Maxima for $\leq_{tc}$ with respect to $\sim_\alpha^c$

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Midwest Computability Seminar, November 15, 2012

Classification

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Turing Computable Embeddings

Knight, et al. considered computable languages, and structures with universes subsets of ω, in order to formulate an effective analogue of the Borel embedding.

Definition

A *Turing computable embedding* of (K, E) into (K', E') is an operator $\Phi = \phi_e$ such that

- for each $\mathcal{A} \in K$ there exists $\mathcal{B} \in K'$ such that $\Phi(\mathcal{A}) = \phi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$, and
- if $\mathcal{A}, \mathcal{A}' \in K$, then $\mathcal{A} E \mathcal{A}' \leftrightarrow \Phi(\mathcal{A}) E' \Phi(\mathcal{A}')$.

This induces a preordering, denoted by $(K, E) \leq_{tc} (K', E')$. If we do not mention the equivalence relation, then it is assumed to be isomorphism.

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Examples

- UG, LO,RCF are universal (every class K of countable structures has $K \leq_{tc} UG$).
- If the isomorphism classes of K are distinguished by computable infinitary sentences, then K cannot be universal.
- Isomorphism for abelian p-groups is not distinguished by sentences at a single level, yet they are not universal.

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$\sim^c_\alpha$

Motivated by the importance of the Σ^c_α sentences in the previous work, we define the following equivalence relation.

Definition

- We say that two structures $\mathcal{A}$ and $\mathcal{B}$ in the same language are Σ^c_α *equivalent* if and only if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same Σ^c_α sentences (denoted $\mathcal{A} \sim^c_\alpha \mathcal{B}$).
- If for all $\alpha < \omega_1^{CK}$, $\mathcal{A} \sim^c_\alpha \mathcal{B}$, then we say that $\mathcal{A}$ and $\mathcal{B}$ are *computably infinitarily equivalent* ($\mathcal{A} \sim^c_{\omega_1^{CK}} \mathcal{B}$).

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Can we find classes which are universal for $\sim^c_\alpha$, but not for $\cong$?

Abelian p-groups

Fix any prime, p. Let $\mathcal{G}$ be a countable Abelian p-group.

- To each group, we associate a sequence from $(\omega + 1)^{<\omega_1}$ (its Ulm sequence).
- For $\alpha < \omega_1^{CK}$ we can say $U_\alpha(\mathcal{G}) \geq k$ with a computable infinitary sentence.

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Analysis of results of Sara Quinn yields the following.

Lemma

For any class K, $(K, \sim_2^c) \leq_{tc} (ApG_\omega, \sim_2^c)$.

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For any class K, $(K, \sim_{2n}^c) \leq_{tc} (ApG_{\omega \cdot n}, \sim_{2n}^c)$.

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For any class K, $(K, \sim_\omega^c) \leq_{tc} (ApG_{\omega^2}, \sim_\omega^c)$.

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For any class K, $(K, \sim_\alpha^c) \leq_{tc} (ApG_\beta, \sim_\alpha^c)$ *if and only if* $\beta = \omega \cdot \gamma$ *and* $\alpha < \gamma$.

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Uniformity of Embeddings

- In each case, we are essentially coding a set (the set of Σ^c_α sentences true in the input structure).
- For *bounded* $\alpha < \omega_1^{CK}$, there is a certain length β such that $(K, \sim^c_\alpha) \leq_{tc} (ApG_\beta, \sim^c_\alpha)$.
- Finding the appropriate β, and a $\Phi = \phi_e$ to witness the embedding seems very uniform.
- Can we 'lace together' all of these embeddings into a 'master procedure,' showing that

$$(K, \sim^c_{\omega_1^{CK}}) \leq_{tc} (ApG, \sim^c_{\omega_1^{CK}})?$$

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‘Lace’ via Recursion

Theorem (V.)

There is a partial computable function $f : \omega \to \omega^2$*, such that if* $a \in \mathcal{O}$*, then* $f(a) = \langle e, b \rangle$*, where* ϕ_e *is the operator witnessing that* $(K, \sim^c_{|a|}) \leq_{tc} (ApG_{|b|}, \sim^c_{|a|})$*.*

The proof is by transfinite induction on ordinal notation, and we will need a few lemmas.

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Lemmas

Lemma (H. Rogers)

There is a partial computable function $\cdot_{\mathcal{O}} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ such that, for all $a, b \in \mathcal{O}$,

$$|a| \cdot |b| = |a \cdot_{\mathcal{O}} b|.$$

Lemma (V.)

There is a computable operator $\Phi = \phi_e$ such that, given an indexed family of groups $(\mathcal{G}_i)_{i\in\omega} = \{\langle i, \psi\rangle : \psi \in D(\mathcal{G}_i)\}$,

$$\Phi((\mathcal{G}_i)_{i\in\omega}) = \phi_e^{(\mathcal{G}_i)_{i\in\omega}} = \chi_{\mathcal{G}}.$$

where $\mathcal{G} \cong \bigoplus_{i\in\omega}\mathcal{G}_i$.

