 1; a) = [1 \wedge x^{1-2a}] G(1, 1; a). \]
\[ G(1, 1; a) = \int_0^\infty \frac{1}{t^{1-a} + \frac{1}{2}} e^{-1/t} h_{1-a}(1/t) \, dt. \]

We now consider the Bessel process starting at \( x > 1 \) stopped at time 1, and let \( q_t(x, y; a) \) be the corresponding density and
\[ \overline{G}(x, y; a) = \int_0^\infty q_t(x, y; a) \, dt, \]
be the corresponding Green’s function. Then, for \( a \neq 1/2 \),
\[ \overline{G}(x, y; a) = G(x, y; a) - \mathbb{P}_x \{ T_1 < \infty \} G(1, y; a). \]
Proof. Suppose \( X_0 = x > 2 \). Let \( \tau_1 = \inf\{t : |X_t - x| = 1\} \), \( \sigma_1 = \inf\{t \geq \tau_1 : X_t = x\} \) and for \( j > 1 \),

\[
\tau_j = \inf\{t > \sigma_{j-1} : |X_t - x| = 1\}, \quad \sigma_j = \inf\{t > \tau_j : X_t = x\}.
\]

Note that \( \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \cdots \) and if any of the stopping times takes on the value infinity, by definition all the later ones are also equal to infinity. Let \( \tilde{q}_t(y) = \tilde{q}_t(x, y; a) \) denote the density of \( X_t \) killed at time \( \tau \) and

\[
\tilde{G}(x) = \tilde{G}(x; a) = \int_0^\infty \tilde{q}_t(x, y; a) \, dt.
\]

Then

\[
G(x, x; a) = \tilde{G}(x) + \mathbb{P}^x\{\sigma_1 < T_1\} G(x, x; a),
\]

that is,

\[
G(x, x; a) = \frac{\tilde{G}(x)}{\mathbb{P}^x\{T_1 < \sigma_1\}}.
\]

Note that

\[
\mathbb{P}^x\{T_1 < \sigma_1\} = \mathbb{P}^x\{\tau_1 = T_x-1\} \mathbb{P}^x\{T_1 < \sigma_1 \mid \tau_1 = T_x-1\} = \mathbb{P}^x\{T_{x-1} < T_{x+1}\} \mathbb{P}^{x-1}\{T_1 < T_x\}
\]

We have

\[
\mathbb{P}^{x-1}\{T_1 < T_x\} = \frac{x^{1-2a} - (x-1)^{1-2a}}{(x+1)^{1-2a} - (x-1)^{1-2a}} = \frac{1 - (1-x^{-1})^{1-2a}}{(1+x^{-1})^{1-2a} - (1-x^{-1})^{1-2a}} = \frac{1}{2} + O(x^{-1}).
\]

(The \( O(\cdot) \) error terms have an implicit \( a \) dependence.)

\[
\mathbb{P}^{x-1}\{T_1 < T_x\} = \frac{x^{1-2a} - (x-1)^{1-2a}}{1 - (1-x^{-1})^{1-2a}} = \frac{1 - (1-x^{-1})^{1-2a}}{1 - x^{2a-1}}
\]

\( \square \)

1.5 Another viewpoint

The Bessel equation is

\[
dX_t = \frac{a}{X_t} \, dt + dB_t,
\]

where \( B_t \) is a Brownian motion and \( a \in \mathbb{R} \). In this section we fix \( a \) and the Brownian motion \( B_t \) but vary the initial condition \( x \). In other words, let \( X_t^x \) be the solution to

\[
dX_t^x = \frac{a}{X_t^x} \, dt + dB_t, \quad X_0 = x,
\]

(24)
which is valid until time $T_0^x = \inf\{ t : X_t^x = 0 \}$. The collection $\{X_t^x\}$ is an example of a stochastic flow. If $t < T_0^x$, we can write

$$X_t^x = x + B_t + \int_0^t \frac{a}{X_s^x} \, ds.$$ 

If $x < y$, then

$$X_t^y - X_t^x = y - x + \int_0^t \left[ \frac{a}{X_s^y} - \frac{a}{X_s^x} \right] \, ds = y - x - \int_0^t \frac{a(X_s^y - X_s^x)}{X_s^x X_s^y} \, ds.$$ 

In other words, if $t < T_0^x \wedge T_0^y$, then $X_t^y - X_t^x$ is differentiable in $t$ with

$$\partial_t [X_t^y - X_t^x] = -[X_t^y - X_t^x] \frac{a}{X_t^x X_t^y},$$

which implies that

$$X_t^y - X_t^x = (y - x) \exp \left\{ -a \int_0^t \frac{ds}{X_s^x X_s^y} \right\}.$$ 

From this we see that $X_t^y > X_t^x$ for all $t < T_0^x$. In particular, $T_0^x \leq T_0^y$ so these equations hold for $t < T_0^x$. Although $X_t^x < X_t^y$ for all $t > T_0^x$, as we will see, it is possible for $T_0^x = T_0^y$.

**Proposition 1.11.** Suppose $0 < x < y < \infty$ and $X_t^x, X_t^y$ satisfy (24) with $X_0^x = x, X_0^y = y$.

1. If $a \geq 1/2$, then $\mathbb{P}\{T_0^x = \infty \text{ for all } x\} = 1$.

2. If $1/4 < a < 1/2$ and $x < y$, then

$$\mathbb{P}\{T_0^x = T_0^y \} > 0.$$ 

3. If $a \leq 1/4$, then with probability one for all $x < y$, $T_0^x = T_0^y$.

**Proof.** If $a \geq 1/2$, then Proposition 1.1 implies that for each $x$, $\mathbb{P}\{T_0^x = \infty \} = 1$ and hence $\mathbb{P}\{T_0^x = \infty \text{ for all rational } x\} = 1$. Since $T_0^x \leq T_0^y$ for $x \leq y$, we get the first assertion.

For the remainder we assume that $a < 1/2$. Let us write $X_t = x_t^x, Y_t = y_t^y, T^x = T_0^x, T^y = T_0^y$. Let $h(x, y) = h(x, y; a) = \mathbb{P}\{T^x = T^y\}$. By scaling we see that $h(x, y) = h(x/y) = h(x/y, 1)$. Hence, we may assume $y = 1$. We claim that $h(0+, 1) = 0$. Indeed, $T^y$ has the same distribution as $r^2 T^1$ and hence for every $\epsilon > 0$ we can find $r, \delta$ such that $\mathbb{P}\{T^y \geq \delta \} \leq \epsilon/2, \mathbb{P}\{T^1 \leq \delta \} \leq \epsilon/2$, and hence $\mathbb{P}\{T^1 = T^y\} \leq \epsilon$.

Let $u = \sup_{t < T^2} Y_t / x_t$. We claim that

$$\mathbb{P}\{T^x < T^1; u < \infty\} = 0.$$ 

$$\mathbb{P}\{T^x = T^1; u = \infty\} = 0.$$ 

The first equality is immediate; indeed, if $Y_t \leq c X_t$ for all $t$, then $T^1 = T^x$. For the second equality, let $\sigma_N = \inf\{ t : Y_t / X_t = N \}$. Then,

$$\mathbb{P}\{u \geq N; T^1 = T^x\} \leq \mathbb{P}\{T^1 = T^x \mid \sigma_N < \infty\} = h(1/N) \rightarrow 0, \quad N \rightarrow \infty.$$
Let

\[ L_t = \log \left( \frac{Y_t}{X_t} - 1 \right) = \log(Y_t - X_t) - \log X_t. \]

Note that

\[ d\log(Y_t - X_t) = -\frac{a}{X_t Y_t} dt, \]

\[ d\log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} dt = \frac{a - \frac{1}{2}}{X_t} dt + \frac{1}{X_t} dB_t, \]

and hence

\[ dL_t = \left[ \frac{1}{2} - a - aX_t \sigma(t) \right] dt - \frac{1}{X_t} dB_t. \]

In order to understand this equation, let us change the time parametrization so that the Brownian term has variance one. More precisely, define \( \sigma(t), W_t \) by

\[ \int_0^{\sigma(t)} \frac{ds}{X_s^2} = t, \quad W_t = -\int_0^{\sigma(t)} \frac{1}{X_s} dB_s. \]

Then \( W_t \) is a standard Brownian motion and \( \tilde{L}_t = L_{\sigma(t)} \) satisfies

\[ d\tilde{L}_t = \left[ \frac{1}{2} - a - \frac{aX_{\sigma(t)}}{Y_{\sigma(t)}} \right] dt + dW_t = \left[ \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} \right] dt + dW_t. \]

For any \( a < 1/2 \), there exists \( u > 0 \) and \( K < \infty \) such that if \( \tilde{L}_t \geq K \), then

\[ \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} > u. \]

Hence, by comparison with a Brownian motion with drift \( u \), we can see that if \( \tilde{L}_t \geq K + 1 \), then with positive probability, \( \tilde{L}_t \to \infty \) and hence \( Y_t/X_t \to \infty \). Hence starting at any initial value \( \tilde{L}_t = l \) there is a positive probability (depending on \( l \)) that \( \tilde{L}_t \to \infty \).

If \( a > 1/4 \), then there exists \( u > 0 \) and \( K < \infty \) such that if \( \tilde{L}_t \leq -K \), then

\[ \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} < -u. \]

Hence by comparison with a Brownian motion with drift \( -u \), we can see that if \( \tilde{L}_t \leq -(K + 1) \), then with positive probability, \( \tilde{L}_t \to -\infty \). Hence starting at any initial value \( \tilde{L}_t = l \) there is a positive probability (depending on \( l \)) that \( \tilde{L}_t \to -\infty \).

If \( a \leq 1/4 \), then

\[ \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} > 0, \]
and hence by comparison with driftless Brownian motion, we see that
\[ \limsup \tilde{L}_t \to \infty. \]  
(25)

But as mentioned before, if \( \tilde{L}_t \geq K + 1 \) for some \( K \) there is a positive probability that \( \tilde{L}_t \to \infty \).

Since (25) shows that we get an “infinite number of tries” we see that \( \tilde{L}_t \to \infty \) with probability one.

\[ \square \]

### 1.6 Functionals of Brownian motion

In the analysis of the Schramm-Loewner evolution, one often has to evaluate or estimate expectations of certain functionals of Brownian motion or the Bessel process. One of the most important functionals is the one that arises as the compensator in the change-of-measure formulas for the Bessel process. These can often be calculated easily from the fact that they arise this way. In this section, we write \( \mathbb{E} \) for Brownian expectations and \( \hat{\mathbb{E}}_a \) for the corresponding expectation with respect to the Bessel process with parameter \( a \).

Suppose \( X_t \) is a Brownian motion with \( X_t = x > 0 \) and let
\[ J_t = \int_0^t \frac{ds}{X_s^2}, \quad K_t = e^{-J_t} = \exp \left\{ -\int_0^t \frac{ds}{X_s^2} \right\}, \]
which are positive and finite for \( 0 < t < T_0 \). Let \( I_t \) denote the indicator function of the event \( \{T_0 > t\} \). The local martingale from (1) as
\[ N_t,a = \left( \frac{X_t}{X_0} \right)^a K_t^{\lambda_a}, \quad \text{where} \quad \lambda_a = \frac{a(a - 1)}{2}. \]

**Proposition 1.12.** If \( \lambda \geq -1/8 \) and
\[ a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8\lambda} \geq \frac{1}{2}, \]
is the larger root of the polynomial \( a^2 - a - 2\lambda \), then
\[ \mathbb{E}^x[K_t^\lambda I_t] = x^a \hat{\mathbb{E}}_a^x \left[ X_t^{-a} I_t \right] = x^a \int_0^\infty q_t(x, y; a) y^{-a} dy. \]

In particular, as \( t \to \infty \),
\[ \mathbb{E}^1[K_t^\lambda I_t] = t^{-\frac{a}{2}} \frac{\Gamma(\frac{a}{2} + \frac{1}{2})}{2^{a/2} \Gamma(a + \frac{1}{2})} [1 + O(t^{-1})]. \]

**Proof.**
\[
\mathbb{E}^x \left[ K_t^\lambda I_t \right] = x^a \mathbb{E}^x \left[ N_{t,a} X_t^{-a} I_t \right]
= x^a \hat{\mathbb{E}}_a^x \left[ X_t^{-a} I_t \right]
= x^a \int_0^\infty q_t(x, y; a) y^{-a} dy.
\]
Since \( q_t(x,y;a) \asymp y^{2a} \) as \( y \downarrow 0 \), the integral is finite. Note that for \( a \geq 1/2 \),

\[
\int_0^\infty q_t(1,y,a) y^{-a} \, dy = t^{-1/2} \int_0^\infty q_1(1/\sqrt{t}, y/\sqrt{t}; a) y^{-a} \, dy \\
= t^{-(a+1/2)} e^{-1/2t} \int_0^\infty y^a e^{-y^2/2t} h \left( \frac{y}{t} \right) \, dy \\
= t^{-(a+1/2)} e^{-1/2t} \int_0^\infty (\sqrt{t} u)^a e^{-u^2/2} h \left( \frac{u}{\sqrt{t}} \right) \sqrt{t} \, du \\
= t^{-(a+1/2)} \left[ 1 + O(t^{-1}) \right] \int_0^\infty \frac{2^{1-a} u^{2a}}{\Gamma(a+1/2)} u^a e^{-u^2/2} \left[ 1 + O(u^2/t) \right] \, du \\
= t^{-(a+1/2)} \frac{\Gamma(a/2 + 1/2)}{2a^2 \Gamma(a + 1/2)} \left[ 1 + O(t^{-1}) \right].
\]

Proposition 1.13. Suppose \( b \in \mathbb{R} \) and \( \lambda + \lambda_b \geq -1/8 \). (26)

Let

\[
a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8(\lambda + \lambda_b)} \geq \frac{1}{2}.
\]

and assume that \( a + b > -1 \). Then,

\[
\hat{E}_b^x [K_t^\lambda I_t] = x^{a-b} \hat{E}_a^x \left[ X_t^{b-a} \right] = x^{a-b} \int_0^\infty y^{b-a} q_t(x,y;a) \, dy.
\]

Note that if \( b > -3/2 \), then the condition \( a + b > -1 \) is automatically satisfied. If \( b \leq -3/2 \), then the condition \( a + b > -1 \) can be considered a stronger condition on \( \lambda \) than (26). If \( b \leq -3/2 \), then the condition on \( \lambda \) is

\[
\lambda > 1 + 2b.
\]

Proof.

\[
\hat{E}_b^x [K_t^\lambda I_t] = x^{a-b} \hat{E}_a^x \left[ K_t^{\lambda + \lambda} X_t^b \right] \\
= x^{a-b} \hat{E}_a^x \left[ M_t,a X_t^{b-a} \right] \\
= x^{a-b} \hat{E}_a^x \left[ X_t^{b-a} \right] \\
= x^{a-b} \int_0^\infty y^{b-a} q_t(x,y;a) \, dy.
\]

The condition \( a + b > -1 \) is needed to make the integral finite.

Proposition 1.14. Let \( \lambda > -1/8 \) and let

\[
a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8\lambda} \leq \frac{1}{2},
\]

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be the smaller root of the polynomial $a^2 - a - 2\lambda$. Then if $y < x$,

$$\mathbb{E}^x[K^{\lambda}_{T_y}] = (x/y)^a.$$

Proof. Let $n > x$ and let $\tau_n = T_y \land T_n$. Note that

$$\mathbb{E}^x[K^{\lambda}_{\tau_n}] = x^a \mathbb{E}^{-a} N_{\tau_n} X_{\tau_n}^{-a} = x^a \mathbb{E}^{-a} X_{\tau_n}^{-a}$$

and similarly,

$$\mathbb{E}^x[K^{\lambda}_{\tau_n}; T_y < T_n] = x^a \mathbb{E}^{-a} X_{T_y}^{-a}; T_y < T_n = (x/y)^a \mathbb{E}^{-a} \{T_y < T_n\},$$

$$\mathbb{E}^x[K^{\lambda}_{\tau_n}; T_y > T_n] = x^a \mathbb{E}^{-a} X_{T_n}^{-a}; T_y > T_n = (x/n)^a \mathbb{E}^{-a} \{T_y > T_n\}.$$ Using (4), we see that

$$\lim_{n \to \infty} \mathbb{E}^x[K^{\lambda}_{T_y}; T_y > T_n] = \lim_{n \to \infty} (x/n)^a \mathbb{E}^{-a} \{T_y > T_n\} = \lim_{n \to \infty} (x/n)^a \frac{x^{1-2a} - y^{1-2a}}{n^{1-2a} - y^{1-2a}} = 0.$$ Therefore,

$$\mathbb{E}^x[K^{\lambda}_{T_y}] = \lim_{n \to \infty} \mathbb{E}^x[K^{\lambda}_{\tau_n}; T_y < T_n] = (x/y)^a \lim_{n \to \infty} \mathbb{E}^{-a} \{T_y < T_n\} = (x/y)^a.$$ The first equality uses the monotone convergence theorem and the last equality uses $a < 1/2$.

\[\square\]

**Proposition 1.15.** Suppose $b \in \mathbb{R}$ and $\lambda + \lambda_b \geq -1/8$. Let

$$a = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 8(\lambda + \lambda_b)} \leq \frac{1}{2},$$

the smaller root of the polynomial $a^2 - a - 2(\lambda + \lambda_b)$. Then if $x < y$,

$$\hat{\mathbb{E}}_b[K^{\lambda}_{T_y}; T_y < \infty] = (x/y)^{a-b}.$$ A special case of this proposition occurs when $b \geq 1/2$, $\lambda = 0$. Then $a = 1 - b$ and

$$\hat{\mathbb{E}}_b[K^{\lambda}_{T_y}; T_y < \infty] = \mathbb{P}_b(T_y < \infty) = (x/y)^{a-b} = (y/x)^{2b-1},$$

which is (5).

