Nonstandard Analysis: A New Way to Compute

Sam Sanders[1]

Midwest Computability Seminar, Nov. 15, 2012

[1]This research is generously supported by the John Templeton Foundation.

How to compute in NSA: Ω-invariance

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(Physics / TCS)

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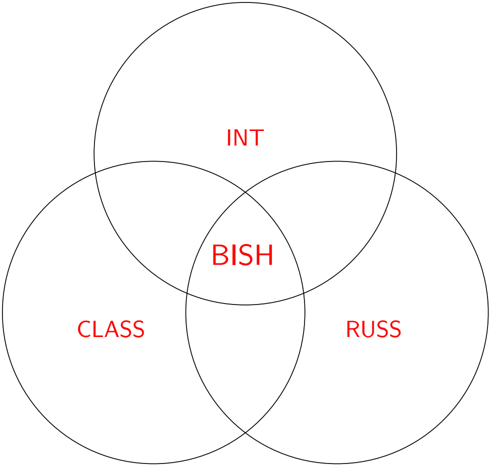

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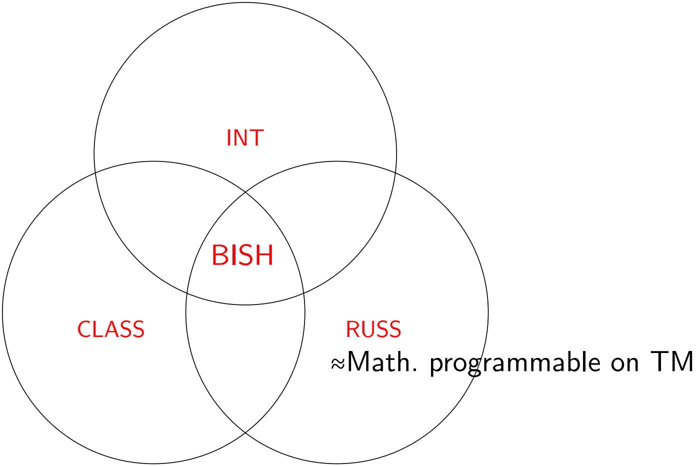

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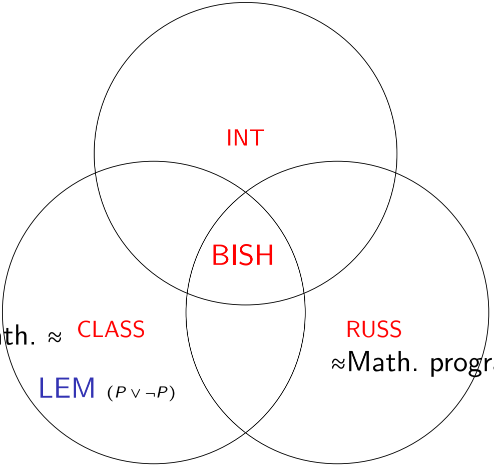

