ZARISKI DENSE RANDOM WALKS ON HOMOGENEOUS SPACES

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Contents

1. Introduction

Measure classification theorems have an important role in dynamics. A milestone is Ratner's theorem [Ra] and related results of Dani and Margulis [DM1] [DM3] [DM2] [DM4] in the context of unipotent flows in homogeneous spaces. Later major advances in the homogeneous setting include the the results of Einsiedler-Katok-Lindenstrauss [EKL] for diagonal actions, and later Bourgain-Furman-Lindenstrauss-Mozes [BFLM] and Benoist-Quint [BQ1] for actions of Zariski dense subgroups of semisimple groups. Benoist-Quint extended their result to subgroups with semisimple Zariski closure in [BQ2]. In [EMi] a Ratner-like theorem was proved for actions of $SL(2,\mathbb{R})$ and its upper triangular subgroup on moduli spaces of abelian differentials. Even though this result was in an inhomogeneous setting, the proof used many ideas from [BQ1] and from the "low entropy method" of [EKL]. In the main paper [EsL], we use, in the homogenious setting, some of the methods developed in [EMi] together with some new ideas, to prove some extensions and improvements to the results of Benoist-Quint [BQ1, BQ2]. See the introduction to [EsL] for the exact statements.

The aim of this (almost entirely expository) note is to serve as in introduction to some of the main ideas of [EsL] and of [EMi] in the simplest possible setting.

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Let G be an Ad-simple noncompact Lie Group with finite center and let $\mathfrak{g}$ denote the Lie algebra of G, and let Γ be a lattice in G. Let μ be a countably supported probablity measure on G. We say that μ has finite first moment if

$$\int_G \log \|g\| \, d\mu(g) < \infty.$$

Let $\mathcal{S}$ denote the support of μ, and let $G_{\mathcal{S}}$ denote the closure of the group generated by $\mathcal{S}$. We say that μ has finite entropy if

$$\sum_{g \in \mathcal{S}} -\mu(g) \log \mu(g) < \infty.$$

Define a measure ν on G/Γ to be μ-stationary if

$$\mu * \nu = \nu, \qquad \text{where } \mu * \nu = \int_G g\nu \, d\mu(g).$$

We will always assume that ν is a probability measure (i.e. $\nu(G/\Gamma) = 1$), and also that ν is ergodic (i.e. is extremal among the μ-stationary measures).

In this note, we use the methods developed in [EMi] and [EsL] to give an alternative proof of a variant of the main theorem of [BQ1]:

Theorem 1.1. *Suppose μ is a countably supported measure on G with finite first moment and finite entropy. Suppose also that the group generated by the support of μ is Zariski dense in G. Let ν be any ergodic μ-stationary probability measure on G/Γ. Then, ν is either Haar measure on G/Γ or is $G_{\mathcal{S}}$-invariant and finitely supported.*

Theorem 1.1 follows from the main result of [BQ1] under the additional assumption that μ is compactly supported. See [EsL] for a much more general setup, including cases not covered by [BQ2].

Notation. Let $\mu^{(n)} = \mu * \mu \cdots * \mu$ (n times). If H is a subgroup of G, we denote the Lie algebra of H by $\mathrm{Lie}(H)$. Let $\mathfrak{g}$ denote the Lie algebra of G, and let Ad denote the adjoint representation. For $g \in G$ and $\mathbf{v} \in \mathfrak{g}$, we will often use the shorthand $(g)_* \mathbf{v}$ for $Ad(g)\mathbf{v}$.

1.1. **Skew Products.** We consider the two sided shift space $\mathcal{S}^{\mathbb{Z}}$. For $x \in \mathcal{S}^{\mathbb{Z}}$, we have $x = (\dots, x_{-1}, x_0, x_1, \dots)$. We write $x = (x^-, x^+)$ where $x^- = (\dots, x_{-1})$ is the "past", and $x^+ = (x_0, x_1, \dots)$ is the "future". Let $T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ denote the left shift i.e. $(Tx)_n = x_{n+1}$. We are thinking of T as "taking one step into the future".

We also have the "skew product" map $\hat{T} : \mathcal{S}^{\mathbb{Z}} \times G \to \mathcal{S}^{\mathbb{Z}} \times G$ given by

$$\hat{T}(x, g) = (Tx, x_0 g), \qquad \text{where } x = (\dots, x_0, \dots).$$

We will often view $\hat{T}$ as a map from $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$ to $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$. For $x \in \mathcal{S}^{\mathbb{Z}}$, and $n \in \mathbb{N}$, write

$$T^n_x = x_{n-1} \dots x_0, \qquad T^{-n}_{T^n x} = (T^n_x)^{-1}$$

so that for $n \in \mathbb{Z}$,

$$\hat{T}^n(x, g) = (T^n x, T_x^n g).$$

Suppose we are given an ergodic μ-stationary measure ν on G/Γ. As in [BQ1], for $x = (\dots, x_{-1}, x_0, x_1, \dots)$, let

$$\nu_{x^-} = \lim_{n\to\infty} x_{-1} \dots x_{-n}\, \nu.$$

The fact that the limit exists follows from the martingale convergence theorem. Then ν_{x^-} is a measure on G/Γ.

Basic Fact: Given a μ-stationary measure ν on G/Γ, we get a $\hat{T}$-invariant measure $\hat{\nu}$ on $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$ given by

$$d\hat{\nu}(x^-, x^+, g\Gamma) = d\mu^{\mathbb{Z}}(x^-, x^+)\, d\nu_{x^-}(g\Gamma) \tag{1.1}$$

It is important that the measure $\hat{\nu}$ is a product of a measure depending on $(x^-, g\Gamma)$ and a measure depending on x^+. (If instead of the two-sided shift space we use the one-sided shift $\mathcal{S}^{\mathbb{N}} \times G/\Gamma$, then $\mu^{\mathbb{N}} \times \nu$ would be an invariant measure for $\hat{T}$.)

Proposition 1.2. *The measure $\hat{\nu}$ is $\hat{T}$-ergodic.*

Proof. Since ν is an ergodic stationary measure, this follows from [Kif, Lemma I.2.4, Theorem I.2.1] □

The "group" $\mathcal{U}_1^+$. We would like to express the fact that the measure ν_{x^-} does not depend on the x^+ coordinate as invariance under the action of a group. The group will be a bit artificial.

Let $P(\mathcal{S})$ denote the permutation group of $\mathcal{S}$, i.e. the set of bijections from $\mathcal{S}$ to $\mathcal{S}$. Let

$$\mathcal{U}_1^+ = P(\mathcal{S}) \times P(\mathcal{S}) \times P(\mathcal{S}) \dots$$

The way $u = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots) \in \mathcal{U}_1^+$ acts on $\mathcal{S}^{\mathbb{Z}}$ is given by

$$u \cdot (\dots, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_{-n}, \dots, x_{-1}, \sigma_0(x_0), \sigma_1(x_1), \dots)$$

We then extend the action of $\mathcal{U}_1^+$ to $\mathcal{S}^{\mathbb{Z}} \times G$ by:

$$u \cdot (x, g) = (ux, g)$$

(So $\mathcal{U}_1^+$ acts by "changing the combinatorial future". $\mathcal{U}_1^+$ fixes x^- and g and changes x^+.) In view of (1.1), the conditional measures of $\hat{\nu}$ along the $\mathcal{U}_1^+$ orbits are the (almost always) the Bernoulli measure $\mu^{\mathbb{N}}$. We refer to this statement as "$\mathcal{U}_1^+$-invariance" of $\hat{\nu}$. In fact $\hat{T}$-invariant measures on the skew-product which come from stationary measures are exactly the $\hat{T}$-invariant measures which are also invariant under $\mathcal{U}_1^+$.

We have a similar group $\mathcal{U}_1^-$ which is changing the combinatorial past. However, in general $\hat{\nu}$ is not $\mathcal{U}_1^-$-invariant.

Stable and unstable manifolds. For $x \in \mathcal{S}^{\mathbb{Z}}$, let

$$W^-[x] = \{y \in \mathcal{S}^{\mathbb{Z}} \quad : \quad \text{for } n \in \mathbb{N} \text{ sufficiently large, } y_n = x_n\}.$$

Then $W^-[x]$ consists of sequences y which eventually agree with x. Clearly, $W^-[x]$ depends only on x^+. We call $W^-[x]$ the "stable leaf" though x. We also have the subset

$$W_1^-[x] = \{y \in \mathcal{S}^{\mathbb{Z}} \;\;:\;\; y^+ = x^+\} \subset W^-[x].$$

Similarly, we define the "unstable leaf"

$$W^+[x] = \{y \in \mathcal{S}^{\mathbb{Z}} \;\;:\;\; \text{for } n \in \mathbb{N} \text{ sufficiently large, } y_{-n} = x_{-n}\}.$$

We also have the subset

$$W_1^+[x] = W_1^+[x^-] = \{y \in \mathcal{S}^{\mathbb{Z}} \;\;:\;\; y^- = x^-\} \subset W^+[x].$$

Let $d_G(\cdot,\cdot)$ be a right invariant Riemannian metric on G. For $\hat{x} = (x,g) \in \mathcal{S}^{\mathbb{Z}} \times G$, let

$$\hat{W}_1^-[\hat{x}] = \{(y,g') \in \mathcal{S}^{\mathbb{Z}} \times G \;\;:\;\; y \in W_1^-[x], \limsup_{n\to\infty} \frac{1}{n} \log d_G(T_x^n g, T_x^n g') < 0\}.$$

Thus, $\hat{W}_1^-[\hat{x}]$ consists of the points $\hat{y}$ which have the same combinatorial future as $\hat{x}$ and such that at $n \to \infty$, $\hat{T}^n\hat{x}$ and $\hat{T}^n\hat{y}$ converge exponentially fast. Similarly, we have a subset

$$\hat{W}_1^+[\hat{x}] = \{(y,g') \in \mathcal{S}^{\mathbb{Z}} \times G \;\;:\;\; y \in W_1^+[x], \quad \limsup_{n\to\infty} \frac{1}{n} \log d_G(T_x^{-n} g, T_x^{-n} g') < 0\},$$

consisting of the points $\hat{y}$ which have the same combinatorial past as $\hat{x}$ and such that at $n \to \infty$, $\hat{T}^{-n}\hat{x}$ and $\hat{T}^{-n}\hat{y}$ converge exponentially fast.

We will show below that that for almost all x there exist unipotent subgroups $N^+(x)$ and $N^-(x)$ so that $N^+(x) = N^+(x^-)$, $N^-(x) = N^-(x^+)$ and

$$\hat{W}_1^+[(x,g)] = W_1^+[x] \times N^+(x)g,$$

and

$$\hat{W}_1^-[(x,g)] = W_1^-[x] \times N^-(x)g,$$

Thus,

$$\hat{W}_1^+[(x^-,x^+,g)] = \{(y^-,y^+,h) \;\;:\;\; y^- = x^-, y^+ \text{ is arbitrary, } h \in N(x^-)g\}.$$

and

$$\hat{W}_1^-[(x^-,x^+,g)] = \{(y^-,y^+,h) \;\;:\;\; y^+ = x^+, y^- \text{ is arbitrary, } h \in N(x^+)g\}.$$

The two cases. In view of (1.1), for almost all $(x, g\Gamma)$, the conditional measure $\hat{\nu}|_{\hat{W}_1^+[(x,g)]}$ is the product of the Bernoulli measure $\mu^{\mathbb{N}}$ on $W_1^+[x] \cong \mathcal{S}^{\mathbb{N}}$ and an unknown measure on $N^+(x)g$. However, we have no such information on the conditional measures $\hat{\nu}|_{\hat{W}_1^-[(x,g)]}$. A priori, we can only make the observation that the entropy $h_{\hat{\nu}}(\hat{T}) > 0$ (since the Bernoulli shift T is a factor), and therefore almost everywhere $\hat{\nu}|_{\hat{W}_1^-[(x,g)]}$ is non-trivial. We distinguish two cases:

- Case I: Not Case II.

- Case II: For almost all $(x, g) \in \mathcal{S}^{\mathbb{Z}} \times G$, the conditional measure $\hat{\nu}|_{\hat{W}_1^-[(x,g)]}$ is supported on a single $\mathcal{U}_1^-$ orbit.

We emphasize that even in Case I, it is possible that all the entropy of $\hat{T}$ comes from the Bernoulli shift (so that the G/Γ fibers do not contrubute to the entropy).

Our proof breaks up into the following statements:

Theorem 1.3. *Suppose μ is a measure on G satisfying the assumptions of Theorem 1.1, and ν is an ergodic μ-stationary probability measure on G/Γ and suppose Case I holds. Then ν is the Haar measure on G/Γ.*

Theorem 1.4. *Suppose μ is a probability measure on G satisying the assumptions of Theorem 1.1, ν is an ergodic μ-stationary probability measure on G/Γ and suppose Case II holds. Then*

(a) *ν is $G_{\mathcal{S}}$-invariant.*
(b) *ν is finitely supported.*

Clearly Theorem 1.1 follows from Theorem 1.3 and Theorem 1.4. We will prove Theorem 1.3 in §2-§6, and we will prove Theorem 1.4 in §7.

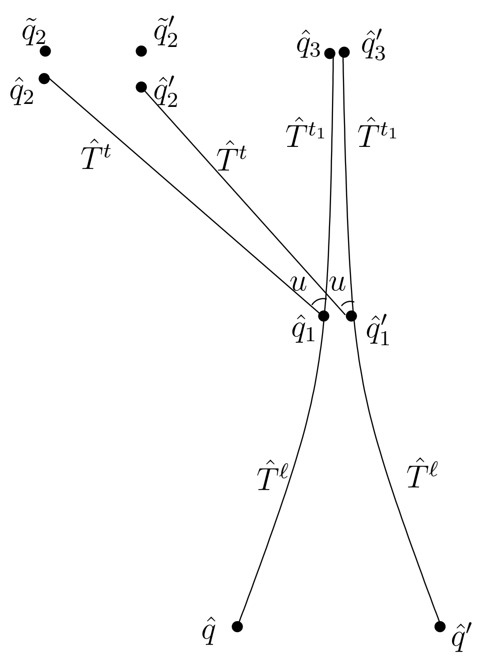

Figure 1. Outline of the proof of Theorem 1.3

1.2. **Outline of the proof of Theorem 1.3.** The assumption of Case I implies that we can find points $\hat{q} = (q, g)$ and $\hat{q}' = (q', g')$ in the support of $\hat{\nu}$, with $\hat{q}' \in \hat{W}_1^-[\hat{q}]$ and $g \neq g'$. (Since $\hat{q}' \in \hat{W}_1^-[\hat{q}]$ we must have $q^+ = (q')^+$, but q^- need not be equal to

$(q')^-)$. Furthermore, we can find such $\hat{q}$ in a set of large measure, and also choose $\hat{q}'$ so that $d_G(g, g') \approx 1$.

(In the rest of the outline, we use a suspension flow construction which will allow us to make sense of expressions like $\hat{T}^t$ where $t \in \mathbb{R}$. This construction is defined in the beginning of §2.)

We now choose an arbitrary large parameter $\ell \in \mathbb{R}^+$, and let $\hat{q}_1 = \hat{T}^\ell \hat{q}$, $\hat{q}_1' = \hat{T}^\ell \hat{q}'$. Since $\hat{q}' \in \hat{W}_1^-[\hat{q}]$, $d(\hat{q}_1, \hat{q}_1')$ is exponentially small in ℓ.

Suppose $u \in \mathcal{U}_1^+$. For most choices of u, $u\hat{q}_1$ and $u\hat{q}_1'$ are no longer in the same stable for $\hat{T}$, and thus we expect $\hat{T}^t u\hat{q}_1$ and $\hat{T}^t u\hat{q}_1'$ to diverge as $t \to \infty$. Fix $0 < \epsilon < 1$ and choose t so that $\hat{q}_2 \equiv \hat{T}^t u\hat{q}_1$ and $\hat{q}_2' \equiv \hat{T}^t u\hat{q}_1'$ satisfy $d(\hat{q}_2, \hat{q}_2') \approx \epsilon$. Write $\hat{q}_2 = (q_2, g_2)$, $\hat{q}_2' = (q_2', g_2')$.

