Given an extension field $E$ of $F$ and an element $\alpha \in E$. We can consider an extension field $F(\alpha)$ of $F$ given by adding $\alpha$ to $F$. Precisely, we have two cases:

**When $\alpha$ is algebraic over $F$:** The kernel of the evaluation homomorphism $\phi_\alpha$ is the maximal ideal $\langle \text{irr}(\alpha, F) \rangle$ generated by the irreducible polynomial for $\alpha$ over $F$. Hence the image $\phi_\alpha(F[x])$ is a subfield of $E$ isomorphic to $F[x]/\langle \text{irr}(\alpha, F) \rangle$, and it is the smallest subfield of $E$ containing $F$ and $\alpha$. We denote this field by $F(\alpha)$.

**When $\alpha$ is transcendental over $F$:** The kernel of $\phi_\alpha$ is trivial in this case and $\phi_\alpha$ gives an isomorphism of $F[x]$ with a subdomain of $E$. Hence $\phi_\alpha[F[x]]$ is an integral domain which we denote by $F[\alpha]$. The quotient field of $F[\alpha]$ is contained in $E$ and we denote this field by $F(\alpha)$.

**Definition 6.3.** An extension field $E$ of a field $F$ is a simple extension of $F$ if $E = F(\alpha)$ for some $\alpha \in E$.

**Theorem 6.4.** Let $E$ be a simple extension $F(\alpha)$ of a field $F$ where $\alpha$ is algebraic over $F$. Let the degree of $\alpha$ over $F$ be $n \geq 1$. Then every element $\beta$ of $F(\alpha)$ can be uniquely written in the from

$$\beta = b_0 + b_1 \alpha + \cdots + b_{n-1} \alpha^{n-1}$$

where the $b_i \in F$, i.e. $F(\alpha)$ is a vector space of dimension $n$ over $F$ with a basis $\{1, \alpha, \alpha^2, \cdots, \alpha^{n-1}\}$.