The Decomposability Problem for Torsion-Free Abelian Groups is Analytic-Complete

Kyle Riggs

Indiana University

April 29, 2014

A group $(G, +)$ is computable if there is an enumeration $\nu : N \to G$ such that the set

$$\{(a, b, c) : \nu(a) + \nu(b) = \nu(c)\}$$

is computable. Such an enumeration is called a *computable presentation* of G.

This means statements of the form

$$x + y = z \text{ and } nx = y$$

are computable for $x, y, z \in G$, $n \in N$.

An abelian group G is *decomposable* if it has two nontrivial subgroups A and B such that $G = A \oplus B$. If not, it is *indecomposable*.

An abelian group G is *decomposable* if it has two nontrivial subgroups A and B such that $G = A \oplus B$. If not, it is *indecomposable*.

We would like to write a group as the direct sum of indecomposable subgroups.

Can we classify the indecomposable groups?

The only indecomposable torsion groups are the cocyclic groups ($Z(p^k)$ for $1 \leq k \leq \infty$).

The only indecomposable torsion groups are the cocyclic groups ($Z(p^k)$ for $1 \leq k \leq \infty$).

Every mixed group is decomposable.

The only indecomposable torsion groups are the cocyclic groups ($Z(p^k)$ for $1 \leq k \leq \infty$).

Every mixed group is decomposable.

Torsion-free groups of rank 1 (subgroups of Q) are indecomposable. There is no other well-known class of indecomposable torsion-free abelian groups. This talk will explain why.

The only indecomposable torsion groups are the cocyclic groups ($Z(p^k)$ for $1 \leq k \leq \infty$).

Every mixed group is decomposable.

Torsion-free groups of rank 1 (subgroups of Q) are indecomposable. There is no other well-known class of indecomposable torsion-free abelian groups. This talk will explain why.

For computable torsion-free groups of finite rank (> 1), being decomposable is a Σ^0_3-complete property. Among computable groups of infinite rank, the property is Σ^1_1-complete.

Here's an example of an indecomposable group of rank 2 (from *Abelian Groups*, L. Fuchs) :

Here's an example of an indecomposable group of rank 2 (from *Abelian Groups*, L. Fuchs) :

Start with the free abelian group generated by two elements: x_1 and x_2.
Add to the group the following elements:

$$\frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \frac{x_1}{13}, \frac{x_2}{17} \ldots$$

Here's an example of an indecomposable group of rank 2 (from *Abelian Groups*, L. Fuchs) :

Start with the free abelian group generated by two elements: x_1 and x_2.
Add to the group the following elements:

$$\frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \frac{x_1}{13}, \frac{x_2}{17} \ldots$$

Finally, add the element $\frac{x_1+x_2}{2}$ to the group. Denote the group generated by these elements as $\mathcal{G}$.

$\{x_1, x_2\}$ is still a basis for $\mathcal{G}$ because for every $g \in \mathcal{G}$, there exist $m, m_1, m_2 \in Z$ such that

$$mg = m_1x_1 + m_2x_2$$

We can also say that there exist $q_1, q_2 \in Q$ such that

$$g = q_1x_1 + q_2x_2$$

However, this does not imply that the elements q_1x_1 and q_2x_2 exist.
For example, $\frac{x_1+x_2}{2} = \frac{1}{2}x_1 + \frac{1}{2}x_2$.

Note that

$$\chi_{\mathcal{G}}(x_1) = (0, 1, 0, 1, 0, 1, 0, ...)$$

and

$$\chi_{\mathcal{G}}(x_2) = (0, 0, 1, 0, 1, 0, 1, ...)$$

But for any $g = q_1 x_1 + q_2 x_2$ with q_1, q_2 nonzero,

$$\mathbf{t}(g) = (0, 0, 0, 0, 0, 0, ...)$$

If there are nontrivial subgroups A, B such that $\mathcal{G} = A \oplus B$, then x_1 must be contained in a direct summand of this decomposition. This also holds for x_2.