Proof.

$$\hat{\mathbb{E}}_b[K^{\lambda}_{T_y}; T_y < \infty] = x^{-b} \mathbb{E}^x[K^{\lambda}_{T_y} K^{\lambda_b}_{T_y} X_{T_y}^b; T_y < \infty] = (y/x)^b \mathbb{E}^x[K^{\lambda_b}_{T_y}] = (x/y)^{a-b}.$$ \[\square\]
It is convenient to view the random variable $J_t$ on geometric scales. Let us assume that $X_0 = 1$ and let
$$\hat{J}_t = J_{e^{-t}}.$$Then if $n$ is a positive integer, we can write
$$\hat{J}_n = \sum_{j=1}^{n} [\hat{J}_j - \hat{J}_{j-1}].$$Scaling shows that the random variables $\hat{J}_j - \hat{J}_{j-1}$ are independent and identically distributed.

More generally, we see that $\hat{J}_t$ is an increasing Lévy process, that is, it has independent, stationary increments. We will assume that $a \leq 1/2$ and write $a = \frac{1}{2} - b$ with $b = -\nu \geq 0$. Let $\Psi_a$ denote the characteristic exponent for this Lévy process, which is defined by
$$E[e^{i\lambda \hat{J}_t}] = \exp\{t\Psi_a(\lambda)\}.$$It turns out that $\nu = a - \frac{1}{2}$ is a nicer parametrization for the next proposition so we will use it.

**Proposition 1.16.** Suppose $b > 0$, $X_t$ satisfies
$$dX_t = \frac{1}{2} - b X_t \, dt + dB_t, \quad X_0 = 1.$$Then if $\lambda \in \mathbb{R}$,
$$E^x \left[ \exp \left\{ i\lambda \int_0^{T_y} \frac{ds}{X_s^2} \right\} \right] = y^{-r},$$where
$$r = b - \sqrt{b^2 - 2i\lambda}.$$is the root of the polynomial $r^2 - 2br + 2i\lambda$ with smaller real part. In other words, if $a \leq 1/2$
$$\Psi_{\frac{1}{2} - b}(\lambda) = b - \sqrt{b^2 - 2i\lambda},$$where the square root denotes the root with positive real part. In particular,
$$E[\hat{J}_t] = \frac{t}{b}.$$**Proof.** We will assume $\lambda \neq 0$ since the $\lambda = 0$ case is trivial. If $r_-, r_+$ denote the two roots of the polynomial ordered by their real part, then $\text{Re}(r_-) < b$, $\text{Re}(r_+) > 2b$; we have chosen $r = r_-$. Let $\tau_k = T_y \wedge T_k$. Using Itô’s formula, we see that $M_{\wedge \tau_k}$ is a bounded martingale where
$$M_t = \exp\left\{ i\lambda \int_0^t \frac{ds}{X_s^2} \right\} X_t^r.$$Therefore,
$$E[M_{\tau_k}] = 1.$$If $b > 0$,
$$E[|M_{\tau_k}|; T_k < T_y] \leq k^{\text{Re}(r)} P\{T_k < T_y\} \leq c(y) k^{\text{Re}(r)} k^{2a-1} = c(r) k^{\text{Re}(r)} k^{-2b},$$
and hence,
\[
\lim_{k \to \infty} E[|M_{\tau_k}|; T_k < T_y] = 0.
\]
(One may note that if \( \lambda \neq 0 \) and used \( r_+ \), then \( \text{Re}(r_+) > 2b \) and this term does not go to zero.) Similarly, if \( b = 0 \),
\[
\lim_{k \to \infty} E[|M_{\tau_k}|; T_k < T_y] = 0.
\]
Therefore,
\[
1 = \lim_{k \to \infty} E[|M_{\tau_k}|; T_k > T_y] = E[M_{T_y}] = y^r E \left[ \exp \left( i\lambda \int_0^{T_y} ds \frac{X_s^2}{X_0^2} \right) \right].
\]
The last assertion follows by differentiating the characteristic function of \( \hat{J}_t \) at the origin. \( \square \)

The moment generating function case is similar but we have to be a little more careful because the martingale is not bounded for \( \lambda > 0 \).

**Proposition 1.17.** Suppose \( b > 0, \ X_t \) satisfies
\[
dX_t = \frac{1}{2} - b X_t \ dt + dB_t, \quad X_0 = 1,
\]
and \( 2\lambda < b^2 \). Then, if \( y < x \),
\[
E^x \left[ \exp \left( \lambda \int_0^{T_y} ds \frac{X_s^2}{X_0^2} \right) \right] = y^{-r},
\]
where
\[
r = b - \sqrt{b^2 - 2\lambda}.
\]
is the smaller root of the polynomial \( r^2 - 2br + 2\lambda \).

**Proof.** By scaling, it suffices to prove this result when \( y = 1 \). Let \( \tau = T_1 \) and let
\[
K_t = \exp \left\{ \lambda \int_0^t ds \frac{X_s^2}{X_0^2} \right\}, \quad M_t = K_t (X_t/X_0)^r.
\]
By Itô’s formula, we can see that \( M_t \) is a local martingale for \( t < \tau \) satisfying
\[
dM_t = \frac{r}{X_t} M_t \ dt, \quad M_0 = 1.
\]
If we use Girsanov and weight by the local martingale \( M_t \), we see that
\[
dX_t = \frac{r + \nu + \frac{1}{2}}{X_t} dt + dW_t, \quad t < \tau
\]
where \( W_t \) is a standard Brownian motion in the new measure which we denote by \( \hat{P} \) with expectations \( \hat{E} \). Since \( r + \nu < 0 \), then with probability one in the new measure \( \hat{P} \{ \tau < \infty \} = 1 \), and hence
\[
E^x [K_\tau; \tau < \infty] = x^r E^x [M_\tau; \tau < \infty] = x^r \hat{E}^x [1\{ \tau < \infty \}] = x^r.
\]
\( \square \)
We can do some “multifractal” or “large deviation” analysis. We start with the moment generating function calculation

\[ \mathbb{E}[e^{\lambda \hat{J}_t}] = e^{k\xi(\lambda)}, \]

where

\[ \xi_b(\lambda) = b - \sqrt{b^2 - 2\lambda}, \quad \xi'(\lambda) = \frac{1}{\sqrt{b^2 - 2\lambda}}, \quad \xi''(\lambda) = \frac{1}{(b^2 - 2\lambda)^{3/2}}. \]

This is valid provided that \( \lambda < b^2/2 \). Recall that \( \mathbb{E}[\hat{J}_t] = t/b \). If \( \theta > 1/b \), then

\[ \mathbb{P}\{\hat{J}_t \geq \theta t\} \leq e^{-\lambda \theta t} \mathbb{E}[e^{\lambda \hat{J}_t}] = \exp\{t[\xi(\lambda) - \lambda \theta]\}, \]

This estimate is most useful for the value \( \lambda \) that minimizes the right-hand side, that is, at the value \( \lambda_\theta \) satisfying \( \xi'(\lambda_\theta) = \theta \), that is,

\[ \lambda_\theta = \frac{1}{2} \left[ b^2 - \theta^{-2} \right], \quad \xi(\lambda_\theta) = b - \frac{1}{\theta}. \]

Therefore,

\[ \mathbb{P}\{\hat{J}_t \geq \theta t\} \leq \exp\{t\rho(\theta)\}, \quad \text{where} \quad \rho(\theta) = b - \frac{1}{2\theta} - \frac{\theta b^2}{2} \]

While this is only an inequality, one can show (using the fact that \( \xi \) is \( C^2 \) and strictly concave in a neighborhood of \( \lambda_\theta \)),

\[ \mathbb{P}\{\hat{J}_t \geq \theta t\} \propto \mathbb{P}\{\theta t \leq \hat{J}_t \leq \theta t + 1\} \propto t^{-1/2} \exp\{t\rho(\theta)\}. \]

Similarly, if \( \theta < 1/b \),

\[ \mathbb{P}\{\hat{J}_t \leq \theta t\} \leq e^{\lambda \theta t} \mathbb{E}[e^{-\lambda \hat{J}_t}] = \exp\{k[\xi(-\lambda) + \lambda \theta]\}, \]

The right-hand side is minimized when \( \xi'(-\lambda) = \theta \), that is, when

\[ \lambda_\theta = \frac{1}{2} \left[ \theta^{-2} - b^2 \right], \quad \xi(-\lambda_\theta) = b - \sqrt{2b^2 - \theta^{-2}} \]

\[ \mathbb{P}\{\hat{J}_t \leq \theta t\} \leq \exp\{t\rho(\theta)\}, \quad \text{where} \quad \rho(\theta) = \frac{1}{2\theta} - \frac{\theta b^2}{2} + b - \sqrt{2b^2 - \theta^{-2}}. \]

### 1.7 The reflected process for \(-1/2 < a < 1/2\)

The Bessel process can be defined with reflection at the origin in this range. Before defining the process formally, let us describe some of its properties. In this section, we assume that \(-1/2 < a < 1/2\).

- The reflected Bessel process \( X_t \) is a strong Markov process with continuous paths taking values in \([0, \infty)\). It has transition density

\[ \psi_t(x, y; a) = \frac{1}{\sqrt{t}} \psi_1 \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}; a \right) = \frac{y^{2a}}{1 + \frac{2}{\mu+2}} \exp\left\{ - \frac{x^2 + y^2}{2t} \right\} h_\alpha \left( \frac{xy}{t} \right). \]
Note that this is exactly the same formula as for $q_t(x, y; a)$ when $a \geq 1/2$. We use a new notation in order not to conflict with our use of $q_t(x, y; a)$ for the density of the Bessel process killed when it reaches the origin. We have already done the calculations that show that

$$\partial_t \psi_t(x, y; a) = L_x \psi_t(x, y; a),$$

and

$$\partial_x \psi_t(x, y; a) \big|_{x=0} = 0.$$

However, if $a \leq 1/2$, it is not the case that

$$\partial_y \psi_t(x, y; a) \big|_{y=0} = 0.$$

In fact, for $a < 1/2$, the derivative does not exist at $y = 0$.

- Note that

$$\psi_1(0, y; a) = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} y^{2a} e^{-y^2/2},$$

and hence

$$\psi_t(0, y; a) = t^{-1/2} \psi_1(0, y/\sqrt{t}; a) = \frac{2^{1-a}}{\Gamma(a + \frac{1}{2})} t^{-\frac{1}{2}-a} y^{2a} e^{-y^2/2t}. \quad (28)$$

- We have the time reversal formula for $x, y > 0$.

$$\psi_t(x, y; a) = (y/x)^{2a} \psi_t(y, z; a). \quad (29)$$

Because of the singularity of the density at the origin we do not write $\psi_t(x, 0; a)$.

- A calculation (see Proposition 5.4) shows that for $x > 0$,

$$\int_0^\infty \psi_t(x, y; a) \, dy = 1. \quad (30)$$

We can view the process as being “reflected at 0” in a way so that the total mass on $(0, \infty)$ is always 1.

- Another calculation (see Proposition 5.6) shows that the $\psi_t$ give transition probabilities for a Markov chain on $[0, \infty)$.

$$\psi_{t+s}(x, y; a) = \int_0^\infty \psi_t(x, z; a) \psi_s(z, y; a) \, dz.$$

Note that this calculation only needs to consider values of $\psi_t(x, y; z)$ with $y > 0$.

- With probability one, the amount of time spent at the origin is zero, that is,

$$\int_0^\infty 1\{X_t = 0\} \, dt = 0.$$

This follows from (30) which implies that

$$\int_0^k 1\{X_t > 0\} \, dt = \int_0^k \int_0^\infty \psi_t(x, y; a) \, dy \, dt = k.$$
For each \( t, x > 0 \), if \( \sigma = \inf\{s \geq t : X_s = 0\} \), the distribution of \( X_s, t \leq s \leq \sigma \), given \( X_t \), is that of a Bessel process with parameter \( \alpha \) starting at \( X_t \) stopped when it reaches the origin.

The process satisfies the Brownian scaling rule: if \( X_t \) is the reflected Bessel process started at \( x \) and \( r > 0 \), then \( Y_t = r^{-1} X_{rt} \) is a reflected Bessel process started at \( x/r \).

To construct the process, we can first restrict to dyadic rational times and use standard methods to show the existence of such a process. With probability one, this process is not at the origin for any dyadic rational \( t \) (except maybe the starting point). Then, as for Brownian motion, one can show that with probability one, the paths are uniformly continuous on every compact interval and hence can be extended to \( t \in [0, \infty) \) by continuity. (If one is away from the origin, one can argue continuity as for Brownian motion. If one is “stuck” near the origin, then the path is continuous since it is near zero.) The continuous extensions do hit the origin although at a measure zero set of times.

Here we explain why we need the condition \( a > -1/2 \). Assume that we have such a process for \( a < 1/2 \). Let \( e(x) = e(x; a) = \mathbb{E}^x[T_x] \) and \( j(x) = j(x; a) = \mathbb{E}^x[T_{2x}] \). We first note that \( e(1) < \infty \); indeed, it is obvious that there exists \( \delta > 0, s < \infty \) such that \( \mathbb{P}^0\{T_1 < s\} \geq \delta \) and hence \( \mathbb{P}^x\{T_x < s\} \geq \delta \) for every \( 0 \leq x < 1 \). By iterating this, we see that \( \mathbb{P}^0\{T_1 \geq ns\} \leq (1 - \delta)^n \), and hence \( \mathbb{E}^0[T_1] < \infty \). The scaling rule implies that \( e(2x) = 4e(x), j(2x) = 4j(x) \). Also, the Markov property implies that

\[
e(2x) = e(x) + j(x) + \mathbb{P}^x\{T_0 < T_{2x}\} e(2x),
\]

which gives

\[
4e(x) = e(2x) = \frac{e(x) + j(x)}{\mathbb{P}^x\{T_0 \geq T_{2x}\}}.
\]

By (6), we know that

\[
\mathbb{P}^x\{T_0 \geq T_{2x}\} = \min\{2^{2a-1}, 1\}.
\]

If \( a \leq -1/2 \), then \( \mathbb{P}^x\{T_0 \geq T_{2x}\} \leq 1/4 \), which is a contradiction since \( j(x) > 0 \).

There are several ways to construct this process. In the next section, we will do one which constructs the process in terms of excursions. In this section, we will not worry about the construction, but rather we will give the properties. We will write the measure as \( \mathbb{P}^x_\alpha \) (this is the same notation as for the Bessel process killed at the origin — indeed, it is the same process just continued onward in time).

If \( x > 0 \), the scaling rule will imply

\[
\psi_t(x, y; a) = t^{-1/2} \psi_1(x/\sqrt{t}, y/\sqrt{t}; a),
\]

so we need only give \( \psi_1(x, y; a) \). What we will show now is that if we assume that (28) holds and gives \( \psi_t(0, y; a) \), then the value \( \psi_t(x, y; a) \) must hold for all \( x \). We will use \( T_0 \), the first time that the process reaches the origin and write

\[
\psi_1(x, y; a) = \tilde{\psi}_1(x, y; a) + q_1(x, y; a)
= \tilde{\psi}_1(x, y; a) + (y/x)^{2a-1} q_1(x, y; 1 - a)
\]

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where
\[
\tilde{\psi}_1(x,y; a) = \int_0^1 \psi_{1-s}(0,y; a) \, dP^x \{ T_0 = s \}.
\] (31)

The term \( q_t(x,y; a) \) gives the contribution from paths that do not visit the origin before time 1, and \( \tilde{\psi}_1(x,y; a) \) gives the contribution of those that do visit. The next proposition is a calculation. The purpose is to show that our formula for \( \psi_1(x,y; a) \) must be valid for \( x > 0 \) provided that it is true for \( x = 0 \).

**Proposition 1.18.** If \(-\frac{1}{2} < a < \frac{1}{2}\), then
\[
\tilde{\psi}_1(x,y; a) = y^{2a} e^{-(x^2+y^2)/2} \left[ h_a(xy) - (xy)^{1-2a} h_{1-a}(xy) \right].
\]

**Proof.** Using (21), we see that
\[
dP^x \{ T_0 = s \} = \frac{2a - \frac{1}{2}}{\Gamma(\frac{1}{2} - a)} x^{1-2a} s^{a - \frac{3}{2}} \exp\{-x^2/2s\},
\]
and hence
\[
\tilde{\psi}_1(x,y; a) = \frac{1}{\Gamma(\frac{1}{2} - a) \Gamma(\frac{1}{2} + a)} \int_0^1 s^{a - \frac{3}{2}} (1 - s)^{-a - \frac{1}{2}} x^{1-2a} y^{2a} e^{-x^2/2s} e^{-y^2/2(1-s)} \, ds
\]
\[
= \frac{x^{1-2a} y^{2a} e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2} - a) \Gamma(\frac{1}{2} + a)} \int_0^1 \left( \frac{1-s}{s} \right)^{-a - \frac{1}{2}} \exp\left\{-\frac{x^2}{2} \frac{1-s}{s}\right\} \exp\left\{-\frac{y^2}{2} \frac{s}{1-s}\right\} s^{-2} \, ds
\]
\[
= \frac{x^{1-2a} y^{2a} e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2} - a) \Gamma(\frac{1}{2} + a)} \int_0^\infty u^{-a - \frac{1}{2}} \exp\left\{-\frac{x^2 u}{2}\right\} \exp\left\{-\frac{y^2}{2u}\right\} \, du
\]
\[
= \frac{x^{1-2a} y^{2a} e^{-(x^2+y^2)/2}}{\Gamma(\frac{1}{2} - a) \Gamma(\frac{1}{2} + a)} \int_0^\infty r^{-a - \frac{1}{2}} \exp\left\{-\frac{xyr}{2}\right\} \exp\left\{-\frac{xy}{2r}\right\} \, dr
\]

Using equation 2.3.16 #1 of Prudnikov-Brychkov-Marychev, Vol. 1 and a well known identity for the Gamma function, we see that
\[
\int_0^\infty r^{-\nu - 1} e^{-rz/2} e^{-z/2r} \, dr = \frac{\pi}{\sin(-\pi \nu)} \left[ I_{\nu}(z) - I_{-\nu}(z) \right]
\]
\[
= \frac{1}{\Gamma(-\nu) \Gamma(1+\nu)} \left[ I_{-\nu}(z) - I_{\nu}(z) \right]
\]
\[
= \frac{1}{\Gamma(a - \frac{1}{2}) \Gamma(\frac{1}{2} - a)} \left[ z^{a-\frac{1}{2}} h_a(z) - z^{\frac{1}{2}-a} h_{1-a}(z) \right].
\]
\[\Box\]

From the expression, we see that \( \tilde{\psi}_1(x,y; a) \) is a decreasing function of \( x \). Indeed, we could give another argument for this fact using coupling. Let us start two independent copies of the process at \( x_1 < x_2 \). We let the processes run until they collide at which time they run together. By the time the process starting at \( x_2 \) has reached the origin, the processes must have collided.
For the remainder of this subsection we assume that \(-1/2 < a < 1/2\). We will describe the measure on paths that we will write as \(\hat{P}_{x}^{a}\). Let \(\psi_{t}(x, y; a), x \geq 0, y > 0, t > 0\) denote the transition probability for the process. We will derive a formula for this using properties we expect the process to have. First, we the reversibility rule (10) will holds: if \(x, y > 0\), then
\[
\psi_{t}(x, y; a) = \left(\frac{y}{x}\right)^{2a} \psi_{t}(y, x; a).
\]
In particular, we expect that \(\psi_{t}(1, x; a) \asymp x^{2a}\) as \(x \downarrow 0\). Suppose that \(X_{t} = 0\) for some \(1 - \epsilon \leq t \leq 1\). By Brownian scaling, we would expect the maximum value of \(X_{t}\) on that interval to be of order \(\sqrt{\epsilon}\) and hence
\[
\int_{1-\epsilon}^{1} 1\{|X_{t}| \leq \sqrt{\epsilon}\} dt \asymp \epsilon.
\]
But,
\[
\mathbb{E}\left[ \int_{1-\epsilon}^{1} 1\{|X_{t}| \leq \sqrt{\epsilon}\} dt \right] = \int_{1-\epsilon}^{1} \int_{0}^{\sqrt{\epsilon}} \psi_{t}(0, x; a) \, dx \, dt \asymp c \epsilon^{a+\frac{3}{2}}.
\]
Hence, we see that we should expect \(\hat{P}_{x}^{a}\{X_{t} = 0\} \asymp c \epsilon^{1/2 + a}\). Brownian scaling implies that \(\hat{P}_{a}^{0}\{X_{t} = 0\} \text{ for some } ru \leq t \leq r\} \) is independent of \(r\) and from this we see that there should be a constant \(c = c(a)\) such that
\[
\hat{P}_{a}^{x}\{X_{t} = 0\} \text{ for some } 1 - \epsilon \leq t \leq 1 \sim c \epsilon^{1/2 + a}.
\]

In fact, our construction will show that we can define a local time at the origin. In other words, there is a process \(L_{t}\) that is a normalized version of “amount of time spent at the origin by time \(t\)” with the following properties. Let \(Z = \{s : X_{s} = 0\}\) be the zero set for the process.

