Proposition 12.3.  
1) If \( 0 \in S \), then \( S \) is linearly dependent.
2) If \( S \) is linearly dependent and \( S \subset T \subset V \), then \( T \) is linearly dependent.
3) If \( S \) is linearly independent and \( U \subset S \subset V \), then \( U \) is linearly independent.
4) \( S \subset V \) is a basis for \( V \) iff each element of \( V \) is a unique linear combination of elements of \( S \).
5) If \( A = \{v_1, \ldots, v_m\} \) spans \( V \), then some subset of \( A \) is a basis for \( V \).
6) Let \( W \) be a subspace of the finite-dimensional vector space \( V \). Then any basis for \( W \) can be extended to a basis for \( V \).

Theorem 12.1. The Rank Theorem. For a linear map \( \phi : V \to W \), we have

\[
\dim V = \dim (\ker \phi) + \dim (\im \phi)
\]

Corollary 12.1. Let \( V \) and \( W \) be finite-dimensional vector spaces. Then we have

1) \( \phi \in \text{End}(V) \) is an isomorphism iff \( \phi \) is injective iff \( \phi \) is surjective.
2) \( V \cong W \) iff \( V \) and \( W \) have the same dimension.

13. DIRECT SUMS, EXACT SEQUENCE, AND QUOTIENTS

Given two vector spaces \( V \) and \( W \), we can form the (external) direct sum of \( V \) and \( W \), denoted \( V \oplus W \): \( V \oplus W \) is a vector space with canonical structure induced from those of \( V \) and \( W \), along with natural injection and projection mappings

\[
\begin{align*}
V & \xrightarrow{i_1} V \oplus W \xrightarrow{p_1} V \\
& \xrightarrow{i_2} V \oplus W \xrightarrow{p_2} W \\
W & \xrightarrow{p_1 \circ i_1 = \text{id}_V} V \oplus W \xrightarrow{p_2 \circ i_2 = \text{id}_W} W
\end{align*}
\]

Proposition 13.1.  
1) If \( \{v_1, \ldots, v_n\} \) is a basis for \( V \) and \( \{w_1, \ldots, w_m\} \) is a basis for \( W \), then \( \{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_m)\} \) is a basis for \( V \oplus W \).
2) \( \dim(V \oplus W) = \dim(V) + \dim(W) \).

Now, given two subspaces \( A \) and \( B \) of a vector space \( V \), we can consider a natural mapping \( \eta : A \oplus B \to V \) defined by \( \eta(a, b) = a + b \). We have

Proposition 13.2.  
1) \( \eta \) is a linear map.
2) \( \eta \) is injective if and only if \( A \cap B = \{0\} \).
3) \( \eta \) is surjective if and only if \( A \cup B \) spans \( V \).
In case \( \eta \) is an isomorphism, we say that \( V \) is an internal direct sum of \( A \) and \( B \). One useful concept is the exact sequence: Given vector spaces \( U, V, W \) and linear maps \( \rho : U \to V \) and \( \sigma : V \to W \), we say that the sequence \( U \xrightarrow{\rho} V \xrightarrow{\sigma} W \) is exact if \( \text{im}(\rho) = \ker(\sigma) \).

Moreover the exact sequence \( 0 \to U \xrightarrow{\rho} V \xrightarrow{\sigma} W \to 0 \) is said to split if there is a linear mapping \( \gamma : W \to V \) such that \( \sigma \circ \gamma \) is the identity mapping on \( W \).

**Proposition 13.3.**

1) For finite-dimensional vector spaces, the above exact sequence splits.

2) In the above situation, \( V \) is represented as an internal direct sum

\[
V = \rho(U) \oplus \gamma(W)
\]

Given a subspace \( W \) of a vector space \( V \), we can form the quotient space \( V/W \) using the equivalence relation. For finite-dimensional vector spaces, we have

**Proposition 13.4.**

1) \( V/W \) is a vector space.

2) The natural projection mapping \( \eta : V \to V/W \) is a surjective linear map.

3) For a linear map \( \phi : U \to V \), we have \( U/\ker(\phi) \cong \phi(U) \)

14. **Eigenvectors and Eigenvalues**

**Definition 14.1.**

1) Let \( \phi : V \to V \) be linear. A subspace \( W \) of \( V \) is said to be \( \phi \)-stable if \( \phi(W) \subset W \). If \( W \) is \( \phi \)-stable, then \( \phi|_W \in \text{End}(W) \).

2) A nonzero vector \( v \in V \) is an eigenvector for \( \phi : V \to V \) if \( \text{Span}(v) \) is \( \phi \)-stable. The unique \( \lambda \) such that \( \phi(v) = \lambda v \) is called an eigenvalue for \( \phi \) corresponding to \( v \).

3) Let \( V(\lambda) = \{ v \in V \mid \phi(v) = \lambda v \} \) and call it the eigenspace belonging to \( \lambda \). The dimension of \( V(\lambda) \) is the geometric multiplicity of \( \lambda \).

For example, the set \( E \) of elementary functions in the real vector space \( D \) of \( C^\infty \)-functions from \([0, 1]\) to \( \mathbb{R} \) is stable under differentiation, but is not stable under integration.

**Proposition 14.1.**

1) If \( v \) is an eigenvector for \( \phi \), then \( rv \) for any scalar \( r \in k \) is an eigenvector for \( \phi \) with the same eigenvalue as \( v \).

2) If \( \lambda, \mu \in k \) and \( \lambda \neq \mu \), then \( V(\lambda) \cap V(\mu) = \{0\} \).

3) There exists a linear map \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( V(\lambda) = \{0\} \) for each \( \lambda \in \mathbb{R} \).

4) If \( v \) is an eigenvector for \( \phi \) with eigenvalue \( \lambda \), then \( \psi(v) \) is an eigenvector for \( \psi \phi \psi^{-1} \) with eigenvalue \( \lambda \).