Integer Valued Randoms

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Martin-Löf randomness-reminder

- A c.e. open set is one of the form $\bigcup_i (q_i, r_i)$ where $\{q_i : i \in \omega\}$ and $\{r_i : i \in \omega\}$ are c.e.. $U = \{[\sigma] : \sigma \in W\}$. Here $[\sigma] = \{\sigma\alpha \mid \alpha \in 2^\omega\}$, and has measure $\mu([\sigma]) = 2^{|\sigma|}$.
- A Martin-Löf test is a uniformly c.e. sequence $U_1, U_2, \ldots$ of c.e. open sets s.t.

$$\forall i(\mu(U_i) \leq 2^{-i}).$$

(Computably shrinking to measure 0)

Definition

α is Martin-Löf random if for every Martin-Löf test,

$$\alpha \notin \bigcap_{i>0} U_i.$$

The martingale view

- Back to von Mises 1919. Think about predicting the next bit of a sequence. Then you bet on the outcome. You should not win!
- (Levy) A martingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all σ,

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$

- the martingale *succeeds* on a real α, if $\limsup_n F(\alpha \upharpoonright n) \to \infty$.

- Think of betting on sequence where you know that every 2nd bit was 1. Then every second bit you could double you stake. This martingale exhibits exponential growth and that can be used to characterize computable reals.
- Ville proved that null sets correspond to success sets for martingales. They were used extensively by Doob in the study of stochastic processes.

- A supermartingale is a function $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ such that for all σ,
$$f(\sigma) \geq \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$
- Schnorr showed that Martin-Löf randomness corresponded to effective (super-)martingales failing to succeed.
- f as being effective or computably enumerable if $f(\sigma)$ is a left-c.e. real, and at every stage we have effective approximations to f in the sense that $f(\sigma) = \lim_s f_s(\sigma)$, with $f_s(\sigma)$ a computable increasing sequence of rationals.

Schnorr

Theorem (Schnorr)

A is Martin-Löf random iff no effective martingale succeeds on it.

- Also lead to variations on randomness:
  1. Schnorr randomness ($\mu(U_n) = 2^{-n}$),
  2. computable randomness (only computable martingales),
  3. Kurtz randomness U_n is a canonical clopen set-or $A \in V_n$ for all open V_n of measure 1.

Integer Valued Randoms

- Many notions of randomness.... I know a good book or two.
- If you actually go to a casino, and try to bet $\$\frac{1}{10000000}$, they will not accept the bet.
- What about if you ask that bets should be in multiples of certain discrete amounts. The canonical example being $\$0, 1, 2, 3, \ldots$.
- By multiplying, we can regard the the values as integers in the martingale.

Definition (Beinvenu, Stephan, Teutsch)

X is IVR iff no computable integer valued martingale succeeds on X.

- We can similarly define single valued if the set of possible bets is a singleton, (either no bet or) a.
- Similarly finitely valued.

Theorem (Bienvenu, Stephan, Teutsch)

1. *Computably random implies IVR implies FVR implies SVR*
2. *Kurtz random implies SVR*
3. *FVR implies bi-immune.*

- As we already know, Computable implies Schnorr implies Kurtz (with no reversals)
- Schnorr implies law of large numbers.

Theorem ([Bienvenu, Stephan, Teutsch)

No other implications hold.

The essential difference

- There are finitary strategies to defeat IV martingales.
- If m bets on an outcome, say $\sigma 1$ in favour of $\sigma 0$, then if we choose $\sigma 0$ he must lose at least \$1.

Definition (Actually a Theorem-Jockusch and Posner)

A is called n-generic if A meets or avoids each Σ^0_n set S of strings. That is either

- $\exists \sigma \in S(\sigma \prec A)$, or
- $\exists \sigma \prec A \forall \tau \in S(\sigma \not\prec \tau)$.

(Kurtz) B is weakly n-generic if A meets all dense S's.

Theorem (BST)

If A is weakly 2-generic then A is IVR. Hence IVR's are co-meager.

However:

Theorem (BST)

There is a 1-generic which is not IRV.

Corollary

IVR does not imply Schnorr randomness.

