Effective Dimension and the Intersection of Random Closed Sets

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Motivation

There have been a number of recent developments concerning effective Hausdorff dimension and classical geometric measure theory.

From this work, we have learned new ways in which effective Hausdorff dimension can yield significant insights about problems in classical mathematics.

Today, I want to offer a new way of understanding the effective Hausdorff dimension of a sequence related to membership in random closed sets.

Previous work by Diamondstone and Kjos-Hanssen

An informative connection between effective dimension and random closed sets was first established by Diamondstone and Kjos-Hanssen, who proved the following:

- If a sequence X has effective Hausdorff dimension $\geq \gamma$, then X is a member of random closed set (with the level of randomness of the closed set determined by the parameter γ).
- If a sequence X is a member of random closed set (again, with its level of randomness determined by the parameter γ), then X has effective Hausdorff dimension $> \gamma$.

As we will see, for a given choice of $\gamma \in [0,1]$, the value $2^{-\gamma}$ is the survival parameter for the process of pruning the full binary tree.

A general consequence

As a consequence of these results of Diamondstone and Kjos-Hanssen, we have:

the collection of sequences in some random closed set given by parameter γ for some $\gamma \in (0,1]$	$=$	the collection of sequences of positive effective Hausdorff dimension

Our goal is to refine this correspondence.

A rough description of our result

The main result that I will report today tells us, roughly:

- The effective Hausdorff dimension of a sequence is inversely proportional to how intersectable the random closed sets containing it must be.

The idea behind this statement is that the more branching there is in a random closed set (determined by the choice of the parameter $\gamma \in [0, 1]$), the larger the number of relatively random closed sets with a non-empty intersection will be.

This is how we will define the degree of intersectability of a family of random closed sets.

Outline

1. Background
2. Intersections of random closed sets
3. Effective dimension and intersectability
4. Future directions

Part 1: Background

Effective Hausdorff dimension

For $X \in 2^\omega$, the *effective Hausdorff dimension* of X is

$$\liminf_{n\to\infty} \frac{K(X\restriction n)}{n}.$$

Intuitively, the effective Hausdorff dimension measures the density of information content in a sequence.

Coding closed subsets of 2^ω

In order to directly apply the machinery of algorithmic randomness for sequences to closed subsets of 2^ω, we will code closed sets as members of 3^ω (as coding scheme due to Barmpalias, Brodhead, Cenzer, Dashti, and Weber).

In particular, as we move node by node through the extendible nodes of our closed set, each value in our sequence encodes one of three outcomes:

- "0" indicates that only the left extension of a given node is extendible;
- "1" indicates that only the right extension of a given node is extendible;
- "2" indicates that both extensions of a given node are extendible.

$X = 2021122012\ldots$

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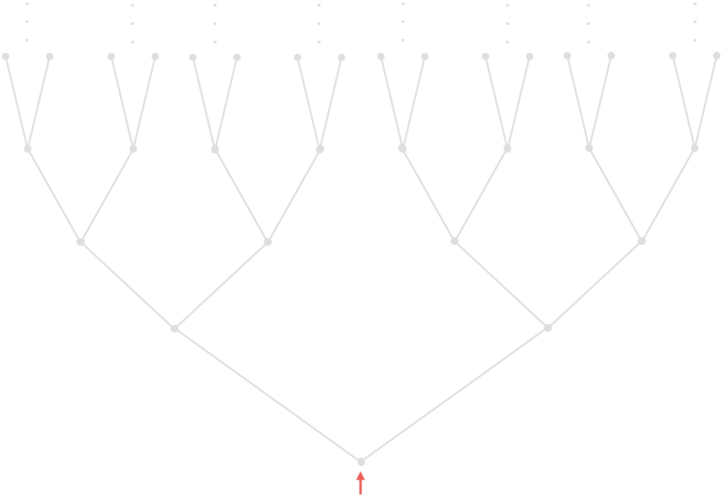

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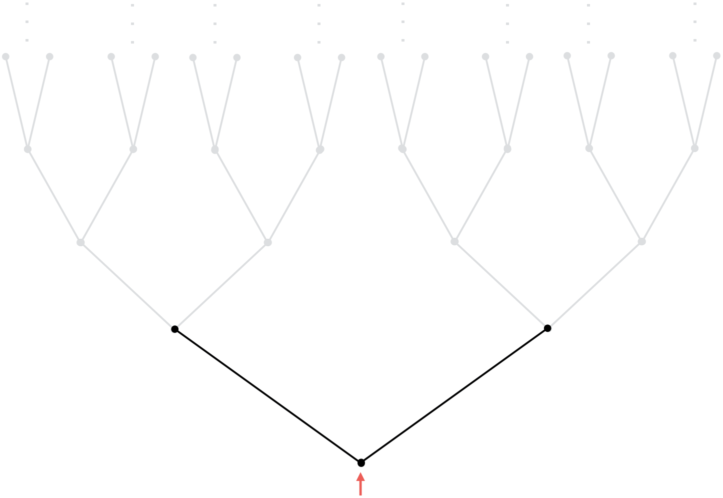

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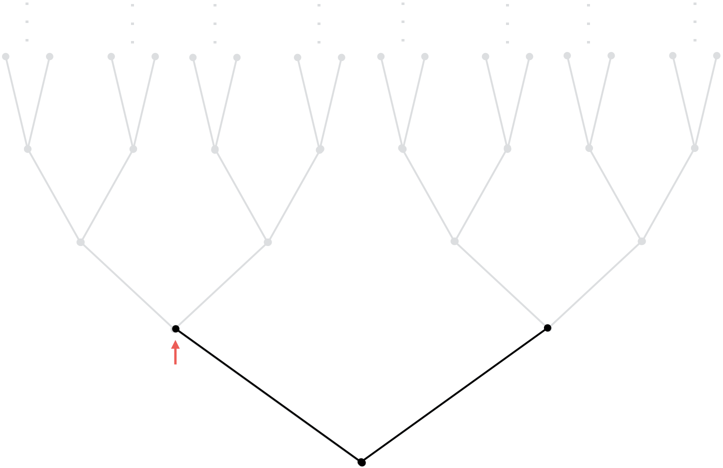