Let $N_1(x) \subset G$ denote the unipotent subgroup whose Lie algebra corresponds to the top Lyapunov exponent λ_1 of μ, i.e.

$$\mathrm{Lie}(N_1)(x) = \left\{ \mathbf{v} \in \mathfrak{g} \;\;:\;\; \lim_{n \to \pm\infty} \frac{1}{n} \log \frac{\|(T_x^n)_* \mathbf{v}\|}{\|\mathbf{v}\|} = \lambda_1 \right\}.$$

Since we are asssuming that $G_{\mathcal{S}}$ is Zariski dense in a simple Lie group G, for a.e. $x \in \mathcal{S}^{\mathbb{Z}}$ and for most choices of u, $\hat{q}_2'$ and $\hat{q}_2$ ‘diverge essentially along N_1, i.e. g_2' is very close to $N_1(q_2)g_2$, with the distance tending to 0 as $\ell \to \infty$. Furthermore, there exists a cocycle $\lambda_1 : \mathcal{S}^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ such that for all $\mathbf{v} \in \mathrm{Lie}(N_1(x))$ and $t \geq 0$, $\|(T_x^t)_* \mathbf{v}\| = e^{\lambda_1(x,t)} \|\mathbf{v}\|$.

Now choose $t_1 > 0$ such that $\lambda_1(q_1, t_1) = \lambda_1(uq_1, t)$, and let $\hat{q}_3 = \hat{T}^{t_1} \hat{q}_1$, $\hat{q}_3' = \hat{T}^{t_1} q_1'$. Then $\hat{q}_3$ and $\hat{q}_3'$ are even closer than $\hat{q}_1$ and $\hat{q}_1'$.

The rest of the setup follows [BQ1] (which only uses the “top half” of Figure 1). For $\hat{x} = (x, g) \in \hat{\Omega}$, let $f_1(\hat{x})$ denote the conditional measure (or more precisely the leafwise measure in the sense of [EiL2]) of $\hat{\nu}$ along $\{x\} \times N_1(x)g$. These measures are only defined up to normalization. Then, since $\hat{\nu}$ is $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant and since $\lambda_1(q_1, t_1) = \lambda_1(uq_1, t)$, we have,

$$f_1(\hat{q}_2) = f_1(\hat{q}_3).$$

Also, since one can show $\lambda_1(uq_1', t) \approx \lambda_1(q_1', t_1)$ we have,

$$f_1(\hat{q}_2') \approx f_1(\hat{q}_3').$$

Since $\hat{q}_3$ and $\hat{q}_3'$ are very close, we can ensure that, $f_1(\hat{q}_3') \approx f_1(\hat{q}_3)$. Then, we get, up to normalization,

$$f_1(\hat{q}_2) \approx f_1(\hat{q}_2').$$

Applying the argument with a sequence of ℓ’s going to infinity, and passing to a limit along a subsequence, we obtain points $\tilde{q}_2 = (z, \tilde{g}_2)$ and $\tilde{q}_2' = (z, \tilde{g}_2')$ with $\tilde{g}_2' \in N_1(z)\tilde{g}_2$, $d_G(\tilde{g}_2, \tilde{g}_2') \approx \epsilon$ and, up to normalization, $f_1(\tilde{q}_2) = f_1(\tilde{q}_2')$. Thus, $f_1(\tilde{q}_2)$ is invariant by a translation of size approximately ϵ. By repeating this argument with a sequence of ϵ’s converging to 0, we show that for almost all $\hat{x} = (x, g) \in \mathcal{S}^{\mathbb{Z}} \times G/\Gamma$, $f_1(\hat{x})$ is invariant under arbitrarily small translations, which implies that there exists a connected non-trivial unipotent subgroup $U^+_{new}(\hat{x}) \subset N_1(x)$ so that $\hat{\nu}$ is "U^+_{new}-invariant" or more precisely, for almost all $\hat{x}$, the conditional measure of $\hat{\nu}$ along $\{x\} \times U^+_{new}(\hat{x})$ is Haar. Since U^+_{new} is unipotent, we can apply Ratner's theorem. The rest of the argument follows closely [BQ1, §8].

To make this scheme work, we need to make sure that all eight points $\hat{q}$,$\hat{q}'$, $\hat{q}_1$, $\hat{q}_1'$, $\hat{q}_2$, $\hat{q}_2'$, $\hat{q}_3$, $\hat{q}_3'$. are in some "good subset" $K_0 \subset \hat{\Omega}$ of almost full measure. (For instance we want the function f_1 to be uniformly continuous on K_0). Showing that this is possible is the heart of the proof. Our strategy for accomplishing this goal is substantially different from that of [BQ1], where a time changed Martingale Convergence argument was used, and from that of [BQ2], where a Local Limit Theorem (proved in [BQ3]) is used. Our strategy is is outlined further in §6.1.

In [EsL] we use a more elaborate version of the argument to handle a more general situation in which (unlike [BQ1] and [BQ2]), the Zariski closure of the group generated by the support of μ is not assumed to be semisimple, and in particular, N_1 and λ_1 with the above properties may not exist. See [EsL, §1.5] for a discussion, and for the relation to other generalizations of the main theorem of [BQ1] and to [EMi].

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2. General cocycle lemmas

Let $\Omega = \mathcal{S}^{\mathbb{Z}} \times [0,1]$. Let T^t denote the suspension flow on on Ω, i.e. T^t is obtained as a quotient of the flow $(x,s) \to (x, t+s)$ on $\mathcal{S}^{\mathbb{Z}} \times \mathbb{R}$ by the equivalence relation $(x, s+1) \sim (Tx, s)$. Let the measure $\tilde{\mu}$ on Ω be the product of the measure $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ and the Lebesgue measure on $[0,1]$.

Let T^n_x be as in §1. We then define

$$T^t_x = T^n_x, \qquad \text{where } n \text{ is the greatest integer smaller than or equal to } t.$$

We define $\hat{\Omega} = \Omega \times G$. We then have a skew-product flow $\hat{T}^t$ on Ω, defined by

$$\hat{T}^t(x,g) = (T^t x, T^t_x g).$$

Also Γ acts on $\hat{\Omega}$ on the right (by right multiplication on the second factor). We also use $\hat{T}$ to denote the induced map on $\hat{\Omega}/\Gamma$. We have an action on the trivial bundle $\Omega \times \mathfrak{g}$ given by

$$T^t(x, \mathbf{v}) = (T^t x, (T^t_x)_* \mathbf{v}).$$

We fix some norm $\|\cdot\|_0$ on $\mathfrak{g}$, and apply the Osceledets multiplicative ergodic theorem to the cocycle $(T^t)_*$. Let λ_i denote the i-th Lyapunov exponent of this cocycle. We always number the exponents so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n.$$

Let

$$\{0\} = \mathcal{V}_{\le 0}(x) \subset \mathcal{V}_{\le 1}(x) \subset \cdots \subset \mathcal{V}_{\le n}(x) = \mathfrak{g} \tag{2.1}$$

denote the backward flag, and let

$$\{0\} = \mathcal{V}_{\ge n+1}(x) \subset \mathcal{V}_{\ge n}(x) \subset \cdots \subset \mathcal{V}_{\ge 1}(x) = \mathfrak{g}$$

denote the forward flag. This means that for almost all $x \in \Omega$ and for $\mathbf{v} \in \mathcal{V}_{\le i}(x)$ such that $\mathbf{v} \notin \mathcal{V}_{\le i-1}(x)$,

$$\lim_{t \to -\infty} \frac{1}{t} \log \frac{\|(T^t_x)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i, \tag{2.2}$$

and for $\mathbf{v} \in \mathcal{V}_{\ge i}(x)$ such that $\mathbf{v} \notin \mathcal{V}_{\ge i+1}(x)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|(T^t_x)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i. \tag{2.3}$$

It follows from (2.2) that for $y \in W_1^+[x]$, we have

$$\mathcal{V}_{\le i}(y) = \mathcal{V}_{\le i}(x). \tag{2.4}$$

Similarly, for $y \in W_1^-[x]$,

$$\mathcal{V}_{\ge i}(y) = \mathcal{V}_{\ge i}(x).$$

By e.g. [GM, Lemma 1.5], we have for a.e. $x \in \Omega$,

$$\mathfrak{g} = \mathcal{V}_{\le i}(x) \oplus \mathcal{V}_{\ge i+1}(x). \tag{2.5}$$

Let

$$\mathcal{V}_i(x) = \mathcal{V}_{\le i}(x) \cap \mathcal{V}_{\ge i}(x).$$

Then, in view of (2.5), for almost all x, we have

$$\mathcal{V}_{\le i}(x) = \bigoplus_{j \le i} \mathcal{V}_j(x), \qquad \mathcal{V}_{< i}(x) = \bigoplus_{j < i} \mathcal{V}_j(x),$$

$$\mathcal{V}_{\ge i}(x) = \bigoplus_{j \ge i} \mathcal{V}_j(x), \qquad \mathcal{V}_{> i}(x) = \bigoplus_{j > i} \mathcal{V}_j(x).$$

We have $\mathbf{v} \in \mathcal{V}_j(x)$ if and only if

$$\lim_{|t| \to \infty} \frac{1}{t} \log \frac{\|(T^t_x)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i.$$

The Lyapunov exponents λ_j and the Lyapunov subspaces $\mathcal{V}_j(x)$ do not depend on the choice of the norm $\|\cdot\|_0$.

It it easy to see that the subspaces

$$\bigoplus_{\lambda_j > 0} \mathcal{V}_j(x) \text{ and } \bigoplus_{\lambda_j < 0} \mathcal{V}_j(x)$$

are both nilpotent subalgebras of $\mathfrak{g}$. We thus define the unipotent subgroups $N^+(x)$ and $N^-(x)$ of G by

$$\mathrm{Lie}(N^+)(x) = \bigoplus_{\lambda_j > 0} \mathcal{V}_j(x), \qquad \mathrm{Lie}(N^-)(x) = \bigoplus_{\lambda_j < 0} \mathcal{V}_j(x).$$

There are the subgroups which appeared in §1.

2.1. **Equivariant measurable flat connections.**

The maps $P^+(x,y)$ and $P^-(x,y)$. For almost all $x \in \Omega$ and almost all $y \in W_1^+[x]$, any vector $\mathbf{v} \in \mathcal{V}_i(x)$ can be written uniquely as

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}'' \qquad \mathbf{v}' \in \mathcal{V}_i(y), \quad \mathbf{v}'' \in \mathcal{V}_{<i}(y).$$

Let $P_i^+(x,y) : \mathcal{V}_i(x) \to \mathcal{V}_i(y)$ be the linear map sending $\mathbf{v}$ to $\mathbf{v}'$. Let $P^+(x,y) : \mathfrak{g} \to \mathfrak{g}$ be the unique linear map which restricts to $P_i^+(x,y)$ on each of the subspaces $\mathcal{V}_i(x)$. (We think of $P^+(x,y)$ as map from T_xG to T_yG). We call $P^+(x,y)$ the "parallel transport" from x to y. The following is immediate from the definition:

Lemma 2.1. *Suppose $x, y \in W_1^+[z]$. Then*
- (a) $P^+(x,y)\mathcal{V}_i(x) = \mathcal{V}_i(y)$.
- (b) $P^+(T^t x, T^t y) = (T_y^t)_* \circ P^+(x,y) \circ (T_x^{-t})_*$.
- (c) $P^+(x,y)\mathcal{V}_{\le i}(x) = \mathcal{V}_{\le i}(y) = \mathcal{V}_{\le i}(x)$. *Thus, the map $P^+(x,y) : \mathfrak{g} \to \mathfrak{g}$ is unipotent.*
- (d) $P^+(x,z) = P^+(y,z) \circ P^+(x,y)$.

If $y \in W_1^-[x]$, then we can define a similar map which we denote by $P^-(x,y)$.

Distance between subspaces. For a subspace V of $\mathfrak{g}$, let SV denote the intersection of V with the unit ball in the $\|\cdot\|_0$ norm. For subspaces V_1, V_2 of $\mathfrak{g}$, we define

$$d_0(V_1, V_2) = \text{The Hausdorff distance between } SV_1 \text{ and } SV_2 \tag{2.6}$$

measured with respect to the distance induced from the norm $\|\cdot\|_0$.

Lemma 2.2. *Fix $\epsilon > 0$ sufficiently small depending on the dimension of G and the Lyapunov exponents. Then there exists a compact subset $\mathcal{C} = \mathcal{C}' \times [0,1) \subset \Omega$ with $\tilde{\mu}(\mathcal{C}) > 0$ and a function $T_0 : \mathcal{C} \to \mathbb{N} \cup \{\infty\}$ with $T_0(c) < \infty$ for $\tilde{\mu}$-a.e. $c \in \mathcal{C}$ and $T_0(c)$ depending only on the projection of c to $\mathcal{S}^{\mathbb{Z}}$, such that the following hold:*
- (a) *There exists $\sigma_0 > 0$ such that for all $c \in \mathcal{C}$, and any subset S of the Lyapunov exponents,*
$$d_0\Big(\bigoplus_{i \in S} \mathcal{V}_i(c), \bigoplus_{j \notin S} \mathcal{V}_j(c)\Big) \ge \sigma_0.$$
- (b) *There exists $\rho > 0$ such that for all $t > T_0(c)$, for all $c \in \mathcal{C}$, for all i and all $\mathbf{v} \in \mathcal{V}_i(c)$,*
$$\rho e^{(\lambda_i - \epsilon)t} \|\mathbf{v}\|_0 \le \|(T_c^t)_* \mathbf{v}\|_0 \le \rho^{-1} e^{(\lambda_i + \epsilon)t} \|\mathbf{v}\|_0.$$

Proof. Part (a) holds since the inverse of the angle between Lyapunov subspaces is finite a.e., therefore bounded on a set of almost full measure. Also, (b) follows immediately from the multiplicative ergodic theorem. □

2.2. **Dynamically defined norms.** The aim of this subsection is to prove the following:

Proposition 2.3. *There exists a T^t-invariant subset $\Psi \subset \Omega$ with $\tilde{\mu}(\Psi) = 1$ and for all $x \in \Psi$ there exists an inner product $\langle \cdot, \cdot \rangle_x$ and a cocycle $\lambda_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ with the following properties:*

(a) *For all $x \in \Psi$, the distinct eigenspaces $\mathcal{V}_i(x)$ are orthogonal.*

(b) *If $\mathbf{v} \in \mathcal{V}_1(x)$, and $t \in \mathbb{R}$, then*
$$\|(T^t_x)_* \mathbf{v}\|_{T^t x} = e^{\lambda_1(x,t)} \|\mathbf{v}\|_x,$$
where $\|\mathbf{v}\|_x$ denotes $\langle \mathbf{v}, \mathbf{v} \rangle_x^{1/2}$.

(c) *There exists a constant $\kappa > 1$ such that for all $x \in \Psi$ and for all $t > 0$,*
$$\kappa^{-1} t \le \lambda_1(x,t) \le \kappa t.$$
Hence, since $\lambda_1(\cdot,\cdot)$ is a cocycle, for all $x \in \Psi$ and for all $t > t'$,
$$\frac{1}{\kappa}(t - t') \le \lambda_1(x,t) - \lambda_1(x,t') \le \kappa(t - t').$$

(d) *There exists a constant $\kappa > 1$ such that for all $x \in \Psi$, for all $\mathbf{v} \in \mathrm{Lie}(N^+)(x)$, and all $t \ge 0$,*
$$e^{\kappa^{-1} t} \|\mathbf{v}\|_x \le \|(T^t_x)_* \mathbf{v}\|_{T^t x} \le e^{\kappa t} \|\mathbf{v}\|_x.$$
Also, for all $x \in \Psi$ and for all $\mathbf{v} \in \mathrm{Lie}(N^-)(x)$, and all $t \ge 0$,
$$e^{-\kappa t} \|\mathbf{v}\|_x \le \|(T^t_x)_* \mathbf{v}\|_{T^t x} \le e^{-\kappa^{-1} t} \|\mathbf{v}\|_x.$$
In addition, for all $\mathbf{v} \in \mathfrak{g}$ and all $t \in \mathbb{R}$,
$$e^{-\kappa|t|} \|\mathbf{v}\|_x \le \|(T^t_x)_* \mathbf{v}\|_{T^t x} \le e^{\kappa|t|} \|\mathbf{v}\|_x.$$
In particular, the map $t \to \|(T^t_x)_* \mathbf{v}\|_{T^t x}$ is continuous.

