# Degree Spectra of Relations on Ordinals

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# Background

Let $\mathcal{A}$ be a structure, and let $R \subseteq A^n$ be an $n$-ary relation on $\mathcal{A}$.

## Definition

A *computable copy* of $\mathcal{A}$ is a copy of $\mathcal{A}$ in which the constants, functions, and relations are uniformly computable.

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### Example

Any finite or cofinite relation has only the computable degree in its spectrum.

### Example

The degree spectrum of the successor relation on $(\omega, <)$ consists of all co-c.e. degrees.

# Unary relations on $(\omega, <)$

**Theorem (Downey, Khoussainov, Miller, and Yu)**

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**Proof.**

Set things up so that $n \in \emptyset' \iff R(2n)$.

Fairly straightforward finite injury.

At the beginning, no elements have entered $\emptyset'$, so we start with even numbers in order, with odd numbers to ensure none are in the relation:

$$0 \qquad \boxed{1} \qquad 2 \qquad 4 \qquad \boxed{3} \qquad \boxed{5} \qquad \cdots$$

At some stage, maybe 1 is enumerated in.

Add odd numbers before its coding element (2) to push it into $R$:

$0 \quad \boxed{1} \quad \mathbf{7} \quad \mathbf{9} \quad \boxed{2} \quad \boxed{4} \quad 3 \quad 5 \quad \cdots$

But there's a problem: 4 has been pushed into the relation, but we don't think that $2 \in \emptyset'$ at this stage!

No big deal. Add more padding elements to push it back out!

$0 \quad \boxed{1} \quad 7 \quad 9 \quad \boxed{2} \quad \boxed{\mathbf{11}} \quad 4 \quad 3 \quad 5 \cdots$

Now we're done for this stage, because everything is where it should be:

$$n \in \emptyset'_s \iff R_s(2n)$$

for all $n$.

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**Remark**

*A small modification to the construction shows that the spectrum consists of exactly the $\Delta_2$ degrees.*

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Main features of the proof in the unary case:

- Designated coding elements (the evens) and other padding elements (the odds)
- A way to keep coding elements out of the relation if they were out (pad below it to push it back out)
- A way to push coding elements into the relation and keep them in (pad below it to push it in)

Designated coding elements: for an $n$-ary relation, we need coding tuples. We'll use tuples of consecutive evens: to code the number $k$ we will use the number

$$\ulcorner k \urcorner = \langle 2nk, 2(nk+1), 2(nk+2), \ldots, 2(n(k+1)-1) \rangle.$$

This, again, leaves the odds free to be used for padding.

To keep coding tuples in or out of the relation:

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2. Using facts about well partial orderings, we can restrict our tuples to a certain computable set in a way that allows us to ensure they stay *out* of the relation at each stage.

We therefore have all of the ingredients for the finite injury construction we did before. We can keep coding elements out of the relation at every stage until they are supposed to enter, and then keep them in the relation after that.

What about ordinals other than $\omega$?

## Definition

We say that a copy $A$ of a computable ordinal is *strongly computable* if the "Cantor-Bendixson rank $\alpha$" relation and the "successor among Cantor-Bendixson rank $\alpha$" relations are computable.

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### Theorem (W.)

*Let $R$ be a computable unary relation on a strongly computable copy $A$ of a computable ordinal $\alpha$. Then for any computable ordinal $\beta$, if there is a computable copy $B$ of $\alpha$ in which $R^B \not\leq_T \emptyset^{(\beta)}$, then there is a computable copy $C$ of $\alpha$ such that $R^C$ has Turing degree $\emptyset^{(\beta+1)}$.*

A similar result holds for limit ordinals.

The proof relies on theorems of Ash and Knight about back and forth relations. As an example, consider coding $\emptyset''$ into $\omega^2$:

### Theorem (Ash, Knight)

*Let $S$ be a $\Sigma^0_2$ set. Then there are uniformly computable sequences of linear orderings $(C_n)_{n\in\omega}$ and $(D_n)_{n\in\omega}$ such that*

$$C_n \cong \begin{cases} \omega & \textit{if } n \in S \\ \omega \times 2 & \textit{if } n \notin S \end{cases}.$$

*and*

$$D_n \cong \begin{cases} \omega \times 2 & \textit{if } n \in S \\ \omega & \textit{if } n \notin S \end{cases}.$$

This lets us put coding elements where we want: we build "pieces" that look like

$$\begin{array}{c} \omega \times 2 \quad + * + \quad \omega \\ \text{or} \\ \omega \quad + * + \quad \omega \times 2 \end{array}$$

depending on whether a given number is in $\emptyset''$ or not, where "*" is our coding element.

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depending on whether a given number is in $\emptyset''$ or not, where "*" is our coding element.

As long as there are infinitely many $n$ such that for some $k_n$ it's the case that $R(\omega \times n + k_n) \neq R(\omega \times (n+1) + k_n)$, then we can code $\emptyset''$ in this way.

We can do a similar construction with

$$\omega + m \quad + * + \quad \omega$$

or

$$\omega + n \quad + * + \quad \omega$$

This construction succeeds as long as there are infinitely many $n$ such that there is a $k_n$ with $R(\omega \times n + k_n) \neq R(\omega \times n + k_n + 1)$.

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What happens if both constructions fail?

1. $R$ is constant on all but finitely many copies of $\omega$.
2. $R$ behaves the same way across all but finitely many copies of $\omega$.

Therefore, $R$ is bounded, and can be computed using the successor relation and finitely many named limit points.
$R$ is therefore intrinsically $\emptyset'$.

The general construction works in a similar way: we attempt to do a coding of $\emptyset^{(\alpha)}$, and show that if it fails we could have computed $R$ using $\emptyset^{(\beta)}$ for some $\beta < \alpha$.

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Open questions:

1. What happens in the case of *n*-ary relations on arbitrary computable ordinals?

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Open questions:

1. What happens in the case of *n*-ary relations on arbitrary computable ordinals?
2. To what extent can these results be generalized to other linear orderings?