# The Finite Intersection Property and Computability Theory

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# FIP

- One equivalent of the axiom of choice
- A family of sets $\mathcal{F} = \{A_i \mid \in Q\}$ has finite intersection property iff for all finite $F \subset Q$, $\cap_{i \in F} A_i \neq \emptyset$.
- The principal says: Any collection of sets has a maximal subfamily with FIP.
- We investigate the computability of this.
- That is computable collections of computable sets.
- First began by Dzharfarov and Mummert.

- The first thing to notice is that it depends on whether you consider the family as set or a sequence
- If as a set then $\emptyset'$ is easily codable into a sequence and the theorem is equivalent to $\mathrm{ACA}_0$. (Namely, have a set $B = B_e$ such that it is initially empty, and if $e \in \emptyset'[s]$ henceforth intersect it with everything, so it must be included. $\emptyset'$ can clearly figure things out.)
- Interesting if a sequence, so that $A_1, A_2, A_3$ is different from $A_2, A_3, A_1$.
- Similarly $\bar{D}_2$IP for for all pairs $A_i \cap A_j \neq \emptyset$. (DM notation)

**Definition**

Say that **a** is FIP (bounding) iff for all computable collections of sets, **a** can compute a solution to the FIP problem.

# A Basic Result

**Theorem (Dzharfarov and Mummert)**

*There is a computable collection of sets with no c.e. subfamily with FIP. So $\mathbf{0}$ is not FIP, or even $\bar{D}_2IP$.*

1. Meet $R_e$ : $W_e$ is not an index for a maximal FIP family.
2. In the below I will use $A_i, \ldots$ and $X_e, B_i$ etc. Of course these are all the same and really are $W_{f(j)}$ for a computable $f$ given by the s-m-n theorem, and I am really concerned with the index $f(i)$. Also we will ensure that each nonzero set has a unique idetifier in it, so these are really streams of numbers under consiferderation.
3. Use a trap set $X_e$.
4. Begin with $A_0, A_1, \ldots$. Wait for $W_e$ to respond.
5. Start intersecting $X_e$ “in the back” . If $W_e$ enumerates it win with finite injury.

**Theorem (Dzharfarov and Mummert)**

*If $\mathbf{a}$ is $\bar{D}_2IP$ then it is hyperimmune. (i.e. not computably dominated for those under 35, or Bob)*

**Theorem (Dzharfarov and Mummert)**

*If $\mathbf{a} \neq \mathbf{0}$ is c.e. then $\mathbf{a}$ is FIP.*

**Theorem (Dzharfarov and Mummert )**

*If $\mathbf{a}$ is $\emptyset'$-hyperimmune then it is FIP.*

- The c.e. noncomputable case below $C \neq_T \emptyset$.
- We are building $A_0, A_1, \ldots A_n$.
- We want to put some element $B$ into this family (with truncation), as we have seen $B$ intersect $A_0, \ldots, A_j$, the first position determined by $B'$s index.
- We then place a permitting challenge to $C$. If later we see $C$ permit $j$, we change the family to $A_0, \ldots A_j, B$.
- When $B$ meets $A_{j+1}[s]$ place another challange on $B$.
- The $\emptyset'$-hyperimmune is because $\emptyset'$ knows if we ever want to put things in, and infinitely often the $C$ can decode this.
- It might seem that the c.e. case would also work for $\Delta^0_2$ $C$, but it fails for a nonuniform reason.
- An earlier promise for a $C$-configuration might force some $D_1$ into the sequence which might be disjoint from the $B$ we are attempting to put in. (board)

**Theorem (DM)**

*There is a computable nontrivial family such that every maximal subfamily with $\bar{D}_2$IP has hyperimmune degree.*

(proof)[DDGT] We will define a computable family of the form

$$\{A_e^i : e \leq i\} \cup \{B_e : e \in \omega\}.$$

We will call sets $A_e^i$ and $B_e$ with subscript $e$ "$e$-sets". We will ensure the following hold.