We often omit the subscript from $\| \cdot \|_x$.

Our proof of Proposition 2.3 relies on the simplicity of G. For a more elaborate version in a more general setting, see [EsL, Proposition 2.14].

Lemma 2.4. *There exists an inner product $\langle \cdot, \cdot \rangle'_x$ on $\mathcal{V}_1(x)$ and a cocycle $\theta : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for $\mathbf{v} \in \mathcal{V}_1(x)$ and $t \in \mathbb{R}$,*
$$\langle (T^t_x)_* \mathbf{v}, (T^t_x)_* \mathbf{v} \rangle'_{T^t x} = e^{\theta(x,t)} \langle \mathbf{v}, \mathbf{v} \rangle'_x.$$

Proof. This is [BQ1, Lemma 5.4], where it is attributed mostly to Furstenberg and Kesten. $\square$

Remark. Lemma 2.4 says that for G simple, the top eigenspace is conformal. In fact, if G is semisimple, then all eigenspaces are block-conformal, see [EMat].

Lemma 2.5. *Suppose $\mathcal{C} = \mathcal{C}' \times [0,1) \subset \Omega$ is such that $\tilde{\mu}(\mathcal{C}) > 0$ and $T_1 : \mathcal{C} \to \mathbb{R}^+$ is a measurable function that is finite a.e. and with $T_1(c)$ depending only on the projection of c to $\mathcal{S}^{\mathbb{Z}}$. Then we can find a $\mathcal{C}_1 \subset \mathcal{C}$ such that for all $c \in \mathcal{C}_1$, for all $0 < t < T_1(c)$, we have $T^t c \notin \mathcal{C}_1$, and also $\bigcup_{t>0} T^t \mathcal{C}_1$ is conull in Ω.*

Proof. We can find $n \in \mathbb{N}$ such that $\mathcal{C}_2' \equiv \{x \in \mathcal{C}' \;:\; T_1(x) < n\}$ has positive $\mu^{\mathbb{Z}}$ measure. Clearly $\mu^{\mathbb{Z}}$ gives 0 measure to the set of periodic points of the shift $T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$. Then, there exists a non-periodic point $x_0 \in \mathcal{C}_2'$ such that every neighborhood of x_0 has positive $\mu^{\mathbb{Z}}$ measure. Since T is continuous, there exists a neighborhood $E \subset \mathcal{S}^{\mathbb{Z}}$ of x_0 with $E, TE, \ldots, T^n E$ pairwise disjoint. Now let $\mathcal{C}_1' = E \cap \mathcal{C}_2'$ and let $\mathcal{C}_1 = \mathcal{C}_1' \times \{0\}$. The last assertion follows from the ergodicity of T. $\square$

Let $\mathcal{C}$ and T_0 be as in Lemma 2.2. Let $T_1 : \mathcal{C} \to \mathbb{R}^+$ be a finite a.e. measurable function to be chosen later. We will choose T_1 so that in particular $T_1(c) > T_0(c)$ for a.e. $c \in \mathcal{C}$. Let $\mathcal{C}_1$ be as in Lemma 2.5. For $c \in \mathcal{C}_1$, let $t(c)$ be the smallest $t > 0$ such that $T^t c \in \mathcal{C}_1$.

We will first define the inner product $\langle , \rangle_c$ for $c \in \mathcal{C}_1$, and then interpolate between $\langle , \rangle_c$ and $\langle , \rangle_{c'}$ where $c' = T^{t(c)} c \in \mathcal{C}_1$.

For c in $\mathcal{C}_1$, we can choose an inner product $\langle \cdot, \cdot \rangle_c$ on $\mathfrak{g}$ such that the following hold:

- For $\mathbf{v}, \mathbf{w} \in \mathcal{V}_1(c)$, $\langle \mathbf{v}, \mathbf{w} \rangle_c = r \langle \mathbf{v}, \mathbf{w} \rangle_c'$ where $r > 0$.
- The distinct $\mathcal{V}_i(c)$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_c$.
- There exists $M = M(\sigma_0)$ (where σ_0 is as in Lemma 2.2), such that for all $\mathbf{v} \in \mathfrak{g}$,
$$M^{-1} \|\mathbf{v}\|_0 \le \langle \mathbf{v}, \mathbf{v} \rangle_c^{1/2} \le M \|\mathbf{v}\|_0.$$

Symmetric space interpretation. For a.e. $x \in \Omega$, we may write x uniquely as $x = T^t c$, where $0 \le t < t(c)$ and $c \in \mathcal{C}_1$. We then define $\langle \cdot, \cdot \rangle_x$ by interpolating between $\langle \cdot, \cdot \rangle_c$ and $\langle \cdot, \cdot \rangle_{c'}$, where $c' = T^{t(c)} c$. To define this interpolation, we recall that the set of inner products on a vector space V is canonically isomorphic to $SO(V) \backslash GL(V)$, where $GL(V)$ is the general linear group of V and $SO(V)$ is the subgroup preserving the inner product on V. In our case, $V = \mathfrak{g}$ with the inner product $\langle \cdot, \cdot \rangle_c$.

Let K_c denote the subgroup of $GL(\mathfrak{g})$ which preserves the inner product $\langle \cdot, \cdot \rangle_c$. Let $\mathcal{Q}$ denote the subgroup of $GL(\mathfrak{g})$ which preserves the splitting $\mathfrak{g} = \bigoplus_i \mathcal{V}_i(c)$, and let $\mathcal{Q}' \subset \mathcal{Q}$ be the subgroup such that the restriction to $\mathcal{V}_1(c)$ is a multiple of the identity. Let $K_c A'$ denote the point in $K_c \backslash GL(\mathfrak{g})$ which represents the inner product $\langle \cdot, \cdot \rangle_{c'}$, i.e.
$$\langle \mathbf{u}, \mathbf{v} \rangle_{c'} = \langle A' \mathbf{u}, A' \mathbf{v} \rangle_c.$$

Then, since $(T_c^{t(c)})_* \mathcal{V}_i(c) = \mathcal{V}_i(c')$, we may assume that $A'(T_c^{t(c)})_* \in \mathcal{Q}$. Furthermore, by the choices of $\langle \cdot, \cdot \rangle_c$ and $\langle \cdot, \cdot \rangle_{c'}$ and Lemma 2.4, we may assume that $A'(T_c^{t(c)})_* \in \mathcal{Q}'$.

Claim 2.6. *We may write*

$$A'(T_c^{t(c)})_* = \Lambda A'',$$

where $\Lambda \in \mathcal{Q}'$ is the diagonal matrix which is scaling by $e^{\lambda_i t(c)}$ on $\mathcal{V}_i(c)$, $A'' \in \mathcal{Q}'$ and $\|A''\| = O(e^{\epsilon t(c)})$, with the implied constant depending only on the constants ϵ, σ_0, ρ from Lemma 2.2.

Proof of claim. By construction, $t(c) > T_0(c)$, where $T_0(c)$ is as in Lemma 2.2. Then, the claim follows from Lemma 2.2. $\square$

Interpolation. We may write $A'' = DA'''$, where D is diagonal, and $\det A''' = 1$. In view of Claim 2.6, $\|D\| = O(e^{\epsilon t})$ and $\|A'''\| = O(e^{\epsilon t})$. We now connect $K_c \backslash K_c A'''$ to the identity by the shortest possible path $\Gamma : [0, t(c)] \to K_c \backslash SL(V)$, which stays in the subset $K_c \backslash K_c \mathcal{Q}'$ of the symmetric space $K_c \backslash SL(V)$. (We parametrize the path so it has constant speed). This path has length $O(\epsilon t)$ where the implied constant depends only on the symmetric space, and the constants σ_0, ρ of Lemma 2.2.

Now for $0 \le t \le t(c)$, let

$$A(t) = (\Lambda D)^{t/t(c)} \Gamma(t). \tag{2.7}$$

Then $A(0)$ is the identity map, and $A(t(c)) = A'(T_c^{t(c)})_*$. Suppose $x = T^t c$, where $0 \le t < t(c)$ and $c \in \mathcal{C}_1$, and $c' = T^{t(c)} c \in \mathcal{C}_1$. Then, we define,

$$\langle \mathbf{u}, \mathbf{v} \rangle_x = \langle A(t)(T_x^{-t})_* \mathbf{u}, A(t)(T_x^{-t})_* \mathbf{v} \rangle_c. \tag{2.8}$$

In particular, since $(T_{c'}^{-t(c)})_* = (T_c^{t(c)})_*^{-1}$, we have, letting $t = t(c)$ in (2.8),

$$\langle \mathbf{u}, \mathbf{v} \rangle_{c'} = \langle A(t(c))(T_{c'}^{-t(c)})_* \mathbf{u}, A(t(c))(T_{c'}^{-t(c)})_* \mathbf{v} \rangle_c = \langle A' \mathbf{u}, A' \mathbf{v} \rangle_c,$$

as required.

Proof of Proposition 2.3. By construction (a) holds. Also, since $A(t) \in \mathcal{Q}'$, (b) holds. From (2.7), we have for $c \in \mathcal{C}_1$, $0 \le t < t(c)$ and $\mathbf{v} \in \mathcal{V}_i(c)$,

$$\frac{d}{dt} \log \|(T_c^t)_* \mathbf{v}\|_{T^t c} = \lambda_i + \gamma_i(c, t),$$

where $\gamma_i(c, t)$ is the contribution of $D^{t/t(c)} \Gamma(t)$. By Claim 2.6,

$$|\gamma_i(c, t)| \le k\epsilon + O(1/t),$$

where k depends only on the dimension, and the implied constant is bounded in terms of the constants σ_0 and ρ in Lemma 2.2. Therefore, if $\epsilon > 0$ in Lemma 2.2 is chosen small enough and $T_1(c)$ in Lemma 2.5 is chosen large enough, $|\gamma_i(c, t)| < |\lambda_i|/2$ and thus (c) and (d) hold. $\square$

Lemma 2.7. *For every $\delta > 0$ there exists a compact subset $K(\delta) \subset \Omega$ with $\tilde{\mu}(K(\delta)) > 1 - \delta$ and a number $C_1(\delta) < \infty$ such that for all $x \in K(\delta)$ and all $\mathbf{v} \in \mathfrak{g}$,*

$$C_1(\delta)^{-1} \leq \frac{\|\mathbf{v}\|_x}{\|\mathbf{v}\|_0} \leq C_1(\delta) \tag{2.9}$$

where $\|\cdot\|_x$ is the dynamical norm defined in this subsection and $\|\cdot\|_0$ is the fixed norm on $\mathfrak{g}$.

Proof. Since any two norms on a finite dimensional vector space are equivalent, there exists a function $\Xi_0 : \Omega \to \mathbb{R}^+$ finite a.e. such that for all $x \in \Omega$ and all $\mathbf{v} \in \mathfrak{g}$,

$$\Xi_0(x)^{-1}\|\mathbf{v}\|_0 \leq \|\mathbf{v}\|_x \leq \Xi_0(x)\|\mathbf{v}\|_0.$$

Since $\bigcup_{N \in \mathbb{N}}\{x \; : \; \Xi_0(x) < N\}$ is conull in Ω, we can choose $K(\delta) \subset \Omega$ and $C_1 = C_1(\delta)$ so that $\Xi_0(x) < C_1(\delta)$ for $x \in K(\delta)$ and $\tilde{\mu}(K(\delta)) \geq (1 - \delta)$. This implies (2.9). □

3. Preliminary divergence estimates

Lemma 3.1. *For a.e. $x \in \Omega$ and a.e. $u \in \mathcal{U}_1^+$, $\mathcal{V}_1(ux) = \mathcal{V}_1(x)$.*

Proof. Write $x = (x^-, x^+)$. Then, $\mathcal{V}_1(x) = \mathcal{V}_{\leq 1}(x)$. But $\mathcal{V}_{\leq 1}(x)$ depends only on x^- (see (2.4)) and the action of u only changes the x^+ coordinate. □

The measure $|\cdot|$. Note that for every $x \in \Omega$, $\mathcal{U}_1^+ x = W_1^+[x] \cong \mathcal{S}^{\mathbb{N}}$, and $\mathcal{S}^{\mathbb{N}}$ supports a Bernoulli measure $\mu^{\mathbb{N}}$. We use the notation $|\cdot|$ to denote the corresponding measure on $\mathcal{U}_1^+ x$.

Lemma 3.2. *For every $\delta > 0$ and every $\eta > 0$ there exists $t_0 = t_0(\delta, \eta) > 0$ and for every $q_1 \in \Omega$ and every $\mathbf{w} \in \mathfrak{g}$ there exists a subset $Q = Q(q_1, \mathbf{w}) \subset \mathcal{U}_1^+$ with $|Q(q_1)q_1| \geq (1 - \delta)|\mathcal{U}_1^+ q_1|$ such that for $u \in Q(q_1)$ and $t > 0$,*

$$\|(T^t_{uq_1})_*\mathbf{w}\| \geq c(\delta)e^{(\lambda_1/2)t}\|\mathbf{w}\|,$$

and for $t > t_0$

$$d\left(\frac{(T^t_{uq_1})_*\mathbf{w}}{\|(T^t_{uq_1})_*\mathbf{w}\|}, \mathcal{V}_1(T^t uq_1)\right) \leq \eta,$$

where $d(\cdot, \cdot)$ is the distance on $\mathfrak{g}$ defined by the dynamical norm $\|\cdot\|_{T^t uq_1}$.

Proof. This is essentially [BQ1, Corollary 5.5]. (To get the second estimate, let $W = \mathcal{V}_1(uq_1) = \mathcal{V}_1(q_1)$ in [BQ1, Corollary 5.5(b)].) □

The map $\mathcal{A}(q_1, u, \ell, t)$. For $q_1 \in \Omega$, $u \in \mathcal{U}_1^+$, $\ell > 0$ and $t > 0$, let $\mathcal{A}(q_1, u, \ell, t) : \mathfrak{g} \to \mathfrak{g}$ denote the map

$$\mathcal{A}(q_1, u, \ell, t)\mathbf{v} = (T^t_{uq_1})_*(T^\ell_{T^{-\ell}q_1})_*\mathbf{v}. \tag{3.1}$$

Proposition 3.3. *For every $\delta > 0$ and any $\eta > 0$ there exists $t_0 = t_0(\delta, \eta) > 0$ such that for almost any $q_1 \in \Omega$ and any $\mathbf{v} \in \mathrm{Lie}(N^-)(T^{-\ell} q_1)$, there exists a subset $Q = Q(q_1, \mathbf{v}) \subset \mathcal{U}_1^+$ with $|Q q_1| \geq (1-\delta)|\mathcal{U}_1^+ q_1|$ such that for $u \in Q(q_1)$ and any $t > 0$,*

$$\|\mathcal{A}(q_1, u, \ell, t)\mathbf{v}\| \geq C(\delta) e^{-\kappa\ell + (\lambda_1/2)t} \|\mathbf{v}\|, \tag{3.2}$$

where κ is as in Proposition 2.3. Also for $t > t_0$,

$$d\left(\frac{\mathcal{A}(q_1, u, \ell, t)\mathbf{v}}{\|\mathcal{A}(q_1, u, \ell, t)\mathbf{v}\|}, \mathcal{V}_1(T^t u q_1) \right) \leq \eta, \tag{3.3}$$

where $d(\cdot, \cdot)$ is the distance on $\mathfrak{g}$ defined by the dynamical norm $\|\cdot\|_{T^t u q_1}$.