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Lemma and Proof

Lemma (V.)

Suppose that α is a computable ordinal of the form $\omega \cdot \beta + n$, and let the function g be defined as

$$g(\omega \cdot \beta + n) = \begin{cases} \omega^2 \cdot \beta + \omega \cdot \frac{n}{2} & \textit{if } n \textit{ is even} \\ \omega^2 \cdot \beta + \omega \cdot \frac{n+1}{2} & \textit{if } n \textit{ is odd} \end{cases}$$

Then, for any K, $(K, \sim_\alpha^c) \leq_{tc} (ApG_{g(\alpha)}, \sim_\alpha^c)$.

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Barwise Compactness

An another approach is to use Barwise-Kreisel compactness.

Theorem (Barwise; Kreisel)

Let Γ be a Π^1_1 set of computable infinitary sentences. If every Δ^1_1 subset of Γ has a model, then Γ also has a model.

How could we use it to lace together operators?

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Operator Existence

Theorem (V.)

For X such that $\omega_1^X = \omega_1^{CK}$, ApG lies on top for $\sim^c_{\omega_1^{CK}}$ amongst classes of X-computable structures.

Sketch: We let Γ_α be a set of sentences axiomatizing an appropriate operator that works for $\sim^c_\alpha$. Namely, sentences like

$$\forall \mathcal{A}, j[(\mathcal{A} \in K) \rightarrow (\mathcal{A} \vDash \psi_j \leftrightarrow U_{h(\beta,j)}(\Phi(\mathcal{A})) = \omega)]$$

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But how do we quantify over structures?

Sketch Continued

- Let $\Gamma = \cup_{\alpha < \omega_1^{CK}} \Gamma_\alpha$, this is a Π^1_1 set.
- Take some Δ^1_1 subset $\Gamma' \subseteq \Gamma$.
- Kleene Bounding theorem tells us that the ordinals mentioned in Γ' have a computable ordinal bound, say β.
- By the earlier work, we can produce an appropriate model (operator) witnessing $(K, \sim^c_\beta) \leq_{tc} (ApG_{g(\beta)}, \sim^c_\beta)$.
- So Barwise-Kreisel compactness gives us a model.
- Barwise compactness works for any X such that $\omega_1^X = \omega_1^{CK}$.

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Toward Consistency Result

That is still not *quite* what we want, so we use new tools.

Definition

A structure $\mathcal{A}$ is *hyperarithmetically saturated* if it satisfies the following conditions

- for all tuples $\bar{a} \in \mathcal{A}$, and all Π^1_1 sets $\Gamma(\bar{a}, x)$ of computable infinitary formulas with parameters $\bar{a}$, if every Δ^1_1 set $\Gamma'(\bar{a}, x) \subseteq \Gamma(\bar{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\bar{a}, x)$ is also satisfied in $\mathcal{A}$,
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The Workhorse

Why investigate hyperarithmetical saturation?

Theorem (Ressayre)

Suppose that $\mathcal{A}$ is a hyperarithmetically saturated structure, and Γ is a Π^1_1 set of sentences involving symbols from the language of $\mathcal{A}$, along with a new symbol. If the consequences of Γ (sentences true in all models of Γ) in the language of $\mathcal{A}$ are all true in $\mathcal{A}$, then $\mathcal{A}$ can be expanded to a model $\mathcal{A}'$ of Γ.

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Suppose that $\mathcal{A}$ *is a hyperarithmetically saturated structure, and* Γ *is a* Π^1_1 *set of sentences involving symbols from the language of* $\mathcal{A}$*, along with a new symbol. If the consequences of* Γ *(sentences true in all models of* Γ*) in the language of* $\mathcal{A}$ *are all true in* $\mathcal{A}$*, then* $\mathcal{A}$ *can be expanded to a model* $\mathcal{A}'$ *of* Γ*.*

Consistency Result

Theorem (V.)

If ZFC has an ω-model, then it has an ω-model, $\mathcal{Z}$ having the following property. There is an $e \in \omega$, such that for all $\alpha < \omega_1^{CK}$ (real world ω_1^{CK}), $\mathcal{Z} \models \psi_\alpha(e)$, where $\psi_\alpha(e)$ says 'ϕ_e is an embedding witnessing that ApG is on top for $\sim_\alpha^c$.'

Subtle point:
There are some weird ordinals in this model, and its interpretation of ω_1^{CK} is bizarre.