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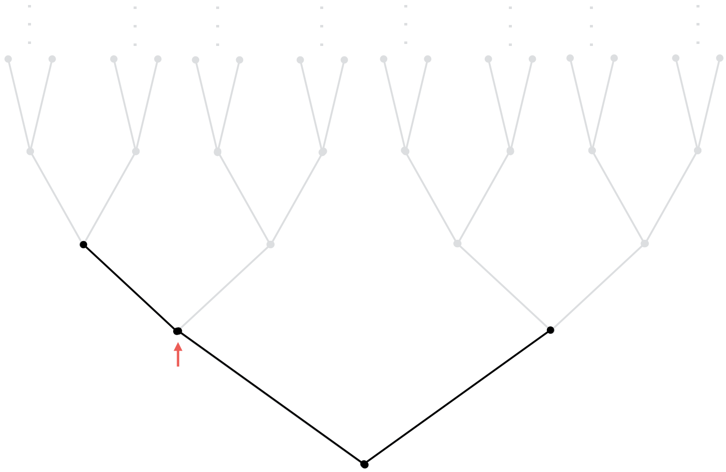

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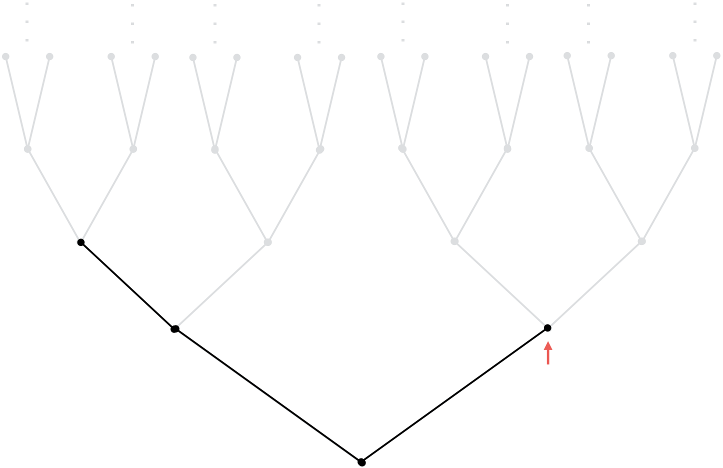

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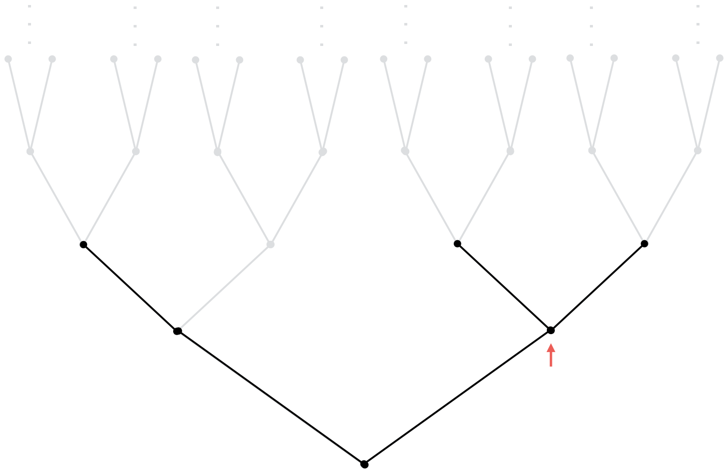

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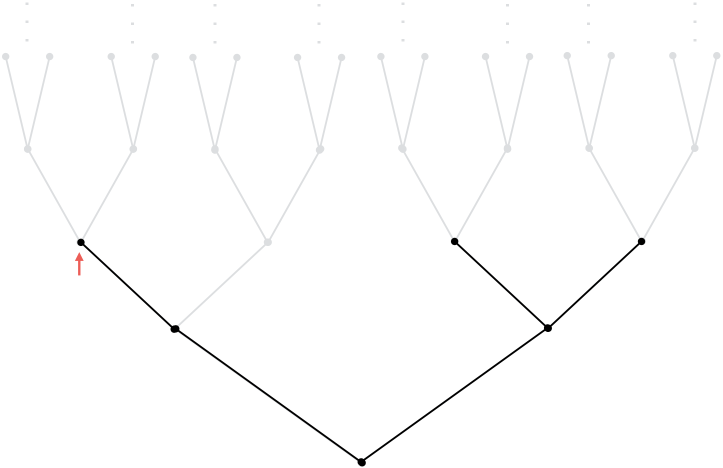
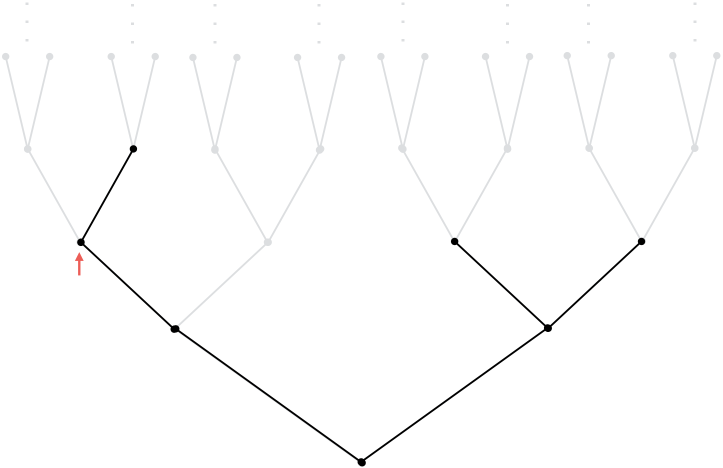
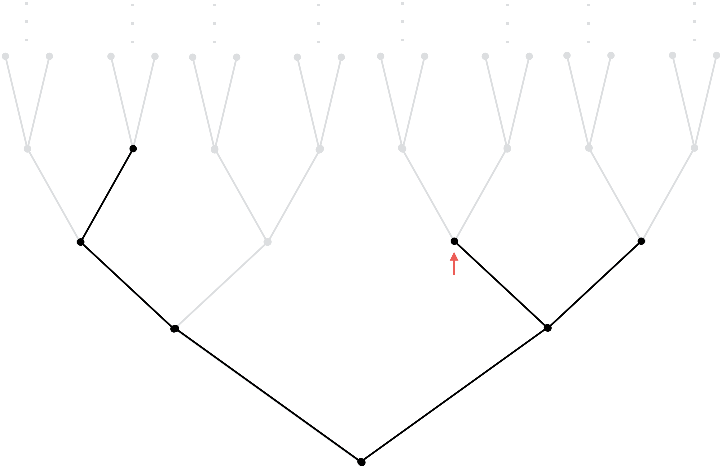