Proof. This is an immediate corollary of Lemma 3.2. □

3.1. **Estimates for nearby points.** Recall that $\|\cdot\|_0$ is a fixed norm on $\mathfrak{g}$.

Lemma 3.4. *There exists $\alpha > 0$ depending only on the Lyapunov spectrum, and for every $\delta > 0$ there exists a subset $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and a constant $C(\delta) > 0$ such that for all $x \in K$, all $y \in W_1^-[x] \cap K$, and all $t > 0$,*

$$\sup_{\mathbf{v} \in \mathfrak{g} - \{0\}} \frac{\|P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \leq C(\delta) e^{-\alpha t}. \tag{3.4}$$

Proof. Pick $\epsilon > 0$ smaller than $\frac{1}{3}\min_{i \neq j} |\lambda_i - \lambda_j|$. As in the proof of Lemma 2.2, there exists $K_1 \subset \Omega$ with $\tilde{\mu}(K_1) > 1 - \delta/2$ and $\sigma_0 = \sigma_0(\delta) > 0$ such that for $x \in K_1$, and any subset S of the Lyapunov exponents and any $t \geq 0$,

$$d_0\Big(\bigoplus_{i \in S} \mathcal{V}_i(T^t x), \bigoplus_{j \notin S} \mathcal{V}_j(T^t x)\Big) \geq \sigma_0 e^{-\epsilon t}. \tag{3.5}$$

(Here $d_0(\cdot, \cdot)$ is a distance on $\mathfrak{g}$ derived from the norm $\|\cdot\|_0$.) Then, (letting $t = 0$ in (3.5)), for all $x \in K_1$, all $y \in W_1^-[x] \cap K_1$, and all $\mathbf{w} \in \mathfrak{g}$,

$$\|P^-(x, y)\mathbf{w}\|_0 \leq C(\delta)\|\mathbf{w}\|_0. \tag{3.6}$$

By the multiplicative ergodic theorem, there exists $K_2 \subset \Omega$ with $\tilde{\mu}(K_2) > 1 - \delta/2$ and $\rho = \rho(\delta) > 0$ such that for $x \in K_2$, any i, any $t > 0$ and any $\mathbf{w}_i \in \mathcal{V}_i(x)$,

$$\rho e^{(\lambda_i - \epsilon)t}\|\mathbf{w}_i\|_0 \leq \|(T_x^t)_* \mathbf{w}_i\|_0 \leq \rho^{-1} e^{(\lambda_i + \epsilon)t}\|\mathbf{w}_i\|_0. \tag{3.7}$$

Now let $K = K_1 \cap K_2$, and suppose $x \in K$, $y \in K$. Let $\mathbf{v}$ be such that the supremum in (3.4) is attained at $\mathbf{v}$. By (3.5) we may assume without loss of generality that $\mathbf{v} \in \mathcal{V}_i(T^t x)$ for some i. Let $\mathbf{w} \in \mathcal{V}_i(x)$ be such that $(T_x^t)_* \mathbf{w} = \mathbf{v}$. By (3.7),

$$\|\mathbf{v}\|_0 \geq \rho e^{(\lambda_i - \epsilon)t}\|\mathbf{w}\|_0. \tag{3.8}$$

Note that

$$P^-(T^t x, T^t y)\mathbf{v} = (T_y^t)_* P^-(x, y)\mathbf{w}.$$

Note that since $y \in W_1^-[x]$ and $t > 0$, $(T_x^t)_* = (T_y^t)_*$. By the definition of $P^-(x,y)$ we have

$$P^-(x,y)\mathbf{w} = \mathbf{w} + \sum_{j>i} \mathbf{w}_j, \qquad \mathbf{w}_j \in \mathcal{V}_j(x).$$

Thus,

$$P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v} = \sum_{j>i} (T_x^t)_* \mathbf{w}_j. \tag{3.9}$$

By (3.6), for all $j > i$,

$$\|\mathbf{w}_j\|_0 \le C_1(\delta)\|\mathbf{w}\|_0,$$

and then, by (3.7),

$$\|(T_x^t)_* \mathbf{w}_j\|_0 \le \rho^{-1} e^{(\lambda_j+\epsilon)t}\|\mathbf{w}_j\|_0 \le C_1(\delta)\rho^{-1}e^{(\lambda_j+\epsilon)t}\|\mathbf{w}\|_0.$$

Now, from (3.9) and (3.8),

$$\|P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v}\|_0 \le \sum_{j>i} C_1(\delta)\rho^{-2}e^{(\lambda_j-\lambda_i+2\epsilon)t}\|\mathbf{v}\|_0,$$

which immediately implies (3.4) since $\lambda_j < \lambda_i$ for $j > i$. $\square$

Lemma 3.5. *For every $\delta > 0$ there exists a compact set $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ such that the following holds: Suppose $t > 0$, $x \in K$, $y \in W_1^-[x] \cap K$, and $T^t x \in K$ and $T^t y \in T^{[-a,a]}K$. Then,*

$$|\lambda_1(x,t) - \lambda_1(y,t)| \le C, \tag{3.10}$$

where C depends only on a and δ.

Proof of Lemma 3.5. Let K be as Lemma 3.4. Suppose $\mathbf{v} \in \mathcal{V}_1(x)$. Let

$$\mathbf{v}' = P^-(x,y)\mathbf{v}.$$

Then, $\mathbf{v}' \in \mathcal{V}_1(y)$. For an invertible linear operator $A : \mathfrak{g} \to \mathfrak{g}$, let $\|A\|_x^y = |A|_x^y + |A^{-1}|_y^x$, where for a linear operator $B : \mathfrak{g} \to \mathfrak{g}$, $|B|_x^y$ denotes operator norm of B relative to the norms $\|\cdot\|_x$ on the domain and $\|\cdot\|_y$ on the range. By Lemma 3.4 and Lemma 2.7, there exist $C = C(\delta)$ and $C_1 = C_1(a,\delta)$ such that

$$\|P^-(x,y)\|_x^y \le C(\delta), \text{ and } \|P^-(T^t x, T^t y)\|_{T^t x}^{T^t y} \le C_1(a,\delta). \tag{3.11}$$

Therefore,

$$C(\delta)^{-1} \le \frac{\|\mathbf{v}'\|_y}{\|\mathbf{v}\|_x} \le C(\delta). \tag{3.12}$$

Note that

$$(T_y^t)_* \mathbf{v}' = P^-(T^t x, T^t y)(T_x^t)_* \mathbf{v}.$$

Then, in view of (3.11), there exists $C_2 = C_2(a, \delta)$ such that

$$C_2(a,\delta)^{-1} \le \frac{\|(T^t_y)_* \mathbf{v}'\|_{T^t y}}{\|(T^t_x)_* \mathbf{v}\|_{T^t x}} \le C_2(a,\delta). \tag{3.13}$$

By the Proposition 2.3(b),

$$\lambda_1(x,t) = \log \frac{\|(T^t_x)_* \mathbf{v}\|_{T^t x}}{\|\mathbf{v}\|_x}, \qquad \lambda_1(y,t) = \log \frac{\|(T^t_y)_* \mathbf{v}'\|_{T^t y}}{\|\mathbf{v}'\|_y}.$$

Now (3.10) follows from (3.12) and (3.13). □

4. Bilipshitz estimates

The subspace $\mathcal{L}^-(\hat{x})$. For $\hat{x} = (x,g) \in \hat{\Omega}$, let $\hat{W}^-_{loc}[\hat{x}] = \{(y,g') \in \hat{W}^-_1[\hat{x}] \;:\; d_G(g,g') < 1\}$. Let $\mathcal{L}^-(\hat{x}) \subset \mathrm{Lie}(N^-)(x) \subset \mathfrak{g}$ denote the smallest subspace of $\mathrm{Lie}(N^-)(x)$ such that the projection to G of the conditional measure $\hat{\nu}|_{\hat{W}^-_{loc}[\hat{x}]}$ is supported on $\exp(\mathcal{L}^-(\hat{x})g)$. The assumption that we are in Case I (see §1) implies $\dim(\mathcal{L}^-(\hat{x})) > 0$ for a.e. $\hat{x}$.

Lemma 4.1. *For almost all $\hat{x} = (x,g) \in \hat{\Omega}$ and all $t \in \mathbb{R}$,*

$$\mathcal{L}^-(\hat{T}^t \hat{x}) = (T^t_x)_* \mathcal{L}^-(\hat{x}). \tag{4.1}$$

Also, for almost all $\hat{x} = (x,g) \in \hat{\Omega}$, $\exp(\mathcal{L}^-)(\hat{x})$ is a subgroup of $N^-(x)$.

Proof. From the definition, for $t > 0$, $(\hat{T}^{-t}_{\hat{x}})_* \mathcal{L}^-(\hat{x}) \subset \mathcal{L}^-(\hat{T}^{-t}\hat{x})$. Let $\phi(\hat{x}) = \dim(\mathcal{L}^-(\hat{x}))$. Then, ϕ is a bounded integer valued function which is increasing under the flow $\hat{T}^{-t}$. Since the flow is ergodic on $\hat{\Omega}/\Gamma$, it follows that ϕ is constant, and therefore (4.1) holds.

For the second assertion, the proof of [EiL1, Proposition 6.2] goes through almost verbatim. □

The function $A(q_1,u,\ell,t)$. For $x \in \Omega$, let $\pi_{\mathcal{V}_1} : \mathfrak{g} \to \mathcal{V}_1(x)$ denote the orthogonal projection using the inner product $\langle \cdot, \cdot \rangle_x$. Suppose $\hat{q}_1 = (q_1, g)$, $u \in \mathcal{U}_1^+$, $\ell > 0$ and $t > 0$. We consider the restriction of $\mathcal{A}(q_1,u,\ell,t)$ to $\mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$, so we are considering $\mathcal{A}(q_1,u,\ell,t)$ as a linear map from $\mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$ to $\mathfrak{g}$. Let $A(\hat{q}_1,u,\ell,t) = \|\pi_{\mathcal{V}_1}\mathcal{A}(q_1,u,\ell,t)\|$ (the norm of the restriction) where the operator norm is with respect to the dynamical norms $\|\cdot\|_{T^{-\ell}q_1}$ and $\|\cdot\|_{T^t u q_1}$.

The function $\tilde{\tau}_{(\epsilon)}(\hat{q}_1,u,\ell)$. For $\epsilon > 0$, almost all $\hat{q}_1 \in \hat{\Omega}$, almost all $u\hat{q}_1 \in \mathcal{U}_1^+\hat{q}_1$ and $\ell > 0$, let

$$\tilde{\tau}_{(\epsilon)}(q_1,u,\ell) = \sup\{t \;:\; t > 0 \text{ and } A(\hat{q}_1,u,\ell,t) \le \epsilon\}.$$

The following easy estimate plays a key role in our proof.

Proposition 4.2. *For almost all $\hat{q}_1 \in \hat{\Omega}$, almost all $u\hat{q}_1 \in \mathcal{U}_1^+\hat{q}_1$, all $\ell > 0$ and all $s > 0$,*

$$\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) + \kappa^{-2}s < \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell + s) < \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) + \kappa^2 s, \tag{4.2}$$

where κ is as in Proposition 2.3(d).

Proof. For $\hat{x} = (x, g) \in \hat{\Omega}$ and $t > 0$ let $\mathcal{A}_+(\hat{x}, t) = \mathcal{A}_+(x, t)$ denote the restriction of $(T_x^t)_*$ to $\mathcal{V}_1(x)$. For $\hat{x} = (x, g) \in \hat{\Omega}$, let $\mathcal{A}_-(\hat{x}, s) : \mathcal{L}^-(\hat{x}) \to \mathcal{L}^-(\hat{T}^s\hat{x})$ denote the restriction of $(T_x^s)_*$ to $\mathcal{L}^-(\hat{x})$. It follows immediately from Proposition 2.3(d) that for some $\kappa > 1$, almost all $\hat{x}$ and $t > 0$,

$$e^{-\kappa^{-1}t} \geq \|\mathcal{A}_-(\hat{x}, t)\| \geq e^{-\kappa t}, \qquad e^{\kappa^{-1}t} \leq \|\mathcal{A}_+(\hat{x}, t)\| \leq e^{\kappa t}. \tag{4.3}$$

and,

$$e^{\kappa t} \geq \|\mathcal{A}_-(\hat{x}, -t)\| \geq e^{\kappa^{-1}t}, \qquad e^{-\kappa t} \leq \|\mathcal{A}_+(\hat{x}, -t)\| \leq e^{-\kappa^{-1}t}. \tag{4.4}$$

Note that by (3.1)

$$\pi_{\mathcal{V}_1}\mathcal{A}(q_1, u, \ell + s, t + \tau) = (T^\tau_{T^t u q_1})_* \pi_{\mathcal{V}_1}\mathcal{A}(q_1, u, \ell, t)(T^s_{T^{-(\ell+s)}q_1})_*$$

Let $t = \tilde{\tau}_{(\epsilon)}(q_1, u, \ell)$, so that $A(q_1, u, \ell, t) = \epsilon$. Therefore, by (4.3) and (4.4),

$$\begin{aligned} A(\hat{q}_1, u, \ell + s, t + \tau) &\leq \|\mathcal{A}_+(\hat{T}^t u\hat{q}_1, \tau)\| A(\hat{q}_1, u, \ell, t)\|\mathcal{A}_-(\hat{T}^{-(\ell+s)}\hat{q}_1, s)\| \leq \\ &\epsilon\|\mathcal{A}_+(\hat{T}^t u\hat{q}_1, \tau)\|\|\mathcal{A}_-(\hat{T}^{-(\ell+s)}\hat{q}_1, s)\| \leq \epsilon e^{\kappa\tau - \kappa^{-1}s}, \end{aligned}$$

where we have used the fact that $A(\hat{q}_1, u, \ell, t) = \epsilon$. If $t + \tau = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell + s)$ then $A(\hat{q}_1, u, \ell + s, t + \tau) = \epsilon$. It follows that $\kappa\tau - \kappa^{-1}s > 0$, i.e. $\tau > \kappa^{-2}s$. Hence, the lower bound in (4.2) holds.

The proof of the upper bound is similar. Note that we have

$$\mathcal{A}(q_1, u, \ell, t) = (T^{-\tau}_{T^{t+\tau}uq_1})_* \pi_{\mathcal{V}_1}\mathcal{A}(q_1, u, \ell + s, t + \tau)(T^{-s}_{T^{-\ell}q_1})_*.$$

Let $t + \tau = \tilde{\tau}_{(\epsilon)}(q_1, u, \ell + s)$. Then, by (4.3) and (4.4),

$$\begin{aligned} A(\hat{q}_1, u, \ell, t) &\leq \|\mathcal{A}_+(\hat{T}^{t+\tau}u\hat{q}_1, -\tau)\| A(\hat{q}_1, u, \ell + s, t + \tau)\|\mathcal{A}_-(\hat{T}^{-\ell}\hat{q}_1, -s)\| \leq \\ &\epsilon\|\mathcal{A}_+(\hat{T}^{t+\tau}u\hat{q}_1, -\tau)\|\|\mathcal{A}_-(\hat{T}^{-\ell}\hat{q}_1, -s)\| \leq \epsilon e^{-\kappa^{-1}\tau + \kappa s}, \end{aligned}$$

where we have used the fact that $A(\hat{q}_1, u, \ell + s, t + \tau) = \epsilon$. Since $A(\hat{q}_1, u, \ell, t) = \epsilon$, it follows that $-\kappa^{-1}\tau + \kappa s > 0$, i.e. $\tau < \kappa^2 s$. It follows that the upper bound in (4.2) holds. □

5. Conditional measures.

We note the following:

Lemma 5.1. *For any $\rho > 0$ there is a constant $c(\rho)$ with the following property: Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between Euclidean spaces. Then there exists a proper subspace $\mathcal{M} \subset \mathcal{V}$ such that for any $v \in \mathcal{V}$ with $\|v\| = 1$ and $d(v, \mathcal{M}) > \rho$, we have*

$$\|A\| \geq \|Av\| \geq c(\rho)\|A\|.$$

Proof of Lemma 5.1. The matrix $A^t A$ is symmetric, so it has a complete orthogonal set of eigenspaces $W_1, \dots, W_m$ corresponding to eigenvalues $\mu_1 > \mu_2 > \dots \mu_m$. Let $\mathcal{M} = W_1^\perp$. $\square$

5.1. Conditional Measure Lemmas.

Let $\mathcal{B}$ be an arbitrary finite measure space.