Classical Math. ≈ CLASS

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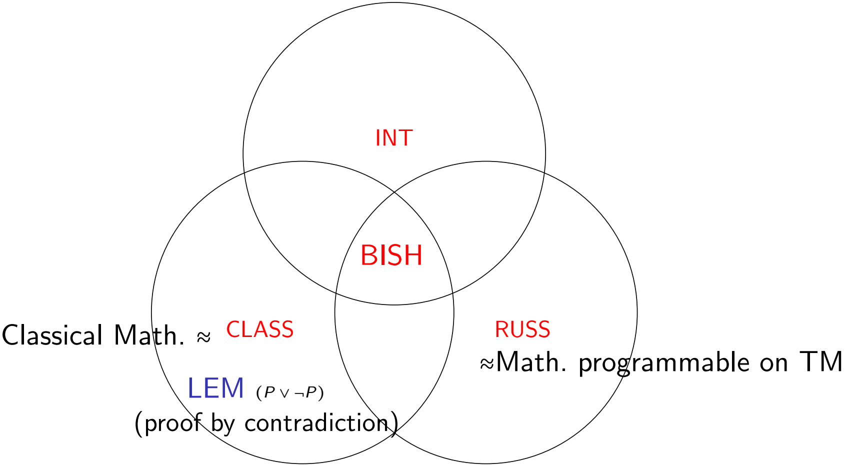
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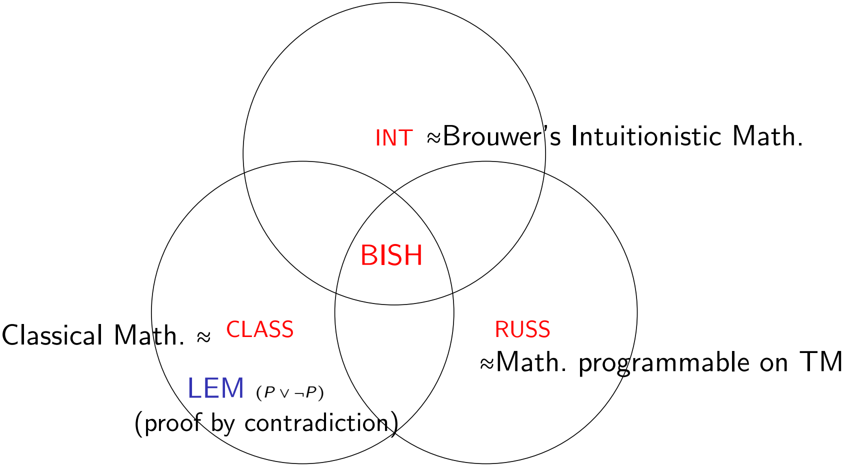
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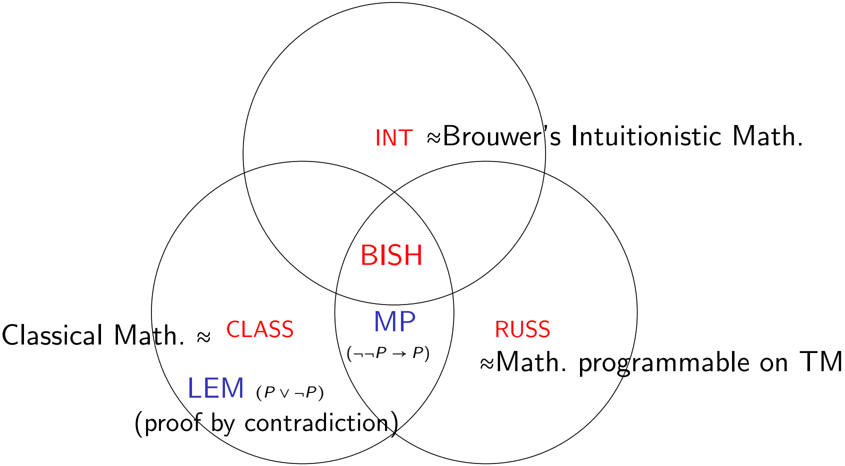
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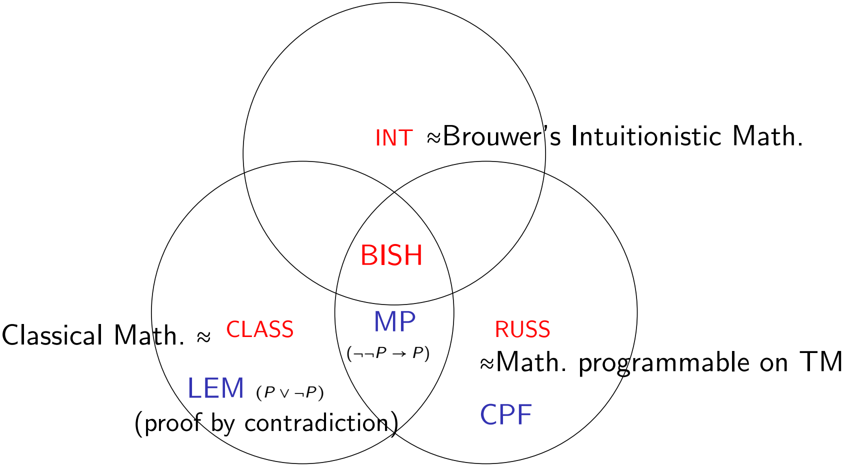