- \(L_{t}\) is continuous, nondecreasing, and has derivative zero on \([0, \infty) \setminus Z\).
- As \(\epsilon \downarrow 0\),
  \[
  \mathbb{P}^{0}\{Z \cap [1 - \epsilon, 1] \neq \emptyset\} = \mathbb{P}^{0}\{L_{1} > L_{1-\epsilon}\} \asymp \epsilon^{1/2 + a}.
  \]
- The Hausdorff dimension of \(Z\) is \(1/2 - a\).
- \[
  \mathbb{E}[L_{t}] = c \int_{0}^{t} s^{-\frac{1}{2} - a} \, ds = \frac{c}{\frac{1}{2} - a} t^{\frac{1}{2} - a}.
  \]

We will use a “last-exit decomposition” to derive the formula for \(\psi_{t}(0, x; a)\).

**Proposition 1.19.** If \(y > 0\), then
\[
\psi_{1}(0, y; a) = \frac{y}{\Gamma(\frac{1}{2} - a) \Gamma(\frac{1}{2} + a)} \int_{0}^{1} s^{-\frac{1}{2} - a} (1 - s)^{a-\frac{3}{2}} e^{-y^{3/2}/(1-s)} \, ds.
\]
(32)
The proof of this proposition is a simple calculation,
\[
\int_0^1 s^{-\frac{1}{2} - a} (1 - s)^{a - \frac{3}{2}} e^{-y^2/2(1-s)} \, ds = \int_0^1 (1 - s)^{-\frac{1}{2} - a} s^{a - \frac{3}{2}} e^{-y^2/2s} \, ds
\]
\[
= e^{-y^2/2} \int_0^1 \left[ \frac{1 - s}{s} \right]^{-\frac{1}{2} - a} \exp \left\{ -\frac{y^2}{2} \frac{1 - s}{s} \right\} s^{-2} \, ds
\]
\[
= e^{-y^2/2} \int_0^\infty u^{-\frac{1}{2} - a} e^{-uy^2/2} \, du
\]
\[
= 2^{a + \frac{1}{2}} y^{2a} e^{-y^2/2} \int_0^\infty (uy^2/2)^{-\frac{1}{2} - a} e^{-uy^2/2} \, d(uy^2/2)
\]
\[
= 2^{\frac{1}{2} - a} y^{2a - 1} \int_0^\infty v^{-\frac{1}{2} - a} e^{-v} \, dv
\]
\[
= 2^{\frac{1}{2} - a} y^{2a - 1} \Gamma \left( \frac{1}{2} - a \right).
\]

We would like to interpret the formula (32) in terms of a “last-exit” decomposition. What we have done is to split paths from 0 to \( t \) at the largest time \( s < t \) at which \( X_s = 0 \). We think of \( s^{-\frac{1}{2} - a} \) as being a normalized version of \( \psi_s(0, 0) \) and then \( t^{-\frac{1}{2} - a} e^{-y^2/2t} \) represents the normalized probability of getting to \( y \) at time \( t \) with no later return to the origin. To be more precise, let
\[
q_t^*(y; a) = \lim_{x \to 0} x^{2a - 1} q_t(x, y; a),
\]
and note that
\[
q_1^*(y; a) = \lim_{x \to 0} x^{2a - 1} q_1(x, y; a)
\]
\[
= \lim_{x \to 0} x^{2a - 1} (y/x)^{2a - 1} q_1(x, y; 1 - a)
\]
\[
= y^{2a - 1} q_1(0, y; 1 - a) = c y e^{-y^2/2}.
\]
\[
q_t^*(y; a) = \lim_{x \to 0} x^{2a - 1} q_t(x, y; a)
\]
\[
= t^{-\frac{1}{2}} \lim_{x \to 0} x^{2a - 1} q_1(x/\sqrt{t}, y/\sqrt{t}; a)
\]
\[
= t^{a - 1} \lim_{z \to 0} z^{2a - 1} q_1(z, y/\sqrt{t}; a)
\]
\[
= t^{a - 1} q_1^*(y/\sqrt{t}; a)
\]
\[
= c t^{\frac{a}{2}} y e^{-y^2/(2t)}.
\]

**Proposition 1.20.** For every \( 0 < t_1 < t_2 < \infty \) and \( y_0 < \infty \), there exists \( c \) such that if \( t_1 \leq t \leq t_2 \) and \( 0 \leq x, y \leq y_0 \), then
\[
c_1 y^{2a} \leq \psi_t(x, y; a) \leq c_2 y^{2a}.
\]

**Proof.** Fix \( t_1, t_2 \) and \( y_0 \) and allow constants to depend on these parameter. It follows immediately from (28) that there exist \( 0 \leq c_1 < c_2 < \infty \) such that if \( t_1/2 \leq t \leq t_2 \) and \( y \leq y_0 \),
\[
c_1 y^{2a} \leq \psi_t(0, y; a) \leq c_2 y^{2a}.
\]
We also know that
\[ \psi_t(x, y; a) = \tilde{\psi}_t(x, y; a) + q_t(x, y; a) \leq \tilde{\psi}_t(0, y; a) + q_t(x, y; a). \]
Using (9) and Proposition 1.7, we see that
\[ q_t(x, y; a) = \left( \frac{y}{x} \right)^2 a - 1 q_t(x, y; 1 - a) \leq cy^2a - 1 y^2(1 - a) = cy^2. \]
Also,
\[ \tilde{\psi}_t(x, y; a) \geq P_x \{ T_0 \leq t_1/2 \} \inf_{t_1/2 \leq s \leq t_2} \psi_s(0, y; a) \geq cy^2a \geq cy^2. \]

For later reference, we prove the following.

**Proposition 1.21.** There exists \( c < \infty \) such that if \( x \geq 3 \pi/4 \) and \( y \leq \pi/2 \), then for all \( t \geq 0 \),
\[ \psi_t(x, y; a) \leq cy^{2a}. \quad (33) \]

**Proof.** Let \( z = 3 \pi/4 \). It suffices to prove the estimate for \( x = z \). By (29),
\[ (z/y)^2a \psi_t(z, y; a) = \psi_t(y, z; a) \leq q_t(y, z; a) + \inf_{0 \leq s < \infty} \psi_t(0, z; a) \leq c. \]

\[ \square \]

### 1.8 Excursion construction of reflected Bessel process

In this section we show how we can construct the reflected Bessel process using excursions. In the case \( a = 0 \) this is the Itô construction of the reflected Brownian motion in terms of local time and Brownian excursions. Let \( 0 < r = a + \frac{1}{2} < 1 \) and let \( K \) denote a Poisson point process from measure
\[ (rt^{-r-1} dt) \times \text{Lebesgue}. \]
Note that the expected number of pairs \((t, x)\) with \( 0 \leq x \leq x_0 \) and \( 2^{-n} \leq t \leq 2^{-n+1} \) is
\[ x_0 \int_{2^{-n}}^{2^{-n+1}} rt^{-r-1} dr = x_0 (1 - 2^{-r}) 2^{rn}, \]
which goes to infinity as \( n \to \infty \). However,
\[ \mathbb{E} \left[ \sum_{(t, x) \in K; x \leq x_0, t \leq 1} t \right] = x_0 \int_0^1 rt^{-r} dr = \frac{r x_0}{1 - r} < \infty. \]
In other words, the expected number of excursions in \( K \) by time one is infinite (and a simple argument shows, in fact, that the number is infinite with probability one), but the expected number by time one of time duration at least \( \epsilon > 0 \) is finite. Also, the expected amount of time spent in excursions by time 1 of time duration at most one is finite. Let
\[ T_x = \sum_{(t, x') \in K; x' \leq x} t. \]

Then with probability one, \( T_x < \infty \). Note that \( T_x \) is increasing, right continuous, and has left limits. It is discontinuous at \( x \) such that \( (t, x) \in K \) for some \( t \). In this case \( T_x = T_{x-} + t \). Indeed, the expected number of pairs \((t, x')\) with \( x' \leq x, t \geq 1 \) is finite and hence with probability one the number of loops of time duration 1 is finite. We define \( L_t \) to be the “inverse” of \( T_x \) in the sense that

\[
L_t = x \text{ if } T_{x-} \leq t \leq T_x.
\]

Then \( L_t \) is a continuous, increasing function whose derivative is zero almost everywhere.

The density \( rt^{-r-1} \) is not a probability density because the integral diverges near zero. However we can still consider the conditional distribution of a random variable conditioned that it is at least \( k \). Indeed we write

\[
P\{ T \leq t \mid T \geq k \} = \int_t^k r s^{-r-1} ds = \frac{k^{-r} - t^{-r}}{k^{-r}},
\]

which means that the “hazard function” is \( r/k \),

\[
P\{ T \leq k + dt \mid T \geq k \} = (r/k) dt + o(dt).
\]

### 1.8.1 Excursions and bridges

Let us consider the measure on a Bessel process with parameter \( a < 1/2 \) started at \( x > 0 \) “conditioned so that \( T_0 = t \).” We write

\[
dX_t = \frac{a}{X_t} dt + dB_t, \quad t < T,
\]

where \( B_t \) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), and \( T = T_0 \) is the first hitting time of the origin. This is conditioning on an event of measure zero, but we can make sense of it using the Girsanov formula. Let

\[
F(x, t) = x^{1-2a} t^{a-\frac{3}{2}} \exp\{-x^2/2t\}.
\]

Up to a multiplicative constant, \( F(x, \cdot) \) is the density of \( T_0 \) given \( X_0 = x \) (see (21)). Let \( M_s = F(X_s, t-s) \); heuristically, we think of \( M_s \) as the probability that \( T = t \) given \( F_s \). Given this interpretation, it is reasonable to expect that \( M_s \) is a local martingale for \( s < t \). Indeed, if we let \( E_t = E_{t, \epsilon} \) be the event \( E_t = \{ t \leq T_0 \leq t + \epsilon \} \), and we weight by

\[
F_s(x, t) = c \hat{P}_a^x(E_t),
\]

then \( F_s(X_s, t-s) = c \hat{P}_a^x(E_t \mid F_s) \) which is a martingale. We can also verify this using Itô’s formula which gives

\[
dM_s = M_s \left[ \frac{1-2a}{X_s} - \frac{X_s}{t-s} \right] dB_s.
\]

Hence, if we tilt by the local martingale \( M_s \), we see that

\[
dX_s = \left[ \frac{1-a}{X_s} - \frac{X_s}{t-s} \right] ds + dW_t,
\]

where \( W_t \) is a Brownian motion in the new measure \( \mathbb{P}^* \).
One may note that this is the same process that one obtains by starting with a Bessel process \(X_t\) with parameter \(1 - a > 1/2\) and weighting locally by \(J_s := \exp\{-X_s^2/2(t-s)\}\). Itô’s formula shows that if \(X_s\) satisfies
\[
dX_s = \frac{1-a}{X_s} \, ds + dB_s,
\]
then
\[
dJ_s = J_s \left[ -\frac{X_s}{t-s} \, dB_s + \frac{a - \frac{3}{2}}{t-s} \, ds \right],
\]
which shows that
\[
N_s = \left( \frac{t}{t-s} \right)^{\frac{3}{2} - a} J_s,
\]
is a local martingale for \(s < t\) satisfying
\[
dN_s = -\frac{X_s}{t-s} \, N_s \, dB_s.
\]
There is no problem defining this process with initial condition \(X_0 = 0\), and hence we have the distribution of a Bessel excursion from 0 to 0.

We can see from this that if \(a < 1/2\), then the distribution of an excursion \(X_s\) with \(X_0 = X_t = 0\) and \(X_s > 0\) for \(0 < s < t\) is the same as the distribution of a Bessel process with parameter \(1 - a\) “conditioned to be at the origin at time \(t\)”. More precisely, if we consider the paths up to time \(t - \delta\), then the Radon-Nikodym derivative of the excursion with respect to a Bessel with parameter \(1 - a\) is proportional to \(\exp\{-X_s^2/2(t-s)\}\).

There are several equivalent ways of viewing the excursion measure. Above we have described the probability measure associated to excursions starting and ending at the origin of time duration \(t\). Let us write \(\mu^#(t; a)\) for this measure. Then the excursion measure can be be given by
\[
c \int_0^\infty \mu^#(t; a) \, t^{a - \frac{3}{2}} \, dt.
\]
The constant \(c\) is arbitrary. This is an infinite measure on paths but can be viewed as the limit of the measure on paths of time duration at least \(s\),
\[
c \int_s^\infty \mu^#(t; a) \, t^{a - \frac{3}{2}} \, dt,
\]
which has total mass
\[
c \int_s^\infty t^{a - \frac{3}{2}} \, dt = \frac{c}{\frac{3}{2} - a} \, s^{a - \frac{1}{2}}.
\]
Another way to get this measure is to consider the usual Bessel process started at \(\epsilon > 0\) stopped when it reaches the origin. This is a probability measure on paths that we will denote by \(\tilde{\mu}^#(\epsilon; a)\). The density of the hitting time \(T\) is a constant times \(\epsilon^{1-2a} \, t^{a - \frac{3}{2}} \exp\{\epsilon^2/2t\}\). Then the excursion measure can be obtained as
\[
\lim_{\epsilon \downarrow 0} \epsilon^{2a-1} \tilde{\mu}^#(\epsilon; a).
\]
From this perspective it is easier to see that in the excursion measure has the following property: the distribution of the remainder of an excursion given that the time duration is at least \( s \) and \( X_s = y \) is that of a Bessel process with parameter \( a \) started at \( y \) stopped when it reaches the origin.

We can also consider \( m_t \) which is the excursion measure restricted to paths with \( T > t \) viewed as a measure on the paths \( 0 \leq X_s \leq t, 0 < s \leq t \). For each \( t \) this is a finite measure on paths, The density of the endpoint at time \( t \) (up to an arbitrary multiplicative constant) is given by

\[
\psi_t(x) = \lim_{\epsilon \downarrow 0} \epsilon^{2a-1} q_t(\epsilon, x; a) = \lim_{\epsilon \downarrow 0} \epsilon^{2a-1} q_t(\epsilon, x; 1-a) = x^{2a-1} q_t(0, x; 1-a) = x t^{1-a} e^{-x^2/2t}.
\]

Note that \( \psi_t \) is not a probability density; indeed,

\[
\int_0^\infty \psi_t(x) \, dx = \int_0^\infty x t^{1-a} e^{-x^2/2t} \, dx = t^{3-a}.
\]

Note that \( \psi_t \) satisfies the Chapman-Kolomogorov equations

\[
\psi_{t+s}(x) = \int_0^\infty \psi_t(y) q_s(y, x; a) \, dy.
\]

For example if \( s = 1 - t \), then this identity is the same as

\[
x e^{-x^2/2} = \int_0^\infty y t^{1-a} (1-t) e^{-x^2/2t} (y/x)^{2a-1} q_1-(x, y; 1-a) \, dy
\]

\[
= \int_0^\infty y t^{1-a} e^{-x^2/2t} (y/x)^{2a-1} \frac{y^{2-2a}}{(1-t)^{2-a}} \exp \left\{ -\frac{x^2 + y^2}{1-t} \right\} h_a(xy/1-t).
\]

### 1.8.2 Another construction

Let us give another description of the reflected Bessel process using a single Brownian motion \( B_t \). Suppose \( B_t \) is a standard Brownian motion and \( X^x_t \) satisfies

\[
dX^x_t = \frac{a}{X^x_t} \, dt + dB_t, \quad X^x_0 = x.
\]  

(34)

This valid up to time \( T^x = \inf\{t : X^x_t > 0\} \). Using the argument in Proposition 1.11, we can see that with probability one, \( x < y \) implies that \( T^x < T^y \), and hence the map \( x \mapsto T^x \) is an increasing one-to-one function from \( (0, \infty) \) to \( (0, \infty) \). However, it is not the case that the function is onto.

