WOMP 2006 Linear Algebra-Rough Outline

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1 References

1. Hoffman and Kunze, Linear Algebra
2. Halmos, Finite Dimensional Vector Spaces
3. Helson, Linear Algebra

2 Outline

I’ll try to use Greek letters for scalars ($\alpha, \beta, ...$) and English letters for vectors ($a, b, v, ...$).

Definition 2.1. A vector space is a set $V$ with an addition operation $+$ and a scalar multiplication over a field $k$ such that

1. $(V, +)$ is a commutative group,
2. $1_k a = a \ \forall a \in V$
3. $(\alpha \beta)a = \alpha(\beta a) \ \forall \alpha, \beta \in k, \forall a \in V$
4. $\alpha(a + b) = \alpha a + \alpha b \ \forall \alpha \in k, \forall a, b \in V$
5. $(\alpha + \beta)a = \alpha a + \beta a \ \forall \alpha, \beta \in k, \forall a \in V$

Definition 2.2. [Briefly]

* Subspace

* Linear combination and Span
  
  Note finiteness in definition.

* Linearly independent set
* Direct sum

* Basis
  Theorem: Every vector space has a basis.

* Dimension
  Theorem: Dimension is well defined.
  Theorem: For vector spaces over the same field, dimension determines isomorphism.

**Definition 2.3.** $T : V \to W$ where $V$ and $W$ are vector spaces over $k$ is a linear transformation if for all $\alpha, \beta \in k$ and $a, b \in V$,

$$T(\alpha a + \beta b) = \alpha T(a) + \beta T(b).$$

The set of all such $T$ forms the vector space $\text{Hom}_k(V, W)$.

If $T : V \to k$ is a linear transformation, $T$ is called a linear functional, and $\text{Hom}_k(V, k)$ is called the dual of $V$.

**Definition 2.4.** [More briefly!]

* $\ker(T) = \{v \in V \mid Tv = 0\}$, $\text{nullity}(T) = \dim(\ker(T))$
* $\im(T) = \{w \in W \mid (\exists v)Tv = w\}$, $\text{rank}(T) = \dim(\im(T))$

Theorem (Rank/Nullity) For finite dimensional $V$,

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

**Theorem 2.5 (Change of Basis).** Suppose $V$ is an $n$-dimensional vector space over $k$. Let $B_1$ and $B_2$ be two bases for $V$. Then there exists an $n$ by $n$ invertible matrix $S$ (called the change of basis matrix) such that for all $v \in V$,

$$[v]_{B_2} = S[v]_{B_1}.$$  

Using the same methods,

**Theorem 2.6.** Let $V$ be an $n$-dimensional vector space and $W$ be an $m$-dimensional one. Then every linear transformation $T : V \to W$ has the form $T_A$ for some $m$ by $n$ dimensional matrix $A$ where $T_A$ is matrix multiplication with respect to given bases for $V$ and $W$.

**Definition 2.7.** $n$ by $n$ matrices $A$ and $B$ (with coefficients in the same field) are similar if there exists an invertible matrix $S$ such that $A = SBS^{-1}$.
Theorem 2.8 (Motivation for Similarity). A and B represent the same linear transformation with respect to different bases for \( V \) if and only if A and B are similar.

We now restrict our attention to \( V = \mathbb{C}^n \) over \( \mathbb{C} \) and to a linear transformation \( T : \mathbb{C}^n \to \mathbb{C}^n \) represented by the matrix \( A \) with respect to the standard basis. Note \( \mathbb{C} \) is algebraically closed.

Definition 2.9. [Again briefly.]
* \( \text{trace}(A) \)
* \( \text{determinant}(A) \)

Theorem 2.10 (Existence and Uniqueness of Determinants).
\[
\det(A) : (\mathbb{C}^n)^n = \text{Mat}_{n,n}(\mathbb{C}) \to \mathbb{C} \text{ is the only complex function of } n \text{ variables (the columns) that is multi-linear, skew-symmetric, and normalized so that } \det(I_n) = 1.
\]

Definition 2.11. [Eigenvalues, Preparation for JCF]
* Eigenvalue: \( \lambda \) is an eigenvalue for \( T \) if \( Tv = \lambda v \) for \( v \neq 0 \). \( v \) is called an eigenvector associated to \( \lambda \).
* Characteristic polynomial: \( p_A(t) = \det(tI - A) = \prod_{i=1}^{k}(t - \lambda_i)^{m_i} \) for \( \lambda_i \) distinct. We call \( m_i \) the algebraic multiplicity of \( \lambda_i \). Cayley’s Theorem: \( p_A(A) = 0 \)
* Eigenspace: \( V_{\lambda} = \{ v \in V | Tv = \lambda v \} \), \( \dim(V_{\lambda}) \) is the geometric multiplicity of \( \lambda \). Note: \( \dim(V_{\lambda}) \leq m_i \).
* Diagonalizable: A is diagonalizable if it is similar to a diagonal matrix. Theorem: A is diagonalizable if and only if A has \( n \) linearly independent eigenvectors. Note: Failure to be diagonalizable is a discrepancy between geometric and algebraic multiplicity.

Theorem: Any A as above is similar to an upper triangular matrix.
* Generalized eigenspace: \( U_{\lambda} = \{ v \in V | (\exists k > 0)(T - \lambda I)^kv = 0 \} \)
* Minimal polynomial: \( m_A(t) = \prod_{i=1}^{k}(t - \lambda_i)^{j_i} \) is the monic polynomial of least degree which annihilates A.
Theorem 2.12 (Jordan Canonical Form). Let $A$ be an $n$ by $n$ complex matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $A$ is similar to a matrix which is the direct sum of Jordan blocks $J_m(\lambda_i)$ (unique up to a reordering of the blocks) with at least one block for each $\lambda_i$ where $J_m(\lambda)$ is an $m$ by $m$ matrix of the form:

\[
\begin{pmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & \ldots & 0 \\
0 & 0 & \lambda & 1 & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & 0 & \lambda
\end{pmatrix}
\]

Summary of Properties of JCF

- $A$ and $B$ are similar iff they have the “same” JCF.
- Algebraic Multiplicity:
  \[m_i = dim U_{\lambda_i} = \text{sum of sizes of all Jordan blocks for } \lambda_i.\]
- Geometric Multiplicity:
  \[dim V_{\lambda_i} = \text{number of Jordan blocks for } \lambda_i.\]
- The exponent of $(t - \lambda_i)$ in $m_A(t)$ is the size of largest Jordan block for $\lambda_i$ (We called this exponent $j_i$). This is also the index of the nilpotent transformation $(A - \lambda_i I)|_{U_{\lambda_i}}$. 