JACOBSON RINGS

Matthew Emerton

The purpose of this note is to develop the elementary theory of Jacobson rings without recourse to Noether's normalization lemma (in contrast to the approach taken by Bourbaki, and hence by Grothendieck in EGA).

All rings will be commutative with unit. If A is a ring, let $max(A)$ denote the set of maximal ideals of A.

Lemma 1. *For any ring A, the following are equivalent:*

(i) for all radical ideals $I = rad(I)$ of A, we have

$$\bigcap_{\substack{\mathfrak{m}\in max(A)\\ I\subset\mathfrak{m}}} \mathfrak{m} = I;$$

(ii) for all prime ideas $\mathfrak{p}$ of A, we have

$$\bigcap_{\substack{\mathfrak{m}\in max(A)\\ \mathfrak{p}\subset\mathfrak{m}}} \mathfrak{m} = \mathfrak{p};$$

(iii) for every integral domain A' which is a homomorphic image of A,

$$\bigcap_{\mathfrak{m}\in \max(A')} \mathfrak{m} = 0.$$

Proof. Since any prime ideal is radical, we see that *(i)* implies *(ii)*, while since any radical ideal is the intersection of the prime ideals which contain it, the converse also holds. Since the integral domain quotients A' of A are precisely the quotients of A by its prime ideals, we see that *(i)* and *(iii)* are equivalent. This completes the proof of the lemma. □

Definition 2. *If A satisfies the equivalent conditions of the preceding lemma, we say that A is Jacobson.*

Lemma 3. *Let A be Jacobson.*

(i) Any homomorphic image of A is Jacobson.

(ii) For any $a \in A$, the localization A_a is Jacobson, and pull-back and push-forward of ideals via the natural map $A \to A_a$ induces a bijection between the maximal ideals of A_a and the maximal ideals of A which do not contain a.

Proof. Part *(i)* is an immediate consequence of the characterization of Jacobson rings provided by part *(iii)* of lemma 1.

For any ring A and element $a \in A$, pull-back and push-forward of ideals via the map $A \to A_a$ establishes an isomorphism (as partially ordered sets) of the set of prime ideals of A_a and the set of prime ideals of A which do not contain a. Since this correspondence respects the partial order of each set, the maximal ideals of A_a will correspond to the maximal elements of the set of prime ideals of A which do not contain a. In particular, any maximal ideal of A which does not contain a corresponds to a maximal ideal of A_a, but for general rings A there will be additional maximal ideals of A_a which do not arise from maximal ideals of A.

However, if A is Jacobson, let $\mathfrak{p}'$ be a prime ideal in A_a, corresponding to a prime ideal $\mathfrak{p}$ of A which does not contain a. Then part *(ii)* of lemma 1 gives

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \in max(A) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m},$$

and so there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$ and not containing a. Thus we see that when A is Jacobson pull-back and push-forward of ideals does induce a bijection between the maximal ideals of A_a and the maximal ideals of A not containing A. This proves the second statement of part *(ii)*.

To complete the proof of part *(ii)* we must show that A_a is Jacobson. Since a is not in $\mathfrak{p}$, note that in fact

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \in max(A) \\ \mathfrak{p} \subset \mathfrak{m}, a \notin \mathfrak{m}}} \mathfrak{m}.$$

(For if some element b of A lies in this intersection, then ab lies in the intersection

$$\bigcap_{\substack{\mathfrak{m} \in max(A) \\ \mathfrak{p} \subset \mathfrak{m}, a \notin \mathfrak{m}}} \mathfrak{m} \quad \cap \quad \bigcap_{\substack{\mathfrak{m} \in max(A) \\ a \in \mathfrak{m}}} \mathfrak{m} \subset \bigcap_{\substack{\mathfrak{m} \in max(A) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m} = \mathfrak{p},$$

and so $b \in \mathfrak{p}$, since $a \notin \mathfrak{p}$.) Since pulling back prime ideals to A_a is an isomorphism of partially ordered sets, we deduce that

$$\mathfrak{p}' = \mathfrak{p}A_a = \bigcap_{\substack{\mathfrak{m} \in max(A) \\ \mathfrak{p} \subset \mathfrak{m}, a \notin \mathfrak{m}}} \mathfrak{m}A_a \overset{(1)}{=} \bigcap_{\substack{\mathfrak{m}' \in max(A_a) \\ \mathfrak{p}' \subset \mathfrak{m}'}} \mathfrak{m}' \supset \mathfrak{p}'.$$

(Equality (1) holds by the result of the preceding paragraph.) Thus this last inclusion is actually an equality, and so A_a satisfies condition *(ii)* of lemma 1, and is Jacobson. □

Recall the following facts about integral extensions:

Lemma 4. *If $A \to B$ is an integral morphism, then*

(i) the pull-back of any maximal ideal of B is a maximal ideal of A;

(ii) if $A \to B$ is injective, then for any maximal ideal $\mathfrak{m}$ of A, there is a maximal ideal of B lying over $\mathfrak{m}$;

(iii) if $A \to B$ is an injection of domains, then for any non-zero ideal I of B, the intersection $A \cap I$ is non-zero.