It is easiest to see this when \( a = 0 \). In this case \( X^x_t = x + B_t \), and

\[T^x = \inf\{t : B_t = -x\} \]

The set of times \( \{T^x : x > 0\} \) are exactly the same as the set of times \( t \) at which the Brownian motion obtains a minimum, that is, \( B_t < B_s, 0 \leq s < t \). Its closure is the set of times \( t \) at which \( B_t \leq B_s, 0 \leq s < t \). The distribution of this latter set is the same as the distribution of the zero set of Brownian motion and is a topological Cantor set of Hausdorff dimension 1/2.

For any \( a \in \mathbb{R} \), and \( X^x_t \) satisfying (34) we can define

\[Y^x_t = \inf\{X^y_t : y > x, t < T^y\}, \quad Y^x_t = Y^0_t \]

If \( a \geq 1/2 \), so that \( T^x = \infty \), we see that \( Y^x_t = X^x_t \). The interesting case is \( a < 1/2 \) which we assume for the remainder of this subsection. Note that this process is coalescing in the sense that if \( T_x < t \), then \( Y^x_t = Y^x_t \). The difference between \( a > -1/2 \) and \( a \leq -1/2 \) can be seen in the next proposition.
Proposition 1.22. If $X_t^x$ is defined as in (34) and $Y_t$ is defined as above, then

- if $a \geq \frac{1}{2}$,
  \[ \mathbb{P}\{Y_t > 0 \text{ for all } t > 0\} = 1; \]

- if $-\frac{1}{2} < a < \frac{1}{2}$, for each $t > 0$,
  \[ \mathbb{P}\{Y_t > 0\} = 1, \]
  but
  \[ \mathbb{P}\{Y_t > 0 \text{ for all } t > 0\} = 0; \]

Before starting the proof, we will set up some notation. If $B_t$ is a standard Brownian motion, and $t \geq 0$, we let $B_{s,t} = B_t + s - B_s$. If $x > 0$ we write $X_{s,t}^x$ for the solution of

\[ dX_{s,t}^x = \frac{a}{X_s^x} ds + dB_{s,t}, \quad X_{0,t}^x = x. \]

This is valid up to time $T_{0,t}^x = \inf\{s : X_{s,t}^x = 0\}$ and for $s < T_{0,t}^x$, we have

\[ X_{s,t}^x = B_{s,t} + a \int_0^s \frac{dr}{X_{r,t}^x}. \]

The Markov property can be written as

\[ X_{s+r,t}^x = X_{r,t+s}^x, \quad s + r < T_{0,t}^x. \]

If $\tau > 0$, we will say that $t$ is an $\tau$-escape time (for parameter $a$) if all $x > 0$,

\[ X_{s,t}^x > 0, \quad 0 \leq s \leq \tau. \]

We say that $t$ is an escape time if it is a $\tau$-escape time for some $\tau > 0$.

Theorem 1.

- If $a \geq 1/2$, then with probability one, all times are escape times.
- If $-1/2 < a < 1/2$, then with probability one, the set of escape times is a dense set of Hausdorff dimension $\frac{1}{2} + a$.

Proof. We first consider the easiest case, $a \geq 1/2$. With probability one, we know that all dyadic times are escape times, and using continuity we can see that this implies that all times are escape times.

We now consider $-1/2 < a < 1/2$; for ease, we will only consider $t \in [0, 1]$ and let $R_\tau$ denote the set of $\tau$-escape times in $[0, 1]$. If $R$ is the set of escape times in $[0, 1]$, then

\[ R = \bigcup_{n=1}^{\infty} R_{1/n}. \]

Let us first fix $\tau$. Let $Q_n$ denote the set of dyadic rationals in $(0, 1]$ with denominator $2^n$,

\[ Q_n = \left\{ \frac{k}{2^n} : k = 1, 2, \ldots, 2^n \right\}. \]
We write $I(k, n)$ for the interval $[(k - 1)2^{-n}, k2^n]$. We say that the interval $I(k, n)$ is good if there exists a time $t \in I(k, n)$ such that $X^{2^{-n/2}}_{s,t} > 0$ for $0 \leq s \leq 1$. Let

$$I_n = \bigcup_{I(k, n) \text{ good}} I(k, n), \quad I = \bigcap_{n=1}^{\infty} I_n.$$ 

Note that $I_1 \supset I_2 \supset \cdots$, and for each $n$, $R_1 \subset I_n$. We also claim that $I \subset R_{1/2}$. Indeed, suppose that $t \notin R_{1/2}$. Then there exists $x > 0$ such that $T^y_{0,t} \leq 1/2$, and hence $T^y_{0,t} \leq 1/2$ for $0 < y \leq x$. Using continuity of the Brownian motion, we see that there exists $y > 0$ and $\epsilon > 0$ (depending on the realization of the Brownian motion $B_t$), such that $T^y_{0,s} \leq 3/4$ for $|t - s| \leq \epsilon$. (The argument is slightly different for $s < t$ and $s > t$.) Therefore, $t \notin I_n$ if $2^{-n} \leq \epsilon$.

Let $J(k, n)$ denote the corresponding indicator function of the event $\{I(k, n) \text{ good}\}$. We will show that there exist $0 < c_1 < c_2 < \infty$ such that

$$c_1 2^{n(a - \frac{1}{2})} \leq \mathbb{E}[J(j, n)] \leq c_2 2^{n(a - \frac{1}{2})}, \quad (35)$$

$$\mathbb{E}[[J(j, n) J(k, n)] \leq c_3 2^{n(a - \frac{1}{2})} [j - k] + 1]^{a - \frac{1}{2}}. \quad (36)$$

Using standard techniques (refer to ???), (35) implies that $\mathbb{P}\{\dim_h(R_1) \leq a + \frac{1}{2}\} = 1$ and (35) and (36) imply that there exists $\rho = \rho(c_1, c_2, c_3, a) > 0$ such that

$$\mathbb{P}\left\{\dim_h(R_{1/2}) \geq a + \frac{1}{2}\right\} \geq \mathbb{P}\left\{\dim_h(I) \geq a + \frac{1}{2}\right\} \geq \rho.$$ 

For the lower bound we use , which implies that for $x \leq \sqrt{t}$,

$$\mathbb{P}\{T^x_0 > t\} \asymp 1 \wedge (x/\sqrt{t})^{1 - 2a}.$$ 

Using stationarity of Brownian increments, it suffices to prove (35) and (36) for $j = 1$ and $k \geq 3$. Let us fix $n$ and write $x = x_n = 2^{-n/2}, t = t_n = 2^{-n}$. The lower bound in (35) follows from

$$\mathbb{E}[J(1, n)] \geq \mathbb{P}\{T^x_{0,0} \geq 1\} \asymp 2^{n(a - \frac{1}{2})}.$$ 

Using $a \leq 1/2$, we can see that $0 \leq s \leq t$,

$$\bar{X} := \max_{0 \leq s \leq t} X^x_{t-s,s} \leq x + \frac{x}{2} + \max_{0 \leq s \leq t} [B_t - B_s].$$

Note that (for $n \geq 1$)

$$\mathbb{E}[J(1, n) \mid \bar{X} = z] \leq c \mathbb{P}\{T^x_{0,t} \geq 1 - t\} \leq c (z \wedge 1)^{1 - 2a}.$$ 

We then get the upper bound in (35) using a standard estimate (say, using the reflection principle) for the distribution of the maximum of a Brownian path.

For the second moment, let us consider the event that $I(1, n)$ and $I(k, n)$ are both good. Let $V_k$ denote the event that there exists $0 \leq s \leq 2^{-n}$ such that $T^x_{s,(k-1)t-s} > 0$. Then $V_k$ is independent of the event $\{I(k, n) \text{ good}\}$ and

$$\{I(1, n) \text{ good}, I(k, n) \text{ good}\} \subset V_k \cap \{I(k, n) \text{ good}\}.$$ 

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Using the argument for the upper bound in the previous paragraph and scaling, we see that
\[ \mathbb{P}(V_k) \leq c k^{a - \frac{1}{2}}. \]

Using Brownian scaling, we see that the upper bound implies that for all \( \tau > 0 \),
\[ \mathbb{P}\{\dim_h(R_\tau) \leq a + \frac{1}{2}\} = 1, \]
and hence with probability one \( \dim_h(R) \leq a + \frac{1}{2} \). We claim that
\[ \mathbb{P}\left\{\dim_h(R) = a + \frac{1}{2}\right\} = 1. \]

Indeed, if we consider the events
\[ E_{j,n} = \left\{ \dim_h[R_{2 - (n+1)/2} \cap I(j,n)] \geq a + \frac{1}{2} \right\}, \quad j = 1, 2, 3, \ldots, 2^{n-1}, \]
then these are independent events each with probability at least \( \rho \). Therefore,
\[ \mathbb{P}\{E_{1,n} \cup E_{3,n} \cup \cdots \cup E_{2n-1,n}\} \geq 1 - (1 - \rho)^{2^{n-1}}. \]

Using this and scaling we see that with probability one for all rationals \( 0 \leq p < q \leq 1 \), \( \dim(R \cap [p, q]) = a + \frac{1}{2} \).

\[ \Box \]

\textbf{Proof of Proposition 1.22.} We follow the same outline as the previous proof, except that we define \( I(k,n) \) to be \( \beta \)-good if if there exists \( t \in I(k,n) \) such that \( X_{s,t}^{2^{-n/2}} > 0 \) for \( 0 \leq s \leq 1 \) and
\[ X_{s,t}^{2^{-n/2}} \geq 2\beta, \quad \frac{1}{4} \leq s \leq 1. \]

Arguing as before, we get the estimates (35) and (36), although the constant \( c_1 \) now depends on \( \beta \).

Let \( R_{1/2,\beta} \) be the set of \( t \in R_{1/2} \) such that
\[ \lim_{x \to 0} X_{1/2,t}^x \geq \beta. \]

Then \( R_{1/2,\beta} \subset I^\beta \) where
\[ I_n^\beta = \bigcup_{I(k,n) \beta \text{-good}} I(K,n), \quad I^\beta = \bigcap_{n=1}^{\infty} I_n^\beta. \]

There exists \( \rho_\beta > 0 \) such that
\[ \mathbb{P}\left\{\dim_h(R_{1/2,\beta}) = \frac{1}{2} + a \right\} \geq \rho_\beta. \]

For each time \( t \in R \), we define
\[ X_{s,t}^0 = \inf\{X_{s,t}^x, x > 0\} \]
where the right-hand side is defined to be zero if \( T_{0,t}^x \leq s \) for some \( x > 0 \). Recall that

\[
\tilde{X}_t = \inf \{ X^x_t : T_{0,t}^x > t \}.
\]

Note that for every \( 0 \leq t \leq 1 \),

\[
\tilde{X}_1 \geq X_{1-t,t}^0.
\]

We claim: with probability one, there exists \( t < 1 \) such that \( X^0_{1-t,t} > 0 \). To see this, consider the events \( V_n \) defined by

\[
V_n = \{ \exists t \in I(2^n - 1, n) \text{ with } X_{1-t,0}^0 > 2^{-n/2} \}.
\]

The argument above combined with scaling shows that \( P(V_n) \) is the same and positive for each \( n \). Also if we choose a sequence \( n_1 < n_2 < n_3 < \cdots \) going to infinity sufficiently quickly, the events \( V_{n_j} \) are almost independent. To be more precise, Let

\[
V^j = \{ \exists t \in I(2^{n_j} - 1, n_j) \text{ with } X_{1-t-2^{-n_j-1},t}^1 > 2 \cdot 2^{-n_j/2} \}.
\]

Then the events \( V^1, V^2, \ldots \) are independent and there exists \( \rho > 0 \) with \( P(V^j) > 0 \). Hence \( P\{ V^j \text{ i.o.} \} = 1 \). If we choose the sequence \( n_j \) to grow fast enough we can see that

\[
\sum_{j=1}^{\infty} P(V^j \setminus V_{n_j}) < \infty,
\]

and hence, \( P\{ V_{n_j} \text{ i.o.} \} > 0 \).

\[ \square \]

2 One more construction

We will give one more construction of the reflected Bessel process. Suppose that \(-\frac{1}{2} < a < \frac{1}{2}\). For each \( \epsilon > 0 \), let \( \Phi_\epsilon = e^{\Psi_\epsilon} : (0, \infty) \to (0, \infty) \), denote a \( C^\infty \) function with

\[
\Phi_\epsilon(x) = x, \quad 0 \leq x \leq \frac{\epsilon}{4},
\]

\[
\Phi_\epsilon(x) = x^a, \quad \frac{\epsilon}{2} \leq x \leq \infty.
\]

Let \( N_t^\epsilon \) denotes the local martingale

\[
N_t^\epsilon = \Psi_\epsilon'(X_t) \exp \left\{ -\frac{1}{2} \int_0^t \frac{\Phi_\epsilon''(X_s)}{\Phi_\epsilon'(X_s)} \, ds \right\},
\]

that satisfies

\[
dN_t^\epsilon = \Psi_\epsilon'(X_t) N_t^\epsilon \, dX_t.
\]

Let \( Q_\epsilon = Q_{\epsilon,a} \) denote the measure obtained from the Girsanov theorem by tilting by \( \Phi_\epsilon \). The Girsanov theorem shows that

\[
dx_t = \Psi_\epsilon'(X_t) \, dt + dB_t,
\]

where \( B_t = B_{t,\epsilon} \) is a standard Brownian motion with respect to \( Q_\epsilon \). If \( T = \inf \{ t : X_t = 0 \} \), then we claim that \( Q_\epsilon \{ T < \infty \} = 0 \). Indeed, on the interval \([0, \epsilon/4]\), this process has the distribution of
a Bessel process with parameter 1. From this we see that \( N_\epsilon \) is, in fact, a martingale and \( Q_\epsilon \) is a probability measure on paths \( X_t, 0 \leq t < \infty \) such that \( X_t > 0 \) for all \( t \).

We define a sequence of stopping times (depending on \( \epsilon \)):

\[
\sigma_1 = \inf \{ t : X_t \geq \epsilon \},
\]

\[
\tau_1 = \inf \{ t \geq \sigma_1 : X_t = \epsilon/2 \},
\]

and recursively,

\[
\sigma_j = \inf \{ t \geq \tau_{j-1} : X_t = \epsilon \},
\]

\[
\tau_j = \inf \{ t \geq \sigma_j : X_t = \epsilon/2 \}.
\]

3 Radial Bessel process

We will now consider a similar process that lives on the bounded interval \([0, \pi]\) that arises in the study of the radial Schramm-Loewner evolution. As in the case of the Bessel process, we will take our initial definition by starting with a Brownian motion and then weighting by a particular function. We will first consider the process restricted to the open interval \((0, \pi)\) and then discuss possible reflections on the boundary. We first discuss such processes in more generality.

3.1 Weighted Brownian motion on \([0, \pi]\)

We will consider Brownian motion on the interval \([0, \pi]\) “weighted locally” by a positive function \( \Phi \). Suppose \( m : (0, \pi) \to \mathbb{R} \) is a \( C^1 \) function and let \( \Phi : (0, \pi) \to (0, \infty) \) be the \( C^2 \) function

\[
\Phi(x) = c \exp \left\{ - \int_{x}^{\pi/2} m(y) \, dy \right\}.
\]

Here \( c \) is any positive constant. Everything we do will be independent of the choice of \( c \) so we can choose \( c = 1 \) for convenience. Note that

\[
\Phi'(x) = m(x) \Phi(x), \quad \Phi''(x) = [m(x)^2 + m'(x)] \Phi(x).
\]

Let \( X_t \) be a standard Brownian motion with \( 0 < X_0 < \pi \), \( T_y = \inf \{ t : X_t = y \} \) and \( T = T_0 \wedge T_\pi = \inf \{ t : X_t = 0 \text{ or } X_t = \pi \} \). For \( t < T \), let

\[
M_{t,\Phi} = \frac{\Phi(X_t)}{\Phi(X_0)} K_{t,\Phi}, \quad K_{t,\Phi} = \exp \left\{ - \frac{1}{2} \int_0^t [m(X_s)^2 + m'(X_s)] \, ds \right\}.
\]

Then Itô’s formula shows that \( M_{t,\Phi} \) is a local martingale for \( t < T \) satisfying

\[
dM_{t,\Phi} = m(X_t) M_{t,\Phi} dX_t.
\]

Using the Girsanov theorem (being a little careful since this is only a local martingale), we get a probability measure on paths \( X_t, 0 \leq t < T \) which we denote by \( \mathbb{P}_\Phi \). To be precise, if \( 0 < \epsilon < \pi/2 \), \( \tau = \tau_\epsilon = \inf \{ t : X_t \leq \epsilon \text{ or } X_t \geq \pi - \epsilon \} \), then \( M_{t,\Phi} \) is a positive martingale with \( M_0 = 1 \). Moreover, if \( V \) is a random variable depending only on \( X_s, 0 \leq s \leq t \wedge \tau \), then

\[
\mathbb{E}_\Phi^x[V] = \mathbb{E}^x[M_{t,\Phi} \wedge \tau V].
\]
The Girsanov theorem implies that
\[ dX_t = m(X_t) \, dt + dB_t, \quad t < T, \]
where \( B_t \) is a standard Brownian motion with respect to \( \mathbb{P}_\Phi \).

- Suppose \( \tau \) is a bounded stopping time for the Brownian motion with \( \tau < T \). Then, considered as measures on paths \( X_t, 0 \leq t \leq \tau, \mathbb{P} \) and \( \mathbb{P}_\Phi \) are mutually absolutely continuous with
  \[ \frac{d\mathbb{P}_\Phi}{d\mathbb{P}} = M_{\tau, \Phi}. \]

Examples
- If \( \Phi(x) = x^a, \quad m(x) = \frac{a}{x}, \)
  then \( X_t \) is the Bessel process with parameter \( a \).
- If \( \Phi(x) = (\sin x)^a, \quad m(x) = a \cot x, \)
  then \( X_t \) is the radial Bessel process with parameter \( a \).

