Notions of independence in computable commutative algebra

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We will give a sufficient condition with applications to abelian groups, ordered abelian groups, DCF_0, RCF, and difference closed fields.

Part 1: Known results

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Example 2 (Folklore)

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Corollary 3 (Mal'cev, ..., ..., Goncharov)

Both V_∞ and $\mathbb{U}$ are not computably categorical, and indeed have auto-dimension ω.

The latter follows because we have a Δ^0_2-isomorphic "bad" copy.

Known results

In contrast to V_∞ and $\mathbb{U}$, torsion-free abelian groups do not have "nice" copies by default.

Theorem 4 (Dobrica, Nurtazin)

Every computable torsion-free abelian group A of infinite $\mathbb{Z}$-rank has two computable copies B and G such that:

- B has no computable $\mathbb{Z}$-base;
- G has a computable $\mathbb{Z}$-base;
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Theorem 6 (Goncharov, Lempp, Solomon)

Every computable Archimedean ordered abelian group A of infinite $\mathbb{Z}$-rank has two computable copies B and G such that:

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So we see a pattern.

Definition 8

We say that a class K of computable structures and the associated notion of independence has

the Mal'cev property

if for every $A \in K$ of infinite dimension there exist computable presentations B and G of A such that:

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Question

Which classes have the Mal'cev property?

Part 2: Computably enumerable pregeometries

Definition 9

Let X be a set and $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ a function on $\mathcal{P}(X)$. We say that cl is a pregeometryif:

- $X \subseteq \mathrm{cl}(X)$ and $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$,
- $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$,
- (finite character) $\mathrm{cl}(X)$ is the union of the sets $\mathrm{cl}(Y)$ where Y is a finite subset of X, and
- (exchange principle) if $x \in \mathrm{cl}(X \cup \{y\})$ and $x \notin \mathrm{cl}(X)$, then $y \in \mathrm{cl}(X \cup \{x\})$.

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Each pregeometry under consideration will be upon the domain of a computable structure $\mathcal{M}$ and will be relatively intrinsically c.e. (i.e., the relations $x \in \mathrm{cl}(\{y_1, \dots, y_n\})$ are relatively intrinsically c.e. in $\mathcal{M}$, uniformly in n.)

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In general, our pregeometry will be computable only within the "good" copy G of $\mathcal{M}$. One can then apply most of the results form [Downey and Remmel] to G.

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Let $\mathcal{M}$ be a structure and $(\mathcal{M}, \mathrm{cl})$ a r.i.c.e. pregeometry.

Condition G: Uniformly in $\bar{c} \in \mathcal{M}$, we can effectively list the existential formulas $\varphi(\bar{c}, \bar{x})$ which have a solution $\bar{a}$ independent over $\bar{c}$.

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Condition B: The existential types of independent elements in $\mathcal{M}$ are non-principal.

(In other words: For any existential formula $\varphi(\bar{c}, x)$ holding of any element a which is independent over $\bar{c}$, there is an element b which satisfies $\varphi(\bar{c}, x)$ and is dependent over $\bar{c}$.)

The metatheorem:

Theorem 10 (Harrison-Trainor, M., Montalban 2014)

Let $\mathcal{M}$ be a structure and $(\mathcal{M}, \mathsf{cl})$ a r.i.c.e. pregeometry of infinite rank that satisfies Condition G and Condition B. Then $(\mathcal{M}, \mathsf{cl})$ has the Mal'cev property.

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In fact, Condition G implies the existence of a Δ^0_2-isomorphic "good" copy where *cl* is computable, and Condition B implies the existence of a Δ^0_2-isomorphic "bad" copy where *cl* is not computable.

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Corollary 11

$\mathcal{M}$ has auto-dimension ω.

Part 3: Applications

Corollary 12 (H.-T., M.,M.)

We get essentially all that was known before *and with nicer proofs*:

- The trivial examples (vector spaces, ACF_0, etc.).
- Torsion-free abelian groups (we use the factorial trick and Rado Lemma).
- Archimedean ordered abelian groups (we use the factorial trick and o-minimality of $\mathbb{R}$).

Definition 13

A *differential field* is a field K together with a derivation operator $\delta : K \to K$ which is linear and which satisfies the Leibniz rule; that is, $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = x\delta(y) + y\delta(x)$.

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Famously, DCF_0 is decidable. Harrington showed that every computable differential field can be embedded into a computable DCF_0. R. Miller has recently showed that DCF_0 has The Low Property.

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Every computable differentially closed field of infinite δ-transcendence degree has the Mal'cev property.

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Interestingly enough, our proof has some technical similarities with the R.Miller's proof of the Low Property for DCF_0. We have no formal explanation for this phenomenon. Something even more general may be going on here.

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Our proof uses cell decompositions etc.

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Definition 17 (Chatzidakis and Hrushovski (?), 1999)

A difference closed field is an existentially closed difference field.

Corollary 18 (H.-T., M., M.)

Every computable difference closed field of characteristic 0 of infinite transformal degree has the Mal'cev property.

Our proof relies on various model theoretic properties of these fields.

Part 4. A negative result

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In the 1960's and 1970's people wondered whether countable abelian *p*-groups admit a good notion of a "basis".

The most well-known attempt is probably Kulikov's *p*-basis. However, this approach has certain limitations.

In effective algebra, we use the notion of a *p*-basic tree (L. Rogers (1977) based on Crawley and Hales (1969)).

Let A be an abelian p-group.

Definition 19 (L. Rogers)

A p-basic tree of A is the collection of elements B of A with the properties:

1. Every $a \in A$ can be uniquely expressed as
$$a = \sum_{b \in B} k_b b,$$
where each $k_b \in \mathbb{Z}_p$;
2. For every $b \in B$, either $pb \in B$ or $pb = 0$.

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Question (Ash, Knight and Oates, $\sim$1990)

Does every computable reduced abelian p-group have a copy with a computable p-basic tree?

The answer is known to be *YES* for every computable reduced p-group of finite Ulm type (to be explained).

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splits into a direct sum of cyclic groups.
We iterate this process:

$$A^0 = A,$$

$$A^{(k+1)} = (A^{(k)})',$$

$$\dots$$

$$A^{(\omega)} = \bigcap_n A^{(n)}$$

$$\dots$$

We also define the Ulm factors

$$A_\beta = A^{(\beta)}/A^{(\beta+1)}.$$

Abelian p-groups

A countable abelian p-group is reduced if there is a (countable) β such that

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The least such β is called the Ulm type of A, written $\mathbb{U}(A)$.

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Theorem 20 (Ulm)

Let $\beta = \mathbb{U}(A)$. Then the isomorphism types of the Ulm factors A_γ, $\gamma < \beta$, completely describe the isomorphism type of A.

Theorem 21 (Ash-Knight-Oates)

Suppose A is a computable reduced abelian p-group and $\mathbb{U}(A) = n < \omega$. Then A has a computable copy with a computable p-basic tree.

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Suppose A is a computable reduced abelian p-group and $\mathbb{U}(A) = n < \omega$. Then A has a computable copy with a computable p-basic tree.

The only proof that we know is non-uniform. For every $k \leq n$ we need to have an access to a certain effective invariant of A_n, namely its limitwise monotonic function.

Computable *p*-groups

If we had a ***uniform*** access to these limitwise monotonic functions, the case of $\mathbb{U}(A) = \omega$ would not be a problem.

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The proof uses elements of the 0''' technique.

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- There is a very little hope of getting a counterexample.

Thanks!