Ordinal numbers and computability

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A word of introduction

I will briefly touch on loosely related projects all involving ordinal numbers.

Naturally, admissible computability (in which ordinals replace the natural numbers) will feature. But I will start fairly low (in fact low_2 c.e. degrees) and then climb higher.

Totally α-computably approximable degrees

Work with Rod Downey

Ershov's hierarchy

Ershov defined a hierarchy of Δ^0_2 sets and functions, based on the complexity of approximating them via the limit lemma. Counting down an ordinal is used to bound the (finitely many) mind-changes. The bigger the ordinal, the more we can change our minds and the more complicated is the function being approximated. The simplest levels are:

- Σ^{-1}_1, the c.e. sets;
- Σ^{-1}_2, the d.c.e. sets, ...,
- Δ^{-1}_ω, the ω-computably approximable sets and functions – the ones approximable with a computable bound on the number of mind-changes.

Lying alert: notations matter.

Ershov's hierarchy, low levels

The first ω powers of ω can be defined inductively:

- A function is ω-c.a. if it can be approximated with a computable bound on the number of mind-changes.
- A function is ω^2-c.a. if it can be approximated with an ω-c.a. bound on the number of mind changes. That is, a computable bound on the number of times we change our mind about how many times we change our mind.
- A function is ω^3-c.a. if it can be approximated with an ω^2-c.a. bound on the number of mind changes...

Ershov's hierarchy and the c.e. Turing degrees

Definition (Originally J. Miller for $\alpha = \omega$)

A Turing degree $\mathbf{d}$ is totally α-c.a. if every function in $\mathbf{d}$ is α-c.a.

Fact

In an analogous way to Ershov's hierarchy, the totally α-c.a. degrees give a hierarchy of complexity within the low_2 c.e. degrees. The array computable c.e. degrees are a uniform version of the totally ω-c.a. degrees.

The hierarchy is not strict at every ordinal. New degrees are obtained at powers of ω.

Dynamic properties of constructions

The hierarchy of totally α-c.a. degrees captures dynamic aspects of permitting arguments. From the point of view of a single requirement,

- Noncomputable c.e. degrees give single permissions.
- High c.e. degrees give cofinally many permissions.
- Array noncomputable c.e. degrees give multiple permissions, but we need to state in advance how many.
- Degrees which are not totally ω^α-c.e. degrees give multiple permissions, with α levels of mind changes about how many permissions we need.

... and natural definability

As a result of this analysis we can give natural definitions to two levels of the hierarchy.

Theorem

- *(Downey,Greenberg,Weber) A c.e. degree is not totally ω-c.a. if and only if it bounds a critical triple in the c.e. degrees.*
- *(Downey,Greenberg) A c.e. degree is not totally $< \omega^\omega$-c.a. if and only if it it bounds a copy of the 1-3-1 lattice in the c.e. degrees.*

Further similar constructions: infing out of a wtt degree within a single Turing degree, computing initial segments of scattered linear orders, computing presentations of left-c.e. reals, computing “multiply generic” sets (McInerney), computing indifferent sets for genericity (Day), ...

Transfinite iterations of the Turing jump

Work with Antonio Montalbán and Ted Slaman

Hyperarithmetic degree spectrum

Taking iterations of the Turing jump along the computable ordinals (and closing downwards in the Turing degrees) gives us the collection of hyperarithmetic sets.

Theorem (Greenberg,Montalbán,Slaman)

There is a countable structure which has isomorphic copies precisely in the non-hyperarithmetic degrees.

A bit on the construction

1. Relativise Slaman-Wehner to $\mathbf{0}^{(\alpha)}$ for every computable α. Obtain a structure $\mathcal{M}_\alpha$ whose degree spectrum consists of the degrees strictly above $\mathbf{0}^{(\alpha)}$.
2. Invert the jump (Goncharov, Harizanov, Knight, McCoy, Miller, Solomon) to obtain a structure $\mathcal{N}_\alpha$ whose degree spectrum is the collection of all non-low$_\alpha$ degrees. [This uses an iterated priority argument of height α.]
3. String these structures together. Use the fact that a degree is hyperarithmetic if and only if it is low$_\alpha$ for some computable α (consider ordinals closed under addition).