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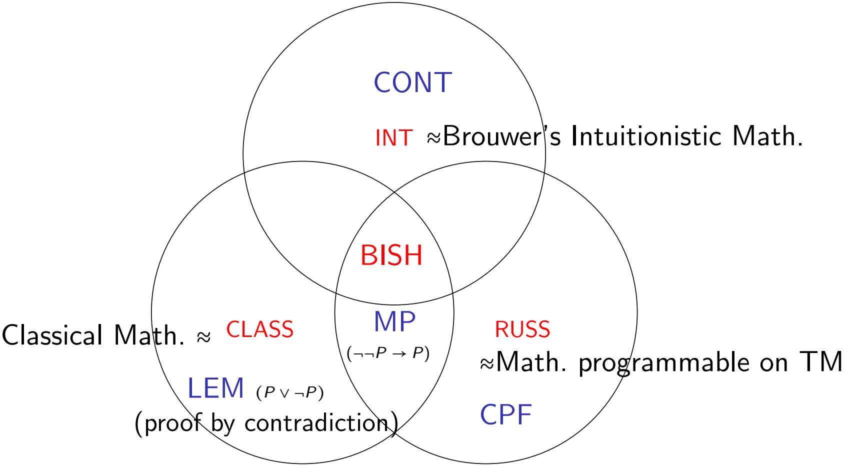
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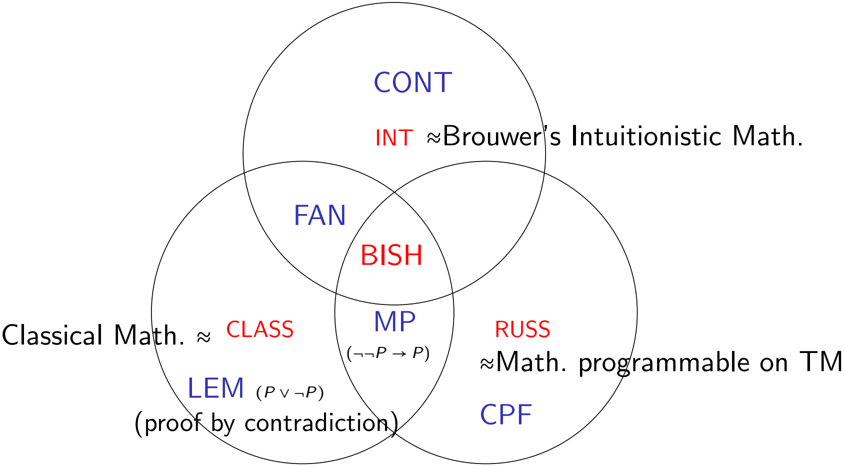
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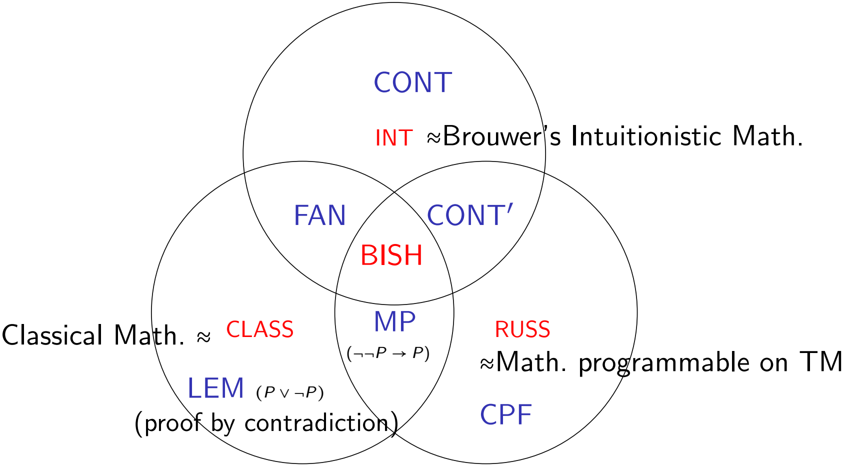
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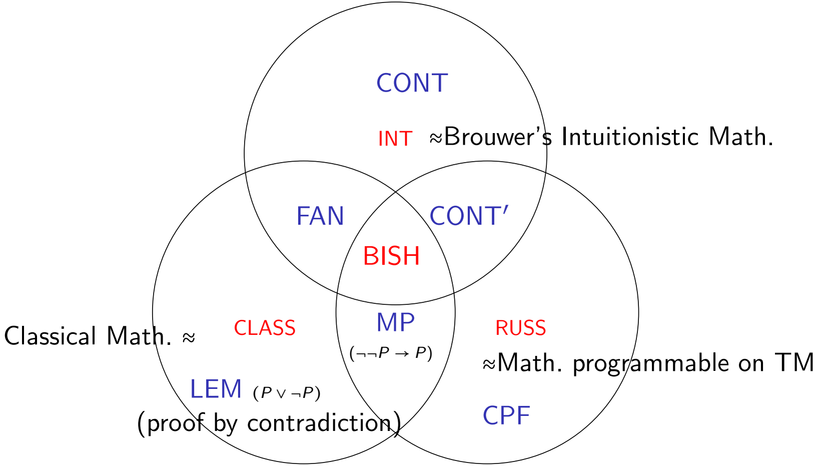