Note that the Bessel process and the radial Bessel process with the same parameter are very similar near the origin. The next definition makes this idea precise.

Definition
- We write \( \Phi_1 \sim_0 \Phi_2 \) if there exists \( c < \infty \) such that for \( 0 < x \leq \pi/2, \)
  \[ |m_1(x) - m_2(x)| \leq c x, \quad |m_1'(x) - m_2'(x)| \leq c. \]
  Similarly, we write \( \Phi_1 \sim_\pi \Phi_2 \) if there exists \( c < \infty \) such that for \( \pi/2 \leq x < \pi, \)
  \[ |m_1(x) - m_2(x)| \leq c (\pi - x), \quad |m_1'(x) - m_2'(x)| \leq c. \]

We write \( \Phi_1 \sim_0 \Phi_2 \) if \( \Phi_1 \sim_0 \Phi_2 \) and \( \Phi_1 \sim_\pi \Phi_2 \). Note that if \( \Phi_1 \sim_0 \Phi_2 \), then there exists \( C > 0 \) such that
  \[ \Phi_1(x) = C \Phi_2(x) \left[ 1 + O(x^2) \right], \quad x \downarrow 0. \]
- We say that \( \Phi \) is asymptotically Bessel-\( a \) at the origin if there exists \( c < \infty \) such that for \( 0 < x \leq \pi/2, \)
  \[ \left| m(x) - \frac{a}{x} \right| \leq c x, \quad \left| m'(x) + \frac{a}{x^2} \right| \leq c. \]
  Similarly, we say that \( \Phi \) is asymptotically Bessel-\( a \) at \( \pi \) if \( \tilde{\Phi}(x) = \Phi(\pi - x) \) is asymptotically Bessel-\( a \) at the origin.

It is easy to see that the radial Bessel-\( a \) process is asymptotically Bessel-\( a \) at both 0 and \( \pi \). This requirement for \( \Phi_1 \sim_0 \Phi_2 \) is strong, but it has the following immediate consequence.
Proposition 3.1. Suppose $\Phi \sim 0 \tilde{\Phi}$. Then there exist $0 < c, \beta < \infty$ such that for all $t$, if $0 < x < \pi/2$ and $\tau = T \wedge T_{x/2}$,

$$M_{t \wedge \tau, \Phi} \leq c e^{\beta t} M_{t \wedge \tau, \tilde{\Phi}}.$$ 

Proof. Assume $X_t = x$ and without loss of generality assume that $\Phi(x) = \tilde{\Phi}(x) = 1$. The proposition follows from the form of $M_{t, \Phi}$ with

$$c = \max_{0 < y \leq \pi/2} \frac{\Phi(y)}{\tilde{\Phi}(y)}, \quad \beta = \max_{0 < y \leq \pi/2} \left[ m(y)^2 - \tilde{m}(y)^2 + m'(y) - \tilde{m}'(y) \right].$$

Example

- Let

$$\Phi(x) = [\sin x]^u [1 - \cos x]^v, \quad m(x) = (u + v) \cot x + \frac{v}{\sin x}. $$

Then $\Phi$ is asymptotically Bessel-$(u + 2v)$ at the origin and asymptotically Bessel-$u$ at $\pi$.

Let

$$F(x) = F_\Phi(x) = \int_{\pi/2}^x \frac{dy}{\Phi(y)^2}. $$

Note that $F$ is strictly increasing on $(0, \pi)$ with $F(\pi/2) = \infty$ and hence $F(0), F(\pi)$ are well defined by limits (perhaps taking on the value $\pm \infty$) Itô's formula shows that $F(X_t)$ is a $\mathbb{P}_\Phi$ local martingale for $t < T$.

Proposition 3.2. If $0 < x < z < \pi$, then

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_\Phi \{ T_\epsilon < T_z \} = 0$$

if and only if $F(0) = -\infty$. Also,

$$\lim_{\epsilon \downarrow 0} \mathbb{P}_\Phi \{ T_{\pi - \epsilon} < T_x \} = 0$$

if and only if $F(\pi) = \infty$.

Proof. We will prove the first; the second follows by symmetry. Let $\tau = T_\epsilon \wedge T_z$. Since $F(X_{t \wedge \tau})$ is a bounded martingale, the optional sampling theorem implies that.

$$F(x) = F(z) \mathbb{P} \{ T_z < T_\epsilon \} + F(\epsilon) \mathbb{P} \{ T_\epsilon < T_z \} = F(\epsilon) \mathbb{P} \{ T_\epsilon < T_z \},$$

$$\lim_{\epsilon \downarrow 0} \mathbb{P} \{ T_\epsilon < T_z \} = \lim_{\epsilon \downarrow 0} \frac{F(z) - F(x)}{F(z) - F(\epsilon)} = \frac{F(z) - F(x)}{F(z) - F(0)}.$$

There are various assumptions that we can make about $\Phi$. The first is a “non-trapping” condition that implies that the process cannot spend a lot of time near the boundary without either getting away or hitting the boundary. There are various ways to phrase it, but the following will do for our purposes.
Definition

- We call \( \Phi \) \textit{(uniformly) non-trapping} if there exist \( \rho > 0, k < \infty \), such that for every \( 0 < x \leq \pi/4 \),
  \[ P^x_\Phi \{ T \wedge T_{2x} \leq kx^2 \} \geq \rho, \quad P^{\pi-x}_\Phi \{ T \wedge T_{\pi-2x} \leq kx^2 \} \geq \rho. \]  

Note that if \( \Phi \in \mathcal{X} \) and \( \tilde{\Phi} \sim \Phi \), then \( \tilde{\Phi} \in \mathcal{X} \).

It is easy to see, by comparison with Brownian motion, that if \( m \) is has a constant sign in a neighborhood of the origin and is also has a constant sign in a neighborhood of \( \pi \), then \( \Phi \) satisfies (37).

Not every \( \Phi \) satisfies (37). If \( \Phi \) is very oscillatory, the process can spend an infinite amount of time in the open interval \((0, \delta)\). We are not interested in such processes, so we will impose the condition (37). The condition (37) is strictly stronger than the condition \( P^x \{ T \wedge T_{\pi/2} < \infty \} \) for all \( x \). Under this weaker condition, we can say that

\[ P^x \{ T \wedge T_{\pi/2} = T < \infty \} = \lim_{\epsilon \downarrow 0} P^{x+\epsilon}_\Phi \{ T_x < T_z \} = 0, \]

and strengthen slightly the previous proposition.

Definition Let \( \mathcal{X} \) denote the collection of \( \Phi \) as above that are (uniformly) nontrapping and \( L^2 \), that is, satisfying (37) and

\[ \| \Phi \|^2_2 = \int_0^\pi \Phi(x)^2 \, dx < \infty. \]

Note that if \( \Phi_1, \Phi_2 \in \mathcal{X} \) and \( \Phi \sim_0 \Phi_1, \Phi \sim_\pi \Phi_2 \), then \( \Phi \in \mathcal{X} \). If \( \Phi \in \mathcal{X} \), we let \( f \) be the probability density

\[ f(x) = f_\Phi(x) = c \Phi(x)^2 \quad c = \left[ \int_0^\pi \Phi(y)^2 \, dy \right]^{-1}. \]

Note that \( f(x) \) satisfies the adjoint equation

\[ -m'(x) f(x) - m(x) f'(x) + \frac{1}{2} f''(x) = 0. \]

Example

- The radial Bessel-\( a \) process with \( \Phi(z) = (\sin x)^a \) is in \( \mathcal{X} \) for \( a > -1/2 \).
- More generally, if \( a, b > -1/2 \) and \( \Phi \) is asymptotically Bessel-\( a \) at zero and asymptotically Bessel-\( b \) at \( \pi \), then \( \Phi \in \mathcal{X} \).
- We remark that for any \( \Phi, a, b \) and \( \epsilon > 0 \) we can find \( \tilde{\Phi} \) that is asymptotically Bessel-\( a \) at zero, asymptotically Bessel-\( b \) at \( \pi \), and such that \( \Phi(x) = \tilde{\Phi}(x), \epsilon \leq x \leq \pi - \epsilon \).
3.2 Conservative case

In this section we let \( X_1 \) denote the set of \( \Phi \in X \) such that if

\[
F(x) = \int_x^{\pi/2} \frac{dy}{\Phi(y)^2},
\]

then \( F(0+) = \infty, F(\pi-) = -\infty \). In particular, for all \( x, \mathbb{P}_\Phi^x \{ T < \infty \} = 0 \). Let \( f(x) = c\Phi(x)^2 \) be the invariant density. Let \( q_t(x, y) = q_t(x, y; \Phi) \) denote the transition probability. This can be given in terms of Brownian motion staying in \((0, \pi)\). Then

\[
q_t(x, y) = \hat{p}_t(x, y) \hat{E}_x^x[M_t, \Phi],
\]

where \( \hat{p}_t(x, y) \) is the density of Brownian motion killed when it leaves \((0, \pi)\) and \( \hat{E}_x^x = \hat{E}_{x,t}^x \) denotes expectation with respect to the corresponding \( h \)-process corresponding to Brownian motion staying in \((0, \pi)\) with \( X_t = y \). We write \( \mathbb{P}_x \) for \( \mathbb{P}_{\Phi}^x \).

**Proposition 3.3.** If \( \Phi \in X_1 \), all \( 0 < x < y < \pi, t > 0 \),

\[
q_t(x, y) = q_t(y, x) \frac{\Phi(y)^2}{\Phi(x)^2} = q_t(y, x) \frac{f(y)^2}{f(x)^2}.
\]

**Proof.** This follows by path reversal. \( \square \)

**Proposition 3.4.** If \( \Phi \in X_1 \), then for every \( t > 0 \), there exists \( \rho > 0 \) such that for all \( 0 < x < \pi \),

\[
\mathbb{P}_x^x \{ T_{\pi/2} \leq t \} \geq \rho.
\]

**Proof.** Without loss of generality, we will assume that \( 0 < x < \pi/2 \). Let \( \tau_n = T_{2^{-n+1}} \). By iterating \( ??? \), we see that there exist \( c, \beta \) such that for \( x \leq 2^{-n} \),

\[
\mathbb{P}_x^x \{ \tau_n \geq j 2^{-2n} \} \leq c e^{-\beta j}.
\]

Choose \( N \) sufficiently large so that

\[
\sum_{j=N}^{\infty} 2^{-j} \leq \frac{t}{2}, \quad \sum_{j=N}^{\infty} \mathbb{P}_x^x \{ \tau_j \geq 2^{-j} \} \leq \frac{1}{2}.
\]

Then we see that for all \( x \leq 2^{-N} \),

\[
\mathbb{P}_x^x \{ T_{2^{-N}} < t/2 \} \geq \frac{1}{2}.
\]

With \( N \) fixed, we can see by comparison to Brownian motion that there exists \( \rho \) such that

\[
\mathbb{P}^{2^{-N}} \{ T_{\pi/2} < t/2 \} \geq 2 \rho,
\]

and hence \( \mathbb{P}_x^x \{ T_{\pi/2} < t \} \geq \rho \). \( \square \)
Proposition 3.5. If $\Phi \in \mathcal{X}_1$ $t_0 > 0$, then there exist $c > 0$ such that for all $0 < x < y < \pi$ and $s \geq t$,

$$q_s(x, y) \geq cf(y).$$

Proof. It suffices to prove this for $s = t$ since

$$q_{t+s}(x, y) = \int_0^\pi q_s(x, z) q_t(z, y) dy.$$

We first consider $-\pi/4 \leq y \leq \pi/4$. By the previous proposition

$$\mathbb{P}^x\{T_{\pi/2} \leq t/2\} \geq \rho,$$

for some $\rho > 0$ independent of $x$. By comparison with Brownian motion, we can see that

$$\min\left\{ q_s\left(\frac{\pi}{2}, y\right) : \frac{t}{2} \leq s \leq t, \frac{\pi}{4} \leq y \leq \frac{3\pi}{4}\right\} > 0.$$

Hence, for every $t_0$

$$\min\left\{ q_t(x, y) : 0 < x < \pi, \frac{\pi}{4} \leq y \leq \frac{3\pi}{4}, t \geq t_0\right\} \geq \rho(t_0) > 0.$$

Using (38), we see that this implies

$$\min\left\{ \frac{q_t(y, x)}{f(x)} : 0 < x < \pi, \frac{\pi}{4} \leq y \leq \frac{3\pi}{4}, t \geq t_0\right\} \geq \rho(t_0) > 0.$$

Combining this with (39), we get the result.

Recall that the variation distance between two probability measures $\mathbb{P}_1, \mathbb{P}_2$ is given by

$$\|\mathbb{P}_1 - \mathbb{P}_2\| = \sup_A |\mathbb{P}_1(A) - \mathbb{P}_2(A)|,$$

where the supremum is over all events. If $\mathbb{P}_1, \mathbb{P}_2$ are measures on the reals with densities $g_1, g_2$, then

$$\|\mathbb{P}_1 - \mathbb{P}_2\| = \frac{1}{2} \int_{-\infty}^{\infty} |g_1(x) - g_2(x)| dx.$$

If $\mu$ is a probability measure on $(0, \pi)$ and $Q_s \mu$ is defined by

$$Q_s \mu = \int_0^\pi q_s(x, y) \mu(dx),$$

then the previous proposition implies that for each $s > 0$, there exists $\rho_s < 1$ such that for every $\mu$,

$$\|Q_s \mu - \tilde{\mu}\| \leq \rho_s.$$

Here $\tilde{\mu}$ denotes the invariant measure $f(x) \, dx$. 

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Proposition 3.6. There exists $\beta > 0$ such that for each $t > 0$ and $0 < x < \pi$,

$$\|Q_t \mu - \bar{\mu}\| \leq e^{\beta(1-t)}.$$ 

Proof. We let $e^{-\beta} = \rho_1$. For any $\mu$, we can write

$$Q_1 \mu = (1 - e^{-\beta}) \bar{\mu} + e^{-\beta} P_1 \mu,$$

where $P_1 u$ is a probability measure. By iterating this we see that for every positive integer $t$,

$$Q_t \mu = [1 - e^{-t\beta}] \bar{\mu} + e^{-t\beta} P_t \mu,$$

for a probability measure $P_t \mu$. In particular,

$$\|Q_t \mu - \bar{\mu}\| \leq e^{-t\beta}.$$ 

If $t = n + s$ where $n$ is an integer and $0 \leq s < 1$, then $Q_t \mu = Q_n[Q_s \mu]$ and hence,

$$\|Q_t \mu - \bar{\mu}\| \leq e^{-n\beta} \leq e^{(1-t)\beta}.$$

We did not need the full power of Proposition 3.5 to conclude the last result. Suppose we assume only that there exists an open interval $I$ such that

$$q_1(x, y) \geq c, \quad 0 < x < \pi, \quad y \in I.$$ 

Then we would have

$$\frac{1}{2} \int_0^\pi |q_1(x, y) - f(y)| \leq 1 - c.$$ 

In other words we could write

$$q_1(x, y) = (1 - \beta) \phi_1(y) + \beta g_1(y), \quad f(y) = Q_1 f(y) = (1 - \beta) \phi_1(y) + \beta h_1(y),$$

for some probability densities $\phi_2, g_1, h_1$. Indeed, $\phi_1$ is the uniform density on $I$ and $(1 - \beta) = c \text{ length}(I)$. Note that $\phi_1$ is the same in both equalities. Probabilists would think of “coupling” the processes with probability $1 - \beta$. If we take another step we would see that we could write

$$q_2(x, y) = (1 - \beta^2) \phi_2(y) + \beta^2 g_2(y), \quad f(y) = Q_1 f(y) = (1 - \beta^2) \phi_2(y) + \beta^2 h_2(y).$$

Here

$$(1 - \beta^2) \phi_2 = (1 - \beta) Q \phi_1 + \beta (1 - \beta) \phi_1.$$ 

The key thing is that the density $\phi_j$ do not depend on the initial condition. This shows that for each $j$ there is a probability distribution $\bar{\mu}_j$ such that for any probability distribution $\mu$,

$$\|Q_n \mu - \bar{\mu}_n\| \leq \beta^n.$$ 

In particular, if $\bar{\mu}$ is an invariant distribution

$$\|\bar{\mu} - \bar{\mu}_n\| \leq \beta^n.$$
This shows that the invariant distribution is unique.
This idea can also be used to construct the distribution. Given any two initial distributions
\( \mu_1, \mu_2 \), we see that
\[
\| Q_n \mu_1 - Q_n \mu_2 \| \leq 2 \beta^n.
\]
In particular, if \( m = n + k \), and \( \mu \) is any initial distribution
\[
\| Q_m \mu - Q_n \mu \| = \| Q_n (Q_k \mu) - Q_n \mu \| \leq c e^{-\beta s},
\]
and hence we have a Cauchy sequence of measures in the topology of weak convergence.

