# Computability in the class of Real Closed Fields

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# Outline

# Real Closed Fields

- All structures have countable universe and all classes are closed under isomorphism.
- $RCF = Mod(Th(\mathbb{R}, +, *, 0, 1, <))$

# Real Closed Fields

- All structures have countable universe and all classes are closed under isomorphism.
- $RCF = Mod(Th(\mathbb{R}, +, *, 0, 1, <))$
- $RCF$ has many nice model theoretic properties
  - Complete, decidable, o-minimal and accepts quantifier elimination.
  - Definable Skolemization which preserves the properties above.

# Skolemization of $RCF$

Let $\mathcal{R} \in RCF$ and $X \subseteq \mathcal{R}^n$ be definable, say by $\varphi(\bar{x}, y)$. Let $X_{\bar{a}} = \{y : (\bar{a}, y) \in X\}$. Then we have:

- if $X_{\bar{a}}$ is empty, then $f_\varphi(\bar{a}) = 0$
- if $X_{\bar{a}}$ has a least element $b$, then $f_\varphi(\bar{a}) = b$
- if the leftmost interval of $X_{\bar{a}}$ is $(c, d)$, then $f_\varphi(\bar{a}) = \frac{d-c}{2}$
- if the leftmost interval of $X_{\bar{a}}$ is $(-\infty, d)$, then $f_\varphi(\bar{a}) = d - 1$
- if the leftmost interval of $X_{\bar{a}}$ is $(c, \infty)$, then $f_\varphi(\bar{a}) = c + 1$

# TC embeddings

**Definition**

Let $K$ and $K'$ be two classes of structures. A *Turing computable embedding* from $K$ to $K'$ is a Turing operator $\Phi = \varphi_e$ such that:

- for each $\mathcal{A} \in K$, there is a $\mathcal{A}' \in K'$ such that $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{A}')}$
- for $\mathcal{A}, \mathcal{B} \in K$ correspond, respectively, to $\mathcal{A}', \mathcal{B}' \in K'$ then $\mathcal{A} \cong \mathcal{B}$ iff $\mathcal{A}' \cong \mathcal{B}'$.

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- Uniform procedure that respects isomorphism types.
- For $K$ and $K^{'}$ as above we write that $K \leq_{tc} K^{'}$, so that TCE induces a preordering or classes.

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- $ARCF$ is a subclass of the class $RCF$ where structures have no infinite elements, i.e only multiplicative classes are $[1]_m$ and $[2]_m$.

# Daisy graphs and ARCF

**Definition**

A *daisy graph* is an undirected graph $\mathcal{G}$ with a distinguished vertex, say $x_0$, and a set of edges $E$, such that every vertex $x \neq x_0$ in the universe of $\mathcal{G}$ is part of a unique loop containing $x_0$.

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- If we are to succeed in $DG \leq_{tc} ARCF$, we must do so by associating families of sets to families of algebraically independent reals.

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**Recall:** A tree $T$ is *perfect* if for every $\sigma \in T$ there is $\sigma \leq \tau$ such that $\tau^\wedge 0, \tau^\wedge 1 \in T$.
**Note:** $T : 2^{<\omega} \to \mathbb{Q} \times \mathbb{Q}$, where $\sigma \mapsto (q, q^*)$ which we interpret as an interval $I_\sigma = [q, q^*]$

# $DG \leq_{tc} ARCF$

**Theorem**

*There is a perfect, computable, binary tree $T$ whose continuum many paths represent algebraically independent reals over $\mathbb{Q}$.*

1. $T(\varnothing) = [0, 1]$
2. If $\sigma \leq \tau$, then $I_\tau \subseteq I_\sigma$.
3. If $length(\sigma) = n$, then $diameter(I_\sigma) \leq 2^{-n}$, where $diameter(a, b)$, for an interval $(a, b)$, is defined to be $b - a$.
4. If $\sigma, \tau$ are incomparable and both of length $n$, $I_\sigma \cap I_\tau = \varnothing$
5. For $f \in 2^\omega$, let $r_f$ be the unique real in $\bigcap_{\sigma \subseteq f} I_\sigma$. Then for distinct $f_1, \ldots, f_n \in 2^\omega$, $r_{f_1}, \ldots, r_{f_n}$ are algebraically independent.

# Further work on TCE

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**Theorem**

*For all* $0 \leq n < \omega$, $VRCF_n \leq_{tc} VRCF_{n+1}$

**Theorem**

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# $ARCF <_{tc} VRCF_1$

**Theorem (Pull-Back Theorem, Knight, Miller, Vanden Boom)**

*If $K \leq_{tc} K'$ via some $\Phi$, then for any computable infinitary sentence $\varphi'$ in the language of $K'$ we can find a computable infinitary sentence $\varphi$ in the language of $K$ such that for all $\mathcal{A} \in K$, $\mathcal{A} \models \varphi$ iff $\Phi(\mathcal{A}) \models \varphi'$. Moreover, if $\varphi'$ is $\Sigma_\alpha$ $(\Pi_\alpha)$ then so is $\varphi$.*

- $\mathcal{A} \cong \mathcal{B} \in ARCF$ iff $\mathcal{A}$ and $\mathcal{B}$ satisfy the same $\Sigma_2^c$ sentences.
- We find $\mathcal{V}$ and $\mathcal{V}' \in VRCF_1$ non-isomorphic satisfying the same $\Sigma_2^c$ sentences.