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6. $(\forall x \in A)P(x)$: for all x, $x \in A \to P(x)$.

Lost in translation

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Central: Ω-invariance and Transfer $(\mathbb{T})$

$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x}, \omega)$ s.t.

$$\psi(\vec{x}, \omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]] \\ \wedge \\ \neg\psi(\vec{x}, \omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$$

$A \Rrightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$

$\sim A$: $A \Rrightarrow (0 = 1)$

$(\exists x)A(x)$: an Ω-inv. proc. computes x_0 such that $A(x_0)$

$\sim[(\forall n \in \mathbb{N})A(n)] \equiv (\exists n \in \mathbb{N}_1)\sim A(n)$

WEAKER than $(\exists n \in \mathbb{N})\sim A(n)$.

Lost in translation

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \to (0 = 1)$	$\sim A$: $A \Rightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: an Ω-inv. proc. computes x_0 such that $A(x_0)$

WHY is this a good/faithful/reasonable/... translation?

Lost in translation

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
Central: algorithm and proof	Central: Ω-invariance and Transfer ($\mathbb{T}$)
$A \vee B$: an algo yields a proof of A or of B	$A \mathbb{V} B$: There is Ω-invariant $\psi(\vec{x},\omega)$ s.t. $\psi(\vec{x},\omega) \to [A(\vec{x}) \wedge [A(\vec{x}) \in \mathbb{T}]]$ $\wedge$ $\neg\psi(\vec{x},\omega) \to [B(\vec{x}) \wedge [B(\vec{x}) \in \mathbb{T}]]$
$A \to B$: an algo converts a proof of A to a proof of B	$A \Rightarrow B$: $\big[A \wedge [A \in \mathbb{T}]\big] \to \big[B \wedge [B \in \mathbb{T}]\big]$
$\neg A$: $A \to (0 = 1)$	$\sim A$: $A \Rightarrow (0 = 1)$
$(\exists x)A(x)$: an algo computes x_0 such that $A(x_0)$	$(\exists x)A(x)$: an Ω-inv. proc. computes x_0 such that $A(x_0)$

WHY is this a good/faithful/reasonable/... translation?

BECAUSE it preserves all essential features of BISH (e.g. CRM)

Constructive Reverse Mathematics

Constructive Reverse Mathematics

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics

BISH (based on BHK) | $\mathbb{NSA}$ (based on CL)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

Constructive Reverse Mathematics

BISH (based on BHK) | $\mathbb{NSA}$ (based on CL)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	
$\updownarrow$	
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	
$\updownarrow$	
MCT: monotone convergence thm	
$\updownarrow$	
CIT: Cantor intersection thm	

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$

$\Longleftrightarrow$

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$

CIT: Cantor intersection thm

$\Longleftrightarrow$

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$

$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

Constructive Reverse Mathematics

BISH (based on BHK)		$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic		Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$		$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbin{\mathbb{V}} {\sim}P$
$\updownarrow$		$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$		$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbin{\mathbb{V}} {\sim}(x > 0))$
$\updownarrow$		$\Longleftrightarrow$
MCT: monotone convergence thm		$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$	$\Longleftrightarrow$	
CIT: Cantor intersection thm		$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} {\sim}(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on BHK)		$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic		Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$		$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$
$\updownarrow$		$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$		$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} {\sim}(x > 0))$
$\updownarrow$		$\Longleftrightarrow$
MCT: monotone convergence thm		$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm		$\mathbb{CIT}$: Cantor intersection thm

Constructive Reverse Mathematics

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$	$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$	$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$	$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$
$\updownarrow$	$\Longleftrightarrow$
MCT: monotone convergence thm	$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo) $\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm	$\mathbb{CIT}$: Cantor intersection thm
(point in intersection computed by algo)	

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$ (limit computed by algo)

CIT: Cantor intersection thm

(point in intersection computed by algo)

$\Longleftrightarrow$

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim(x > 0))$

$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

(limit computed by Ω-inv. proc.)

