We will give another definition of the determinant using the permutation group: Let $S_n$ be the permutation group of $n$-elements. The sign of a permutation $\sigma \in S_n$ is given by $sgn(\sigma) = (-1)^k$ where $k$ is the number of transpositions in $\sigma$. The determinant can be expressed as

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

This expression tells us that $\det(A) = \det(A^t)$, i.e. the transpose of $A$. We will show some important properties of determinant as well as computational methods.

**Proposition 16.6.** 1) $\det(I_n) = 1$.

2) If the rows (or columns) of $A$ are linearly dependent, then $\det(A) = 0$.

3) For $\lambda \in k$, $\det(\lambda A) = \lambda^n (\det A)$.

4) Let $A, B$ be $n \times n$-matrices, then $\det(AB) = (\det(A))(\det(B))$.

5) $A$ is nonsingular if and only if $\det(A) \neq 0$.

6) The determinant is constant on an equivalence class of conjugate matrices.

In order to actually compute the determinant of any given square matrix $A$. We can use the Laplace expansion formula, also known as the row expansion formula.

**Theorem 16.1.** For any given $n \times n$-matrix $A = (a_{ij})$ over a field $k$, we have for any $i = 1, \cdots, n$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = (-1)^{i+1} [a_{i1}\det(A_{i1}) - \cdots \pm a_{in}\det(A_{in})]$$

where $A_{ij}$ is the $(n-1) \times (n-1)$-matrix obtained by removing $i$-th row and $j$-th column of $A$.

Another way to compute the determinant, or a linear system in general, is through row-reduction:

**Proposition 16.7.** Let $A$ be an $n \times n$-matrix.

1) If $A'$ is a matrix obtained from $A$ by adding a scalar multiple of $i$-th row to $j$-th row where $i \neq j$, then $\det(A) = \det(A')$.

2) If $A'$ is a matrix obtained from $A$ by switching two rows, then $\det(A) = -\det(A')$.

3) If $A'$ is a matrix obtained from $A$ by multiplying a nonzero scalar $\lambda \in k$ to a row, then $\det(A) = \lambda^{-1}\det(A')$.

The row operations described in the previous proposition is often called elementary row operations. It can also be realized as a matrix multiplication with elementary matrices. Recall from the Calculus classes that the determinant appeared in the Change of Variables formula as the proportion of volume elements corresponding to two coordinate systems.