$X = 202\textcolor{red}{1}122012\ldots$



$$X = 20211122012\ldots$$



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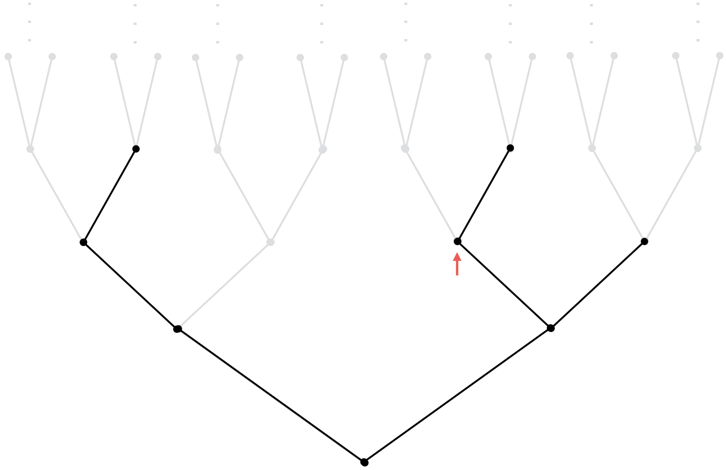

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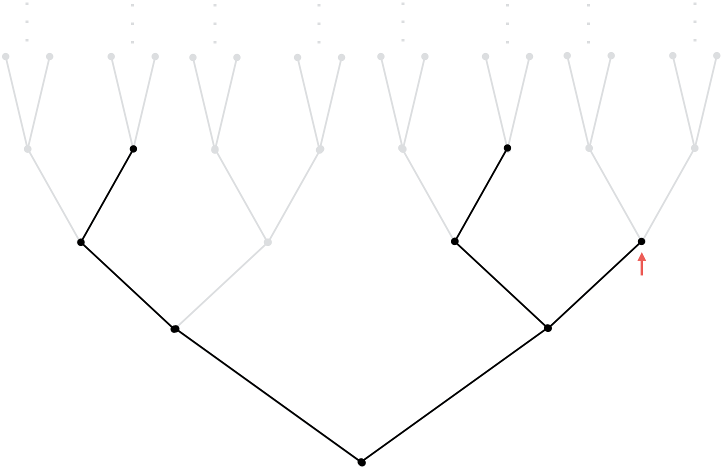

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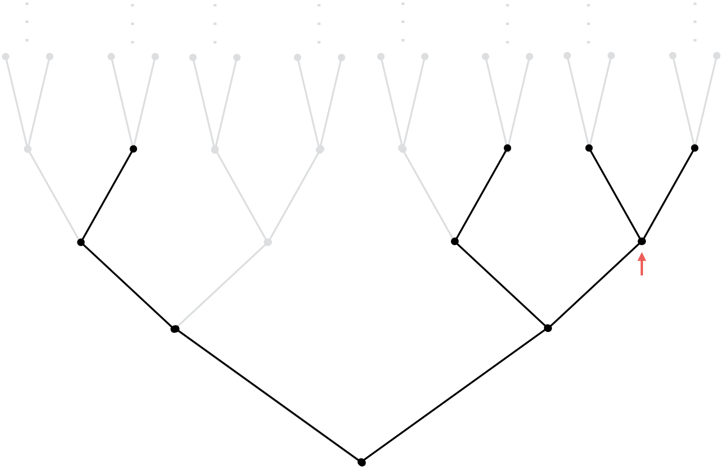

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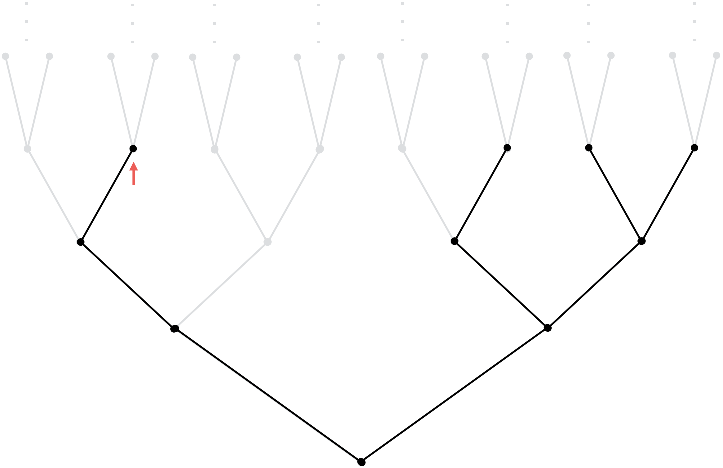

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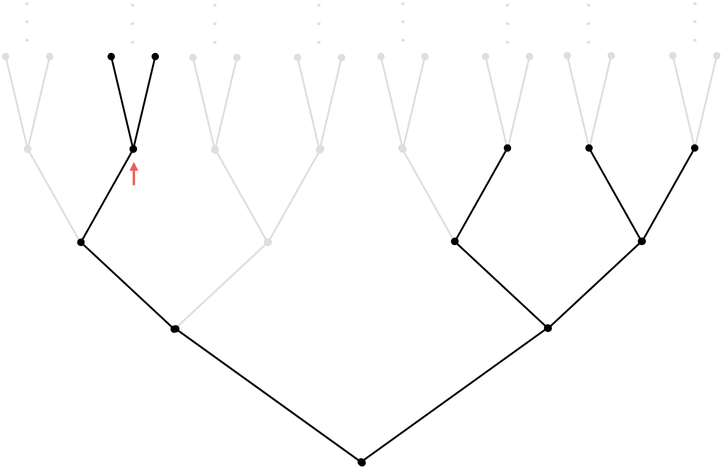