If there are nontrivial subgroups A, B such that $\mathcal{G} = A \oplus B$, then x_1 must be contained in a direct summand of this decomposition. This also holds for x_2.

Because we added the element $\frac{x_1+x_2}{2}$ to the group, but neither $\frac{x_1}{2}$ nor $\frac{x_2}{2}$, then x_1 and x_2 must be in the same direct summand.

If there are nontrivial subgroups A, B such that $\mathcal{G} = A \oplus B$, then x_1 must be contained in a direct summand of this decomposition. This also holds for x_2.

Because we added the element $\frac{x_1+x_2}{2}$ to the group, but neither $\frac{x_1}{2}$ nor $\frac{x_2}{2}$, then x_1 and x_2 must be in the same direct summand.

Thus, $\mathcal{G} = A$ (or B), so the group is indecomposable. We say the element $\frac{x_1+x_2}{2}$ serves as a "link" connecting x_1 and x_2.

In this example, x_1 and x_2 are elements of *strictly maximal type*. An element x has strictly maximal type if, for any y linearly independent from x, $\mathbf{t}(x) \not\leq \mathbf{t}(y)$ (there are infinitely many pairs $\langle p, k \rangle \in P \times N$ such that $p^k | x$ but $p^k \nmid y$).

In this example, x_1 and x_2 are elements of *strictly maximal type*. An element x has strictly maximal type if, for any y linearly independent from x, $\mathbf{t}(x) \not\leq \mathbf{t}(y)$ (there are infinitely many pairs $\langle p, k \rangle \in P \times N$ such that $p^k | x$ but $p^k \nmid y$).

As we have seen, any element of strictly maximal type in a decomposable group must be contained in one of the direct summands.

In this example, x_1 and x_2 are elements of *strictly maximal type*. An element x has strictly maximal type if, for any y linearly independent from x, $\mathbf{t}(x) \not\leq \mathbf{t}(y)$ (there are infinitely many pairs $\langle p, k \rangle \in P \times N$ such that $p^k | x$ but $p^k \nmid y$).

As we have seen, any element of strictly maximal type in a decomposable group must be contained in one of the direct summands.

A group is indecomposable if it has a basis of elements of strictly maximal type with a link (or a chain of links) between every pair of basis elements. The converse of this statement does not hold.

If G is a torsion-free group of finite rank and $G = A \oplus B$, then for any bases $\bar{a}, \bar{b}$ of A, B (resp.) the set $\bar{a} \sqcup \bar{b}$ is a basis for G with the following property:

For every $g \in G$, if $g = \sum_i q_i a_i + \sum_j r_j b_j$, then there are elements a, b such that $a = \sum_i q_i a_i$ and $b = \sum_j r_j b_j$

If G is a torsion-free group of finite rank and $G = A \oplus B$, then for any bases $\bar{a}, \bar{b}$ of A, B (resp.) the set $\bar{a} \sqcup \bar{b}$ is a basis for G with the following property:

For every $g \in G$, if $g = \sum_i q_i a_i + \sum_j r_j b_j$, then there are elements a, b such that $a = \sum_i q_i a_i$ and $b = \sum_j r_j b_j$

This allows us to talk about decomposability in terms of finite bases instead of subgroups.

For a computable group G there is a Π^0_2 formula $BASIS$ on finite sets such that

$$BASIS(\bar{x}) \Leftrightarrow \bar{x} \text{ is a basis of } G$$

For a computable group G there is a Π^0_2 formula *BASIS* on finite sets such that

$$BASIS(\bar{x}) \Leftrightarrow \bar{x} \text{ is a basis of } G$$

This gives us the following Σ^0_3 formula for decomposable torsion-free groups of finite rank: G is decomposable iff

$$(\exists\, \bar{a}, \bar{b} \in [G]^{<\omega}) \left\{ BASIS(\bar{a} \sqcup \bar{b}) \wedge \bar{a} \neq \emptyset \wedge \bar{b} \neq \emptyset \right.$$
$$\wedge (\forall y \in G)\, (\forall \bar{q} \in Q^{<\omega})(\exists w \in G) \big[(|\bar{q}| = |\bar{a}| + |\bar{b}|$$
$$\left. \wedge\ y = \sum_i q_i a_i + \sum_j q_j b_j) \Rightarrow\ w = \sum_i q_i a_i \big] \right\}$$

To see this property is Σ^0_3-complete, we describe a computable function $f : \omega \to \{\text{torsion-free abelian groups of rank } 2\}$ such that $\mathcal{G}_n$ is decomposable iff W_n is cofinite.