Proposition 5.2. *For every $\delta > 0$ there exist constants $c_1(\delta) > 0$, $\epsilon_1(\delta) > 0$ with $c_1(\delta) \to 0$ and $\epsilon_1(\delta) \to 0$ as $\delta \to 0$, and also constants $\rho(\delta) > 0$ and $\rho'(\delta) > 0$, such that the following holds:*

For any Γ-invariant subset $K' \subset \hat{\Omega}$ with $\hat{\nu}(K'/\Gamma) > 1-\delta$, there exists a Γ-invariant subset $K \subset K'$ with $\hat{\nu}(K/\Gamma) > 1 - c_1(\delta)$ such that the following holds: suppose for each $\hat{q} \in \hat{\Omega}$ we have a measurable map from $\mathcal{B}$ to proper subspaces of $\mathcal{L}^-(\hat{q})$, written as $u \to \mathcal{M}_u(\hat{q})$. Then, for any $\hat{q} = (q, g) \in K$ there exists $\hat{q}' = (q', \exp(\mathbf{w})g) \in K'$ with $q' \in W_1^-[q]$, $\mathbf{w} \in \mathcal{L}^-(\hat{q})$,

$$\rho'(\delta) \leq \|\mathbf{w}\|_0 \leq 1/100 \tag{5.1}$$

and

$$d_0(\mathbf{w}, \mathcal{M}_u(q)) > \rho(\delta) \qquad \text{for at least } (1 - \epsilon_1(\delta))\text{-fraction of } u \in \mathcal{B}. \tag{5.2}$$

In the rest of this subsection we will prove Proposition 5.2.

Notation. For $\hat{x} = (x, g) \in \hat{\Omega}$, let $\hat{\nu}|_{\hat{W}_1^-[\hat{x}]}$ denote conditional measure of $\hat{\nu}$ on $\hat{W}_1^-[\hat{x}]$. Let $\tilde{\nu}_{\hat{x}}$ denote the projection of $\hat{\nu}|_{\hat{W}_1^-[\hat{x}]}$ to the G factor. By abuse of notation, we think of $\tilde{\nu}_{\hat{x}}$ as a measure on $\mathfrak{g}$. Then, by the definition of $\mathcal{L}^-(\hat{x})$, $\tilde{\nu}_{\hat{x}}$ is supported on $\mathcal{L}^-(\hat{x})$. Recall that by Lemma 4.1, $\mathcal{L}^-(\hat{x})$ is a subalgebra of $\mathfrak{g}$.

Lemma 5.3. *(cf. [EiL1, Corollary 6.4]) For $\hat{\nu}$-almost all $\hat{x} = (x, g) \in \hat{\Omega}$, for any $\epsilon > 0$ (which is allowed to depend on $\hat{x}$), the restriction of the measure $\tilde{\nu}_{\hat{x}}$ to the ball $B(0, \epsilon) \subset \mathcal{L}^-(\hat{x})$ is not supported on a finite union of proper affine subspaces of $\mathcal{L}^-(\hat{x})$.*

Outline of proof. Suppose not. Let $N(\hat{x})$ be the minimal integer N such that for some $\epsilon = \epsilon(\hat{x}) > 0$, the restriction of $\tilde{\nu}_{\hat{x}}$ to $B(0, \epsilon)$ is supported on N affine subspaces. Since $\mathcal{L}^-(\hat{x}) \subset \mathrm{Lie}(N^-)(x)$, the induced action on on $\mathcal{L}^-$ of T^{-t} for $t \geq 0$ is expanding.

Then $N(\hat{x})$ is invariant under T^{-t}, $t \geq 0$. This implies that $N(\hat{x})$ is constant for $\hat{\nu}$-almost all $\hat{x}$, and also that the only affine subspaces of $\mathcal{L}^-(\hat{x})$ which contribute to $N(\cdot)$ pass through the origin. Then, $N(\hat{x}) > 1$ almost everywhere is impossible. Indeed, suppose $N(\hat{x}) = k$ a.e., then for $\hat{x} = (x, g)$ pick $\hat{y} = (y, \exp(\mathbf{w})g) \in \hat{W}_1^-[\hat{x}]$ near $\hat{x}$ such that $\mathbf{w}$ is in one of the affine subspaces through 0; then there must be exactly k affine subspaces of non-zero measure passing though $\mathbf{w}$, but then at most one of them passes through 0. Thus, the measure restricted to a neighborhood of 0 gives positive weight to at least $k+1$ subspaces, contradicting our assumption. Thus, we must have $N(\hat{x}) = 1$ almost everywhere; but then (after flowing by $\hat{T}^{-t}$ for sufficiently large $t > 0$) we see that for almost all $\hat{x}$, $\tilde{\nu}_{\hat{x}}$ is supported on a proper subspace of $\mathcal{L}^-(\hat{x})$, which contradicts the definition of $\mathcal{L}^-(\hat{x})$. $\square$

The partitions $\hat{\mathfrak{B}}^-$ and $\mathfrak{B}^-$. We may choose a Γ-invariant partition of $\hat{\mathfrak{B}}^-$ of $\hat{\Omega}$ subordinate to $\hat{W}_1^-$, so that for each $\hat{x} = (x^+, x^-, g)$ the atom $\hat{\mathfrak{B}}^-[\hat{x}]$ containing $\hat{x}$ is of the form $W_1^-[x^+] \times \mathfrak{B}^-[x^+, g]$, where $\mathfrak{B}^-[x^+, g] \subset N^-(x)g$. Following our conventions, we will write $\mathfrak{B}^-[x^+, g]$ as $\mathfrak{B}^-[\hat{x}]$. We may also assume that the diameter of each $\mathfrak{B}^-[\hat{x}]$ is at most $1/100$.

The measure $\nu'_{\hat{x}}$. For $x \in \hat{\Omega}$, let $\nu'_{\hat{x}} = \tilde{\nu}_{\hat{x}}|_{\mathfrak{B}^-[\hat{x}]})$, i.e. $\nu'_{\hat{x}}$ is the restriction of $\tilde{\nu}_{\hat{x}}$ (which is a measure on $N^-(x)g)$ to the subset $\mathfrak{B}^-[\hat{x}]$. Then, for $\hat{y} \in \hat{\mathfrak{B}}^-[\hat{x}]$, $\nu'_{\hat{y}} = \nu'_{\hat{x}}$.

Lemma 5.4. *For every $\eta > 0$ and every $N > 0$ there exists $\beta_1 = \beta_1(\eta, N) > 0$, $\rho_1 = \rho_1(\eta, N) > 0$ and a Γ-invariant subset $K_{\eta,N}$ with $K_{\eta,N}/\Gamma$ compact and of measure at least $1-\eta$ such that for all $\hat{x} \in K_{\eta,N}$, and any proper subspaces $\mathcal{M}_1(\hat{x}), \dots, \mathcal{M}_N(\hat{x}) \subset \mathcal{L}^-(\hat{x})$,*

$$\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}_k(\hat{x}), \rho_1)) \geq \beta_1 \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{5.3}$$

Outline of Proof. By Lemma 5.3, there exist $\beta_{\hat{x}} = \beta_{\hat{x}}(N) > 0$ and $\rho_{\hat{x}} = \rho_{\hat{x}}(N) > 0$ such that for any subspaces $\mathcal{M}_1(\hat{x}), \dots \mathcal{M}_N(\hat{x}) \subset \mathcal{L}^-(\hat{x})$,

$$\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}(\hat{x}), \rho_{\hat{x}})) \geq \beta_{\hat{x}} \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{5.4}$$

Let $E(\rho_1, \beta_1)$ be the set of $\hat{x}$ such that (5.3) holds. By (5.4),

$$\hat{\nu}\left(\bigcup_{\substack{\rho_1 > 0 \\ \beta_1 > 0}} E(\rho_1, \beta_1) \right) = 1.$$

Therefore, we can choose $\rho_1 > 0$ and $\beta_1 > 0$ such that $\hat{\nu}(E(\rho_1, \beta_1)) > 1 - \eta$. $\square$

Lemma 5.5. *For every $\eta > 0$ and every $\epsilon_1 > 0$ there exists $\beta = \beta(\eta, \epsilon_1) > 0$, a Γ-invariant set $K_\eta = K_\eta(\epsilon_1) \subset \hat{\Omega}$ with K_η/Γ compact and of measure at least $1 - \eta$, and $\rho = \rho(\eta, \epsilon_1) > 0$ such that the following holds: Suppose for each $u \in \mathcal{B}$ let $\mathcal{M}_u(\hat{x})$ be a proper subspace of $\mathcal{L}^-(\hat{x})$. Let*

$$E_{good}(\hat{x}) = \{v \in \mathfrak{B}^-[\hat{x}] \ : \ \text{for at least } (1-\epsilon_1)\text{-fraction of } u \text{ in } \mathcal{B}, \quad d_0(v, \mathcal{M}_u(\hat{x})) > \rho/2\}.$$

Then, for $\hat{x} \in K_\eta$,

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq \beta \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{5.5}$$

Proof. Let $n = \dim \mathcal{L}^-[\hat{x}]$. By considering determinants, it is easy to show that for any $C > 0$ there exists a constant $c_n = c_n(C) > 0$ depending on n and C such that for any $\eta > 0$ and any points $v_1, \dots, v_n$ in a ball of radius C with the property that $\|v_1\| \geq \eta$ and for all $1 < i \leq n$, v_i is not within η of the subspace spanned by $v_1, \dots, v_{i-1}$, then $v_1, \dots, v_n$ are not within $c_n \eta^n$ of any $n-1$ dimensional subspace. Let $k_{max} \in \mathbb{N}$ denote the smallest integer greater then $1 + n/\epsilon_1$, and let $N = N(\epsilon_1) = \binom{k_{max}}{n-1}$. Let β_1, ρ_1 and $K_{\eta,N}$ be as in Lemma 5.4. Let $\beta = \beta(\eta, \epsilon_1) = \beta_1(\eta, N(\epsilon_1))$, $\rho = \rho(\eta, \epsilon_1) = c_n \rho_1(\eta, N(\epsilon_1))^n$, $K_\eta(\epsilon_1) = K_{\eta, N(\epsilon_1)}$. Let $E_{bad}(\hat{x}) = \mathfrak{B}^-[\hat{x}] \smallsetminus E_{good}(\hat{x})$. To simplify notation, we choose coordinates so that $\hat{x} = 0$. We claim that $E_{bad}(\hat{x})$ is contained in the union of the ρ_1-neighborhoods of at most N subspaces. Suppose this is not true. Then, for $1 \leq k \leq k_{max}$ we can inductively pick points $v_1, \dots, v_k \in E_{bad}(\hat{x})$ such that v_j is not within ρ_1 of any of the subspaces spanned by $v_{i_1}, \dots, v_{i_{n-1}}$ where $i_1 \leq \dots \leq i_{n-1} < j$. Then, any n-tuple of points $v_{i_1}, \dots, v_{i_n}$ is not contained within $\rho = c_n \rho_1$ of a single subspace. Now, since $v_i \in E_{bad}(\hat{x})$, there exists $U_i \subset \mathcal{B}$ with $|U_i| \geq \epsilon_1 |\mathcal{B}|$ such that for all $u \in U_i$, $d_0(v_i, \mathcal{M}_u) < \rho/2$. We now claim that for any $1 \leq i_1 < i_2 < \dots < i_n \leq k$,

$$U_{i_1} \cap \dots \cap U_{i_n} = \emptyset. \tag{5.6}$$

Indeed, suppose u belongs to the intersection. Then each of the $v_{i_1}, \dots v_{i_n}$ is within $\rho/2$ of the single subspace $\mathcal{M}_u$, but this contradicts the choice of the v_i. This proves (5.6). Now,

$$\epsilon_1 k_{max} |\mathcal{B}| \leq \sum_{i=1}^{k_{max}} |U_i| \leq n \left| \bigcup_{i=1}^{k_{max}} U_i \right| \leq n |\mathcal{B}|.$$

This is a contradiction, since $k_{max} > 1 + n/\epsilon_1$. This proves the claim. Now (5.3) implies that

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}_k(\hat{x}), \rho_1)) \geq \beta \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]).$$

$\square$

Proof of Proposition 5.2. Let $\hat{\nu}|_{\hat{W}^-[\hat{x}]}$ denote the conditional measure of $\hat{\nu}$ on the stable leaf $\hat{W}^-[\hat{x}]$. Let

$$K'' = \{\hat{x} \in \hat{\Omega} \ : \ \hat{\nu}|_{\hat{W}^-[\hat{x}]}(K' \cap \hat{\mathfrak{B}}^-[\hat{x}]) \geq (1-\delta^{1/2})\hat{\nu}|_{\hat{W}^-[\hat{x}]}(\hat{\mathfrak{B}}^-[\hat{x}])\}.$$

Since $\hat{\mathfrak{B}}^-$ is a partition, we have $\hat{\nu}(K'') \geq 1-\delta^{1/2}$.

Let π_G denote the projection $\hat{\Omega} \to G$. We have, for $\hat{x} \in K''$,

$$\nu'_{\hat{x}}(\pi_G(K') \cap \mathfrak{B}^-[\hat{x}]) \geq (1-\delta^{1/2})\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{5.7}$$

Let $\beta(\eta, \epsilon_1)$ be as in Lemma 5.5. Let

$$c(\delta) = \delta + \inf\{(\eta^2+\epsilon_1^2)^{1/2} \ : \ \beta(\eta,\epsilon_1) \geq 8\delta^{1/2}\}.$$

We have $c(\delta) \to 0$ as $\delta \to 0$. By the definition of $c(\delta)$ we can choose $\eta = \eta(\delta) < c(\delta)$ and $\epsilon_1 = \epsilon_1(\delta) < c(\delta)$ so that $\beta(\eta,\epsilon_1) \geq 8\delta^{1/2}$.

By (5.5), for $\hat{x} \in K_\eta$,

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq 8\delta^{1/2}\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{5.8}$$

Let $K = K' \cap K'' \cap K_\eta$. We have $\hat{\nu}(K/\Gamma) \geq 1 - \delta - \delta^{1/2} - c(\delta)$, so $\hat{\nu}(K/\Gamma) \to 1$ as $\delta \to 0$. Also, if $\hat{q} \in K$, by (5.7) and (5.8),

$$\pi_G(K') \cap \mathfrak{B}^-[\hat{q}] \cap E_{good}(\hat{q}) \neq \emptyset.$$

Thus, we can choose $\hat{q}' \in K' \cap \hat{\mathfrak{B}}^-[\hat{q}]$ such that $\pi_G(\hat{q}') \in E_{good}(\hat{q})$. Then (5.2) holds with $\rho = \rho(\eta(\delta), \epsilon_1(\delta)) > 0$. Also the upper bound in (5.1) holds since $\mathfrak{B}^-[\hat{q}]$ has diameter at most $1/100$. Since all $\mathcal{M}_u(\hat{q})$ contain the origin, the lower bound in (5.1) follows from (5.2). □

5.2. **Conditional measures on $\hat{W}^+$.** Note that for a.e. $x \in \Omega$, $\mathcal{V}_1(x)$ is the Lie algebra of a subgroup $N_1(x)$.