- Every $A_e^i$ is nonempty.
- $B_e$ is nonempty iff $\phi_e(e)\downarrow$, and contains only numbers larger than the stage when $\phi_e(e)$ converges.
- If $i \neq e$, then every nonempty $e$-set intersects every nonempty $i$-set.
- For all $i, j \geq e$, $A_e^i$ intersects $A_e^j$.
- $A_e^i$ intersects $B_e$ iff $\phi_e(x)\downarrow$ for all $x \leq i+1$. Moreover, the intersection only contains elements larger than the least stage $s$ such that $\phi_e(x)\downarrow[s]$ for all $x \leq i+1$.

We can assume the nonempty sets also code their indices, so that for every subfamily $\mathcal{C} = \{C_n \mid n \in \omega\}$ which does not contain the empty set, we can compute from $C_n$ which set $A_e^i$ or $B_e$ is equal to $C_n$.

Let $\mathcal{C}$ be a maximal subfamily with $\bar{D}_2$IP, and let $\mathcal{C}_s$ denote $\{C_n \mid n \leq s\}$. Since $\mathcal{C}$ does not contain the empty set, for each $e$, if $B_e \notin \mathcal{C}$, then $A_e^i \in \mathcal{C}$ for every $i \geq e$, since $A_e^i$ intersects every nonempty set in our family, except perhaps $B_e$.

Now if $\phi_e(x)$ is total, then $B_e$ must be in the family. From the family, we can compute the least number $q$ with $q \in B_e \cap A_i^x$ for $x \geq e$, and this will exceed $\phi_e(x)$. We need to make the function essentially coding this total whether or not $\phi_e$ is total.

Let $g$ be defined by

$$g(x) = (\mu s)\forall e \leq i \leq x\ A_e^i \in \mathcal{C}_s \vee B_e \in \mathcal{C}_s.$$

Let $f$ be defined by

$$f(x) = (\mu n)\forall i, j \leq g(x)\ C_i \cap C_j \cap [0, n] \neq \emptyset.$$

Observe that $f \leq_T \mathcal{C}$.
We will show $f$ is not majorized by any computable function. Suppose $\phi_e$ is total. Then every $e$-set intersects every nonempty set in the family we built, so the maximal subfamily $\mathcal{C}$ must contain $B_e$ and every $A_e^i$. Let $x \geq e$ be minimal such that $A_e^x$ appears after $B_e$ in $\mathcal{C}$. We claim $f(x) > \phi_e(x)$. Notice $g(x)$ bounds the position that $B_e$ appears. If $x = e$, then $B_e \cap [0, f(x)]$ is nonempty and therefore $f(x) > \phi_e(e)$. If $x > e$, then $g(x)$ also bounds the position $A_e^{x-1}$ appears, and therefore $B_e \cap A_e^{x-1} \cap [0, f(x)]$ is nonempty. Thus $f(x) > \phi_e(x)$.

# 1-Generices, again

**Theorem (DDGT)**

*If* **a** *bounds a 1-generic then* **a** *is FIP.*

The main idea: Think about the proof that if **a** is c.e. then it is FIP. If we want to add some $B$ to $A_0, A_1, \ldots,$ then we put up a permitting challenge to **a**$a$ and if permission occurs slot $B$ in, and truncate the family.
If we need to add some $B$ in then it will be dense in the construction so a permission occurs. For a 1-generic construction, for finite partial families, we will see such $B$ occur and challenge generics to include $B$ by the enumeration of a c.e. set of strings (thinking of sequences as strings, and the family as coding the generic). If this is dense then the generic will meet the condition.

In fact:
Call a tree $T \subset \omega^{<\omega}$ set-like if:
1) if $\sigma \in T$, $\sigma$ is injective; and
2) if $\sigma \in T$, then every permutation of $\sigma$ is in $T$.
Call A FIP-generic if for every c.e. set-like tree T, A computes a path f through T such that for every n, it is not the case that $(f \restriction j) * n \in T$ for all $j$.
Then FIP-generic is equivalent to FIP. If we drop the requirement that T is set-like, this is precisely 1-generic.
Actually it is enough to be a path to be FIP since there's a univeral family. (as we see later)