Corollary 5. *If $A \to B$ is an integral morphism and A is Jacobson then B is Jacobson.*

Proof. Let B' be a domain which is a homomorphic image of B, and let A' be the image of A in B'. Then $A' \to B'$ is an integral injection of domains. Let

$$I = \bigcap_{\mathfrak{m}' \in max(B')} \mathfrak{m}'.$$

Then

$$I \cap A' = \bigcap_{\mathfrak{m}' \in max(B')} \mathfrak{m}' \cap A = \bigcap_{\mathfrak{m} \in \max(A)} \mathfrak{m} = 0,$$

by parts *(i)* and *(ii)* of lemma 4. Thus part *(iii)* of lemma 4 shows that $I = 0$, and so we see that B satisfies condition *(iii)* of lemma 1. □

Lemma 6. *If $A \to B$ is an injection of domains such that A is Jacobson, and for some $a \in A \setminus \{0\}$, the induced morphism $A_a \to B_a$ is integral, then*

$$\bigcap_{\mathfrak{m} \in max(B)} \mathfrak{m} = 0.$$

Proof. Lemma 3, part *(ii)*, shows that A_a is Jacobson, and so by corollary 5, B_a is Jacobson.

Let $\mathfrak{m}'$ be maximal in B_a. Then by lemma 4, part *(i)*, $\mathfrak{m}' \cap A_a$ is maximal in A_a, and so by lemma 3, part *(ii)*, $\mathfrak{m}' \cap A = \mathfrak{m}' \cap A_a \cap A$ is maximal in A, and so $A/(\mathfrak{m}' \cap A) \to A_a/(\mathfrak{m}' \cap A_a) = (A/\mathfrak{m}' \cap A)_a$ is an isomorphism. Thus we have the commutative diagram of injections

$$\begin{array}{ccc} A/(\mathfrak{m}' \cap A) & \longrightarrow & B/(\mathfrak{m}' \cap B) \\ \downarrow \sim & & \downarrow \\ A_a/(\mathfrak{m}' \cap A_a) & \xrightarrow{\text{integral}} & B_a/\mathfrak{m}', \end{array}$$

and so we see that $B/(\mathfrak{m}' \cap B)$ sits within the algebraic extension of fields $A/(\mathfrak{m}' \cap A) \to B_a/\mathfrak{m}'$, and so is a field. Thus $\mathfrak{m}' \cap B$ is a maximal ideal of B, for any maximal ideal $\mathfrak{m}'$ of B_a.

Now we compute

$$\bigcap_{\mathfrak{m} \in max(B)} \mathfrak{m} \subset \bigcap_{\mathfrak{m}' \in max(B_a)} \mathfrak{m}' \cap B \subset \bigcap_{\mathfrak{m}' \in max(B_a)} \mathfrak{m}',$$

and this last intersection vanishes, since B_a is a Jacobson domain. □

Lemma 7. *If A is a ring such that*

$$\bigcap_{\mathfrak{m} \in max(A)} \mathfrak{m} = 0,$$

then

$$\bigcap_{\mathfrak{m}' \in max(A[X])} \mathfrak{m}' = 0.$$

Proof. Note that the hypothesis implies that

$$\bigcap_{\mathfrak{m} \in max(A)} \mathfrak{m}A[X] = 0.$$

Thus it suffices to show that for each $\mathfrak{m}$ maximal in A,

$$\bigcap_{\substack{\mathfrak{m}' \in max(A[X]) \\ \mathfrak{m}A[X] \subset \mathfrak{m}'}} \mathfrak{m}' = \mathfrak{m}A[X],$$

or equivalently,

$$\bigcap_{\mathfrak{m}'' \in max((A/\mathfrak{m})[X])} \mathfrak{m}'' = 0.$$

But this is immediate, since $A/\mathfrak{m}$ is a field. □

Theorem 8. *If A is Jacobson, then $A[X]$ is Jacobson.*

Proof. Let B be an integral domain image of $A[X]$, and let A' be the image of A in B, so that we have the sequence of morphisms $A' \to A'[X] \to B$. We have to show that

$$\bigcap_{\mathfrak{m} \in max(B)} \mathfrak{m} = 0.$$

If $A'[X] \to B$ is an isomorphism, this follows from lemma 7, since by assumption A is Jacobson and so A' satisfies the hypothesis of that lemma.