Other strangenesses of IVR's

- One of the classic methods for martingales is called the savings trick.
- This says, given a martingale m convert it into a new one n by; after you win a dollar, put it in the bank, and bet the same proportion as m, till you make another dollar.
- The result has the property that it is (up to ± 1) nondecreasing.

Theorem (Teutsch)

There is an X which is not IRV but is IVR for martingales with the savings property.

- Of course, the reason is that proportions are not available. His method exploits this.

Theorem (Chalcraft, Dougherty, Freiling, Teutsch)

If B and A are finite integer value sets, then every B-IVR is A-IVR iff there is a c with $B \supseteq c \cdot A$ some $c \in \mathbb{Q}$.

- The proof of the easy direction is that if $B \supseteq c \cdot A$ then any bet A makes B can simply make a multiple of, so "essentially" the same bet up to scaling.
- The hard direction observes that if $B \not\supseteq c \cdot A$, then we construct a real that S_A succeeds on and B does not.
- S_A's strategy is to always bet on 0, and cycle through its values, $\{a_1, \ldots, a_n\}$
- We can always assume that B bets on 0 else we makre easy progress.
- We compare $\frac{S_A\text{'s capital}}{M_B\text{'s capital}}$ with $\frac{S_A\text{'s bet}}{M_B\text{'s bet}}$.
- If the first $\geq$ second bet 0 S wins more proportionally to it capital, if the second exceeds first bet 1 M loses more proportionally to its capital.
- It can be proven that $\frac{S_A\text{'s capital}}{M_B\text{'s capital}} \to \infty$ unless it has a limit C, and then $c = \frac{1}{C}$ is the multiple
- Recently (as best I undestand the proof) this was extended to computable A and B by Bavly and Peretz. Stephan and I independently observed that it cannot be extended to all infinite A and B using immune sets, which is cheating.

Our questions

- What kinds degrees have/bound IVR's?
- Are these degree notions?
- What kinds of genericity might align itself to IVR's. We know weak 2- is enough, and 1- is not enough.
- Does it matter for left c.e. reals?
- What about partial IVR's? That is in a casino I won't tell you what I would bet in advance. So consider the definition with m's partial computable.
- What about c.e. degrees? Do they relate to any known degree class?

Multiply generic reals

- A new genericity notion. Recall that a Σ^0_1 set is the range of a computable f. What about if this is ω-c.a., and with the property that it has an approximation g such that for all n, $g(n,s) \preceq g(n,s+1)$. We call this proper.
- Recall that an order is a computable function h with $h(n+1) \geq h(n)$ and $h(n) \to \infty$.
- For an order h, say that f is h-c.a. if the approximation has $g(n,s+1) \neq g(n,s)$ at most $h(n)$ many times.

Definition

Let h be an order. We say that A is h-multiply generic if A meets or avoids all sets S which are the ranges of proper h-c.a. functions. Similarly for weakly and dense sets.

- Evidently for any computable h, $\emptyset'$ can compute an h-multiply generic.

Theorem

If h and h' are orders and G is h-(weakly) multiply generic, then it is also $h'-$ (weakly) multiply generic. Thus A is multiply generic iff A is h-multiply generic for some h.

Lemma

If G is weakly multiply generic then it is also IVR. The converse is not true since MLR's are not weakly 1-generic.

Feebleness

- Actually a weaker notion of randomness is enough for IVR.

Definition

- A computable function $g : 2^{<\omega} \times \mathbb{N} \to 2^{<\omega}$ is acceptable if
  1. $g(\sigma, s) = \sigma$ and
  2. $g(\sigma, s) \preceq g(\sigma, s+1)$.
  3. $\lim_s g(\sigma, s)$ exists; we call this the range.
- X is feebly multiply generic if X meets the range of each acceptable g with at most $h(n)$ many mind changes on each argument n.

- The independence of order etc goes thru as does the lemma

Lemma

If G is feebly multiply generic then it is also IVR. The converse is not true since MLR's are not weakly 1-generic.

- The proof is modelled on that by Bienvenu et. al. noting that all that is needed is the weak multiple genericity.