Start with the group $\mathcal{G}$ in the previous example.

$$\mathcal{G} = \langle \frac{x_1 + x_2}{2}, \frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \ldots \rangle$$

Start with the group $\mathcal{G}$ in the previous example.

$$\mathcal{G} = \langle \frac{x_1 + x_2}{2}, \frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \ldots \rangle$$

To create the group $\mathcal{G}_n$, we add to $\mathcal{G}$ the element $\frac{x_2}{p_{2k+1}}$ for every k such that $\Phi_n(k) \downarrow$.

Start with the group $\mathcal{G}$ in the previous example.

$$\mathcal{G} = \langle \frac{x_1 + x_2}{2}, \frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \ldots \rangle$$

To create the group $\mathcal{G}_n$, we add to $\mathcal{G}$ the element $\frac{x_2}{p_{2k+1}}$ for every k such that $\Phi_n(k)\downarrow$.

If W_n is coinfinite, there are still infinitely many primes dividing x_1 but not x_2, so the group remains indecomposable.

Start with the group $\mathcal{G}$ in the previous example.

$$\mathcal{G} = \langle \frac{x_1 + x_2}{2}, \frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \ldots \rangle$$

To create the group $\mathcal{G}_n$, we add to $\mathcal{G}$ the element $\frac{x_2}{p_{2k+1}}$ for every k such that $\Phi_n(k)\downarrow$.

If W_n is coinfinite, there are still infinitely many primes dividing x_1 but not x_2, so the group remains indecomposable.

If W_n is cofinite, then x_1 no longer has strictly maximal type, and a decomposition of $\mathcal{G}_n$ exists.

Start with the group $\mathcal{G}$ in the previous example.

$$\mathcal{G} = \langle \frac{x_1 + x_2}{2}, \frac{x_1}{3}, \frac{x_2}{5}, \frac{x_1}{7}, \frac{x_2}{11}, \ldots \rangle$$

To create the group $\mathcal{G}_n$, we add to $\mathcal{G}$ the element $\frac{x_2}{p_{2k+1}}$ for every k such that $\Phi_n(k) \downarrow$.

If W_n is coinfinite, there are still infinitely many primes dividing x_1 but not x_2, so the group remains indecomposable.

If W_n is cofinite, then x_1 no longer has strictly maximal type, and a decomposition of $\mathcal{G}_n$ exists.

Thus, the set of decomposable torsion-free groups of finite rank (> 1) is Σ^0_3-complete.

If we adapt our previous formula to describe decomposable groups of infinite rank, we get the following:

$$(\exists\,(\bar{a} \sqcup \bar{b}) \in [G]^{\omega})\ [BASIS(\bar{a} \sqcup \bar{b}) \wedge \bar{a} \neq \emptyset \wedge \bar{b} \neq \emptyset \wedge (\forall y \in G)$$
$$(\forall \bar{q} \in Q^{<\omega})(\exists w \in G)\ (y = \sum_i q_i a_i + \sum_j q_j b_j \Rightarrow\ w = \sum_i q_i a_i)]$$

This is a Σ^1_1 formula, and in fact we will show that this property is Σ^1_1-complete for computable groups, and analytic-complete for groups in general.

We describe a (computable) function from (computable) trees in $\omega^{<\omega}$ to torsion-free abelian groups of infinite rank that gives us a group G_T which is decomposable iff there is an infinite path through the tree T.

We start with a countable basis

$$\{x_i\}_{i\geq 1}, \{x_\sigma\}_{\sigma\in\omega^{<\omega}}, \{y_j\}_{j\geq 1}$$

and make each element of this basis infinitely divisible by a unique prime.