The issue of course is that a non-hyperarithmetic degree cannot necessarily list all computable ordinals. The main work is bypassing this problem by considering pseudo-ordinals.

A limiting result

The theorem shows that the analogue of the Slaman-Wehner theorem (all nonzero degrees form a degree spectrum) holds in the hyperdegrees as well. Such an analogue fails in the degrees of constructibility. The reason is essentially:

Theorem (Greenberg,Montalbán,Slaman; Kalimullin,Nies)

If a degree spectrum is co-null then it contains Kleene's O (the complete Π^1_1 set).

Π^1_1 sets and equivalence relations

Work with Dan Turetsky

Π^1_1 sets are c.e.

Σ^1_1 sets of reals are the effective analogue of analytic sets: they are the images of computable real-valued functions.
However their complements, the Π^1_1 sets, admit an ordinal analysis which makes them behave like c.e. sets.

- A Π^1_1 set $A \subseteq 2^\omega$ is the union $\bigcup_{\alpha<\omega_1} A_\alpha$, where the sets A_α are (uniformly) Borel.
- A Π^1_1 set $A \subseteq \omega$ is the union $\bigcup_{\alpha<\omega_1^{ck}} A_\alpha$, where the sets A_α are uniformly hyperarithmetic.

Think of A_α as the collection of elements of A which have been enumerated into A by stage α.

Another way to see that Π^1_1 sets are c.e. is to consider the Spector-Gandy theorem: any Π^1_1 set can be defined by an existential quantifier, ranging over the hyperarithmetic sets.

Π^1_1 equivalence relations

The study of Borel equivalence relations has been effectivised. For example:

Theorem (Fokina,Friedman,Harizanov,Knight,McCoy,Montalbán)

Isomorphism of computable structures is a universal Σ^1_1 equivalence relation on ω.

Claim (Greenberg,Turetsky - unwritten, so...)

The existence of hyperarithmetic isomorphisms is a universal Π^1_1 equivalence relation on ω.

Main idea: we start by diagonalising. If at stage α we discover that $j \equiv k$ then we build a $\mathbf{0}^{(\alpha)}$-computable isomorphism from $\mathcal{M}_k$ to $\mathcal{M}_j$. Again to make the structures computable we need to use pseudo-ordinals.

Π^1_1 equivalence relations

In greater detail. Let δ be a pseudo-ordinal.

- Use component (e, i, j) to diagonalise against F_e (the e^{th} Π^1_1 partial function) being an isomorphism between $\mathcal{M}_i$ and $\mathcal{M}_j$. Each component will have two a-priori indistinguishable parts (part A and part B), each linearly ordered in ordertype ω^α or $\omega^\alpha \cdot 2$ for some $\alpha < \delta$.
- Suppose that at stage α we discover that F_e converges on $(A(e, i, j))^{\mathcal{M}_i}$ and maps it to $(A(e, i, j))^{\mathcal{M}_j}$ or to $(B(e, i, j))^{\mathcal{M}_j}$. Define the components to be either ω^α or $\omega^\alpha \cdot 2$, so as to defy F_e.
- Unless... by stage α we have already discovered that $j \equiv k$ for some $k < j$. In that case we copy what $\mathcal{M}_k$ does. If $\mathcal{M}_j$ made some decisions on some components at earlier stages, $\mathbf{0}^{(\alpha)}$ can see what happened and compute the isomorphisms anyway.
- Think a little about what happens if α is nonstandard (in both cases).

Use the Ash-Knight machinery to approximate $\mathbf{0}^{(\delta)}$ and so make the structures computable.

