Fourier Series and the Fourier Transform

Ariel Barton

September 18, 2006

1 Fourier series

Consider periodic functions, functions defined on some finite interval, or functions defined on the unit circle. All are equivalent; I prefer to think of functions defined on [0, 1].

We observe:

\[
\int_0^1 e^{2\pi inx}e^{2\pi imx} \, dx = 1, \quad \text{if } n = m; \quad 0, \quad \text{if } n \neq m.
\]

So \(\{e^{2\pi inx}\}\) is an orthonormal set in the Hilbert space \(L^2([0,1])\). Let \(U_n(x) = e^{2\pi inx}\).

**Question.** Is \(\{U_n\}\) an orthonormal basis for \(L^2([0,1])\)?

**Answer.** Yes!

**Sketch of proof** It’s a basis if for any \(f \in L^2\), there exists a \(f : \mathbb{Z} \rightarrow \mathbb{C}\) such that in \(L^2\) norm, \(f\) is the limit of the following sequence of trigonometric polynomials:

\[
\left\{ \sum_{|n| \leq N} \hat{f}(n)U_n \right\}_{N=1}^\infty
\]

I will only show that every \(L^2\) function is the limit of some sequence of trigonometric polynomials.

**Useful fact.** If \(f \in L^2(\mathbb{R})\) or if \(f \in L^2([0,1])\), then for every \(\epsilon > 0\), there exists an \(h \in L^2\) such that \(h\) is uniformly continuous and \(||f - h||_{L^2} < \epsilon\). (You will prove this fact in first-quarter analysis.)

If \(f\) is continuous, let

\[
F_k(x) = \int_0^1 f(y)Q_k(x-y) \, dy
\]

where \(Q_k(x) = c_k[1 + \cos(2\pi x)]^k = c_k[1 + \frac{1}{2}U_1(x) + \frac{1}{2}U_{-1}(x)]^k\), \(c_k\) chosen so \(\int_0^1 Q_k = 1\).
Then we can write
\[ Q_k(x - y) = \sum_{|n| \leq k, |m| \leq k} C_{k,n,m} U_n(x) U_m(y) \]
for some constants \( C_{k,n,m} \). So \( F_N(x) \) is a trigonometric polynomial.

But as \( k \to \infty \), \( Q_k \) becomes very small away from the integers, and so \( F_N(x) \to f \) pointwise if \( f \) is continuous.

So the \( \{U_n\} \) are a basis; if \( f \in L^2 \), then \( f = \sum_n \hat{f}(n) U_n \), for some \( \hat{f}(n) \in \mathbb{R} \). Note that convergence is in \( L^2 \), and in \( L^2 \) only. Fourier series in general do not converge pointwise. (They do converge pointwise if \( f \) is, for example, differentiable.)

We can write down a formula for the \( \hat{f}(n) \):
\[ \hat{f}(n) = \langle f, U_n \rangle = \int_0^1 e^{-2\pi i nx} f(x) \, dx. \]

**Parseval’s Inequality:** Since \( \{U_n\} \) is a basis,
\[ \sum_n |\hat{f}(n)|^2 = \left\langle \sum_n \hat{f}(n) U_n, \sum_m \hat{f}(m) U_m \right\rangle = \langle f, f \rangle = ||f||^2_{L^2}. \]

In particular, \( \hat{f}(n) \to 0 \) as \( n \to \infty \) for any \( f \in L^2 \).

Note that we may define \( \hat{f}(n) \) for \( f \in L^1([0,1]) \), via the above integral. In this case, we still have that \( \hat{f}(n) \to 0 \) as \( n \to \infty \). (This is the Riemann-Lebesgue Lemma.)

### 2 Fourier Transform

We now move on to functions defined on all of \( \mathbb{R} \), rather than just \([0,1]\). If \( f \in L^1(\mathbb{R}) \), we define the Fourier transform \( \hat{f} \) by
\[ \hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x) \, dx. \]

(Unless otherwise indicated, all integrals in this section are over the real number line \( \mathbb{R} \).)

The Fourier transform has many nice properties. Assume that \( f, g \in L^1 \).

Then:

- If \( h(x) = f * g(x) \), then
  \[ \hat{h}(\xi) = \int e^{-2\pi i x \xi} \int f(y) g(x - y) \, dy \, dx \]
  \[ = \left( \int e^{-2\pi i y \xi} f(y) \, dy \right) \left( \int e^{-2\pi i z \xi} g(z) \, dz \right) \]
  \[ = \hat{f}(\xi) \hat{g}(\xi). \]

(Here \( * \) denotes convolution, that is, \( f * g(x) = \int f(y) g(x - y) \, dy \).)
• If \( h(x) = f'(x) \), then \( \hat{h}(\xi) = 2\pi i \xi \hat{f}(\xi) \).

• \( r\hat{f}(\xi) = \hat{r}\hat{f}(\xi) \), if \( r \in \mathbb{R} \); \( \hat{f}(\xi) + \hat{g}(\xi) = \hat{f}(\xi) + \hat{g}(\xi) \), and so the Fourier transform is a linear operator.

• If \( h(x) = f(x - \alpha) \), then \( \hat{h}(\xi) = \hat{f}(\xi)e^{2\pi i \alpha \xi} \).

• If \( h(x) = \frac{1}{r}f \left( \frac{x}{r} \right) \), \( r > 0 \), then \( \hat{h}(\xi) = \hat{f}(r\xi) \).

• If \( f \in L^1 \), \( \hat{f} \) is continuous. ¹

• If \( \phi(x) = e^{-\pi x^2} \), then \( \hat{\phi} = \phi \).