We let \( \mathcal{X}_2 \) be the set of \( \Phi \in \mathcal{X}_2 \) such that for all \( t > 0 \) there exists \( c_t < \infty \) such that for all
\( 0 < x < y < \pi \).
\[
q_t(x, y) \leq (1 + c_1) f(y),
\]
(40)

**Proposition 3.7.** Suppose \( \Phi \in \mathcal{X}_2 \) satisfying (40) Then there exist \( 0 < \beta, c_1 < \infty \) such that for every \( t \geq t_0 \) and every \( 0 < x < y < \pi \),
\[
[1 - e^{\beta (1-t)}] f(y) \leq q_t(x, y) \leq [1 + c_1 e^{\beta (1-t)}] f(y).
\]

**Proof.** If \( \mu_t \) is the measure with density \( q_t(x, y) \), then we know that we can write
\[
\mu_t = (1 - e^{\beta (1-t)}) \bar{\mu} + e^{\beta (1-t)} \nu
\]
for some probability measure \( \nu \). \( \square \)

**Corollary 3.8.** If \( \Phi \in \mathcal{X}_2 \), then for every \( t_0 > 0 \), there exist \( 0 < c < \beta \) such that for all \( 0 < x < y < \infty \), and \( t \geq t_0 \),
\[
[1 - c e^{-\beta t}] f(y) \leq q_t(x, y) \leq [1 + c e^{-\beta t}] f(y).
\]

In particular, if \( g \) is a nonnegative function with
\[
\int_0^\pi g(x) \Phi(x)^2 dx < \infty,
\]
then for all \( 0 < x < \pi \) and \( t \geq t_0 \),
\[
[1 - c e^{-\beta t}] \bar{g} \leq \mathbb{E}_\Phi^x [g(X_t)] \leq [1 + c e^{-\beta t}] \bar{g},
\]
where
\[
\bar{g} = \int_0^\pi g(x) f(x) dx = \frac{\int_0^\pi g(x) \Phi(x)^2 dx}{\int_0^\pi \Phi(x)^2 dx}.
\]

As shorthand we write
\[
\mathbb{E}_\Phi^x [g(X_t)] = \bar{g} [1 + O(e^{-\beta t})],
\]
to mean that there exists \( 0 < c, \beta, t_0 < \infty \) such that for all \( 0 < x < \pi \) and all \( t \geq t_0 \),
\[
[1 - c e^{-\beta t}] \bar{g} \leq \mathbb{E}_\Phi^x [g(X_t)] \leq [1 + c e^{-\beta t}] \bar{g},
\]
Proposition 3.9. Suppose $a, b \geq 1/2$ and $\Phi$ is asymptotically Bessel-$a$ at the origin and $\Phi$ is asymptotically Bessel-$b$ at $\pi$. Then $\Phi \in X_2$.

Proof. We fix $t$ and allow constants to depend on $t$. By symmetry we can restrict our consideration to $y \leq \pi/2$, and by comparison with Brownian motion, we need only consider $y$ near zero, and we need to show

$$q_t(x, y) \leq cy^{2a}, \quad 0 < x < 2\pi.$$ 

If $x = \pi/4$ and $0 \leq s \leq t$,

$$q_s(x, y) = \frac{\Phi(y)^2}{\Phi(x)^2} q_s(y, x) \leq c \Phi(y)^2 \leq cy^{2a}.$$ 

Let $\tilde{q}_t(x, y)$ denote the density of the process killed at time $T_{\pi/8}$. Then using the above we see that it suffices to show that

$$\tilde{q}_t(x, y) \leq cy^{2a}.$$ 

This follows by comparison with a Bessel-$a$ process using Proposition 1.7 and Proposition 3.1.

□

Proposition 3.10. Suppose $\Phi \in X_2$. If $X_t$ is a standard Brownian motion, then there exists $\beta > 0$ such that

$$\mathbb{E}^x[K_t, \Phi; T > t] = c_* \Phi(x) [1 + O(e^{-\beta t})], \quad c_* = \frac{\int_0^\pi \Phi(y) \, dy}{\int_0^\pi \Phi(y)^2 \, dy}.$$ 

Proof.

$$\mathbb{E}^x[K_t, \Phi; T > t] = \Phi(x) \mathbb{E}^x[\Phi(X_t)^{-1} M_t, \Phi; T > t]$$

$$= \Phi(x) \mathbb{E}^x[\Phi(X_t)^{-1}; T > t]$$

$$= \Phi(x) \mathbb{E}^x[\Phi(X_t)^{-1}]$$

$$= c_* \Phi(x) [1 + O(e^{-\beta t})].$$

The third equality uses $\mathbb{P}_\Phi^x \{ T > t \} = 1$. □

Example. Suppose $\Phi(x) = (\sin x)^a$ with $a \geq 1/2$. Then,

$$m(x)^2 + m'(x) = a^2 \cot^2 x - \frac{a}{\sin^2 x} = \frac{a(a - 1)}{\sin x} + a^2,$$

$$K_t, \Phi = e^{-a^2 t/2} \exp \left\{ \frac{a(1 - a)}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}.$$ 

We therefore get

$$\mathbb{E}^x \left[ \exp \left\{ \frac{a(1 - a)}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}; T > t \right] = e^{-a^2 t/2} c_* [\sin x]^a [1 + O(e^{-\beta t})].$$ 

where

$$c_* = \frac{\int_0^\pi [\sin y]^a \, dy}{\int_0^\pi [\sin y]^{2a} \, dy}.$$ 

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3.3 Reflected process

It is possible to define a reflected process for any $\Phi \in \mathcal{X}$, we will restrict ourselves to the case of asymptotically Bessel processes for which we can give a more detailed description.

**Definition** Let $\mathcal{X}_B$ denote the set of $\Phi \in \mathcal{X}$ that are asymptotically Bessel-$a$ at the origin and asymptotically Bessel-$b$ at $\pi$ for some $a, b > -1/2$.

We technically did not have to include $a, b > -1/2$ in the definition, since this follows from the fact that $\Phi \in \mathcal{X}$ and hence $\int_0^{\pi} \Phi(x)^2 \, dx < \infty$.

The process can be constructed from the reflected Bessel-$a$ and Bessel-$b$ processes. More precisely, if we define

$$
\sigma_0 = 0,
\tau_0 = \min \left\{ t : X_t \leq \frac{\pi}{8} \right\},
$$

and recursively,

$$
\sigma_j = \min \left\{ t > \tau_{j-1} : X_t = \frac{7\pi}{8} \right\},
\tau_j = \min \left\{ t > \sigma_j : X_t = \frac{\pi}{8} \right\},
$$

then the excursions $X[\tau_j, \sigma_{j+1}]$ are absolutely continuous with respect to the reflected Bessel-$a$ process and $X_{\sigma_j, \tau_j}$ is absolutely continuous with respect to the (reflection about $\pi/2$) of the reflected Bessel-$b$ process. For example, if $x < 7\pi/8$, then for every $t < \infty$,

3.4 $-1/2 < a < 1/2$

The radial Bessel process can also be defined by reflection after hitting the boundary in the range $-1/2 < a < 1/2$. The quickest way to define it is in terms of the reflected Bessel process defined in Section 1.7. Indeed, if $X_t$ is a reflected Bessel process with parameter $a$, we let

$$
K_t = \frac{\mathcal{M}_t}{\mathcal{N}_t} \equiv e^{-\frac{a^2 t}{2}} \left( \frac{S_t}{X_t} \right)^a \exp \left\{ \frac{a(1-a)}{2} \int_0^t \left[ \frac{1}{S_s^2} - \frac{1}{X_s^2} \right] \, ds \right\},
$$

where, as before, $S_t = \sin X_t$. Assume $X_0 \leq \pi/2$. The key fact is the following.

For future reference, we note that the unique cubic polynomial $g(x)$ satisfying $g(0) = 0, g'(0) = 0, g(\epsilon) = \gamma, g'(\epsilon) = \theta/\epsilon$ is

$$
g(x) = \left[ \theta - 2\gamma \right] (x/\epsilon)^3 + [3\gamma - \theta] (x/\epsilon)^2.
$$

Note that for $|x| \leq \epsilon$,

$$
|g(x)| + \epsilon |g'(x)| \leq 17|\gamma| + 7|\theta|.
$$

(41)
Proposition 3.11. Suppose \(-1/2 < a < 1/2\), and \(X_t\) is a reflected Bessel process with parameter \(a\), \(0 < z < \pi\) and with \(X_0 \leq z\). Let \(\tau = T_z\). Then \(K_{t \wedge \tau}\) is a martingale with \(E[K_\tau] = E[K_0] = 1\).

Proof. Roughly speaking, this hold by the Itô formula calculation we did earlier. However, this calculation is not valid (at least without more explanation) at the boundary. We are starting with a Bessel process \(X_t\) and then tilting by \(\phi(X_t)\) where

\[
\phi(x) = \phi(x; a) = \left( \frac{\sin x}{x} \right)^a = \left( \frac{2}{\pi} \right)^a \exp \left\{ -\int_x^{\pi/2} \left[ \cot y - \frac{1}{y} \right] dy \right\}.
\]

Note that \(\phi\) is \(C^2\) on \([0, z]\) with

\[
\phi(x) = 1 + O(x^2), \quad \phi'(x) = O(x), \quad \phi''(x) = O(1), \quad x \downarrow 0.
\]

(Here and below constants, including those implicit in the \(O(\cdot)\) notation, can depend on \(a\).) For \(0 < \epsilon < 1/2\), define

\[
\phi_\epsilon(x) = \left( \frac{2}{\pi} \right)^a \exp \left\{ -a \int_x^{\pi/2} h_\epsilon(y) dy \right\},
\]

where

\[
h_\epsilon(y) = 0, \quad 0 \leq y \leq \frac{\epsilon}{2},
\]

\[
h_\epsilon(y) = \cot y - \frac{1}{y}, \quad \epsilon \leq y < \pi,
\]

\[
h_\epsilon(y) = g \left( y - \frac{\epsilon}{2} \right), \quad \frac{\epsilon}{2} \leq y \leq \epsilon,
\]

where \(g(y)\) is the unique cubic polynomial with \(g(0) = g'(0) = 0\),

\[
g \left( \frac{\epsilon}{2} \right) = \cot(\epsilon/2) - \frac{2}{\epsilon} = O(\epsilon), \quad g' \left( \frac{\epsilon}{2} \right) = \frac{2}{\epsilon^2} - \frac{1}{2 \sin^2(\epsilon/2)} = O(1),
\]

Using (41), we see that that

\[
\max_{0 \leq x \leq \epsilon} \left[ |h_\epsilon(x)| + \epsilon |h_\epsilon'(x)| \right] \leq c \epsilon.
\]

By interpolating by an appropriate polynomial, we can see that for each \(0 < \epsilon < 1/2\) we can find a \(C^2\) nonnegative function \(\Phi_\epsilon(x) = \Phi_\epsilon(x; a)\) and \(C\) (independent of \(\epsilon\)) such that

\[
\Phi_\epsilon(x) = 1, \quad 0 \leq x \leq \epsilon,
\]

\[
\Phi_\epsilon(x) = \Phi(x), \quad 2\epsilon \leq x \leq z
\]

\[|\Phi_\epsilon''(x)| \leq C, \quad 0 \leq x \leq z.
\]

Suppose that we tilt by \(\Phi_\epsilon\) rather than \(\Phi\), and therefore have a Radon-Nikodym derivative given by the corresponding \(K^\epsilon\). Since the martingale does not change when one is near the origin, the Itô formula is valid to see that \(K^\epsilon_{t \wedge \tau}\) is a \(\mathbb{P}_a\) local martingale, and, in fact, for each \(t\), \(K^\epsilon_{s \wedge \tau}, 0 \leq s \leq t\) is a uniformly bounded martingale. Hence for \(s < t\), \(E[K^\epsilon_{t \wedge \tau} | \mathcal{F}_s] = K^\epsilon_{s \wedge \tau}\).

Then the function equals one near the origin, so there is no change in the martingale then. For \(\epsilon \leq X_t \leq z\), the Itô calculation is valid. Using properties of the reflected Bessel process (essentially
the fact that it spends zero time at the origin and hence the amount of time that it spends in \([0, 2\epsilon]\) goes to zero as \(\epsilon \to 0\), one can see that with \(\hat{P}_a\) probability one, \(K_{t\wedge \tau} \to K_{t\wedge \tau}\), and since this is uniformly bounded we can conclude that
\[
E[K_{t\wedge \tau} \mid F_s] = K_{s\wedge \tau}.
\]

It is easy to see that for a process satisfying the radial Bessel process, there is a positive probability that within time one it will hit \(z\). Therefore, the probability of avoiding \(z\) up to time \(t\) decays exponentially in \(t\),
\[
E[K_{t\wedge \tau} \mid \tau > t] \leq c e^{-\beta t}.
\]
Using this we see that
\[
\lim_{t \to \infty} E[K_{t\wedge \tau} \mid \tau \leq t] = E[K_0] = 1,
\]
and hence by the monotone convergence theorem,
\[
\]

One can check that the proof required more than the fact that \(x \sim \sin x\) near the origin. We needed that
\[
\sin x = x \left[1 - O(x^2)\right].
\]
An error term of \(O(x)\) would have not been good enough.

We can now describe how to construct the reflecting radial Bessel process.

- Run the paths until time \(\tau = T_{3\pi/4}\). Then
\[
\frac{d\hat{P}_a}{d\hat{P}_0} = K_\tau.
\]

- Let \(\sigma = \inf\{t \geq \tau : X_t = \pi/4\}\). The measure on \(X_t, \tau \leq t \leq \sigma\) is defined to be the measure obtained in the first step by reflecting the paths around \(x = \pi/2\).

- Continue in the same way.

Let \(\phi_t(x, y; a)\) denote the transition probability for the reflected process which is defined for \(0 \leq x, y < \pi\) and \(t > 0\). This is also defined for \(x = 0\) and \(x = \pi\) by taking the limit, but we restrict to \(0 \leq y < \pi\). We will use \(p_t(x, y; a)\) for the transition probability for the process killed at 0.

If \(\mu_0\) is any probability distribution on \([0, \pi]\), let \(\Phi_t\mu\) denote the distribution of \(X_t\) given \(X_0\) has distribution \(\mu_0\).

Lemma 3.12. If \(-1/2 < a < 1/2\), there exists \(c, \beta\) and a probability distribution \(\mu\) such that if \(\mu_0\) is any initial distribution and \(\mu_t = \Phi_t\mu_0\), then
\[
\|\mu - \mu_t\| \leq c e^{-\beta t}.
\]

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Proof. This uses a standard coupling argument. The key fact is that there exists \( \rho > 0 \) such that for every \( x \in [0, \pi] \), the probability that the process starting at \( x \) visits 0 by time 1 is at least \( \rho \).

Suppose \( \mu^1, \mu^2 \) are two different initial distributions. We start processes \( X_1, X_2 \) independently with distributions \( \mu^1, \mu^2 \). When the particles meet we coalesce the particles and they run together. If \( X_1 \leq X_2 \), then the coalescence time will be smaller than the time for \( X_2 \) to reach the origin. If \( X_1 \geq X_2 \), the time will be smaller than the time for \( X_1 \) to reach the origin. Hence the coalescence time is stochastically bounded by the time to reach the origin. Using the strong Markov property and the previous paragraph, the probability that \( T > n \) is bounded above by \( (1 - \rho)^n = e^{-\beta n} \) and \( \| \mu^1 - \mu^2 \| \) is bounded above by the probability that the paths have not coalesced by time \( t \). If \( s > t \), we can apply the same argument using initial probability distributions \( \mu^1_{s-t}, \mu^2_0 \) to see that

\[
\| \mu^1_{s-t} - \mu^2_t \| \leq c e^{-\beta t}, \quad s \geq t.
\]

Using completeness, we see that the limit measure

\[
\mu = \lim_{n \to \infty} \mu^1_n
\]

exists and satisfies (42).

The construction of the reflected process shows that \( \{ t : \sin X_t = 0 \} \) has zero measure which shows that the limiting measure must be carried on \( (0, \pi) \).

We claim that the invariant density is given again by \( f_a = C_{2a} \left( \sin x \right)^{2a} \). As mentioned before, it satisfies the adjoint equation

\[
-[m(x)f_a(x)]' + \frac{1}{2} f_a''(x) = 0, \quad \text{where } m(x) = a \cot x.
\]

Another way to see that the invariant density is proportional to \( (\sin x)^{2a} \) is to consider the process reflected at \( \pi/2 \). Let \( p_t(z, x) = p_t(z, x) + p_t(z, \pi - x) \) be the probability density for this reflected process. Suppose that \( 0 < x < y < \pi/2 \) and consider the relative amounts of time spent at \( x \) and \( y \) during an excursion from zero. If an excursion is to visit either \( x \) or \( y \), it must start by visiting \( x \). Given that it is \( x \), the amount of time spent at \( x \) before the excursion ends is

\[
\int_0^\infty p_t(x, x; a) \, dt,
\]

and the amount of time spent at \( y \) before the excursion ends is

\[
\int_0^\infty p_t(x, y; a) \, dt = \left[ \frac{\sin y}{\sin x} \right]^{2a} \int_0^\infty p_t(y, x; a) \, dt.
\]

The equality uses (??). The integral on the right-hand side gives the expected amount of time spent at \( x \) before reaching zero for the process starting at \( y \). However, if it starts at \( y \) it must hit \( x \) before reaching the origin. Hence by the strong Markov property,

\[
\int_0^\infty p_t(y, x; a) \, dt = \int_0^\infty p_t(x, x; a) \, dt,
\]

and hence,

\[
\int_0^\infty p_t(x, y; a) \, dt = \left[ \frac{\sin y}{\sin x} \right]^{2a} \int_0^\infty p_t(x, x; a) \, dt.
\]

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An important property of the radial Bessel process is the exponential rate of convergence to the equilibrium density. The next proposition gives a Harnack-type inequality that states within time one that one is within a multiplicative constant of the invariant density.

**Proposition 3.13.** For every $-1/2 < a < 1/2$ and $t_0 > 0$, there exists $c = c < \infty$ such that for every $0 < x, y < 2\pi$ and every $t \geq t_0$,

$$c^{-1} [\sin x]^{2a} \leq \phi_t(y, x; a) \leq c [\sin x]^{2a}.$$ 

**Proof.** By the Markov property it suffices to show that for each $s > 0$ there exists $c = c(s) < \infty$ such that

$$c^{-1} [\sin x]^{2a} \leq \phi_s(y, x; a) \leq c [\sin x]^{2a}. $$

We fix $s$ and allow constants to depend on $s$. We write $z = \pi/2$. By symmetry, we may assume that $x \leq \pi/2$, for which $\sin x \propto x$. 

By comparison with Brownian motion, it is easy to see that

$$\inf \{ \phi_t(z, y) : s/3 \leq t \leq s, \pi/4 \leq y \leq 3\pi/4 \} > 0.$$ 

Therefore, for any $0 \leq x \leq \pi/2$, $2s/3 \leq t \leq s$, $\pi/4 \leq y \leq 3\pi/4$,

$$\phi_t(x, y; a) \geq \mathbb{P}_a^x \{ T_z \leq s/3 \} \inf \{ \phi_r(z, y) : s/3 \leq r \leq s, \pi/4 \leq y \leq 3\pi/4 \} \geq c,$$

and hence for such $t$, using ???,

$$\phi_t(z, x; a) = (x/z)^{2a} \phi_t(x, z; a) \geq c x^{2a}.$$ 

Hence, for every $0 \leq y \leq \pi$,

$$x^{-2a} \phi_s(y, x; a) \geq \mathbb{P}_a^y \{ T_z \leq s/3 \} \inf \{ x^{-2a} \phi_r(z, x) : s/3 \leq r \leq s, 0 \leq y \leq \pi \} \geq c.$$ 

This gives the lower bound.