Π^1_1 randomness

Work with Benoit Monin

Π^1_1 randomness

Back in the 1960s Martin-Löf considered Δ^1_1-randomness: avoiding all hyperarithmetic null sets. Nies and Hjorth considered other "higher" analogues of notions from algorithmic randomness. For example there is a higher analogue of Martin-Löf randomness: Π^1_1-MLR – the open sets are Π^1_1 rather than c.e.
They considered another strengthening: avoiding Π^1_1 null sets, not necessarily low in the Borel hierarchy.

Theorem (Kechris;Hjorth,Nies)

There is a largest null Π^1_1 set.

Theorem (Chong,Nies,Yu)

A real x is Π^1_1-random if and only if it is Δ^1_1 random and $\omega_1^x = \omega_1^{\mathrm{ck}}$.

Remark (Nies,Kalimullin)

If a degree spectrum of a structure is co-null, then it contains every Π^1_1-random real.

The Borel rank

Some Π^1_1 sets are not Borel. Some are Borel but have high rank:

Theorem (Steel)

The Borel rank of the set of reals which collapse ω_1^{ck} (the reals x such that $\omega_1^x > \omega_1^{\mathrm{ck}}$) is $\omega_1^{\mathrm{ck}} + 2$.

Randomness smoothes things a bit.

Theorem (Greenberg,Monin)

The set of Π^1_1 random reals is $\mathbf{\Pi}^0_3$.

Question: nonetheless we have the intuition that the set of Π^1_1-random reals is complicated. We have a higher analogue of the arithmetical hierarchy (and beyond); it would be nice to know if the set of Π^1_1 random reals lies in this hierarchy or not.

The Borel rank

Here is a sketch of the argument.

For any set G let G^* be the union of all Δ^1_1 closed sets which are subsets of G.

Remark

If G is higher Π^0_2 then G^* is the union of all Σ^1_1 closed sets which are subsets of G. Why? If $P \subseteq U_n$ we see this at some computable stage (compactness); if $P \subseteq \bigcap U_n$ we see this at some computable stage (admissibility).

Claim

For any Π^1_1 set G, $G - G^*$ is null. Why?
$\lambda(G) = \lambda(G_{\omega_1^{\mathrm{ck}}}) = \sup_{\alpha < \omega_1^{\mathrm{ck}}} \lambda(G_\alpha)$. *For each α we can find a Δ^1_1 closed subset of close measure.*

The Borel rank

Claim

A real x is not Π^1_1-random if and only if it is an element of $G - G^*$ for some higher Π^0_2 set G.

$\Longleftarrow$ Suppose that $x \in G - G^*$. If $x \in G_\alpha$ for some $\alpha < \omega_1^{\mathrm{ck}}$, then $x \in G_\alpha - G^*_\alpha$ which is a Δ^1_1 null set. If $x \in G - G_{\omega_1^{\mathrm{ck}}}$ then $\omega_1^x > \omega_1^{\mathrm{ck}}$.

$\Longrightarrow$ Suppose that x computes $f^x : \omega \to \omega_1^{\mathrm{ck}}$. Let $P_{n,\alpha}$ be the set of oracles y such that $f^y \upharpoonright_n : n \to \alpha$. Approximate the sets P_n from above by $U_{n,\epsilon}$. Let $G = \bigcap_{n,\epsilon} U_{n,\epsilon}$. If $x \in G^*$ then again by compactness and admissibility $x \in G_\alpha$ for some $\alpha < \omega_1^{\mathrm{ck}}$. If x is Δ^1_1-random then it is in P_α as well.

Generic analogues

Theorem (Greenberg,Monin)

A Δ^1_1-Cohen-generic real x preserves ω_1^{ck} if and only if it meets every dense Σ^1_1 sets of strings.

(Meeting or avoiding Π^1_1 sets of strings is not enough.) Thus the collection of generics which preserve ω_1^{ck} is G_δ.