We start with a countable basis

$$\{x_i\}_{i\geq 1}, \{x_\sigma\}_{\sigma\in\omega^{<\omega}}, \{y_j\}_{j\geq 1}$$

and make each element of this basis infinitely divisible by a unique prime.

Then, for each pair of x-elements we create a link using a unique prime. We do the same for each pair of y-elements.

We start with a countable basis

$$\{x_i\}_{i\geq 1}, \{x_\sigma\}_{\sigma\in\omega^{<\omega}}, \{y_j\}_{j\geq 1}$$

and make each element of this basis infinitely divisible by a unique prime.

Then, for each pair of x-elements we create a link using a unique prime. We do the same for each pair of y-elements.

For every n we also add an element of the form

$$\frac{y_1 + y_2 + ... + y_n}{p_{f(n)}}$$

At this point, the group decomposes as $A \oplus B$, where A is the pure subgroup containing all the x-elements, and B the pure subgroup containing all the y-elements.

Then, for every i we create a link between x_i and y_i. The group (denoted G) is now indecomposable.

Idea: Given a tree T, we add elements to G to create a group G_T.

If T has no infinite paths, G_T is still indecomposable.

If T has an infinite path π, $G_T = A_T \oplus B_\pi$, where A_T is the pure subgroup containing all the x-elements and B_π is the pure subgroup containing all the elements of the form $y_n + x_{\pi\restriction n}$

$$B_\pi = \langle y_1 + x_{\sigma_1}, y_2 + x_{\sigma_2}, ...\rangle_*$$

where $|\sigma_n| = n$ and $\sigma_n \prec \sigma_{n+1}$ for all n.

Sketch: Let σ be a string of length n, and let p be the prime that infinitely divides y_n. Whenever a string $\tau \succ \sigma$ is found on T, we make x_σ divisible by another power of p.

Sketch: Let σ be a string of length n, and let p be the prime that infinitely divides y_n. Whenever a string $\tau \succ \sigma$ is found on T, we make x_σ divisible by another power of p.

If there is a string of length n with infinitely many strings above it, then y_n no longer has strictly maximal type. However, the set

$$\{y_n\} \cup \{x_\sigma : |\sigma| = n \wedge (\exists^\infty \tau \in T)\, \tau \succ \sigma\}$$

does have strictly maximal type. This means that if y_n decomposes as $y_n = a + b$, then a and b must be in the span of this set.

When we find σ on T $(|\sigma| = n)$, then we break some links to allow $y_n + x_\sigma$ to be contained in a direct summand. For example:

$$\frac{x_n + y_n}{q} = \frac{x_n - x_\sigma}{q} + \frac{y_n + x_\sigma}{q}$$

When we find σ on T $(|\sigma| = n)$, then we break some links to allow $y_n + x_\sigma$ to be contained in a direct summand. For example:

$$\frac{x_n + y_n}{q} = \frac{x_n - x_\sigma}{q} + \frac{y_n + x_\sigma}{q}$$

$$\frac{y_1 + y_2 + \ldots + y_n}{p_{f(n)}} = -\frac{x_{\sigma\restriction 1} + x_{\sigma\restriction 2} + \ldots + x_\sigma}{p_{f(n)}} + \frac{(y_1 + x_{\sigma\restriction 1}) + (y_2 + x_{\sigma\restriction 2}) + \ldots + (y_n + x_\sigma)}{p_{f(n)}}$$

We denote the finished group G_T.
If there is an infinite path π through T, $G_T = A_T \oplus B_\pi$.
If G_T is decomposable, we can use the increasing sequence of links to find an infinite path through T.

We denote the finished group G_T.
If there is an infinite path π through T, $G_T = A_T \oplus B_\pi$.
If G_T is decomposable, we can use the increasing sequence of links to find an infinite path through T.

Thus, the set of computable decomposable torsion-free abelian groups of infinite rank is Σ^1_1-complete, and the set of decomposable torsion-free abelian groups of infinite rank is $\mathbf{\Sigma^1_1}$-complete.

Thank you!