Our next claim is if $w = 3\pi/4$ and

$$\theta_1 := \sup \{ x^{-2a} \phi_t(w, x) : 0 \leq t \leq s, 0 \leq x \leq \pi/2 \},$$

then $\theta_1 < \infty$. To see this let $\phi_t^*(y, x)$ be the density of the process $X_{t\land T_\pi/w}$. Using (33) and absolute continuity, we can see that

$$\phi_t^*(w, x) \leq c x^{2a}.$$ 

However, by the strong Markov property, we can see that

$$x^{-2a} \phi_t(y, x) \leq x^{-2a} \phi_t^*(y, x) + \mathbb{P}_{x/8}^x \{ T_w \leq s \} \theta_1 w \leq x^{-2a} \phi_t^*(w, x) + (1 - \rho) \theta_1,$$

for some $\rho > 1$. Hence $\theta_1 \leq x^{-2a} \phi_t^*(w, x) \leq c/\rho < \infty$.

We now invoke Proposition 1.20 and absolute continuity, to see that for all $0 \leq y \leq 3\pi/4$ $\phi_s^*(y, x) \leq c x^{2a}$. Hence, by the Markov property,

$$\phi_s(y, x) \leq \phi_s^*(y, x) + \sup \{ \phi_t(w, x) : 0 \leq t \leq s \} \leq c x^{2a}.$$ 

\[\square\]
Proposition 3.14. For every $-1/2 < a < 1/2$, there exists $\beta > 0$ such that for all $t \geq 1$ and all $0 < x, y < \pi$,

$$\phi_t(x,y;a) = f_a(y) \left[ 1 + O(e^{-t\beta}) \right].$$

More precisely, for every $t_0 > 0$, there exists $c < \infty$ such that for all $x, y$ and all $t \geq t_0$,

$$f_a(y) \left[ 1 - ce^{-\beta t} \right] \leq \phi_t(x,y;a) \leq f_a(y) \left[ 1 + ce^{-\beta t} \right].$$

Proof. Exactly as in Proposition 3.5.

3.5 Functionals of Brownian motion

Now suppose $X_t$ is a Brownian motion with $X_0 = x \in (0,\pi)$. Let $T = \inf\{t : \Theta_t = 0 \text{ or } \pi\}$ and let $I_t$ denote the indicator function of the event $\{T > t\}$. Suppose that $\lambda > -1/8$ and let

$$a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8\lambda} \geq \frac{1}{2},$$

be the larger root of the polynomial $a^2 - a - 2\lambda$. Let

$$J_t = \exp \left\{ - \int_0^t \frac{ds}{S_s^2} \right\}.$$

If $M_{t,a}$ denotes the martingale in (??), then we can write

$$M_{t,a} = \left[ \frac{S_t}{S_0} \right]^a J^\lambda_a, \quad \text{where } \lambda_a = \frac{a(a-1)}{2}.$$

Proposition 3.15. Suppose $\lambda \geq -1/8$. Then there exists $\beta = \beta(\lambda) > 0$ such that

$$\mathbb{E}^x[J^\lambda_t I_t] = \left[ C_{2a}/C_a \right] (\sin x)^a e^{-at} \left[ 1 + O(e^{-\beta t}) \right],$$

where

$$a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8\lambda} \geq \frac{1}{2}. \quad (43)$$

Proof. Let $a$ be defined as in (43). Then,

$$\mathbb{E}^x[J^\lambda_t I_t] = (\sin x)^a e^{-at} \mathbb{E}^x \left[ M_{t,a} I_t S_t^{-a} \right]$$

$$= (\sin x)^a e^{-at} \mathbb{E}^x \left[ I_t S_t^{-a} \right]$$

$$= (\sin x)^a e^{-at} \int_0^\pi p_t(x,y;a) [\sin y]^{-a} dy.$$

$$= c' (\sin x)^a e^{-at} \left[ 1 + O(e^{-\beta t}) \right].$$

Here $\beta = \beta_a$ is the exponent from Proposition 3.14 and

$$c' = \int_0^\pi f_a(y) [\sin y]^{-a} dy = C_{2a}/C_a.$$

Note that in the third line we could drop the $I_t$ term since $\mathbb{P}_x\{I_t = 1\} = 1$. \qed
Proposition 3.16. Suppose $b \in \mathbb{R}$ and
\[
\lambda + \lambda_b \geq -\frac{1}{8}.
\] (44)

Let
\[
a = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8(\lambda + \lambda_b)} \geq \frac{1}{2},
\] (45)

and assume that $a + b > -1$. Then, there exists $\beta = \beta(\lambda, b) > 0$ such that
\[
\mathbb{E}_b^x[J_t^\lambda I_t] = [C_{2a}/C_{a+b}] (\sin x)^{a-b} e^{(b-a)t} [1 + O(e^{-\beta t})].
\]

Proof. Let $a$ be as in (45) and note that $\lambda_a = \lambda + \lambda_b$.
\[
\mathbb{E}_b^x[J_t^\lambda I_t] = \mathbb{E}_x^x[M_{t,b} J_t^\lambda I_t]
\]
\[
= (\sin x)^{-b} e^{bt} \mathbb{E}_x^x[S_t^b J_t^{\lambda+b} I_t]
\]
\[
= (\sin x)^{a-b} e^{(b-a)t} \mathbb{E}_x^x[S_t^{b-a} M_{t,a} I_t]
\]
\[
= (\sin x)^{a-b} e^{(b-a)t} \mathbb{E}_a^x[S_t^{b-a}]
\]
\[
= (\sin x)^{a-b} e^{(b-a)t} \int_0^\pi p_t(x,y; a) [\sin y]^{b-a} dy
\]
\[
= c'(\sin x)^{a-b} e^{(b-a)t} [1 + O(e^{-\beta t})].
\]

Here $\beta = \beta_a$ is the exponent from Proposition ?? and
\[
c' = \int_0^\pi f_a(y) [\sin y]^{b-a} dy = C_{2a}/C_{a+b}.
\]

The fourth equality uses the fact that $\mathbb{P}_a^x \{I_t = 1\} = 1$. \hfill $\square$

4 General process

We will consider general processes of the form
\[
dX_t = m(X_t) \, dt + dB_t, \quad m(x) = \frac{\Phi'(x)}{\Phi(x)}, \quad 0 < t < \pi.
\]

Here we assume that $\Phi : (0, \pi) \to (0, \infty)$ is a $C^2$ function. Equivalently, we assume that $m : (0, \pi) \to (-\infty, \infty)$ is a $C^1$ function and let
\[
\Phi(x) = \exp \left\{ - \int_x^{\pi/2} m(y) \, dy \right\}.
\]

Such an equation is obtained by starting with a standard Brownian motion $X_t$ and then “weighting locally” by $\Phi(X_t)$. To be more precise, suppose that with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
\( X_t \) is a standard Brownian motion with filtration \( \{ \mathcal{F}_t \} \) with \( X_0 \in (0, \pi) \) and \( \Phi(X_0) > 0 \). Let \( T = \inf\{ t : X_t \in \{0, \pi\} \} \). If

\[
M_t = \Phi(X_t) A_t, \quad A_t = \exp \left\{ -\frac{1}{2} \int_0^t \frac{\Phi''(X_s)}{\Phi'(X_s)} \, ds \right\},
\]

then Itô’s formula shows that \( M_t, t < T \) is a local martingale in \([0, T)\) satisfying

\[
dM_t = m(X_t) M_t \, dX_t, \quad t < T.
\]

If \( \epsilon > 0 \), \( T_\epsilon = \inf\{ t : \sin X_t \leq \epsilon \} \), and \( \tau \) is a stopping time with \( \tau < T_\epsilon \), then \( M_t \wedge \tau \) is a martingale that is uniformly bounded for \( 0 \leq t \leq t_0 \) (with a \( (t_0, \epsilon) \)-dependent bound). Let \( P_\Phi, E_\Phi \) denote probabilities and expectations obtained by tilting by the local martingale \( M_t \); more precisely, if \( Y \) is \( \mathcal{F}_\tau \)-measurable, then

\[
E_\Phi[Y] = M_0^{-1} E[M_\tau Y] = \Phi(X_0)^{-1} E[M_\tau Y].
\]

The Girsanov theorem implies that

\[
dX_t = m(X_t) \, dt + dB_t, \quad 0 < t < T,
\]

where \( B_t \) is a standard Brownian motion with respect to the measure \( P_\Phi \).

We will start by putting a strong assumption on \( \Phi \) which is too strong for applications but it is a good starting point.

**Assumption 1.** \( \Phi \) is a \( C^2 \) function on \( \mathbb{R} \) that is strictly positive on \( (0, \pi) \) such that for all \( \epsilon \) sufficiently small,

\[
\Phi(\epsilon) \leq 2 \epsilon \Phi'(\epsilon), \quad \Phi(\pi - \epsilon) \leq -2 \epsilon \Phi'(\pi - \epsilon).
\]

In particular, \( \Phi(0) = \Phi(\pi) = 0 \). Equivalently,

\[
m(x) \geq \frac{1}{2x}, \quad m(\pi - x) \leq -\frac{1}{2x} \quad 0 < x < \epsilon,
\]

Let \( T = \inf\{ t : X_t \in \{0, \pi\} \} \). We claim that \( P_\Phi \{ T < \infty \} = 0 \). Indeed, this follows immediately by comparison with a Bessel process.

For \( 0 < x, y < \pi \), let \( p(t, x, y) \) denote the density at time \( t \) of a Brownian motion starting at \( x \) that is killed when it leaves \((0, \pi)\), that is

\[
P^x \{ y_1 < X_t < y_2; T > t \} = \int_{y_1}^{y_2} p(t, x, y) \, dy.
\]

Let \( p_\Phi(t, x, \cdot) \) be the corresponding density for \( X_t \) under the tilted measure \( P_\Phi \). Since \( P_\Phi \{ T = \infty \} = 0 \), \( p_\Phi(t, x, \cdot) \) is a probability density,

\[
P^x_\Phi \{ y_1 < X_t < y_2 \} = P^x_\Phi \{ y_1 < X_t < y_2; T < \infty \} = \int_{y_1}^{y_2} p_\Phi(t, x, y) \, dy.
\]

We note that \( P^x_\Phi \ll P^x \) with Radon-Nikodym derivative

\[
\frac{dP^x_\Phi}{dP^x} = M_{t \wedge T} = \mathbf{1}\{T > t\} \frac{\Phi(X_t)}{\Phi(X_0)} A_t,
\]

where \( A_t \) is as in (46).
Lemma 4.1. If $0 < x < y < \pi$ and $t > 0$, then under Assumption 1,

$$p_\Phi(t, x, y) \Phi(x)^2 = p_\Phi(t, y, x) \Phi(y)^2.$$  

In particular,

$$\int_0^\pi \Phi(x)^2 p_\Phi(t, x, y) \, dy = \Phi(y)^2.$$  

The second conclusion can be restated by saying that the invariant probability density for the diffusion given by (47) is $c \Phi^2$ where $c$ is chosen to make this a probability density.

Proof. For each path $\omega : [0, t] \to (0, \pi)$ with $\omega(0) = x, \omega(t) = y$, let $\omega^R$ denote the reverse path $\omega^R(s) = \omega(t - s)$ which goes from $y$ to $x$. The Brownian measure on paths from $y$ to $x$ staying in $(0, \pi)$ can be obtained from the corresponding measure for paths from $x$ to $y$ by the transformation $\omega \mapsto \omega^R$. The compensator term $A_t$ is a function of the path $\omega$; indeed

$$A_t(\omega) = \exp \left\{ -\frac{1}{2} \int_0^t \Phi''(\omega(s)) \Phi(\omega(s)) \, ds \right\}.$$  

Then

$$p_\Phi(t, x, y) = p_\Phi(t, x, y) \frac{\Phi(y)}{\Phi(x)} \mathbb{E}_{x,y,t}[A_t], \quad p_\Phi(t, y, x) = p_\Phi(t, y, x) \frac{\Phi(x)}{\Phi(y)} \mathbb{E}_{y,x,t}[A_t],$$

where $\mathbb{E}_{x,y,t}$ denotes the probability measure associated to Brownian bridges started at $x$, at $y$ at time $t$, and staying in $(0, \pi)$. But the path reversal argument shows that $\mathbb{E}_{x,y,t}[A_t] = \mathbb{E}_{y,x,t}[A_t]$. □

For later we give the following.

Proposition 4.2. Suppose $\Phi$ satisfies Assumption 1, and define

$$F(x) = \int_x^{\pi/2} \frac{dy}{\Phi(y)^2}.$$  

If $\epsilon < X_0 < \pi/2$, and $\tau = T_\epsilon \wedge T_{\pi/2}$, then $F(X_{t \wedge \tau})$ is a $\mathbb{P}_\Phi$-martingale. In particular,

$$\mathbb{P}_\Phi \{ T_\epsilon < T_{\pi/2} \} = \frac{F(x)}{F(\epsilon)}. \quad (48)$$

Proof. Note that

$$F'(x) = -\frac{1}{\Phi(x)^2}, \quad F''(x) = \frac{2 \Phi'(x)}{\Phi(x)^3} = \frac{2 m(x)}{\Phi(x)^2}$$

and hence

$$dF(X_t) = F'(X_t) dX_t + \frac{1}{2} F''(X_t) dt = -\frac{1}{\Phi(X_t)^2} dB_t.$$  

Using this we see that $F(X_{t \wedge \tau})$ is $\mathbb{P}_\Phi$-martingale, and the optional sampling theorem implies that

$$F(x) = \mathbb{E}_\Phi[F(X_\tau)] = F(\epsilon) \mathbb{P}_\Phi \{ T_\epsilon < T_{\pi/2} \},$$

from which (48) follows. □
Suppose $\Phi_1, \Phi_2$ are two functions satisfying Assumption 1 with corresponding drifts $m_1, m_2$, and suppose that $\Phi_1(x) = \Phi_2(x)$ (and, hence, $m_1(x) = m_2(x)$) for $\epsilon \leq x \leq \pi - \epsilon$. Let us view the process $X_t$ only during the excursions from $2\epsilon$ or $\pi - 2\epsilon$ until it reaches $\epsilon$ or $\pi - \epsilon$. To be more precise, suppose that $X_0 = x \in [2\epsilon, \pi - 2\epsilon]$. Let $\sigma_1 = 0$ and recursively,

$$\tau_j = \inf \{ t \geq \sigma_j : X_j \in \{ \epsilon, \pi - \epsilon \} \},$$

$$\sigma_{j+1} = \inf \{ t \geq \tau_j : X_j \in \{ 2\epsilon, \pi - 2\epsilon \} \}.$$

We define the $j$th excursion by $X_{t+\sigma_j}, \ 0 \leq t \leq \tau_j - \sigma_j$.

It is easy to see that the distribution of the excursions is the same whether we choose $\Phi_1$ or $\Phi_2$. This will tell us how to define a process for a positive $C^2$ function that does not satisfy Assumption 1. Given $m(x)$, it is easy to see that we can find a $C_1$ function $m_1(x)$ such that $m_1(x) = m(x)$ for $\epsilon \leq x \leq \pi - \epsilon$ and $m_1(\pi - x) = 1/x$ for $0 < x < \epsilon/2$. This constructs the process during the excursions. Describing the process for all the excursions will define the process provided that no time is spent on the boundary (or, essentially equivalently, the fraction of time spend with distance $\delta$ of the boundary converges to zero as $\delta \to 0$). The invariant density describes the fraction of time spent at a point in the interior and the ratio of time spent at $x$ to that spent at $y$ is $\Phi(x)^2 \Phi(y)^2$.

Our next assumption will guarantee that we spend zero time at the boundary, and will allow us to define a reflected version of the process.

- **Assumption 2.** $\Phi$ is a positive $C^2$ function on $(0, \infty)$ with

$$\int_0^\pi \Phi(x)^2 \, dx < \infty,$$

and for $\epsilon$ sufficiently small,

$$\Phi'(\epsilon) \geq -2\epsilon \Phi(\epsilon), \quad \Phi'(\pi - \epsilon) \leq 2\epsilon \Phi(\pi - \epsilon).$$

If, for example, one assumes that $\Phi(x) \sim x^\xi, \Phi'(x) \sim \xi x^{\xi-1}$ as $x \downarrow 0$, the first condition requires that $\xi > -1/2$ and the second condition would hold automatically.

For the remainder, we assume that $\Phi$ satisfies Assumption 2 and let

$$F(x) = \int_x^{\pi/2} \frac{dy}{\Phi(y)^2}.$$ 

for $0 < \epsilon < \pi/4$, we define

$$\Phi_\epsilon(x) = \Phi(x) \phi_\epsilon(x) \phi_\epsilon(\pi - x).$$

It is easy to check that $\Phi_\epsilon$ satisfies Assumption 1; $\Phi_\epsilon \leq \Phi$; and

$$\Phi(x) = \Phi_\epsilon(x), \quad 2\epsilon \leq x \leq \pi - 2\epsilon.$$

Then the measure on paths $\Phi$ is the weak limit of the measure on paths induced by $\Phi_\epsilon$.