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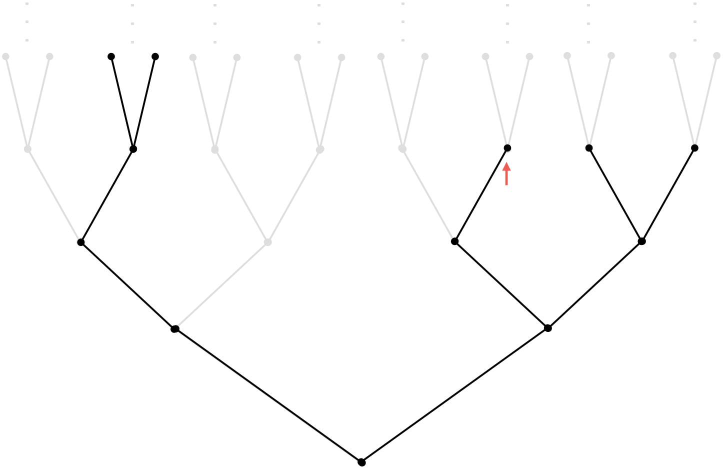
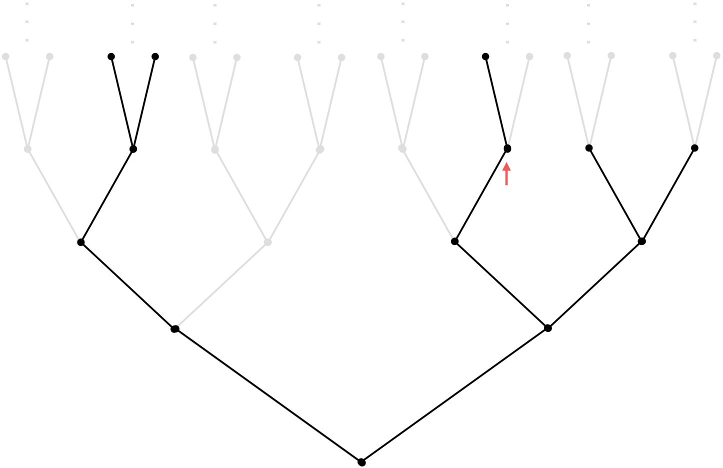
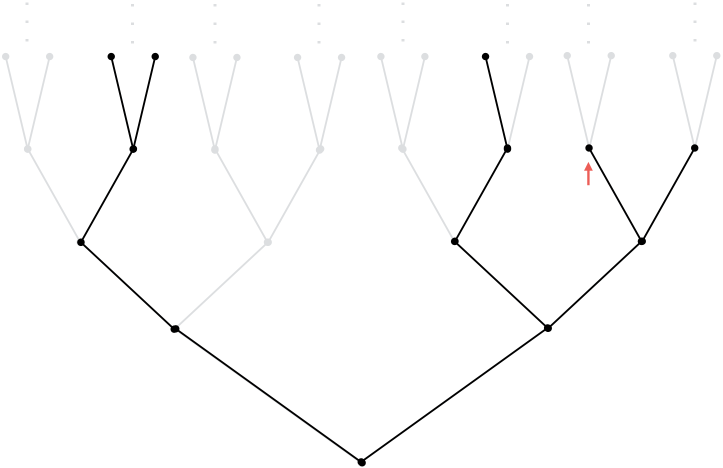


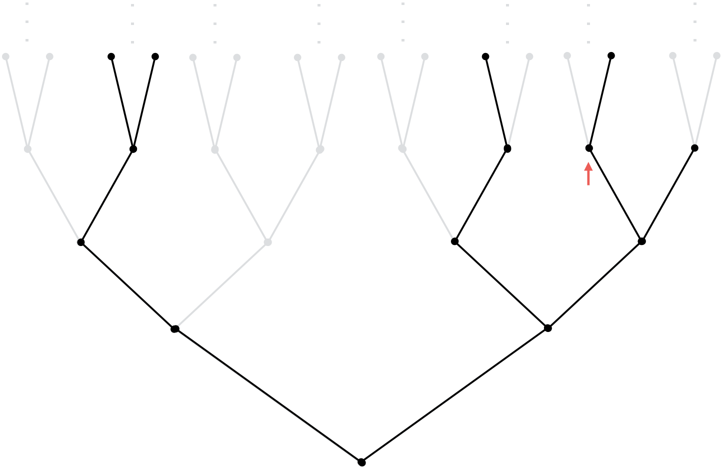
$X = 2021122\textcolor{red}{0}12\ldots$



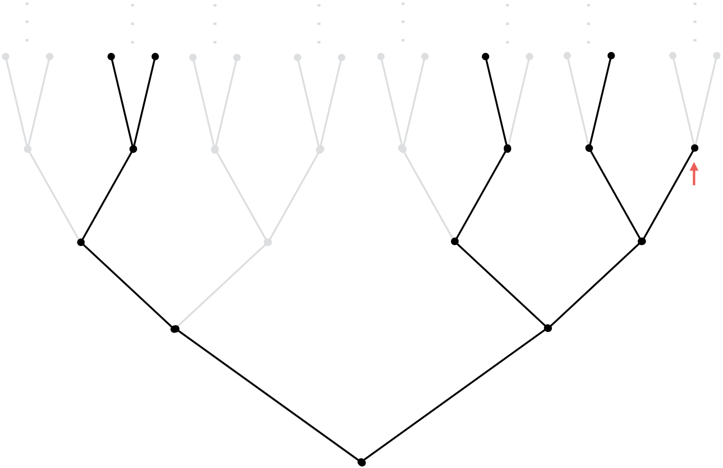
$$X = 2021122\textcolor{red}{0}12\ldots$$