$\mathbb{CIT}$: Cantor intersection thm

(point in intersection computed by Ω-inv. proc.)

Constructive Reverse Mathematics

BISH (based on BHK)		$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic		Transfer Principle
LPO: For $P \in \Sigma_1$, $P \vee \neg P$		$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} \sim P$
$\updownarrow$		$\Longleftrightarrow$
LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$		$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \mathbb{V} \sim (x > 0))$
$\updownarrow$		$\Longleftrightarrow$
MCT: monotone convergence thm		$\mathbb{MCT}$: monotone convergence thm
$\updownarrow$ (limit computed by algo)	$\Longleftrightarrow$	(limit computed by Ω-inv. proc.)
CIT: Cantor intersection thm		$\mathbb{CIT}$: Cantor intersection thm
		$\Longleftrightarrow$
		Π_1-TRANS^{SET}
		$(\forall n \in \mathbb{N})\varphi(n, \vec{X}) \to (\forall n \in {}^*\mathbb{N})\varphi(n, {}^*\vec{X})$

Constructive Reverse Mathematics

BISH (based on BHK)

non-constructive/non-algorithmic

LPO: For $P \in \Sigma_1$, $P \vee \neg P$

$\updownarrow$

LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$

$\updownarrow$

MCT: monotone convergence thm

$\updownarrow$ (limit computed by algo)

CIT: Cantor intersection thm

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LPO}$: For $P \in \Sigma_1$, $P \mathbb{V} {\sim} P$

$\Longleftrightarrow$

$\mathbb{LPR}$: $(\forall x \in \mathbb{R})(x > 0 \,\mathbb{V} {\sim}(x > 0))$

$\Longleftrightarrow$

$\mathbb{MCT}$: monotone convergence thm

$\Longleftrightarrow$ (limit computed by Ω-inv. proc.)

$\mathbb{CIT}$: Cantor intersection thm

$\Longleftrightarrow$

Π_1-TRANS^{SET}

$(\forall n \in \mathbb{N})\varphi(n, \vec{X}) \to (\forall n \in {}^*\mathbb{N})\varphi(n, {}^*\vec{X})$

$\mathbb{NSA}$ does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \Rightarrow (x > 0) \,\mathbb{V} (x < \delta)\big]$.

BISH does prove $(\forall \delta \in \mathbb{R})\big[\delta > 0 \to (x > 0) \vee (x < \delta)\big]$.

Constructive Reverse Mathematics II

Constructive Reverse Mathematics II

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Transfer Principle

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rrightarrow \sim P \,\mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\Longleftrightarrow$

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \mathbb{V} y = 0)$

$\Longleftrightarrow$

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \,\mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \,\mathbb{V}\, y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics II

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\Longleftrightarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\Longleftrightarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem (int. value computed by algo)	$\mathbb{IVT}$: Intermediate value theorem

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

(int. value computed by algo)

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \,\mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \,\mathbb{V}\, y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

(int. value computed by Ω-inv. proc.)

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \to \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \to x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\updownarrow$ (int. value computed by algo)

WKL

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \,\mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \,\mathbb{V}\, x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \,\mathbb{V}\, y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

(int. value computed by Ω-inv. proc.)

$\Longleftrightarrow$ $\mathbb{WKL}$

Constructive Reverse Mathematics II

BISH (based on BHK)	$\mathbb{NSA}$ (based on CL)
non-constructive/non-algorithmic	Transfer Principle
LLPO	$\mathbb{LLPO}$
For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$	For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$
$\updownarrow$	$\Longleftrightarrow$
LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$	$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$
$\updownarrow$	$\Longleftrightarrow$
NIL	$\mathbb{NIL}$
$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$	$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \mathbb{V} y = 0)$
$\updownarrow$	$\Longleftrightarrow$
IVT: Intermediate value theorem	$\mathbb{IVT}$: Intermediate value theorem
$\updownarrow$ (int. value computed by algo)	(int. value computed by Ω-inv. proc.)
WKL	$\Longleftrightarrow$ $\mathbb{WKL}$ $\Longleftrightarrow$ $\vee$-Transfer