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Sketch

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- Ressayre's results imply that if *ZFC* has an ω-model, then it has a hyperarithmetically saturated ω-model.
- We take the language of set theory, and add a single new constant, *e*, to be interpreted as an index for an operator.
- So, let Γ say statements similar to those in the previous proof, but now add $e \in \omega$.
- Since we are in set theory, the following sentence is now acceptable

$$\forall \mathcal{A}, j[(\mathcal{A} \in K) \rightarrow (\mathcal{A} \vDash \psi_j \leftrightarrow U_{h(\beta, j)}(\Phi(\mathcal{A})) = \omega)]$$

- Apply the theorem.

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Questions

- Is it true that *ApG* is universal for $\leq_{tc}$ with respect to $\sim^{c}_{\omega_1^{CK}}$ in *any* model of *ZFC*?
- What natural classes can be used to *tightly* fill in the odd levels?
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Thank You!

(References available upon request)

Abelian p-groups

Fix any prime, p. Recall that an *Abelian p-group* is an Abelian group where each element has order a power of p. Let $\mathcal{G}$ be a countable Abelian p-group.

- Define inductively: $\mathcal{G}_0 = \mathcal{G}$, $\mathcal{G}_{\beta+1} = p\mathcal{G}_\beta$, and $\mathcal{G}_\lambda = \cap_{\gamma<\beta}\mathcal{G}_\gamma$.
- $\mathcal{G}$ is *divisible* if every $x \in \mathcal{G}$ is divisible by p^n for all n.
- For each $\mathcal{G}$, there is a length, λ such that $\mathcal{G}_\lambda = \mathcal{G}_{\lambda+1}$.
- If $\mathcal{G}_\lambda = \{0\}$, we call $\mathcal{G}$ *reduced*.
- Each element obtains a *height* in this way. The height of x is the unique β such that $x \in \mathcal{G}_\beta$ but not in $\mathcal{G}_{\beta+1}$.

These can be visualized as trees–this is very useful.

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P-Groups

- To each (reduced) Abelian p-group, we associate an *Ulm sequence*, an element of $(\omega + 1)^{<\omega_1}$.
- The αth element of the sequence associated to $\mathcal{G}$ is denoted $U_\alpha(\mathcal{G}$.
- $P_\alpha(\mathcal{G}) = \{x \in \mathcal{G}_\alpha : px = 0\}$, and $U_\alpha(\mathcal{G}) = \dim(P_\alpha(\mathcal{G})/P_{\alpha+1}(\mathcal{G}))$.
- We can say $U_\alpha(\mathcal{G}) \geq k$ with a computable infinitary sentence.

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Results of S. Quinn

Let ApG_α be the class of countable reduced abelian p-groups of length α.

Theorem (S. Quinn)

For any K, $K \leq_{tc} ApG_\omega$ if and only if there is a computable sequence of Σ^c_2 sentences $(\psi_n)_{n\in\omega}$ such that for $\mathcal{A} \not\cong \mathcal{B} \in K$, there is an n such that ψ_n is true in exactly one of $\mathcal{A}$ and $\mathcal{B}$.

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For any K, $K \leq_{tc} ApG_{\omega\cdot m}$ if and only if there is a computable sequence $(\psi_n)_{n\in\omega}$ of Σ^c_{2m} sentences in the language of K such that for $\mathcal{A} \not\cong \mathcal{B} \in K$, there is an n such that ψ_n is true in exactly one of $\mathcal{A}$ and $\mathcal{B}$.

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Suppose that $\alpha = \omega \cdot \beta$, and $(\beta_n)_{n\in\omega}$ has limit β. For any K, $K \leq_{tc} ApG_\alpha$ if and only if there is a computable sequence $(\psi_{\beta_n})_{n\in\omega}$ of $\Sigma^c_{\beta_n}$ sentences in the language of K such that for $\mathcal{A} \not\cong \mathcal{B} \in K$, there is an n such that ψ_n is true in exactly one of $\mathcal{A}$ and $\mathcal{B}$.

Despite these theorems, not all K embed into *ApG* (daisy graphs, linear orders, etc.).

Goal:Is *ApG* universal amongst classes of structures where isomorphism types are distinguished by Σ^c_β sentences?

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Toward Consistency Result

Definition

A structure $\mathcal{A}$ is *hyperarithmetically saturated* if it satisfies the following conditions

- for all tuples $\bar{a} \in \mathcal{A}$, and all Π^1_1 sets $\Gamma(\bar{a}, x)$ of computable infinitary formulas with parameters $\bar{a}$, if every Δ^1_1 set $\Gamma'(\bar{a}, x) \subseteq \Gamma(\bar{a}, x)$ is satisfied in $\mathcal{A}$, then $\Gamma(\bar{a}, x)$ is also satisfied in $\mathcal{A}$,
- for all tuples $\bar{a} \in \mathcal{A}$ and all Π^1_1 sets Λ of pairs $(i, \gamma(\bar{a}))$, where $i \in \omega$ and $\gamma(\bar{a})$ is a computable infinitary sentence with parameters $\bar{a}$, if for every Δ^1_1 set $\Lambda' \subseteq \Lambda$,

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