Array noncomputable degrees

Definition (Downey, Joskusch, Stob)

A degree $\mathbf{a}$ is called array noncomputable iff for all functions $f \leq_{wtt} \emptyset'$ there is a a function g computable from $\mathbf{a}$ such that

$$\exists^{\infty} x(g(x) > f(x)).$$

- Looks like "non-low$_2$."
- Indeed many nonlow$_2$ constructions can be run with only the above. For example, every anc degree bounds a generic.
- The c.e. ones are of special interest.

A ubiquitous class

- c.e. anc degree are those that:
- (Kummer) Contain c.e. sets of infinitely often maximal Kolmogorov complexity $C(A \upharpoonright n) = 2 \log n$.
- Have effective packing dimension 1 (Downey and Greenberg)
- Compute left c.e. reals α (halting probabilities) and $B <_T \alpha$ such that if V is a presentation of α (that is V is prefix free, c.e. and $\alpha = \mu(V)$), then $V \leq_T B$. (Downey and Greenberg)
- (Downey, Jockusch, and Stob) bound disjoint c.e. sets A and B such that every separating set for A and B computes the halting problem
- have strong minimal covers (Ishmukhametov).
- (Cholak, Coles, Downey, Herrmann) The array noncomputable c.e. degrees form an invariant class for the lattice of Π^0_1 classes via the thin perfect classes

Theorem

1. *Every array noncomputable degree* $\mathbf{a}$ *bounds bounds a feebly multiply generic and hence a IVR.*
2. *If* $\mathbf{a}$ *is c.e. and bounds a IVR then it is ANC.*

The proof models itself on the classical one that each c.e. degree bounds a 1-generic.

What about multiply generics?

- These need another concept.

Definition (Downey, Greenberg, Weber)

We say that a c.e. degree $\mathbf{a}$ is totally ω-c.e. iff for all functions $g \leq_T \mathbf{a}$, g is ω-c.e.. That is, there is a computable approximation $g(x) = \lim_s g(x, s)$, and a computable function h, such that for all x,

$$|\{s : g(x, s) \neq g(x, s+1)\}| < h(x).$$

- array computability is a uniform version of this notion where h can be chosen independent of g.
- Every array computable degree (and hence every contiguous degree) is totally ω-c.e..
- Can extend to computable ordinals (Downey and Greenberg).
- These degrees are definable in the c.e. degrees (DGW), and (Downey and Ng) correspond to the degree containing computably finitely random reals.

Theorem

- *Every not totally ω-c.a. c.e. set computes a multiply generic set.*
- *Every not totally ω^2-c.a. set computes a multiply generic set.*
- *If $\mathbf{a}$ is c.e. and bounds a weakly multiply generic then it is not totally ω-ca.*

- The left c.e. case.
- Notably, there are left c.e. computably random reals (and hence IVR) in each high c.e. degree.

Theorem

If X is left c.e. and IVR then X is of high degree.

- We want to construct a procedure $\Gamma^X(e,s)$ such that $\lim_s \Gamma^X(e,s) = \text{Tot}(e)$.
- The cycle for (x,s) is to use the smallest new place in the approximation of X_s with $X_s(n) = 1$. For $s' \geq s$ we set $\Gamma^X(e,s')[t'] = 0$ until $\text{Tot}(e)$ looks more total as more arguments converge. We have inductively defined a partial computable $m = m_e$ up to strings of length s.
- If $\text{Tot}(e)$ looks more total at stage $t > s$ we play m conservatively except on $n(t)$ playing to win \$1 there on the 1 outcome.
- The key is that almost always he must move away, and he cannot play where we have lost.

Theorem

- *There are IVR's that are not partial IVR.*
- *There are partial IVR's that are not partial computably random.*

Theorem

If $\mathbf{a}$ *c.e. and high, then it contains a partial IVR iff it contains a IVR.*

Theorem

There are Δ^0_2 *degrees* $\mathbf{b}$ *containing IVR's bounding no degree containing a PIVR.*

- There are low c.e. degrees containing IVR's.
- This class jump inverts.
- There are high_2 c.e. degrees containing no IVR's.

Thank You