$$X = 20211220\textcolor{red}{1}2\ldots$$



$X = 2021122012\ldots$



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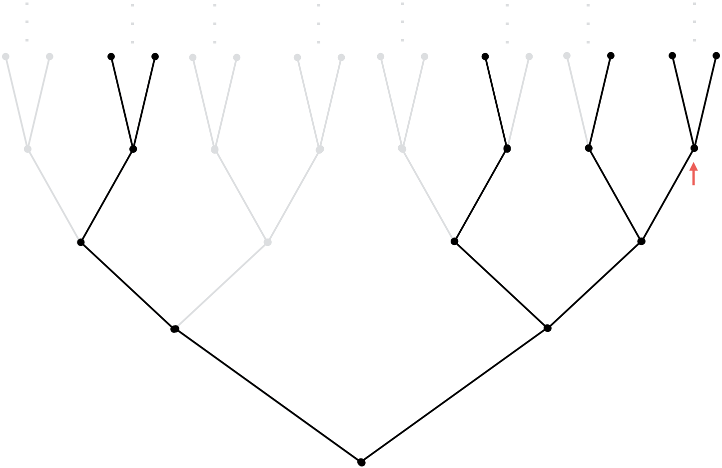

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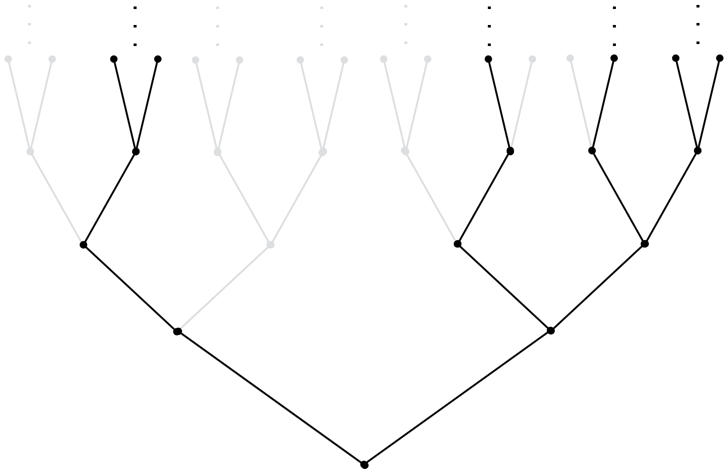


$X = 2021122012\ldots$

Computable Probability Measures on 3^ω

Definition

A probability measure μ on 3^ω is *computable* if $\sigma \mapsto \mu(\llbracket\sigma\rrbracket)$ is computable as a real-valued function.

Hereafter, for $i \in \{0, 1, 2\}$ and $\sigma \in 3^{<\omega}$, I will use the shorthand

$$\mu(\sigma i \mid \sigma) = \frac{\mu(\llbracket\sigma i\rrbracket)}{\mu(\llbracket\sigma\rrbracket)}.$$

Let us consider some examples.

Bernoulli measures on 3^ω

Let $p, q \in [0,1]$ satisfy $p + q \leq 1$. Then the measure $\mu_{\langle p,q \rangle}$ defined by the conditional probabilities

- $\mu_{\langle p,q \rangle}(\sigma 0 \mid \sigma) = p$
- $\mu_{\langle p,q \rangle}(\sigma 1 \mid \sigma) = q$
- $\mu_{\langle p,q \rangle}(\sigma 2 \mid \sigma) = 1 - p - q$

for $\sigma \in 3^{<\omega}$ defines a *Bernoulli measure* on 3^ω.

$\mu_{\langle p,q \rangle}$ is a computable measure if and only if p and q are both computable.

Symmetric Bernoulli measures on 3^ω

Let $p \in (0, \frac{1}{2})$. Then the measure μ_p defined by the conditional probabilities

- $\mu_p(\sigma 0 \mid \sigma) = p$
- $\mu_p(\sigma 1 \mid \sigma) = p$
- $\mu_p(\sigma 2 \mid \sigma) = 1 - 2p$

for $\sigma \in 3^{<\omega}$ defines a *symmetric Bernoulli measure* on 3^ω.

μ_p is a computable measure if and only if p is computable.

Martin-Löf randomness

Let μ be a computable measure on 3^ω.

Definition

A *μ-Martin-Löf test* is a uniformly Σ^0_1 sequence $(\mathcal{U}_i)_{i\in\omega}$ of subsets of 3^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

A sequence $X \in 3^\omega$ *passes the μ-Martin-Löf test* $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_i \mathcal{U}_i$.

$X \in 3^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathrm{MLR}_\mu$, if X passes *every* μ-Martin-Löf test.

Note that we can relative this definition to any oracle.

Algorithmically random closed sets

Let $\mathcal{K}(2^\omega)$ be the collection of closed subsets of 2^ω.

One way to define an algorithmically random closed subset of 2^ω, due to Barmpalias, Brodhead, Cenzer, Dashti, and Weber:

- A closed set $\mathcal{C} \subseteq 2^\omega$ is random if it can be coded by an algorithmically random sequence $X \in 3^\omega$.

Uniformly random closed sets

This definition was originally given for the case $p = q = \frac{1}{3}$, i.e., with respect to the Lebesgue measure on 3^ω.

It was later extended to more general measures on 3^ω in a number of other Cenzer-led projects.

Robustness of this definition

The resulting definition of randomness is equivalent to one obtained by

- defining Martin-Löf random closed sets in a way that is "native" to $\mathcal{K}(2^\omega)$, which uses the hit-or-miss or Fell topology on 2^ω (Axon); and
- defining Martin-Löf random closed sets in terms of Galton-Watson processes (Diamondstone, Kjos-Hanssen).

Convention about measures on $\mathcal{K}(2^\omega)$

If μ is a measure on 3^ω, then we write μ^* to stand for the corresponding measure on $\mathcal{K}(2^\omega)$.

That is, μ^*-random closed sets are those closed sets coded by a μ-random sequence in 3^ω.

Symmetric Bernoulli random closed sets

Random closed sets with respect to a symmetric Bernoulli measure μ_p^* for $p \in [0, \frac{1}{2}]$ are particularly nice.

The symmetry of the measure manifests itself in the fact that the probability of having only a left branch at a given node is equal to the probability of having only a right branch at that node.