Suppose $0 < y_0 < y_1 < y_2 < y_3 < \pi$. Assume we start at $y_1$ and let $\sigma_1 = 0$,

$$\tau_j = \inf \{ t > \sigma_j : X_t = y_2 \}. $$
\[ \sigma_j = \inf \{ t > \tau_{j-1} : X_t = y_1 \} , \]

Let

\[ Y_j = \int_{\sigma_j}^{\tau_j} 1\{ y_0 \leq Y_t \leq y_1 \} \, dt, \quad Z_j = \int_{\tau_j}^{\sigma_{j+1}} 1\{ y_2 < X_t < y_3 \} \, dt. \]

Note that the random variables \( Y_1, Y_2, \ldots \) and \( Z_1, Z_2, \ldots \) are mutually independent. The amount of time spent in \([y_0, y_1]\) by time \( \tau_n \) is \( Y_1 + \cdots + Y_n \) and the amount of time spent in \([y_2, y_3]\) by time \( \tau_n \) is \( Z_1 + \cdots + Z_n \). We know that the ratio of these two quantities must have a limit of

\[ \frac{\int_{y_0}^{y_1} \Phi(x)^2 \, dx}{\int_{y_2}^{y_3} \Phi(x)^2 \, dx}, \]

and hence (using standard renewal arguments), we see that

\[ \frac{\mathbb{E}[Y_1]}{\mathbb{E}[Z_1]} = \frac{\int_{y_0}^{y_1} \Phi(x)^2 \, dx}{\int_{y_2}^{y_3} \Phi(x)^2 \, dx}. \]

This is true for any \( \epsilon \) “cutoff” with \( \epsilon \leq y_0 \). We can therefore, let \( \epsilon \to 0 \), and see that for the process with no cutoff, if \( y_0 = 0 \), we still have that the ratio is

\[ \frac{\int_{y_0}^{y_1} \Phi(x)^2 \, dx}{\int_{y_2}^{y_3} \Phi(x)^2 \, dx}. \]

Here we have used the fact that \( \Phi(x)^2 \) is integrable on \((0, \pi)\). Therefore, the amount of time spent in \([0, y_1]\) tends to zero as \( y_1 \to 0 \).

We can build the reflected process as follows. For every \( \epsilon < 1/4 \), let \( Y_1, Y_2, \ldots \) be independent random variables each distributed as \( T_{2\epsilon} \) assuming \( X_0 = \epsilon \). Similarly, let \( Z_1, Z_2, \ldots \), be independent random variables each distributed at \( T_{\pi-2\epsilon} \) assuming \( X_0 = \pi - \epsilon \). Suppose \( 2\epsilon < X_0 < \pi - 2\epsilon \). As before let \( \sigma_0 = 0 \),

\[ \tau_j = \inf \{ t > \sigma_j : X_t \in \{ \epsilon, \pi - \epsilon \} \}, \]

\[ \sigma_{j+1} = \inf \{ t > \tau_j : X_t \in \{ 2\epsilon, \pi - 2\epsilon \} \}. \]

We now imagine that we only observe the excursion times \( \tau_0, \sigma_1 - \tau_0, \tau_1 - \sigma_1, \ldots \) and the excursions \( X_t, \tau_j \leq t \leq \sigma_{j+1} \).

5 Appendix

5.1 Asymptotics of \( h_a \)

Suppose \( a > -1/2 \) and

\[ h_a(z) = \sum_{k=0}^{\infty} c_k z^{2k} \text{ where } c_k = c_{k,a} = \frac{1}{2^{a+2k+1/2} k! \Gamma(k+a+1/2)}. \]

We note that the modified Bessel function of the first kind of order \( \nu \) is given by \( I_\nu(z) = z^\nu h_{\nu+\frac{1}{2}}(z) \). What we are discussing in this appendix are well known facts about \( I_\nu \), but we will state and prove
them for the analytic function $h_a$. Since $c_k$ decays like $[2^k k!]^{-2}$, is easy to see that the series has an infinite radius of convergence, and hence $h_a$ is an entire function. Note that the $c_k$ are given recursively by

$$
c_0 = \frac{2^{\frac{1}{2} - a}}{\Gamma(a + \frac{1}{2})}, \quad c_{k+1} = \frac{c_k}{(2k + 2)(2k + 2a + 1)}. \tag{49}
$$

**Proposition 5.1.** $h_a$ is the unique solution to

$$zh''(z) + 2ah'(z) - zh(z) = 0. \tag{50}$$

with

$$h(0) = \frac{2^{\frac{1}{2} - a}}{\Gamma(a + \frac{1}{2})}, \quad h'(0) = 0.$$

**Proof.** Using term-by-term differentiation and (49), we see that $h_a$ satisfies (50). A second, linearly independent solution of (50) can be given by

$$\tilde{h}_a(z) = \sum_{k=1}^\infty \tilde{c}_{k-1} z^{2k-1},$$

where $\tilde{c}_k$ are defined recursively by

$$\tilde{c}_0 = 1, \quad \tilde{c}_k = \frac{\tilde{c}_{k-1}}{(2k + 1)(2k + 2a)}.$$

Note that $\tilde{h}_a(0) = 0, \tilde{h}_a'(0) = 1$. By the uniqueness of second-order linear differential equations, every solution to (50) can be written as $h(z) = \lambda h_a(z) + \tilde{\lambda} \tilde{h}_a(z)$, and only $\lambda = 1, \tilde{\lambda} = 0$ satisfies the initial condition. \qed

**Proposition 5.2.** Suppose $h$ satisfies (50), and

$$\phi(x, y) = y^{2a} \exp \left\{ -\frac{x^2 + y^2}{2} \right\} h(xy).$$

Let

$$q_t(x, y; a) = \frac{1}{\sqrt{t}} \phi(x/\sqrt{t}, y/\sqrt{t}).$$

Then for every $t$,

$$\partial_t q_t(x, y; a) = L_x q_t(x, y; a) = L^* q_t(x, y; a),$$

where

$$Lf(x) = \frac{a}{x} f'(x) + \frac{1}{2} f''(x),$$

$$L^* f(x) = \frac{a}{x^2} f(x) - \frac{a}{x} f'(x) + \frac{1}{2} f''(x).$$
Proof. This is a straightforward computation. We first establish the equalities at $t = 1$. Note that

$$\partial_t q_t(x, y; a) \big|_{t=1} = -\frac{1}{2} \left[ \phi(x, y) + x \phi_x(x, y) + y \phi_y(x, y) \right].$$

Hence we need to show that

$$\phi_{xx}(x, y) + \left[ \frac{2a}{x} + x \right] \phi_x(x, y) + y \phi_y(x, y) + \phi(x, y) = 0,$$

$$\phi_{yy}(x, y) + x \phi_x(x, y) + \left[ y - \frac{2a}{y} \right] \phi_y(x, y) + \left[ \frac{2a}{y^2} + 1 \right] \phi(x, y) = 0.$$

Direct computation gives

$$\phi_x(x, y) = \left[ -x + y \frac{h'(xy)}{h(xy)} \right] \phi(x, y),$$

$$\phi_{xx}(x, y) = \left[ -1 + \frac{h''(xy)}{h(xy)} y^2 - 2xy \frac{h'(xy)}{h(xy)} + x^2 \right] \phi(x, y),$$

$$\phi_y(x, y) = \left[ \frac{2a}{y} - y + x \frac{h'(xy)}{h(xy)} \right] \phi(x, y),$$

$$\phi_{yy}(x, y) = \left[ -1 - 4a + \frac{4a^2 - 2a}{y^2} + y^2 + x^2 \frac{h''(xy)}{h(xy)} + \left( \frac{4ax}{y} - 2xy \right) \frac{h'(xy)}{h(xy)} \right] \phi(x, y).$$

If $h$ satisfies (15), then

$$\frac{h''(xy)}{h(xy)} = 1 - \frac{2a}{xy} \frac{h'(xy)}{h(xy)},$$

so we can write

$$\phi_{xx}(x, y) = \left[ -1 + x^2 + y^2 + \left( -2xy - \frac{2ay}{x} \right) \frac{h'(xy)}{h(xy)} \right] \phi(x, y),$$

$$\phi_{yy}(x, y) = \left[ -1 - 4a + x^2 + y^2 + \frac{4a^2 - 2a}{y^2} + \left( \frac{4ax}{y} - 2xy \right) \frac{h'(xy)}{h(xy)} \right] \phi(x, y).$$

This gives the required relation.

For more general $t$, note that

$$\partial_t q_t(z, w; a) = \frac{1}{2t^{3/2}} \left[ \phi(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) - \phi_y(z/\sqrt{t}, w/\sqrt{t}; a) \right].$$

$$\partial_x q_t(z, w; a) = \frac{1}{t} \phi_x(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_{xx} q_t(z, w; a) = \frac{1}{t^{3/2}} \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_y q_t(z, w; a) = \frac{1}{t} \phi_y(z/\sqrt{t}, w/\sqrt{t}; a),$$

$$\partial_{yy} q_t(z, w; a) = \frac{1}{t^{3/2}} \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a).$$
\[ L_x q_t(z, w; a) = \frac{a}{(z/\sqrt{t})} 1_{\frac{3}{2}} \phi_x(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2} 1_{\frac{3}{2}} \phi_{xx}(z/\sqrt{t}, w/\sqrt{t}; a), \]

\[ L^*_w q_t(z, w; a) = \frac{a}{(w/\sqrt{t})} 1_{\frac{3}{2}} \phi(z/\sqrt{t}, w/\sqrt{t}; a) - \frac{a}{(w/\sqrt{t})} 1_{\frac{3}{2}} \phi_y(z/\sqrt{t}, w/\sqrt{t}; a) + \frac{1}{2} 1_{\frac{3}{2}} \phi_{yy}(z/\sqrt{t}, w/\sqrt{t}; a), \]

\[ \text{Proposition 5.3. If } h \text{ satisfies (50), then exist an analytic function } u \text{ with } u(0) \neq 0 \text{ such that for all } x > 0, \]

\[ h(x) = x^{-a} e^x u(1/x). \]

Proof. Let

\[ v(x) = e^{-x} x^a h_a(x). \]

Then,

\[ v'(x) = v(x) \left[ -1 + \frac{a}{x} + \frac{h'_a(x)}{h_a(x)} \right], \]

\[ v''(x) = v(x) \left[ -1 + \frac{a}{x} + \frac{h'_a(x)}{h_a(x)} \right]^2 - \frac{a}{x^2} + \frac{h''_a(x)}{h_a(x)} - \frac{h''_a(x)}{h_a(x)^2} \]

\[ = v(x) \left[ 1 + \frac{a^2 - a}{x^2} - \frac{2a}{x} + \left( \frac{2a}{x} - 2 \right) \frac{h'_a(x)}{h_a(x)} + \frac{h''_a(x)}{h_a(x)} \right] \]

\[ = v(x) \left[ 2 + \frac{a^2 - a}{x^2} - \frac{2a}{x} - \frac{2h'_a(x)}{h_a(x)} \right] \]

\[ = -2v'(x) + \frac{a^2 - a}{x^2} v(x). \]

The third equality uses the fact that \( h_a \) satisfies (50). If \( u_a(x) = v(1/x) \), then

\[ u'(x) = -\frac{1}{x^2} v'(1/x), \]

\[ u''_a(x) = \frac{2}{x^4} v'(1/x) + \frac{1}{x^4} v''(1/x) \]

\[ = \left[ \frac{2}{x^3} - \frac{2}{x^4} \right] v'(1/x) + \frac{a^2 - a}{x^2} v(1/x) \]

\[ = \left[ \frac{2}{x^2} - \frac{2}{x} \right] u'_a((x) + \frac{a^2 - a}{x^2} u_a((x). \]

In other words, \( u \) satisfies the equation

\[ x^2 u''(x) + (2 - 2x) u'(x) + (a^2 - a) u(x) = 0. \]

We can find two linearly independent entire solutions to this equation of the form

\[ u(z) = \sum_{k=0}^{\infty} b_k z^k \]
by choosing \( b_0 = 1, b_1 = 0 \) or \( b_0 = 0, b_1 = 1 \), and the recursively,
\[
b_{k+2} = \frac{(2k + a - a^2) b_k - 2(k + 1)b_{k+1}}{(k+1)(k+2)}.
\]

Then,
\[
u(z) = \sum_{k=0}^{\infty} b_k z^k,
\]
\[
u'(x) = \sum_{k=0}^{\infty} (k+1)b_{k+1} z^k,
\]
\[
z
\nu'(z) = \sum_{k=1}^{\infty} k b_k z^k,
\]
\[
u''(z) = \sum_{k=0}^{\infty} (k+1)(k+2)b_{k+2} z^k,
\]
then the differential equation induces the relation
\[
b_{k+2} = \frac{(2k + a - a^2) b_k - 2(k + 1)b_{k+1}}{(k+1)(k+2)}.
\]

Note that
\[
|b_{k+2}| \leq \frac{2}{k+2} (|b_k| + |b_{k+1}|),
\]
from which we can conclude that the power series converges absolutely for all \( z \). By uniqueness \( u_a(x) \) must be a linear combination of these solutions and hence must be the restriction of an entire function to the real line.

5.2 Some integral identities

In this subsection we establish two “obvious” facts about the density by direct computation. We first prove that \( \psi_t(x, \cdot; a) \) is a probability density.

**Proposition 5.4.** For every \( a > -1/2 \) and \( x > 0 \),
\[
\int_0^{\infty} \psi_t(x, y; a) \, dy = 1.
\]

We use a known relation about special functions, that we state here.

**Lemma 5.5.** If \( a > -1/2 \) and \( x > 0 \),
\[
\int_0^{\infty} z^{2\alpha} \exp\left\{ -\frac{z^2}{2x^2} \right\} h_a(z) \, dz = x^{2\alpha-1} e^{x^2/2}.
\]
Proof. If we let \( \nu = a - \frac{1}{2} \) and \( r = x^2 \), we see this is equivalent to
\[
\int_{0}^{\infty} z^{\nu+1} \exp \left\{ -\frac{z^2}{2r} \right\} I_{\nu}(z) \, dz = r^{\nu+1} e^{r/2}.
\]

Volume 2 of Prudnikov-Bryckov-Marychev has the formula
\[
\int_{0}^{\infty} x^{b-1} e^{-px^2} I_{\nu}(x) \, dx = 2^{-\nu-1} p^{-\frac{b+\nu}{2}} \Gamma\left(\frac{b+\nu}{2}\right) \frac{\Gamma\left(\nu + 1\right)}{\Gamma(\nu + 1)} M\left(b + 1, \nu + 1, \frac{1}{4p}\right),
\]
where \( M(a, b, z) \) is the confluent hypergeometric function. In our case, \( b = \nu + 2, p = 1/(2r) \), so the right-hand side equals
\[
2^{-\nu-1} (1/2r)^{-\nu-\frac{1}{2}} {}_1F_1(\nu + 1, \nu + 1, r/2) = r^{\nu+1} M(\nu + 1, \nu + 1, r/2) = r^{\nu+1} e^{r/2}.
\]
The last equality is a well known identity about confluent hypergeometric functions. \( \square \)

**Proof of 5.4.** Since
\[
\int_{0}^{\infty} \psi_{t}(x, y; a) \, dy = \int_{0}^{\infty} t^{-1/2} \psi_{1}\left(x/\sqrt{t}, y/\sqrt{t}; a\right) \, dy = \int_{0}^{\infty} \psi_{1}(x/\sqrt{t}, z; a) \, dz,
\]
it suffices to show that for all \( x \),
\[
\int_{0}^{\infty} \psi(x, y) \, dy = 1
\]
where
\[
\psi(x, y) = \psi_{1}(x, y; a) = y^{2a} \exp \left\{ -\frac{x^2 + y^2}{2} \right\} h_{a}(xy).
\]
Using the substitution \( xy = z \), we see that
\[
\int_{0}^{\infty} \psi(x, y) \, dy = e^{-x^2/2} \int_{0}^{\infty} y^{2a} \exp \left\{ -\frac{y^2}{2} \right\} h_{a}(xy) \, dy = x^{-2a-1} e^{-x^2/2} \int_{0}^{\infty} z^{2a} \exp \left\{ -\frac{z^2}{2x^2} \right\} h_{a}(z) \, dz = 1
\]
\( \square \)

We now show that \( \psi_{t}(x, y; a) \) satisfies the Chapman-Kolomogrov equations.

**Proposition 5.6.** if \( a > -1/2, 0 < t < 1 \) and \( x > 0 \), then
\[
\int_{0}^{\infty} \psi_{t}(x, z; a) \psi_{1-t}(z, y; a) \, dz = \psi_{1}(xy; a).
\]

**Proof.** This is equivalent to
\[
t^{-a-\frac{1}{2}} (1-t)^{-a-\frac{1}{2}} \int_{0}^{\infty} z^{2a} \exp \left\{ -\frac{x^2 + z^2}{2t} \right\} \exp \left\{ -\frac{z^2 + y^2}{2(1-t)} \right\} h_{a}\left(\frac{xz}{t}\right) h_{a}\left(\frac{zy}{1-t}\right) \, dz = \exp \left\{ -\frac{x^2 + y^2}{2} \right\} h_{a}(xy),
\]
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which is equivalent to saying that $h_a(xy)$ equals
\[
\exp \left\{ -\frac{x^2(1-t)}{2t} - \frac{y^2t}{2(1-t)} \right\} [t(1-t)]^{-a-\frac{1}{2}} \int_0^\infty z^{2a} \exp \left\{ -\frac{z^2}{2t(1-t)} \right\} h_a \left( \frac{xz}{t} \right) \ h_a \left( \frac{zy}{1-t} \right) \ dz.
\]
If we set $\nu = a - \frac{1}{2}$,
\[
\int_0^\infty z^{2a} \exp \left\{ -\frac{z^2}{2t(1-t)} \right\} h_a \left( \frac{xz}{t} \right) \ h_a \left( \frac{zy}{1-t} \right) \ dz
= \left( \frac{t}{x} \right)^{a-\frac{1}{2}} \left( \frac{1-t}{y} \right)^{a-\frac{1}{2}} \int_0^\infty z \exp \left\{ -\frac{z^2}{2t(1-t)} \right\} I_\nu \left( \frac{xz}{t} \right) I_\nu \left( \frac{zy}{1-t} \right) \ dz
= \left( \frac{t}{x} \right)^{a-\frac{1}{2}} \left( \frac{1-t}{y} \right)^{a-\frac{1}{2}} \exp \left\{ \frac{x^2(1-t)}{2t} + \frac{y^2t}{2(1-t)} \right\} I_\nu(xy)
= [t(1-t)]^{a-\frac{1}{2}} \exp \left\{ \frac{x^2(1-t)}{2t} + \frac{y^2t}{2(1-t)} \right\} h_a(xy)
\]
The second equality is Equation 2.15.20 #8 on p.321 of Prudnikov-Bryckov-Marychev, Vol. 2.