# $ARCF <_{tc} VRCF_1$

- $\mathcal{V} = RC(\mathbb{Q}(g, a_0, \ldots, a_n, \ldots))$ and $\mathcal{V}' = RC(\mathcal{V}(b))$, where
  - $b = \Sigma_i a_i g^{q_i}$
  - $(q_i)_{i<\omega}$ decreasing sequence converging to an irrational.

**Lemma**

*For any $\Pi^0_2$ set $S$, we can uniformly produce a sequence of structures $(F_n)_{n<\omega}$ such that $F_n \cong \mathcal{V}'$ if $n \in S$ and $F_n \cong \mathcal{V}$ otherwise.*

# Relative Categoricity

**Definition (Relatively $\Delta^0_\gamma$-categorical)**

A structure $\mathcal{A}$ is relatively $\Delta^0_\gamma$-categorical if for all structures $\mathcal{B} \cong \mathcal{A}$, there is some isomorphism $F : \mathcal{A} \to \mathcal{B}$ such that $F$ is $\Delta^0_\gamma(\mathcal{B})$.

# Relative Categoricity

A structure $\mathcal{A}$ is relatively $\Delta^0_\gamma$ categorical iff $\mathcal{A}$ has a formally $\Sigma^c_\gamma$ Scott family.

**Definition**

A formally $\Sigma^c_\gamma$ Scott family for a structure $\mathcal{A}$ is a set $\Phi$ of formulas, with fixed parameters $\bar{c}$ from $\mathcal{A}$, such that:

- for each tuple $\bar{a}$ of elements of $\mathcal{A}$, there is a formula $\varphi(\bar{x}, \bar{c}) \in \Phi$, such that $\mathcal{A} \models \varphi(\bar{a}, \bar{c})$
- if two tuples $\bar{a}$ and $\bar{b}$ from $\mathcal{A}$ satisfy the same formula from $\Phi$, then there is an automorphism of $\mathcal{A}$ mapping $\bar{a}$ to $\bar{b}$

# Motivation

**Theorem (Corollary from work of Nurtazin)**

*Let $\mathcal{R}$ be a computable RCF, then $\mathcal{R}$ is computably categorical if and only if $\mathcal{R}$ has finite transcendence degree.*

**Theorem (Calvert)**

*If $\mathcal{R}$ is a computable archimedean RCF, then $\mathcal{R}$ is $\Delta^0_2$-categorical.*

Moral: The complexity is in the infinite elements.

# Two Results

**Theorem (Ash)**

*Suppose $\alpha$ is a computable ordinal, with $\omega^{\delta+n} \leq \alpha < \omega^{\delta+n+1}$, $\delta$ is either $0$ or a limit ordinal, and $n < \omega$. Then $\alpha$ is $\Delta^0_{\delta+2n}$-stable but not $\Delta^0_\beta$-stable for $\beta < \delta + 2n$.*

**Theorem**

*Let $\alpha$ be a computable well-order and let $\mathcal{R}_\alpha$ be the RCF constructed around $\alpha$. Then $\mathcal{R}_\alpha$ is relatively $\Delta^0_\gamma$-categorical and not $\Delta^0_\beta$-categorical for $\beta < \gamma$, where $n < \omega$ and*

$$\gamma = \begin{cases} 2n+1, \text{ if } \omega^n \leq \alpha < \omega^{n+1} \\ \delta + 2n, \text{ if } \omega^{\delta+n} \leq \alpha < \omega^{\delta+n+1} \end{cases}$$

# Relative categoricity

- $LO$ :
  $\lambda_0(x) \equiv (\forall y)(x \leq y)$

- $RCF$ :
  $x \approx_m y \equiv \bigvee\!\!\!\bigvee_{n \in \mathbb{N}} x < y^n \;\&\; \bigvee\!\!\!\bigvee_{n \in \mathbb{N}} y < x^n$

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  $\lambda_0^*(x) \equiv INF(x) \ \& \ (\forall y)(x \approx_m y \vee x <_m y)$

# Scott family for $\mathcal{R}_\alpha$

For all $\beta < \alpha$, we take all formulas of the form:

- $\varphi(\bar{x}) \equiv (\exists y_1)\cdots(\exists y_i)(\lambda^*_{\beta_1}(y_1)\ \&\ \cdots\ \&\ \lambda^*_{\beta_i}(y_i)\ \&\ \psi(\bar{x},\bar{y}))$, where $\psi(\bar{x},\bar{y})$ is quantifier free.

THANK YOU