Consequently, the collection of μ_p^*-random closed sets is closed under mirror images (swapping 0s and 1s in the code).

Extreme instances of symmetric Bernoulli random closed sets

The parameter p for a μ_p^*-random closed set determines the amount of branching in the closed sets.

The smaller p is, the more branching there is.

We have the following extreme cases:

- If $p = 0$, then the infinite sequence of 2s is the only μ_p-random sequence, and thus the only μ_p^*-random closed set is 2^ω.
- If $p = \frac{1}{2}$, then μ_p-random sequences contain no 2s, and thus each μ_p^*-random closed set has the form $\{X\}$, where $X \in 2^\omega$ is Martin-Löf random.

Part 2: Intersections of Random Closed Sets

Cenzer/Weber on the intersection of random closed sets

Theorem (Cenzer, Weber)

Suppose that $p, q, r, s \geq 0$ *are computable,* $0 \leq p + q \leq 1$ *and* $0 \leq r + s \leq 1$.

Suppose that $P \in \mathcal{K}(2^\omega)$ *is* $\mu^*_{\langle p,q \rangle}$*-random relative to* $Q \in \mathcal{K}(2^\omega)$ *and that* Q *is* $\mu^*_{\langle r,s \rangle}$*-random relative to* P.

Then one of three possibilities occurs:

The first possibility

- If $p + q + r + s \geq 2 + pr + qs$, then $P \cap Q = \emptyset$.

This technical condition guarantees that neither P nor Q have a sufficient amount of branching to guarantee a non-empty intersection.

The second possibility

- If $p+q+r+s<1+pr+qs$, then $P \cap Q = \emptyset$ with probability

$$\frac{ps+qr}{(1-p-q)(1-r-s)}.$$

In this case, there may be a sufficient amount of branching in P and Q, but we see that the intersection is empty due to some finite level of both P and Q.

The third possibility

- If $p + q + r + s < 1 + pr + qs$ and $P \cap Q \neq \emptyset$, then $P \cap Q$ is Martin-Löf random with respect to the measure $\mu^*_{\langle p+r-pr, q+s-qs \rangle}$.

Now we have a sufficient amount of branching in P and Q and some infinite path in their intersection.

The amount of branching in the resulting closed set is computable in the Bernoulli parameters of both P and Q.

A side question: The converse?

Cenzer and Weber left open the question as to whether the converse is true:

In the case that $p + q + r + s < 1 + pr + qs$, can every $\mu^*_{\langle p+r-pr, q+s-qs\rangle}$-random closed set be obtained as the intersection of $P, Q \in \mathcal{K}(2^\omega)$, where P is $\mu^*_{\langle p,q\rangle}$-random relative to $Q \in \mathcal{K}(2^\omega)$ and Q is $\mu^*_{\langle r,s\rangle}$-random relative to P?

We answered the question in the affirmative.

A corollary of the intersection theorem

Corollary (Cenzer, Weber)

For $p \in (0, 1/2)$, *let* $P, Q \in \mathcal{K}(2^\omega)$ *be relatively* μ_p^**-random.*

1. *If* $p \geq 1 - \frac{\sqrt{2}}{2}$, *then* $P \cap Q = \emptyset$.
2. *If* $p < 1 - \frac{\sqrt{2}}{2}$, *then* $P \cap Q = \emptyset$ *with probability* $\frac{2p^2}{(1-2p)^2}$.
3. *If* $p < 1 - \frac{\sqrt{2}}{2}$ *and* $P \cap Q \neq \emptyset$, *then* $P \cap Q$ *is Martin-Löf random with respect to the measure* $\mu_{2p-p^2}^*$.

Jointly random closed sets

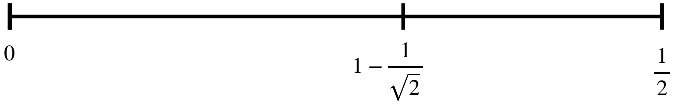
We would like to extend this to the intersection of more than two random closed sets.

To do so, we need to make sure that any closed set in our intersection is relatively random to all of the others (taken together).

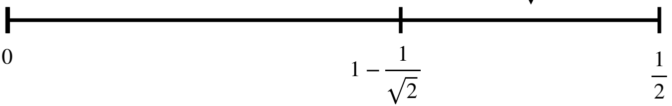
A sequence of random closed sets $P_1, P_2, \ldots P_n \subseteq 2^\omega$ with codes $X_1, X_2, \ldots, X_n \in 3^\omega$ is *jointly random* if for each $j \in n$

$$X_j \text{ is random relative to } \bigoplus_{i \neq j} X_j.$$

By van Lambalgen's theorem, this is equivalent to requiring that $\bigoplus_{i=1}^n X_j$ be random.


0
$1 - \frac{1}{\sqrt{2}}$
$\frac{1}{2}$

For any parameter p in this interval,
the intersection of relatively μ_p^*-random closed
sets is **always** empty.



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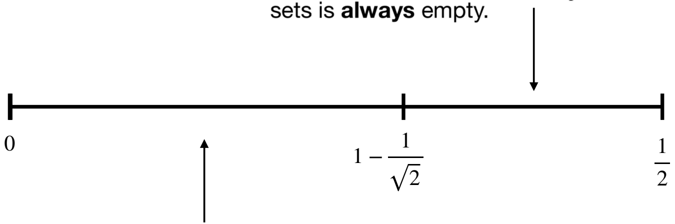
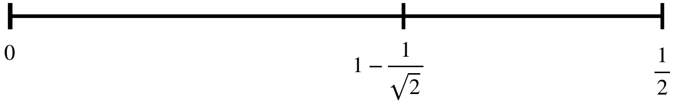
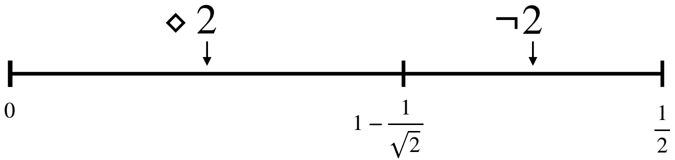


For any parameter p in this interval,
the intersection of relatively μ_p^*-random closed
may be non-empty.