Lemma 5.6. *Suppose $x \in \hat{\Omega}$, $t \in \mathbb{R}$ and $u \in \mathcal{U}_1^+$.*

(a) $N_1(T^t x) = \mathrm{Ad}(T_x^t) N_1(x)$.
(b) *For a.e. $u \in \mathcal{U}_1^+$, $N_1(ux) = N_1(x)$.*

Proof. Part (a) follows from the equivariance of $\mathcal{V}_1(x)$. Part (b) follows from Lemma 3.1. □

The measures $f_1(\hat{x})$. Write $\hat{x} = (x, g)$. Recall that $N_1(x)$ is a unipotent subgroup of G. We now apply the leafwise measure construction described in [EiL2] to get leafwise measures $f_1(\hat{x})$ of $\hat{\nu}$ on $N_1(x)$. (Roughly speaking, $f_1(\hat{x})$ is the pullback to $N_1(x)$ of the "conditional measure of $\hat{\nu}$ along $N_1(x)g$"). The measure $f_1(\hat{x})$ is only defined up to normalization. We view $f_1(\hat{x})$ as a measure on G which happens to be supported on the subgroup $N_1(x)$.

Lemma 5.7. *We have for a.e. $\hat{x} = (x, g) \in \hat{\Omega}$, $u \in \mathcal{U}_1^+$ and s and t in $\mathbb{R}$,*

$$f_1(\hat{T}^t u \hat{T}^{-s} \hat{x}) \propto (T^t_{ux})_* (T^{-s}_x)_* f_1(\hat{x}).$$

Proof. See [EiL2, Lemma 4.2(iv)]. □

6. The Eight Points

Let $\pi_\Omega : \hat{\Omega} \to \Omega$ denote the forgetful map. If $f(\cdot)$ is a function on Ω, and $\hat{x} \in \hat{\Omega}$, we will often write $f(\hat{x})$ instead of $f(\pi_\Omega(\hat{x}))$. Let $\pi_G : \hat{\Omega} \to G$ be the projection to the second factor.

We will derive Theorem 1.3 from the following:

Proposition 6.1. *Suppose μ satisfies the assumptions of Theorem 1.1, and $\hat{\nu}$ is a $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant measure on $\hat{\Omega}/\Gamma$. Suppose also that Case I holds (see §1). Then for almost all $x \in \hat{\Omega}/\Gamma$ there exists a nontrivial unipotent subgroup $U^+_{new}(\hat{x}) \subset N_1(\hat{x})$ such that the following hold:*

- (a) *For almost all $\hat{x} = (x, g) \in \hat{\Omega}$ and all $t \in \mathbb{R}$, $U^+_{new}(\hat{T}^t \hat{x}) = \operatorname{Ad}(T^t_x) U^+_{new}(\hat{x})$ and for almost all $u \in \mathcal{U}_1^+$, $U^+_{new}(u\hat{x}) = U^+_{new}(\hat{x})$.*
- (b) *For almost all $\hat{x} = (x, g) \in \hat{\Omega}$, the leafwise measure of $\hat{\nu}$ along $N_1[\hat{x}] = \{x\} \times N_1(x) g$ (which is by definition a measure on $N_1(x)$) is right invariant under $U^+_{new}(\hat{x}) \subset N_1(x)$.*

Most of the rest of §6 will consist of the proof of Proposition 6.1. The argument has been outlined in §1.2, and we have kept the same notation (in particular, see Figure 1).

Proposition 6.1 will be derived from the following:

Proposition 6.2. *Suppose μ and $\hat{\nu}$ are as in Proposition 6.1. Then there exists $0 < \delta_0 < 0.1$, a Γ-invariant subset $K_* \subset \hat{\Omega}$ with K_*/Γ compact and $\hat{\nu}(K_*/\Gamma) > 1 - \delta_0$ such that f_1 is uniformly continuous on K_*, and $C > 1$ (depending on K_*) such that for every $\epsilon > 0$ there exists a Γ-invariant $E \subset K_*$ with $\hat{\nu}(E/\Gamma) > \delta_0$, such that for every $\hat{x} \in E$ there exists $\hat{y} \in N_1[\hat{x}] \cap K_*$ with*

$$C^{-1}\epsilon \le d_G(\pi_G(\hat{x}), \pi_G(\hat{y})) \le C\epsilon \tag{6.1}$$

and

$$f_1(\hat{y}) \propto f_1(\hat{x}). \tag{6.2}$$

6.1. **Outline of the proof of Proposition 6.2.** We use the same notation as in §1.2. A simplified scheme for choosing the eight points is as follows:

- (i) Choose $\hat{q}_1$ in some good set, so that in particular, for most t, $\hat{T}^t \hat{q}_1 \in K_*$ and $\hat{T}^{-t} \hat{q}_1 \in K_*$ and for most u and most t, $\hat{T}^t u \hat{q}_1 \in K_*$.

(ii) Let $\mathcal{A}(\hat{q}_1, u, \ell, t)$ be as in §3, so that if we write $\pi_G(\hat{q}') = \exp(\mathbf{w})\pi_G(\hat{q})$ then $\pi_G(\hat{q}_2') = \exp(\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w})\pi_G(\hat{q}_2)$. Let $\hat{q} = \hat{T}^{-\ell}\hat{q}_1$ and let $\hat{q}_2 = \hat{T}^t u\hat{q}_1$, where $t = \tau_{(\epsilon)}(\hat{q}_1, u, \ell)$ is the solution to the equation $\|\mathcal{A}(\hat{q}_1, u, \ell, t)\| = \epsilon$. Since by Proposition 4.2, for fixed $\hat{q}_1$, u, ϵ, $\tau_{(\epsilon)}(\hat{q}_1, u, \ell)$ is bilipshitz in ℓ, for most choices of ℓ, we have $\hat{q} \in K^*$ and $\hat{q}_2 \in K^*$.

(iii) Let $t_1 = t_1(\hat{q}_1, u, \ell)$ be defined by the equation $\lambda_1(u\hat{q}_1, t) = \lambda_1(\hat{q}_1, t_1)$. Since $\lambda_1(x, t)$ is bilipshitz in t, the same argument shows that for most choices of ℓ, $\hat{q}_3 \equiv \hat{T}^{t_1}\hat{q}_1 \in K_*$.

(iv) Let $M_u \subset \mathcal{L}^-(\hat{q})$ be the subspace of Lemma 5.1 for the linear map $\mathcal{A}(\hat{q}_1, u, \ell, t)$ restricted to $\mathcal{L}^-(\hat{q})$. By Proposition 5.2, we can choose $\hat{q}' \in K_*$ with $\pi_G(\hat{q}') = \exp(\mathbf{w})\pi_G(\hat{q})$ with $\|\mathbf{w}\| \approx 1$ and so that $\mathbf{w}$ avoids most of the subspaces M_u as u varies over $\mathcal{U}_1^+$. Then, for most u,

$$d_G(\pi_G(\hat{q}_2), \pi_G(\hat{q}_2')) \approx \|\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}\| \approx \|\mathcal{A}(\hat{q}_1, u, \ell, t)\|\|\mathbf{w}\| \approx \epsilon,$$

as required.

(v) In view of Proposition 3.3, we can choose u so that $\hat{q}_2'$ is close to $N_1[\hat{q}_2]$ as required.

(vi) We now proceed as in §1.2. Let $\hat{q}_1' = \hat{T}^\ell\hat{q}'$, $\hat{q}_2' = \hat{T}^t u\hat{q}_1'$ where $t = \tau_{(\epsilon)}(\hat{q}_1, u, \ell)$, and let $\hat{q}_3' = \hat{T}^{t_1}\hat{q}_1'$. Since $\hat{\nu}$ is $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant and since $\lambda_1(\hat{q}_1, t_1) = \lambda_1(u\hat{q}_1, t)$, we have,

$$f_1(\hat{q}_2) = f_1(\hat{q}_3).$$

Also, since one can show $\lambda_1(uq_1', t) \approx \lambda_1(q_1', t_1)$ we have,

$$f_1(\hat{q}_2') \approx f_1(\hat{q}_3').$$

Since $\hat{q}_3$ and $\hat{q}_3'$ are very close, we can ensure that, $f_1(\hat{q}_3') \approx f_1(\hat{q}_3)$. Then, we get, up to normalization,

$$f_1(\hat{q}_2) \approx f_1(\hat{q}_2').$$

Applying the argument with a sequence of ℓ's going to infinity, and passing to a limit along a subsequence, we obtain points $\hat{x}$, $\hat{y}$ satisfying (6.1) and (6.2).

(In the above outline we also conflated $\mathcal{A}(\cdot, \cdot, \cdot)$ with $\pi_{\mathcal{V}_1} \circ \mathcal{A}(\cdot, \cdot, \cdot, \cdot)$ but this is a very minor issue).

In fact our proof uses the same ideas, but we need to take a bit more care, mostly because we also need to make sure that $\hat{q}_2'$ and $\hat{q}_3'$ belong to K_*. We now give a brief outline of the strategy.

We define a Y*-configuration* $Y = Y(\hat{q}_1, u, \ell)$ depending on the parameters $\hat{q}_1 \in \hat{\Omega}$, $u \in \mathcal{U}_1^+$, $\ell > 0$ to be a quadruple of points $\hat{q}$, $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$ such that $\hat{q}$, $\hat{q}_2$, $\hat{q}_3$ are chosen as in (ii) and (iii) (depending on $\hat{q}_1$, u, ℓ). Given a Y-configration Y, we refer to its points as $q(Y)$, $q_1(Y)$, etc. A Y-configuration Y is *good* if $\hat{q}(Y)$ $\hat{q}_1(Y)$, $\hat{q}_2(Y)$, and $\hat{q}_3(Y)$ all belong to some good set K_*. The argument of (i),(ii), (iii) and Fubini's theorem show that for an almost full density set of ℓ, there are very many good Y-configurations with that value of ℓ. See Claim 6.4 below for the exact statement.

We say that two Y-configurations $Y = Y(\hat{q}_1, u, \ell)$ and $Y' = Y(\hat{q}'_1, u', \ell')$ are *coupled* if $\ell = \ell'$, $u = u'$, $\hat{q}(Y') \in \hat{W}_1^-[\hat{q}(Y)]$, and also if we write $\hat{q}(Y) = (q, g)$, $\hat{q}(Y') = (q', \exp(\mathbf{w})g)$ then $\|\mathbf{w}\| \approx 1$ and also $\mathbf{w}$ avoids the subspace M_u of (iv). Then the argument of (iv) shows that we can (for most values of ℓ) choose points $\hat{q}_1$, $\hat{q}'_1$ such that for most u, the Y-configurations $Y(\hat{q}_1, u, \ell)$ and $Y(\hat{q}'_1, u, \ell)$ are both good and also are coupled. (see "Choice of parameters #2" below for the precise statement).

We then choose u as in (v). (See Claim 6.7 and "Choice of parameters #3"). We are now almost done, except for the fact that the lengths of the legs of $Y = Y(\hat{q}_1, u, \ell)$ and $Y' = Y(\hat{q}'_1, u, \ell)$ are not same. (The bottom leg of Y has length ℓ, and so does the bottom leg of Y', but the two top legs of Y can potentially have different lengths than the corresponding legs of Y'). We show that the lengths of the corresponding legs are close (see Claim 6.8 and (6.19)) then make some corrections using (6.4). We then proceed to (vi).

6.2. Choosing the eight points.

We now begin the formal proof of Proposition 6.2.

Choice of parameters #1. We then choose $\delta > 0$ sufficiently small; the exact value of δ will be chosen at the end of this section. All subsequent constants will depend on δ. Let $\epsilon > 0$ be arbitrary and $\eta > 0$ be arbitrary; however, we will always assume that ϵ and η are sufficiently small depending on δ.

We will show that Proposition 6.2 holds with $\delta_0 = \delta/10$. Let $K_* \subset \hat{\Omega}$ be any Γ-invariant subset with K_*/Γ compact and $\hat{\nu}(K_*/\Gamma) > 1 - \delta_0$ on which the function f_1 are uniformly continuous. It is enough to show that there exists $C = C(\delta)$ such that for any $\epsilon > 0$ and for an arbitrary Γ-invariant set $K_{00} \subset \hat{\Omega}$ with K_{00}/Γ compact and $\hat{\nu}(K_{00}/\Gamma) \geq (1 - 2\delta_0)$, there exists $\hat{x} \in K_{00} \cap K_*$ and $\hat{y} \in N_1[\hat{x}] \cap K_*$ satisfying (6.1) and (6.2). Thus, let $K_{00} \subset \hat{\Omega}$ be an arbitrary Γ-invariant set with K_{00}/Γ compact and $\hat{\nu}(K_{00}/\Gamma) > 1 - 2\delta_0$.

Let $\epsilon' > 0$ be a constant which will be chosen later depending only on the Lyapunov exponents. Then, by the multiplicative ergodic theorem, for any $\delta > 0$ there exists a Γ-invariant set $K'_0 \subset \hat{\Omega}$ with K'_0/Γ compact and $\hat{\nu}(K'_0/\Gamma) > 1 - \delta$ and $T'_0 = T'_0(\delta) > 0$ such that for $t > T'_0$, $\hat{x} \in K'_0$ and $\mathbf{v} \in \mathcal{V}_1(\hat{x})$,

$$e^{-(\lambda_1+\epsilon')t}\|\mathbf{v}\| \leq \|(\hat{T}_{\hat{x}}^{-t})_*\mathbf{v}\| \leq e^{-(\lambda_1-\epsilon')t}\|\mathbf{v}\|. \tag{6.3}$$

Let $K_0 = K_{00} \cap K_* \cap K'_0$.

Let $\kappa > 1$ be as in Proposition 4.2, and so that Proposition 2.3(c) holds. Without loss of generality, assume $\delta < 0.01$. We now choose a Γ-invariant subset $K \subset \hat{\Omega}$ with $\hat{\nu}(K/\Gamma) > 1 - \delta$ such that the following hold:

- There exists a number $T_0(\delta)$ such that for any $\hat{x} \in K$ and any $T > T_0(\delta)$,

$$\{t \in [-T/2, T/2] \ : \ \hat{T}^t\hat{x} \in K_0\} \geq 0.9T. \tag{6.4}$$

  (This can be done by the Birkhoff ergodic theorem).
- Lemma 2.7 holds for $K(\delta) = \pi_\Omega(K)$ and $C_1 = C_1(\delta)$.

For $u \in \mathcal{U}_1^+$ and $\hat{q}_1 \in \hat{\Omega}$ and $t > 0$, let $t_1 = t_1(\hat{q}_1, u, t)$ be the unique solution to

$$\lambda_1(\hat{q}_1, t_1) = \lambda_1(u\hat{q}_1, t)$$

Then, in view of Proposition 2.3(c), for fixed $\hat{q}_1, u$, $t_1(q_1, u, t)$ is κ-bilipshitz in t. Let $\tilde{\tau}_{(\epsilon)}(q_1, u, \ell)$ be as in §4. Let

$$E_2(\hat{q}_1, u) = E_2(\hat{q}_1, u, K_{00}, \delta, \epsilon, \eta) = \{\ell \ : \ \hat{T}^{\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)} u\hat{q}_1 \in K\},$$

$$E_3(\hat{q}_1, u) = E_3(\hat{q}_1, u, K_{00}, \delta, \epsilon, \eta) = \{\ell \in E_2(\hat{q}_1, u) \ : \ T^{t_1(q_1, u, \tilde{\tau}_{(\epsilon)}(q_1, u, \ell))} q_1 \in K\}.$$

Note that if we make choices as in §6.1 (ii) and (iii), then if $\ell \in E_3(\hat{q}_1, u)$ then $\hat{q}_2 \in K$ and $\hat{q}_3 \in K$.