One part of the proof for FIP generic is clear. for 1-generic:
You build a tree T and a reduction $\Gamma$. At every stage, for every $\sigma \in T$, you enumerate $\sigma * 0$ into T and define $\Gamma^{\sigma * 0} = (\Gamma^\sigma) * 0$.
Whenever you see strings $\sigma$ and $\tau$ with $\sigma \in T, \tau \in W_e$ and $\Gamma^\sigma \preceq \tau$, enumerate $\sigma * (e+1)$ into the tree and define $\Gamma^{\sigma * (e+1)} = \tau$.
If $\Gamma^A$ neither meets nor avoids $W_e$, then for every $j, (A \restriction j) * (e+1)$ will be enumerated into the tree. So A will neither meet nor avoid $e+1$.

# The $\Delta^0_2$ Case

**Theorem (DDGT)**

*If $X$ is $\Delta^0_2$ and of FIP degree, then $X$ computes a 1-generic.*

The theorem is aided by the fact that there is a universal family.

**Theorem (DDGT)**

*There is a computable instance of FIP named $\mathcal{U}$ which is universal in the sense that any maximal solution for $\mathcal{U}$ computes a maximal solution for every other computable instance of FIP. Further, this reduction is uniform—if $\mathcal{A}$ is a computable instance of FIP, then from an index for $\mathcal{A}$, one can effectively obtain an index for a reduction that computes a maximal solution for $\mathcal{A}$ from a maximal solution for $\mathcal{U}$. Thus FIP for $\mathcal{U}$ is Medvedev-above all other computable FIPs.*

The idea for the proof is "intersect a lot, in a recoverable way."

# The $\Delta^0_2$ case

- Given $Q$ of FIP degree, we build 1-generic $G \leq_T Q$, and a family. (NB nonuniformity or use the recursion theorem)
- At some stage have $X_0, X_1, \ldots$ and $G \leq_T Q[s]$.
- Want to make $G$ meet $V_e$, say. Use a auxilairy set $B = B_e$.
- Make it meet, say, $X_0, \ldots, X_e$ (but not the rest) (A permitting challenge). Repeat with $X_{e+1}$ etc.
- If at some stage we get permission, then want to have, say, $X_0, \ldots, X_j, B_e$ want to block this from going back (For the principle all families representing the same collections of sets should give the same 1-generic) using bocker $Z_{e,j}$

## The general case

**Theorem (DDGT)**

*There is a minimal FIP* $\mathbf{a}$ *in* $\Delta^0_3$.

The proof is a tricky full approximation argument.

- Image we have so far $A_0, A_1, \ldots, A_n$ and wish to slot in $B_2$, postion determined by index and "state".
- Presumably we have enumerated some description of $\Phi^{\langle A_0, A_1, \ldots, A_n \rangle}(j)$ for $j \leq p$.
- We can move $A_0 A_1 B \ldots$ for one step seeking agreeing computations.
- Then we can go back. If $B$ stops intersecting, then who cares? If $B$ intersects more, repeat.
- If we get a split we can change state.
- A split must generate equivalent families. $A_0 A_1 B \ldots A_j$ and $A_0 A_1 \ldots A_j B$ and this forces lots of pain when interactions are considered.

- Notably, priorities ensure that you need to force many splits before you believe "split", as places for entry of high priority sets.
- These are "left hanging" which is why the trees are partial.
- That is, we might have $A_0A_1B\dots A_j$ and $A_0A_1\dots A_jB$, being the place where we promise we would introduce $C$, but this intersection might never occur, so we force another split (at least).
- Matters can be arranged to make sure that the first splits split with the second, arguing about uses.
- Then we would work on the second split unless we need to introduce $C$.
- Interactions are intricate.

# Finite variations

- Do the same but use only families of finite sets.
- Computably true if given as either canonical finite sets, or with a bound on the number.
- FIP is computably true (look at the big intersection)
- If only finite and weak indices:

**Theorem (DDGT)**

*$\overline{D}_2 IP_{finite}$ and $\Delta^0_2$ iff it bounds a 1-generic.*

- The proof is similar but uses more initialization and priority.

Thank You