$\neg n$: the intersection of n jointly random closed sets is empty

$\Diamond\, n$: the intersection of n jointly random closed sets may be non-empty



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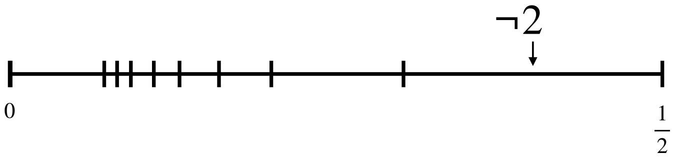

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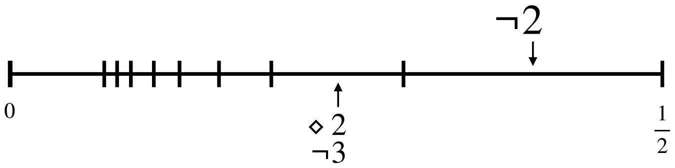

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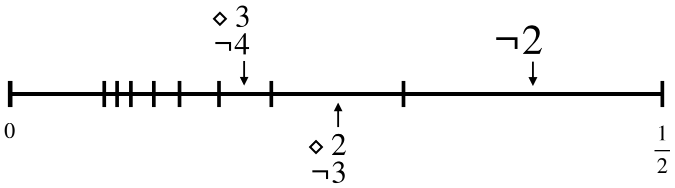

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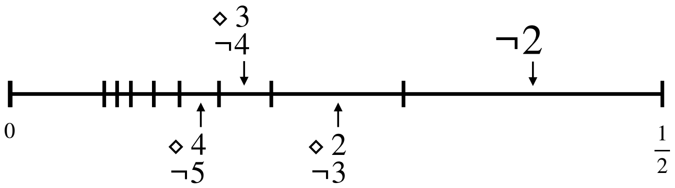

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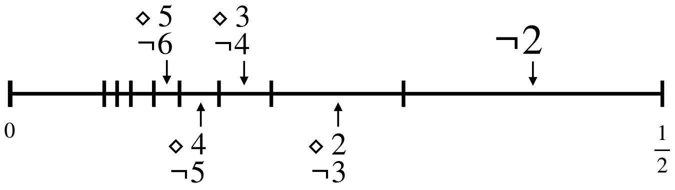

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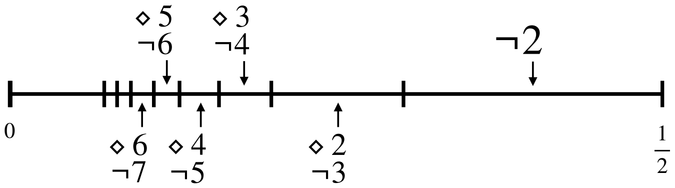

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Intersecting 3 jointly random closed sets

Given jointly μ_p^*-random closed sets P_1, P_2, and P_3, then assuming that $\bigcap_{i=1}^3 P_i \neq \emptyset$, we can conclude:

- $P_1 \cap P_2$ is $\mu^*_{2p-p^2}$-random (by the Cenzer/Weber corollary),
- $(P_1 \cap P_2) \cap P_3$ is $\mu^*_{3p-3p^2+p^3}$-random (by the original Cenzer/Weber intersection theorem)

  Moreover, setting $p = q$ and $r = s = 2p - p^2$, from the condition $p + q + r + s < 1 + pr + qs$, we can conclude

$$3p - 3p^2 + p^3 < \frac{1}{2}.$$

More generally

For $n \geq 1$, there is a sequence of polynomials $(f_n)_{n\in\omega}$ such that given $n+1$ jointly μ^*_p-random closed sets $P_1, \ldots, P_{n+1}$, then assuming that $\bigcap_{i=1}^{n+1} P_i \neq \emptyset$, we can conclude:

- $\bigcap_{i=1}^{n} P_i$ is $\mu^*_{f_n(p)}$-random,
- $\left(\bigcap_{i=1}^{n} P_i\right) \cap P_{n+1}$ is $\mu^*_{p+f_n(p)-pf_n(p)}$-random.

  Thus we have

  $$f_{n+1}(p) = p + f_n(p) - pf_n(p)$$

  and we require that, for every n,

  $$f_n(p) < \frac{1}{2}.$$

Analyzing the f_n's, Part 1

From the conditions

$$f_0(p) = p$$

and

$$f_{n+1}(p) = p + f_n(p) - pf_n(p),$$

we can show via induction that

$$f_n(p) = 1 - (1 - p)^n.$$

Analyzing the f_n's, Part 2

From

$$f_n(p) = 1 - (1 - p)^n$$

and the requirement for a non-empty intersection of n jointly random μ_p^*-random closed sets, we have

$$f_n(p) < \frac{1}{2},$$

we can derive the equivalent requirement that

$$p < 1 - \frac{1}{\sqrt[n]{2}}.$$

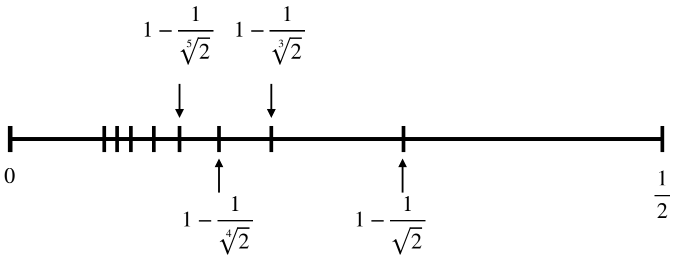
The full result

For $n \in \omega$, let $f_n(p) = 1 - (1-p)^n$.