Claim 6.3. *There exists $\ell_3 = \ell_3(K_{00}, \delta, \epsilon, \eta) > 0$, a Γ-invariant set $K_3 = K_3(K_{00}, \delta, \epsilon)$ with $K_3 \subset K$ and K_3/Γ compact and of measure at least $1 - c_3(\delta)$ and for each $\hat{q}_1 \in K_3$ a subset $Q_3 = Q_3(\hat{q}_1\Gamma, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{U}_1^+$ with $|Q_3\hat{q}_1| \geq (1 - c_3'(\delta))|\mathcal{U}_1^+\hat{q}_1|$ such that for all $\hat{q}_1 \in K_3$ and $u \in Q_3$, $u\hat{q}_1 \in K$, and for $\ell > \ell_3$, $|E_3(\hat{q}_1, u) \cap [0, \ell]| > (1 - c_3''(\delta))\ell$. Also we have $c_3(\delta)$, $c_3'(\delta)$ and $c_3''(\delta) \to 0$ as $\delta \to 0$.*

Proof of claim. By the ergodic theorem, for any $\delta > 0$ there exists a Γ-invariant set $K_2(\delta) \subset \hat{\Omega}$ with K_2/Γ compact and $\hat{\nu}(K_2/\Gamma) > 1 - \delta$ and $\ell_2 > 0$ such that for any $\hat{q}_1 \in K_2$, and $L > \ell_2$ the measure of $\{t \in [0, L] \ : \ \hat{T}^t\hat{q}_1 \in K\}$ is at least $(1 - \delta)L$. We choose

$$K_3 = K_2 \cap \{\hat{x} \in \hat{\Omega} \ : \ |\mathcal{U}_1^+\hat{x} \cap K_2| > (1 - \delta)|\mathcal{U}_1^+\hat{x}|\}.$$

Suppose $\hat{q}_1 \in K_3$, and $u\hat{q}_1 \in K_2$.

Let

$$E_{bad} = \{t \ : \ \hat{T}^t u\hat{q}_1 \in K^c\}.$$

Then, since $u\hat{q}_1 \in K_2$, for $\ell > \ell_2$, the density of E_{bad} is at most δ. We have

$$E_2(\hat{q}_1, u)^c = \{\ell \ : \ \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) \in E_{bad}\}.$$

Then, by Proposition 4.2, for $\ell > \kappa\ell_2$, the density of $E_2(\hat{q}_1, u)$ is at least $1 - 4\kappa^2\delta$. Similarly, since the function $\ell \to t_1(\hat{q}_1, u, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$ is κ^2-bilipshitz (since it is the compostion of two κ-bilipshitz functions), for $\ell > \kappa^2\ell_2$, the density of $E_3(\hat{q}_1, u)$ is at least $1 - 8\kappa^4\delta$.

The last assertion follows from Lemma 2.7. □

The following claim states that good Y-configurations are plentiful for an almost full density set of ℓ.

Claim 6.4. *There exists a set $\mathcal{D}_4 = \mathcal{D}_4(K_{00}, \delta, \epsilon, \eta) \subset \mathbb{R}^+$ and a number $\ell_4 = \ell_4(K_{00}, \delta, \epsilon, \eta) > 0$ so that $\mathcal{D}_4$ has density at least $1 - c_4(\delta)$ for $\ell > \ell_4$, and for $\ell \in \mathcal{D}_4$ a Γ-invariant subset $K_4(\ell) = K_4(\ell, K_{00}, \delta, \epsilon) \subset \hat{\Omega}$ with $K_4 \subset K$ and $\hat{\nu}(K_4(\ell)/\Gamma) > 1 - c_4'(\delta)$, such that for any $\hat{q}_1 \in K_4(\ell)$ there exists a subset $Q_4 = Q_4(\hat{q}_1\Gamma, \ell) \subset Q_3 \subset \mathcal{U}_1^+$*

with $|Q_4\hat{q}_1| \ge (1 - c_4''(\delta))|\mathcal{U}_1^+\hat{q}_1|$ *so that for all* $\ell \in \mathcal{D}_4$*, for all* $\hat{q}_1 \in K_4(\ell)$ *and all* $u \in Q_4$*,*

$$\ell \in E_3(\hat{q}_1, u) \tag{6.5}$$

(We have $c_4(\delta)$*,* $c_4'(\delta)$ *and* $c_4''(\delta) \to 0$ *as* $\delta \to 0$*).*

Proof of Claim. This follows from Claim 6.3 by applying Fubini's theorem to $\hat{\Omega}_{\mathcal{B}} \times [0, L]$, where $\hat{\Omega}_{\mathcal{B}} = \{(\hat{x}, u\hat{x}) \;:\; \hat{x} \in \Omega,\; ux \in \mathcal{U}_1^+ x\}$ and $L \in \mathbb{R}$. □

Choice of parameters #2: Choice of $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$ (depending on δ, ϵ, $\hat{q}_1$, ℓ). Suppose $\ell \in \mathcal{D}_4$. Let $\mathcal{A}(\hat{q}_1, u, \ell, t)$ be as in (3.1). (Note that following our conventions, we use the notation $\mathcal{A}(\hat{q}_1, u, \ell, t)$ for $\hat{q}_1 \in \hat{\Omega}$, even though $\mathcal{A}(\hat{q}_1, u, \ell, t)$ was originally defined for $\hat{q}_1 \in \Omega$) and for $u \in Q_4(\hat{q}_1\Gamma, \ell)$ let $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the restriction of the linear map $\pi_{\mathcal{V}_1}\mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$ to $\mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$.

We now apply Proposition 5.2 with $K' = \hat{T}^{-\ell}K_4(\ell)$. We denote the resulting set K by $K_5(\ell) = K_5(\ell, K_{00}, \delta, \epsilon)$. We have $\nu(K_5(\ell)) \ge 1 - c_5(\delta)$, where $c_5(\delta) \to 0$ as $\delta \to 0$. Let $K_6(\ell) = \hat{T}^{\ell}K_5(\ell)$.

Suppose $\ell \in \mathcal{D}_4$ and $\hat{q}_1 \in K_6(\ell)$. Let $\hat{q} = \hat{T}^{-\ell}\hat{q}_1$. Then, $\hat{q} \in K_5(\ell)$. Write $\hat{q} = (q, g)$ where $q = \pi_\Omega(\hat{q}) \in \Omega$. By Proposition 5.2 and the definition of $K_5(\ell)$, we can choose

$$\hat{q}' = (q', \exp(\mathbf{w})g) \in \hat{T}^{-\ell}K_4(\ell) \tag{6.6}$$

so that $q' \in W_1^-[q]$, and $\mathbf{w} \in \mathcal{L}^-(\hat{q})$ with $\rho'(\delta) \le \|\mathbf{w}\| \le 1/100$ and so that (5.2) holds with $\epsilon_1(\delta) \to 0$ as $\delta \to 0$. Let $\hat{q}_1' = \hat{T}^{\ell}\hat{q}'$. Then $\hat{q}_1' \in K_4(\ell)$.

Standing Assumption. We assume $\ell \in \mathcal{D}_4$, $\hat{q}_1 \in K_6(\ell)$ and $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$ are as in Choice of parameters #2. (This means that in the language of §6.1, for most u, the Y configurations $Y(\hat{q}_1, u, \ell)$ and $Y(\hat{q}_1', u, \ell)$ are both good and are coupled).

Notation. For $u \in \mathcal{U}_1^+$, let

$$\tau(u) = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell), \qquad \tau'(u) = \tilde{\tau}_{(\epsilon)}(\hat{q}_1', u, \ell),$$

Claim 6.5. *For* $u \in Q_4(\hat{q}_1\Gamma, \ell) \cap Q_4(\hat{q}_1'\Gamma, \ell)$*,*

$$\hat{T}^{\tau(u)}u\hat{q}_1 \in K, \quad \text{and} \quad \hat{T}^{\tau'(u)}u\hat{q}_1' \in K. \tag{6.7}$$

Proof of Claim. Suppose $u \in Q_4(\hat{q}_1\Gamma, \ell)$. Since $\hat{q}_1 \in K_4$ and $\ell \in \mathcal{D}_4$, it follows from (6.5) that $\ell \in E_2(\hat{q}_1, u)$, and then from the definition of $E_2(\hat{q}_1, u)$ is follows that $\hat{T}^{\tau(u)}u\hat{q}_1 \in K$. Similarly, since $\hat{q}_1' \in K_4$, we have for $u \in Q_4(\hat{q}_1'\Gamma, \ell)$ we have $\hat{T}^{\tau'(u)}u\hat{q}_1' \subset K$. This completes the proof of (6.7). □

The numbers t_1 and t_1'. Suppose $u \in Q_4(\hat{q}_1\Gamma, \ell)$. Let t_1 be defined by the equation

$$\lambda_1(\hat{q}_1, t_1) = \lambda_1(u\hat{q}_1, \tau(u)). \tag{6.8}$$

Then, since $\ell \in \mathcal{D}_4$ and in view of (6.5), we have $\ell \in E_3(\hat{q}_1, u)$. In view of the definition of E_3, it follows that

$$\hat{T}^{t_1}\hat{q}_1 \in K. \tag{6.9}$$

Similarly, suppose $u \in Q_4(\hat{q}_1'\Gamma, \ell)$. Let t_1' be defined by the equation

$$\lambda_1(\hat{q}_1', t_1') = \lambda_1(u\hat{q}_1', \tau'(u)). \tag{6.10}$$

By the same argument,

$$\hat{T}^{t_1'}\hat{q}_1' \in K. \tag{6.11}$$

The map $\mathbf{v}(u)$. For $u \in \mathcal{U}_1^+$, let

$$\mathbf{v}(u) = \mathbf{v}(\hat{q}, \hat{q}', u, \ell, t) = \mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w} \tag{6.12}$$

where $t = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)$, $\mathbf{w}$ is as in (6.6) and $\mathcal{A}(\cdot,\cdot,\cdot,\cdot)$ is as in (3.1).

Claim 6.6. *There exists a subset $Q_5 = Q_5(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon) \subset Q_4(\hat{q}_1\Gamma, \ell) \subset \mathcal{U}_1^+$ with $|Q_5\hat{q}_1| \geq (1 - c_5''(\delta))|\mathcal{U}_1^+\hat{q}_1|$ (with $c_5''(\delta) \to 0$ as $\delta \to 0$), and a number $\ell_5 = \ell_5(\delta, \epsilon)$ such that if $\ell > \ell_5$, for all $u \in Q_5$,*

$$C'(\delta)^{-1}\epsilon \leq \|\pi_{\mathcal{V}_1}(\mathbf{v}(u))\| \leq C'(\delta)\epsilon. \tag{6.13}$$

Proof of claim. Let $\mathcal{M}_u$ be the subspace of Lemma 5.1 applied to the restriction to $\mathcal{L}^-(\hat{q})$ of linear map $(\pi_{\mathcal{V}_1} \circ \mathcal{A})(\hat{q}_1, u, \ell, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$, where $\mathcal{A}(\,,\,,\,)$ is as in (3.1). Let $Q_5 \subset Q_4(\hat{q}_1\Gamma) \cap Q_4(\hat{q}_1'\Gamma)$ be such that for all $u \in Q_5$,

$$d(\mathbf{w}, \mathcal{M}_u) \geq \beta(\delta)$$

Then, by (5.2),

$$|Q_5\hat{q}_1| \geq |(Q_4(\hat{q}_1\Gamma) \cap Q_4(\hat{q}_1'\Gamma))\hat{q}_1| - \epsilon_1(\delta)|\mathcal{U}_1^+\hat{q}_1| \geq (1 - \epsilon_1(\delta) - c_4''(\delta))|\mathcal{U}_1^+\hat{q}_1|.$$

We now apply Lemma 5.1 to the linear map $(\pi_{\mathcal{V}_1} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)$. Then, for all $u \in Q_5$,

$$c(\delta)\|(\pi_{\mathcal{V}_1} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\| \leq \|(\pi_{\mathcal{V}_1} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\mathbf{w}\| \leq \|(\pi_{\mathcal{V}_1} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\|.$$

Therefore, since $t = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)$, (6.13) holds. $\square$

Standing assumption: We assume $C(\delta)\epsilon < 1/100$ for any constant $C(\delta)$ arising in the course of the proof. In particular, this applies to $C_2(\delta)$ and $C_2'(\delta)$ in the next claim.

Claim 6.7. *There exists constants $c_6(\delta)$ and $c_6'(\delta) > 0$ with $c_6(\delta)$ and $c_6'(\delta) \to 0$ as $\delta \to 0$, a Γ-invariant subset $K_6' = K_6'(\ell, K_{00}, \delta, \epsilon) \subset K_5$ with $\hat{\nu}(K_6'/\Gamma) > 1 - c_6(\delta)$, for each $\hat{q}_1 \in K_6'$ a subset $Q_6 = Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon) \subset \mathcal{U}_1^+$ with $|Q_6\hat{q}_1| \geq (1 -$*

$c_6'(\delta))|\mathcal{U}_1^+\hat{q}_1|$ *and for any* $\eta > 0$ *a number* $\ell_6 = \ell_6(\delta,\eta)$ *such that for* $\ell > \ell_6$, $\hat{q}_1 \in K_6'$, $u \in Q_6$,

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathcal{V}_1(\hat{T}^{\tau(u)}u\hat{q}_1)\right) \leq \eta, \tag{6.14}$$

$$C_1(\delta)\epsilon \leq d(\hat{T}^{\tau(u)}u\hat{q}_1, \hat{T}^{\tau(u)}u\hat{q}_1') \leq C_2(\delta)\epsilon, \tag{6.15}$$

$$C_1'(\delta)\epsilon \leq \|\mathbf{v}(u)\| \leq C_2'(\delta)\epsilon, \tag{6.16}$$

and

$$\alpha_3^{-1}\ell \leq \tau(u) \leq \alpha_3\ell. \tag{6.17}$$

where $\alpha_3 > 1$ *depends on the Lyapunov spectrum.*

Proof. Let Q be as in Proposition 3.3 for $\mathbf{v} = \mathbf{w}$, and let $Q_6 = Q_5 \cap Q$. Then, (6.14) follows immediately from (3.3) and the definition of $\mathbf{v}(u)$. This immediately implies (6.15) and (6.16), in view of (6.13). Now the upper bound in (6.17) follows easily from (3.2). The lower bound in (6.17) follows from Proposition 2.3(d). $\square$

Standing Assumption. We assume $\hat{q}_1 \in K_6'$ and $\ell > \ell_6$.

Claim 6.8. *Suppose* $u \in Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell)$. *Then, there exists* $C_0 = C_0(\delta)$ *such that*

$$|\tau(u) - \tau'(u)| \leq C_0(\delta). \tag{6.18}$$

Proof of claim. Note that $\hat{q} = (q, g)$, $\hat{q}' = (q', g')$ where $q' \in W_1^-[q]$ and $g' \in \exp(\mathcal{L}^-[q'])g$. This implies in particular that $N^-(q') = N^-(q)$, and

$$\mathcal{A}(\hat{q}_1, u, \ell, t) = \mathcal{A}(\hat{q}_1', u, \ell, t).$$

By Lemma 4.1, we have $\mathcal{L}^-(\hat{q}') = \mathcal{L}^-(\hat{q})$. Thus, in view of Lemma 2.7 and (6.14),

$$|\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) - \tilde{\tau}_{(\epsilon)}(\hat{q}_1', u, \ell)| \leq C_0(\delta)$$

i.e. (6.18) holds. $\square$

Choice of parameters #3: Choosing u**,** $\hat{q}_2$**,** $\hat{q}_2'$**,** $\hat{q}_3$**,** $\hat{q}_3'$ **(depending on** $\hat{q}_1$**,** $\hat{q}_1'$**,** u**,** ℓ**).** Choose $u \in Q_6(\hat{q}_1\Gamma, \ell) \cap Q_6(\hat{q}_1'\Gamma, \ell)$ so that (6.15) holds. We have $\hat{T}^{\tau(u)}u\hat{q}_1 \in K$ and $\hat{T}^{\tau'(u)}u\hat{q}_1' \in K$. By (6.18),

$$|\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) - \tilde{\tau}_{(\epsilon)}(\hat{q}_1', u, \ell)| \leq C_0(\delta),$$

therefore,

$$\hat{T}^{\tau(u)}u\hat{q}_1' \in T^{[-C,C]}K,$$

where $C = C(\delta)$.