That last example allows us to prove the Fourier inversion formula.

**Theorem 1** If \( g \in L^1 \) is continuous at \( x \in \mathbb{R} \), and if either

• \( \hat{g} \) is also in \( L^1 \), or

• \( \hat{g} \geq 0 \) everywhere and \( x = 0 \),

then

\[
g(x) = \int e^{2\pi ix\xi} \hat{g}(\xi) \, d\xi.
\]

**Proof** Let \( \phi(x) = e^{-\pi x^2} \), \( \phi_r(x) = \frac{1}{r}\phi \left( \frac{x}{r} \right) \). Note that \( \int \phi_r = 1 \) for all \( r > 0 \).

Then if \( g \) is continuous, \( g(x) = \lim_{r \to 0} g \ast \phi_r(x) \).

So:

\[
\int e^{2\pi ix\xi} \hat{g}(\xi) \, d\xi = \lim_{r \to 0} \int e^{2\pi ix\xi} \hat{g}(\xi)\phi(r\xi) \, d\xi = \lim_{r \to 0} \int e^{2\pi ix\xi} \hat{g}(\xi)\phi(r\xi) \, d\xi.
\]

This is where we use our conditions on \( \hat{g} \). Switching limits with integrals is an interesting subject you will look at in first-quarter analysis.

Now,

\[
\int e^{2\pi ix\xi} \phi(r\xi)\hat{g}(\xi) \, d\xi = \int e^{2\pi ix\xi} \phi(r\xi) \int e^{-2\pi iy\xi} g(y) \, dy \, d\xi = \int g(y) \int \phi(r\xi)e^{-2\pi i(y-x)\xi} \, d\xi \, dy = \int g(y)\phi_r(x-y) \, dy.
\]

So

\[
g(x) = \lim_{r \to 0} \int g(y)\phi_r(x-y) \, dy = \int e^{2\pi ix\xi} \hat{g}(\xi) \, d\xi.
\]

¹We will later define \( f \) for \( f \in L^2 \) as well as \( L^1 \). This result will not hold there.
In the section on Fourier series, it was the $L^2$ theory that was interesting. Unfortunately, we can only define the Fourier transform for $f \in L^1$. So now we look at functions in $L^1 \cap L^2$.

We have a very useful and interesting result:

**Theorem 2 (Plancherel’s Theorem)** If $f \in L^1 \cap L^2$, then \( \hat{f} \in L^2 \) as well, with 
\[
||f||_{L^2} = ||\hat{f}||_{L^2}.
\]

**Proof** Let \( \tilde{f}(x) = \overline{f(-x)} \), and \( g(x) = f \ast \tilde{f}(x) \).

Now,
\[
g(0) = \int f(x)\overline{f(-0-x)} \, dx = ||f||_{L^2}^2
\]
and
\[
\hat{g}(x) = \hat{f}(\xi)\hat{\tilde{f}}(\xi) = |\hat{f}(\xi)|^2,
\]
so if we could apply our previous theorem (with \( x = 0 \)), we would be done.

We need only show that \( g \in L^1 \) and \( g \) continuous. But
\[
\int |g(x)| \, dx = \int \left| \int f(y)f(y-x) \, dy \right| \, dx \\
\leq \int |f(y)| \int |f(y-x)| \, dx \, dy \leq ||f||_{L^1}^2
\]
and so \( g \in L^1 \).

Recall our useful fact: if \( f \in L^2 \), then for every \( \epsilon > 0 \), there is some \( h \in L^2 \) such that \( h \) is uniformly continuous and \( ||f-h||_{L^2} < \epsilon \). Let \( \delta \) be such that \( |h(x + \delta) - h(x)| < \epsilon \) for all \( x \).

So
\[
|g(x + \delta) - g(x)| = \left| \int f(y)[f(y-x-\delta) - f(y-x)] \, dy \right| \\
\leq \int |f(y)||h(y-x-\delta) - h(y-x)| \, dy \\
+ \int |f(y)||f(y-x-\delta) - h(y-x-\delta)| \, dy \\
+ \int |f(y)||f(y-x) - h(y-x)| \, dy \\
\leq \epsilon ||f||_{L^1} + 2\epsilon ||f||_{L^2}
\]
and so \( g \) is continuous. Thus we are done. \( \blacksquare \)

We can use these to extend the Fourier transform (and its inverse) to all of $L^2$. 

4
If \( f \in L^2 \), there is some \( \{f_n\} \subset L^1 \cap L^2 \) such that \( f_n \rightarrow f \) in \( L^2 \). For example, let \( f_n(x) = f(x) \) if \( |x| < n \) and 0 otherwise; clearly \( f_n \rightarrow f \) in \( L^2 \), and since \( |f| \leq \max(1, |f|^2) \),

\[
\int |f_n(x)| \, dx \leq \int_{-n}^{n} 1 \, dx + \int |f|^2 \, dx \leq 2n + ||f||^2_{L^2}
\]

and so \( f_n \in L^1 \).

Then by Plancherel’s theorem, \( \{\hat{f}_n\} \) is a Cauchy sequence in \( L^2 \), and since \( L^2 \) is a complete metric space, \( \lim_{n \to \infty} \hat{f}_n \) exists in \( L^2 \)-norm, and so we can define \( \hat{f} = \lim_{n \to \infty} \hat{f}_n \).

Note that this means the Fourier transform of an \( L^2 \) function is an \( L^2 \) function. If \( f \notin L^1 \), then \( \hat{f} \) may not be continuous, and its value at a given point is completely arbitrary.