Constructive Reverse Mathematics II

BISH (based on BHK)

non-constructive/non-algorithmic

LLPO

For $P, Q \in \Sigma_1$, $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

$\updownarrow$

LLPR: $(\forall x \in \mathbb{R})(x \geq 0 \vee x \leq 0)$

$\updownarrow$

NIL

$(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$

$\updownarrow$

IVT: Intermediate value theorem

$\updownarrow$ (int. value computed by algo)

WKL

$\mathbb{NSA}$ (based on CL)

Transfer Principle

$\mathbb{LLPO}$

For $P, Q \in \Sigma_1$, $\sim(P \wedge Q) \Rightarrow \sim P \mathbb{V} \sim Q$

$\Longleftrightarrow$

$\mathbb{LLPR}$: $(\forall x \in \mathbb{R})(x \geq 0 \mathbb{V} x \leq 0)$

$\Longleftrightarrow$

$\mathbb{NIL}$

$(\forall x, y \in \mathbb{R})(xy = 0 \Rightarrow x = 0 \mathbb{V} y = 0)$

$\Longleftrightarrow$

$\mathbb{IVT}$: Intermediate value theorem

(int. value computed by Ω-inv. proc.)

$\Longleftrightarrow$ $\mathbb{WKL}$ $\Longleftrightarrow$ $\vee$-Transfer

BISH and $\mathbb{NSA}$ can prove $(\forall k \in \mathbb{N})(\exists x_0 \in [0,1])(|f(x_0)| < \frac{1}{k})$.

Constructive Reverse Mathematics III

Constructive Reverse Mathematics III

BISH (based on BHK)

non-constructive/non-algorithmic

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics III

BISH (based on BHK)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics III

BISH (based on BHK)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$

$\updownarrow$

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics III

BISH (based on BHK)

non-constructive/non-algorithmic

MP: For $P \in \Sigma_1$, $\neg\neg P \to P$

$\updownarrow$

MPR: $(\forall x \in \mathbb{R})(\neg\neg(x > 0) \to x > 0)$

$\updownarrow$

EXT: the extensionality theorem

$\mathbb{NSA}$ (based on CL)

Constructive Reverse Mathematics III

BISH (based on BHK)

non-constructive/non-algorithmic

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Transfer Principle

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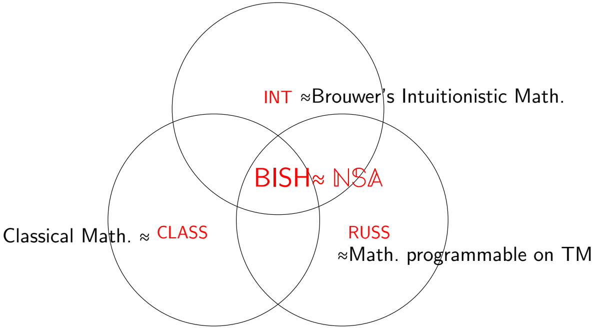

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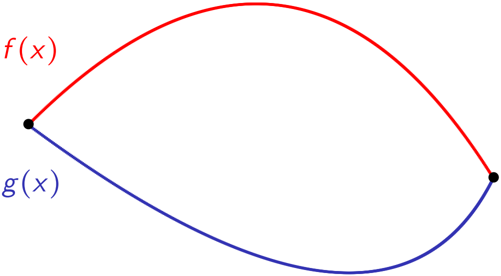

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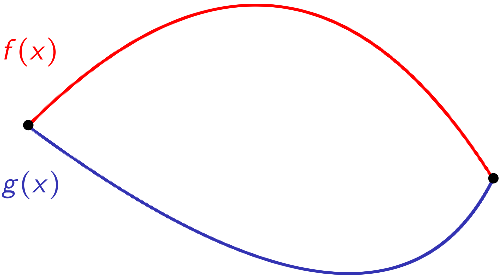

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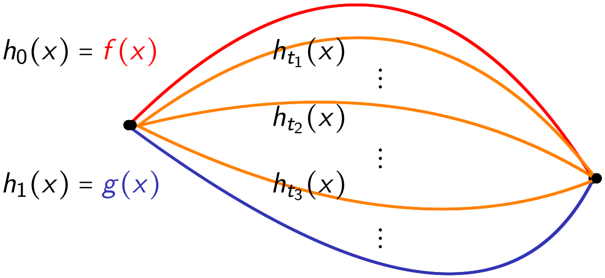