Suppose now that the kernel $\mathfrak{p}$ of $A'[X] \to B$ is non-zero. Then we may find a non-zero element a of A' such that after inverting a, $\mathfrak{p}$ is generated by a monic polynomial $f \in A'_a[X]$. Thus $B_a = A'_a[X]/f$ is finite over A'_a, and so lemma 6 shows that again

$$\bigcap_{\mathfrak{m} \in max(B)} \mathfrak{m} = 0.$$

□

Corollary 9. *If A is Jacobson then any finite-type A-algebra is Jacobson.*

Proof. It follows from theorem 8 together with induction that any polynomial ring $A[X_1, \ldots, X_n]$ over A is Jacobson. The corollary now follows by part *(i)* of lemma 3. □

Theorem 10. *If A is Jacobson and K is a finite-type A-algebra which is a field, then K is a finite A-algebra. Conversely, any ring satisfying this condition is Jacobson.*

Proof. We will prove the first claim by induction on the number of elements required to generate K as an A-algebra. Write $K = A[X_1, \ldots, X_n]/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. We will show that K is finite over $A[X_1, \ldots, X_{n-1}]/(A[X_1, \ldots, X_{n-1}] \cap \mathfrak{m})$. It then follows that this latter A-algebra is also a field, and so by induction is a finite A-algebra. Thus K is a finite A-algebra.

Since corollary 9 shows that $A[X_1, \ldots, X_{n-1}]/(A[X_1, \ldots, X_n] \cap \mathfrak{m})$ is Jacobson, we see that is suffices to show that if A is a Jacobson domain and $\mathfrak{m}$ is a maximal ideal of $A[X]$ such that A injects into $A[X]/\mathfrak{m}$, then this latter ring is a finite A-algebra.

The same argument used to prove theorem 8 shows that there exists a non-zero $a \in A$ such that $A_a \to (A[X]/\mathfrak{m})_a$ is finite. Since $A[X]/\mathfrak{m}$ is a field, we see that $(A[X]/\mathfrak{m})_a = A[X]/\mathfrak{m}$ is a field, and that A_a is thus a field. Thus the zero ideal of A_a is maximal, and lemma 3, part *(ii)*, shows that the zero ideal of A is also maximal, so that A is a field. Thus $A = A_a$, and we have shown that $A[X]/\mathfrak{m}$ is finite over A.

Now suppose that A is a ring such that every finite-type A-algebra which is a field is in fact a finite A-algebra. We must show that A is Jacobson. Since this condition is clearly inherited by arbitrary quotients of A, it will suffice to show, under the additional hypothesis that A is a domain, that

$$\bigcap_{\mathfrak{m} \in max(A)} \mathfrak{m} = 0.$$

Equivalently, for any non-zero $a \in A$ we must find a maximal ideal $\mathfrak{m}$ which does not contain a.

Let a be a non-zero element of A. Then A_a is non-zero finite-type A-algebra, and so contains a maximal ideal $\mathfrak{m}'$; thus $A_a/\mathfrak{m}'$ is a finite-type A-algebra which is a field, and so by assumption is finite over A. Thus, if $\mathfrak{m} = A \cap \mathfrak{m}'$, the field $A_a/\mathfrak{m}'$ is a finite extension of the integral domain $A/\mathfrak{m}$, and thus $A/\mathfrak{m}$ is also a field. Hence $\mathfrak{m}$ is a maximal ideal of A which does not contain a, and we are done. □

Since any field is obviously Jacobson, Hilbert's Nullstellensatz is a corollary of theorem 10. Here is another:

Corollary 11. *If A is a finite-type $\mathbb{Z}$-algebra and $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite field.*

Proof. First observe that $\mathbb{Z}$ is Jacobson (as is any one-dimensional domain possessing infinitely many maximal ideals), so that $A/\mathfrak{m}$, being a finite-type $\mathbb{Z}$-algebra which is also a field, is a finite $\mathbb{Z}$-algebra. Thus $A/\mathfrak{m}$ is a finite extension of the image of $\mathbb{Z}$ in $A/\mathfrak{m}$. Hence this image is a field, and so must equal $\mathbb{Z}/p$ for some prime p. Thus $A/\mathfrak{m}$ is finite over $\mathbb{Z}/p$, and so is a finite field. □