Theorem (Case, Porter)

For $p \in (0, \frac{1}{2})$ and $n \geq 2$, given n jointly μ_p^*-random closed sets $P_1, \ldots, P_n$, the following hold:

1. *If $p \geq 1 - \frac{1}{\sqrt[n]{2}}$, then $\bigcap_{i=1}^n P_i = \emptyset$.*
2. *If $p < 1 - \frac{1}{\sqrt[n]{2}}$, then $\bigcap_{i=1}^n P_i = \emptyset$ with probability $1 - \frac{1-2f_n(p)}{(1-2p)^n}$.*
3. *If $p < 1 - \frac{1}{\sqrt[n]{2}}$ and $\bigcap_{i=1}^n P_i \neq \emptyset$, then $\bigcap_{i=1}^n P_i$ is Martin-Löf random with respect to the measure $\mu_{f_n(p)}^*$.*


$1-\frac{1}{\sqrt[5]{2}}$
$1-\frac{1}{\sqrt[3]{2}}$
0
$1-\frac{1}{\sqrt[4]{2}}$
$1-\frac{1}{\sqrt{2}}$
$\frac{1}{2}$

Another side question: The converse?

Can a μ_p^*-random closed set be obtained as the intersection of n jointly random closed sets of the appropriate type?

The answer is affirmative:

Theorem (Case, Porter)

For $p \in (0, 1/2)$ and $n \geq 2$, every μ_p^*-random closed set can be obtained as the intersection of n jointly random $\mu^*_{f_n^{-1}(p)}$-random closed sets.

Part 4: Effective dimension and intersectability

Applying Diamondstone/Kjos-Hanssen's result

We now want to apply the result of Diamondstone and Kjos-Hanssen on the effective dimension of members of random closed sets mentioned earlier.

To do so, we first have to translate their approach to random closed sets, via Galton-Watson trees, to the approach we are taking.

Idea: For a fixed $\gamma \in [0, 1]$, starting with the full binary tree, we move edge by edge length-lexicographically and remove an edge with probability $1 - 2^{-\gamma}$.

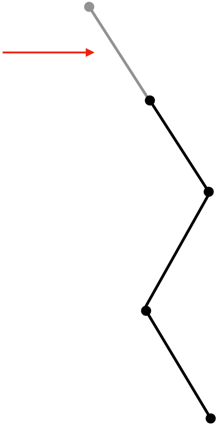
The probability that an edge remains in the tree is $2^{-\gamma}$.

Translating between the two approaches

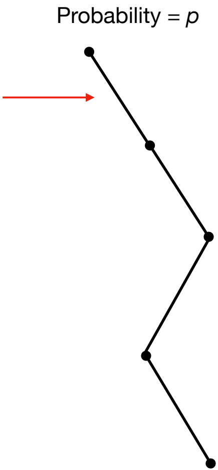
For the measure μ_p^*, the probability that a given edge remains in the tree (assuming that it is still connected to the root), is equal to

$$p + (1 - 2p) = 1 - p.$$

This edge remains
in the tree
if and only if
the corresponding
code for the resulting
closed set has
a 0 or a 2 in the
relevant position

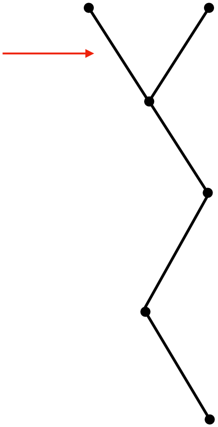


This edge remains
in the tree
if and only if
the corresponding
code for the resulting
closed set has
a 0 or a 2 in the
relevant position



This edge remains
in the tree
if and only if
the corresponding
code for the resulting
closed set has
a 0 or a 2 in the
relevant position

Probability = 1- 2*p*



Translating between the two approaches

For the measure μ_p^*, the probability that a given edge remains in the tree (assuming that it is still connected to the root), is equal to

$$p + (1 - 2p) = 1 - p.$$

Setting $1 - p = 2^{-\gamma}$, this gives us

$$\gamma = -\log(1 - p).$$

Stating the Diamondstone/Kjos-Hanssen result

Theorem (Diamondstone/Kjos-Hanssen)

Let $\gamma \in [0, 1]$*.*

- *If* X *is a member of a random closed set with survival parameter* $2^{-\gamma}$*, then*

$$\dim(X) \geq \gamma.$$

- *If*

$$\dim(X) > \gamma,$$

then X *is a member of a random closed set with survival parameter* $2^{-\gamma}$*.*

In the case that $p = 1 - \frac{1}{\sqrt[n]{2}}$, we get

$$-\log(1-p) = -\log\left(1 - (1 - \frac{1}{\sqrt[n]{2}})\right) = \frac{1}{n}.$$

Thus in our context, setting $p_n = 1 - \frac{1}{\sqrt[n]{2}}$, we get:

- If X is a member of a $\mu^*_{p_n}$-random closed set, then
$$\dim(X) \geq \frac{1}{n}.$$
- If
$$\dim(X) > \frac{1}{n},$$
then X is a member of a $\mu^*_{p_n}$-random closed set.

Degrees of intersectability

Let us set the *degree of intersectability* of a μ_p^*-random closed set to be the unique n such that (i) n jointly μ_p^*-random closed sets can have a non-empty intersection and (ii) $n+1$ jointly μ_p^*-random closed sets always have an empty intersection.

We know such an n exists by our theorem on multiple intersections.

Moreover, the μ_p^*-random closed sets with degree of intersectability equal to n are precisely those satisfying

$$p \in \left[1 - \frac{1}{\sqrt[n+1]{2}}, 1 - \frac{1}{\sqrt[n]{2}}\right).$$

Dimension and intersectability

If P is a symmetric Bernoulli random closed set with degree of intersectability n, then for every $X \in P$, we have $\dim(X) \geq \dfrac{1}{n+1}$.

Let $s \in [0,1]$ satisfying $s \neq \frac{1}{n}$ for $n \in \omega$. If $\dim(X) = s$, then

1. X is contained in a symmetric Bernoulli random closed set of degree of intersectability k for all $k \geq \lfloor \frac{1}{s} \rfloor$, but
2. X is not contained in any symmetric Bernoulli random closed set of degree of intersectability k for $k < \lfloor \frac{1}{s} \rfloor$.

Part 4: Future directions

Where do we go from here?

Two possible directions for further investigation are:

1. Study the relationship between energy randomness and degree of intersectability of the members of random closed sets.
2. Apply these ideas to study the intersectability of random subfractals of self-similar fractals.

Thank you!