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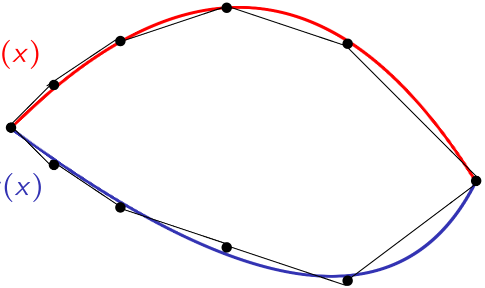

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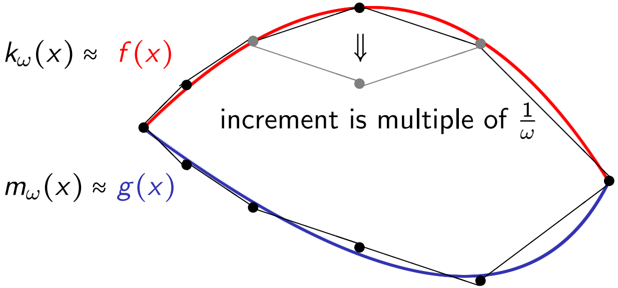

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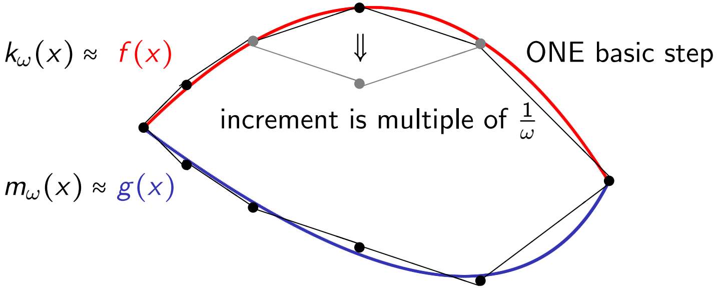

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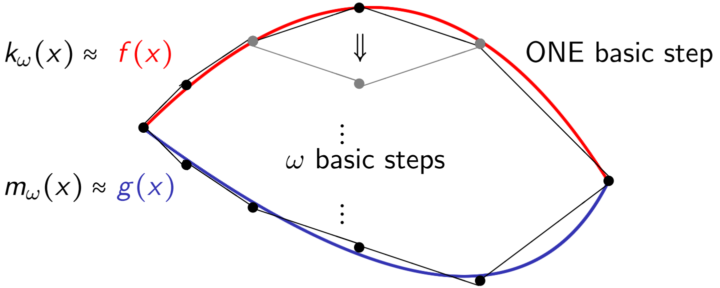

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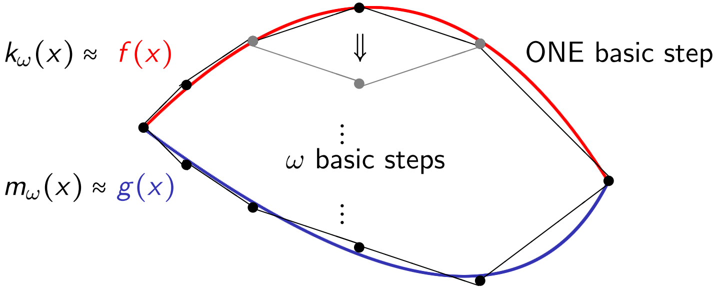

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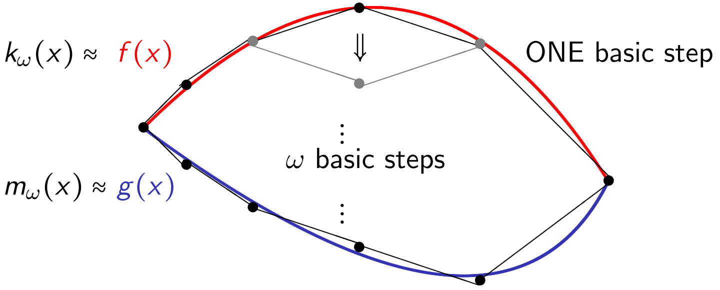

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A mathematical result with physical meaning will not depend on the choice of infinite number/infinitesimal used, i.e. it is Ω-invariant.

Final Thoughts

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The two eyes of exact science are mathematics and logic, the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two.

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Any questions?