A corollary

A corollary of the investigation of the Borel rank of Π^1_1 random reals shows that if x is sufficiently generic for the partial ordering of closed Σ^1_1 sets of positive measure (higher analogues of Π^0_1 classes) is Π^1_1 random.

Corollary (Greenberg,Monin)

A real is low for Π^1_1 randomness if and only if it is hyperarithmetic.

Admissible computability

Computability on ordinals

One of the motivations for admissible computability is the understanding of Π^1_1 sets as c.e. The set-theoretic version of the Spector-Gandy theorem is:

- A subset of ω is Π^1_1 if and only if it is Σ_1-definable over $L_{\omega_1^{\mathrm{ck}}}$.

Another motivation comes from Jensen's fine structure. Also from Takeuti's work on computability on the class of ordinals. The idea is to treat an ordinal α as "the new ω", so ordinals $\beta < \alpha$ correspond to natural numbers.

Definition

Let α be an ordinal. A subset of α is α-c.e. if it is Σ_1-definable over L_α.

Note that in this notation, c.e. is the same as ω-c.e; and for subsets of ω, Π^1_1 is ω_1^{ck}-c.e.

Admissibility

Using the notion of α-c.e. as the basic one we can define (partial) α-computable functions and so on. For most ordinals this does not behave well (consider for example $\alpha = \omega + 5$ or $\alpha = \omega + \omega$).

Fact

The following are equivalent for a limit ordinal α:

1. *For all $\beta < \alpha$ and all α-computable functions f, $f[\beta]$ is bounded below α.*
2. *We can define α-computable functions by recursion (in α many steps).*

Such ordinals are called admissible.

Uncountable structures

Work with Knight, Kach, Lempp, Turetsky, Melnikov. And thanks to Denis.

Effective properties of uncountable structures

Any cardinal is admissible, and regular cardinals are particularly nice. We can use κ-computability to consider notions analogous to ones of computable algebra and computable model theory. I will just mention a couple of results.

- (Greenberg,Kach,Lempp,Turetsky, following an idea of Knight's) Characterising the ω_1-computably categorical linear orderings. [Note: we have no clue about $\kappa \geqslant \omega_2$.]
- (Greenberg,Knight,Melnikov) Work on relative computable categoricity and Scott families. [Continuity is a required extra ingredient.]

Ordinals x 2

Dan Turetsky and I developed uncountable analogues of the Ash-Knight machinery (using a presentation by Montalbán). We used it to study uncountable linear orderings in the spirit of Hausdorff and Watnick.

- We isolate a derivative operation that can be iterated transfinitely (through ω_1-computable ordinals). There are several options to choose from; the nicest one generalises the Cantor-Bendixon (rather than Hausdorff's) derivative.
- This derivative can be inverted using a layered priority argument. This gives a sharp bound on the complexity of the iterated derivative operation.

α-c.e. degrees

α-c.e. degrees

Many researchers looked at the α-c.e. degrees and the lattice of α-c.e. sets for a variety of admissible ordinals α (mostly in the 1960s and 70s). Some basic questions are still open, for example the existence of minimal pairs.

Partly using coding techniques:

Theorem (Greenberg)

For any admissible $\alpha > \omega$, the partial orderings of the α-c.e. degrees and the c.e. degrees are not elementarily equivalent.

A natural difference

Another part of the argument looked at embeddings of the 1-3-1 lattice. Following work of Shore's I showed that copies of the 1-3-1 lattice in the α-c.e. degrees (for $\alpha > \omega$) have to have a high top. In particular, the fine distinctions between totally β-c.a. degrees for various small ordinals β disappear (recall that they are all low$_2$). The only thing that matters is the existence of a $\mathbf{d}$-computable counting of α.

Corollary (Downey,Greenberg)

There is a single, natural first-order sentence which holds in the c.e. degrees and fails in the α-c.e. degrees for all admissible $\alpha > \omega$. It is: "there is an incomplete degree which bounds a critical triple but not the 1-3-1 lattice."

Thank you