Note that $\pi_\Omega(u\hat{q}_1') \in W_1^-[\pi_\Omega(u\hat{q}_1)]$ and $\lambda_1(x,t) = \lambda_1(\pi_\Omega(x),t)$. Then, by Lemma 3.5,

$$|\lambda_1(u\hat{q}_1, \tau(u)) - \lambda_1(u\hat{q}_1', \tau(u))| \leq C_4'(\delta).$$

Then, by (6.18) and Proposition 2.3(c),

$$|\lambda_1(u\hat{q}_1, \tau(u)) - \lambda_1(u\hat{q}_1', \tau'(u))| \le C_4''(\delta).$$

Let t_1, t_1' be as in (6.8) and (6.10). Then, the above equation can be rewritten as

$$|\lambda_1(\hat{q}_1, t_1) - \lambda_1(\hat{q}_1', t_1')| \le C_4''(\delta).$$

Then, by Lemma 3.5 (applied to the points $\pi_\Omega(\hat{q}_1)$ and $\pi_\Omega(\hat{q}_1') \in W_1^-[\pi_\Omega(\hat{q}_1)]$) we have

$$|\lambda_1(\hat{q}_1, t_1) - \lambda_1(\hat{q}_1, t_1')| \le C_4'''(\delta).$$

Hence, by Proposition 2.3(c),

$$|t_1 - t_1'| \le C_5(\delta). \tag{6.19}$$

Therefore, by (6.9) and (6.11), we have

$$\hat{T}^{t_1}\hat{q}_1 \in K, \qquad \text{and} \qquad \hat{T}^{t_1}\hat{q}_1' \in \hat{T}^{[-C_5(\delta), C_5(\delta)]}K.$$

Thus, at this point, the situation is as in Figure 1, except that $T^{\tau(u)}u\hat{q}_1'$ (which should be $\hat{q}_2'$) is in $\hat{T}^{[-C(\delta),C(\delta)]}K$ instead of K, and $\hat{T}^{t_1}\hat{q}_1'$ (which should be $\hat{q}_3'$) is in $\hat{T}^{[-C_5(\delta),C_5(\delta)]}K$ instread of K. This is rectified as follows. By the definition of K we can find $C_4(\delta)$ and $s \in [0, C_4(\delta)]$ such that

$$\hat{q}_2 \equiv T^s T^{\tau(u)} u\hat{q}_1 \in K_0, \qquad \hat{q}_2' \equiv T^s T^{\tau(u)} u\hat{q}_1' \in K_0,$$

Similarly, by the definition of K, we can find $s'' \in [0, C_5''(\delta)]$ such that

$$\hat{q}_3 \equiv \hat{T}^{s''+t_1}\hat{q}_1 \in K_0, \qquad \text{and} \qquad \hat{q}_3' \equiv \hat{T}^{s''+t_1}\hat{q}_1' \in K_0.$$

Let $\tau = s + \tau(u)$, $\tau' = s'' + t_1$. Then we have

$$\hat{q}_2 = \hat{T}^\tau u\hat{q}_1, \quad \hat{q}_2' = \hat{T}^\tau u\hat{q}_1', \quad \hat{q}_3 = \hat{T}^{\tau'}\hat{q}_1, \quad \hat{q}_3' = \hat{T}^{\tau'}\hat{q}_1'.$$

Thus, at this point the situation is as in Figure 1, with τ in place of t, τ' in place of t_1, $\hat{q}_2, \hat{q}_2', \hat{q}_3, \hat{q}_3' \in K_0$, and (in particular), $\hat{q}_1$, $\hat{q}_1'$, $u\hat{q}_1$, $u\hat{q}_1' \in K_3$.

6.3. **Completing the proofs.** We continue the proof of Proposition 6.2.

For the next claim, we need a metric on the leafwise measures. By [EiL2, Theorem 6.30], there exists a function $\rho : G \to \mathbb{R}^+$ which is integrable with respect to any leafwise measure. Let $\mathcal{M}_\rho$ denote the space of positive Radon measures ω on G for which $\int_G \rho \, d\omega \le 1$ equipped with the weakest topology for which for any continuous compactly supported ϕ the function $\omega \to \int_G \phi \, d\omega$ is continuous. Then, $\mathcal{M}_\rho$ is compact and metrizable, by some metric d' (see e.g. [Kal, Theorem 4.2]). Then, if ω_1 and ω_2 are leafwise measures, we can define $d(\omega_1, \omega_2) = d'(c_1\omega_1, c_2\omega_2)$, where $c_i^{-1} = \int_G \rho \, d\omega_i$.

Claim 6.9. *There exists $c_{10}(\delta, \ell)$, with $c_{10}(\delta, \ell) \to 0$ as $\ell \to \infty$ such that*

$$d(f_1(\hat{q}_2), f_1(\hat{q}_2')) \le c_{10}(\delta, \ell). \tag{6.20}$$

In (6.20) we consider $f_1(x)$ to be a measure on G which happens to be supported on the subgroup $N_1(x)$.

Proof of claim. Let

$$R = (T^{\tau}_{uq_1})_*(T^{-\tau'}_{q_3})_* = (T^{\tau}_{uq'_1})_*(T^{-\tau'}_{q'_3})_*.$$

We may view R as a map from the Lie algebra at $\hat{q}_3$ to the Lie algebra at $\hat{q}_2$ or as a map from the Lie algebra at $\hat{q}'_3$ to the Lie algebra at $\hat{q}'_2$.

Let $B : \mathcal{V}_1(\hat{q}_3) \to \mathcal{V}_1(\hat{q}_2)$ denote the restriction of R to $\mathcal{V}_1(\hat{q}_3)$, and let $B' : \mathcal{V}_1(\hat{q}'_3) \to \mathcal{V}_1(\hat{q}'_2)$ denote the restriction of R to $\mathcal{V}_1(\hat{q}'_3)$. By the construction of τ and τ', there exists $C = C(\delta)$ such that

$$\max(\|B\|, \|B^{-1}\|) \le C(\delta) \quad \text{and} \quad \max(\|B'\|, \|(B')^{-1}\| \le C(\delta). \tag{6.21}$$

By Lemma 5.7,

$$f_1(\hat{q}_2) \propto B_* f_1(\hat{q}_3), \qquad f_1(\hat{q}'_2) \propto B'_* f_1(\hat{q}'_3). \tag{6.22}$$

Since $\pi_\Omega(\hat{q}'_3) \in W^-[\pi_\Omega(\hat{q}_3)]$ and $\pi_\Omega(\hat{q}'_2) \in W^-[\pi_\Omega(\hat{q}_2)]$, in view of Lemma 2.1(a),

$$\mathcal{V}_1(\hat{q}'_2) = P^-(\hat{q}_2, \hat{q}'_2)\mathcal{V}_1(\hat{q}_2),$$

and

$$\mathcal{V}_1(\hat{q}'_3) = P^-(\hat{q}_3, \hat{q}'_3)\mathcal{V}_1(\hat{q}_3).$$

By Lemma 3.4 and (6.17), there exists $C = C(\delta)$ such that

$$\|P^-(\hat{q}_2, \hat{q}'_2) - I\| \le C(\delta) e^{-\alpha\alpha_3^{-1}\ell} \tag{6.23}$$

and

$$\|P^-(\hat{q}_3, \hat{q}'_3) - I\| \le C(\delta) e^{-\alpha\alpha_3^{-1}\ell}. \tag{6.24}$$

Suppose $\mathbf{v} \in \mathcal{V}_1(\hat{q}'_3)$, and let $\mathbf{w} = (P^-(\hat{q}_3, \hat{q}'_3) - I)\mathbf{v}$. Then, by (6.3),

$$\|(T^{-\tau'}_{\hat{q}'_3})_* \mathbf{w}\|_{\hat{q}'_1} \le e^{(-\lambda_1+\epsilon')\tau'} \|\mathbf{w}\|_{\hat{q}'_3}.$$

Hence, by Lemma 2.7,

$$\|(T^{-\tau'}_{\hat{q}'_3})_* \mathbf{w}\|_{u\hat{q}'_1} \le C e^{(-\lambda_1+\epsilon')\tau'} \|\mathbf{w}\|_{\hat{q}'_3}, \tag{6.25}$$

and then, since by (6.3) and Proposition 2.3(a), the norm of $(T^{\tau}_{u\hat{q}'_1})_*$, is at most $e^{(\lambda_1+\epsilon')\tau}$, we have, for large enough ℓ,

$$\|R\mathbf{w}\|_{q'_2} = \|(T^{\tau}_{u\hat{q}'_1})_*(T^{-\tau'}_{\hat{q}'_3})_* \mathbf{w}\|_{\hat{q}'_2} \le e^{(\lambda_1+\epsilon')\tau} e^{(-\lambda_1+\epsilon')\tau'} \|\mathbf{w}\|_{\hat{q}'_3}. \tag{6.26}$$

Choose $\epsilon' = \alpha\alpha_3^{-1}/100$. Note that for ℓ large enough, in view of (6.17), the definitions of τ and τ' and (6.3), we have $|\tau - \tau'| \le 4\epsilon'\ell$. Therefore by (6.21), (6.23), (6.24) and (6.25), for all $\mathbf{v} \in \mathcal{V}_1(\hat{q}_3)$,

$$\|B' P^-(\hat{q}_3, \hat{q}'_3)\mathbf{v} - P^-(\hat{q}_2, \hat{q}'_2) B\mathbf{v}\| \le C_2(\delta) e^{-(\alpha\alpha_3^{-1}/2)\ell} \|\mathbf{v}\|. \tag{6.27}$$

By Lemma 5.7,

$$f_1(\hat{q}_2) \propto B_* f_1(\hat{q}_3), \qquad f_1(\hat{q}_2') \propto B'_* f_1(\hat{q}_3'). \tag{6.28}$$

Since $\hat{q}_3 \in K_0$ and $\hat{q}_3' \in K_0$,

$$d(f_1(\hat{q}_3), f_1(\hat{q}_3')) \to 0 \text{ as } \ell \to \infty.$$

Then, also by (6.24),

$$d(P^-(\hat{q}_3, \hat{q}_3')_* f_1(\hat{q}_3), f_1(\hat{q}_3')) \to 0 \text{ as } \ell \to \infty.$$

Then, applying B' to both sides and using (6.21) and (6.28), we get

$$d(B' P^-(\hat{q}_3, \hat{q}_3')_* f_1(\hat{q}_3), f_1(\hat{q}_2')) \to 0 \text{ as } \ell \to \infty.$$

Using (6.27), we get

$$d(P^-(\hat{q}_2, \hat{q}_2') B_* f_1(\hat{q}_3), f_1(\hat{q}_2')) \to 0 \text{ as } \ell \to \infty.$$

Then, by (6.28) and (6.23), (6.20) follows. □

Taking the limit as $\eta \to 0$. For fixed δ and ϵ, we now take a sequence of $\eta_k \to 0$ (this forces $\ell_k \to \infty$) and pass to limits (mod Γ) along a subsequence. Let $\tilde{q}_2 \in K_0$ be such that the the limit of the $\hat{q}_2\Gamma$ is $\tilde{q}_2\Gamma$, and and let $\tilde{q}_2' \in K_0$ be such that the limit of the $\hat{q}_2'\Gamma$ is $\tilde{q}_2'\Gamma$. We get (after possibly replacing $\tilde{q}_2'$ by $\tilde{q}_2'\gamma$ for some $\gamma \in \Gamma$),

$$\frac{1}{C(\delta)}\epsilon \le d(\tilde{q}_2, \tilde{q}_2') \le C(\delta)\epsilon,$$

and in view of (6.14),

$$\tilde{q}_2' \in N_1[\tilde{q}_2].$$

Now, by (6.20), we have

$$f_1(\tilde{q}_2) \propto f_1(\tilde{q}_2').$$

We have $\tilde{q}_2 \in K_0 \subset K_{00} \cap K_*$, and $\tilde{q}_2' \in K_0 \subset K_*$. This concludes the proof of Proposition 6.2. □

Proof of Proposition 6.1. Take a sequence $\epsilon_m \to 0$. We now apply Proposition 6.2 with $\epsilon = \epsilon_m$. We get, for each m a Γ-invariant set $E_m \subset K_*$ with $\hat{\nu}(E_m/\Gamma) > \delta_0$ and with the property that for every $\hat{x} \in E_m$ there exists $\hat{y} \in N_1[\hat{x}] \cap K_*$ such that (6.1) and (6.2) hold for $\epsilon = \epsilon_m$. Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m \subset K_*,$$

(so F consists of the points which are in infinitely many E_m). Suppose $\hat{x} \in F$. Then there exists a sequence $\hat{y}_m \to \hat{x}$ such that $\hat{y}_m \in N_1[x]$, $\hat{y}_m \neq \hat{x}$, and so that $f_1(y_m) \propto f_1(\hat{x})$. We may write $\hat{x} = (x, g)$, $\hat{y}_m = (y_m, \gamma_m g)$. Since $\hat{y}_m \in N_1[\hat{x}]$, $y_m = x$ and $\gamma_m \in N_1(x)$. By (6.1), γ_m tends to the identity of G as $m \to \infty$.

By (6.2)

$$f_1(\hat{x}) \propto (r_{\gamma_m})_* f_1(\hat{x}), \tag{6.29}$$

where $(r_g)_*$ denotes the action on measures induced by right multiplication by g. For $\hat{x} \in F$ let $U^+_{new}(\hat{x})$ denote the maximal connected subgroup of $N_1(x)$ such that for $n \in U^+_{new}(\hat{x})$,

$$(r_n)_* f_1(\hat{x}) \propto f_1(\hat{x}). \tag{6.30}$$

The set of $n \in N_1(x)$ satisfying (6.30) is closed, and by (6.29) is not discrete. Therefore, for $\hat{x} \in F$, $U^+_{new}(\hat{x})$ is non-trivial. By construction, the subgroup $U^+_{new}(\hat{x})$ is constant as $\hat{x}$ varies over $N_1[\hat{x}] = \{x\} \times N_1(x)g$, where we wrote $\hat{x} = (x, g)$.

Suppose $\hat{x} \in F$ and $u \in \mathcal{U}_1^+$. Then, since $f_1(u\hat{x}) = f_1(\hat{x})$, we have that (6.30) holds for $n \in U^+_{new}(u\hat{x})$. Therefore, by the maximality of $U^+_{new}(\hat{x})$, for $\hat{x} \in F$, $u \in \mathcal{U}_1^+$ such that $u\hat{x} \in F$,

$$U^+_{new}(u\hat{x}) = U^+_{new}(\hat{x}). \tag{6.31}$$

Suppose $\hat{x} \in F$, $t < 0$ and $\hat{T}^t\hat{x} \in F$. Then, since the $N_1[\hat{x}]$ are $\hat{T}^t$-equivariant (see Lemma 5.6) we have that (6.30) holds for $n \in \hat{T}^{-t} U^+_{new}(\hat{T}^t\hat{x})$. Therefore, by the maximality of $U^+_{new}(\hat{x})$, for $\hat{x} \in F$, $t < 0$ with $\hat{T}^t\hat{x} \in F$ we have

$$\hat{T}^{-t} U^+_{new}(\hat{T}^t\hat{x}) = U^+_{new}(\hat{x}), \tag{6.32}$$

and (6.30) and (6.31) still hold.

From (6.30), we get that for $\hat{x} \in F$ and $n \in U^+_{new}(\hat{x})$,

$$(r_n)_* f_1(\hat{x}) = e^{\beta_{\hat{x}}(n)} f_1(\hat{x}), \tag{6.33}$$

where $\beta_{\hat{x}} : U^+_{new}(\hat{x}) \to \mathbb{R}$ is a homomorphism. Since $\nu(F/\Gamma) > \delta_0 > 0$ and $\hat{T}^t$ is ergodic, for almost all $\hat{x} \in \hat{\Omega}$ there exist arbitrarily large $t > 0$ so that $\hat{T}^{-t}\hat{x} \in F$. Then, we define $U^+_{new}(\hat{x})$ to be $\hat{T}^t U^+_{new}(\hat{T}^{-t}\hat{x})$. (This is consistent in view of (6.32)). Then, (6.33) holds for a.e. $\hat{x} \in \hat{\Omega}$. It follows from (6.33) that for a.e. $\hat{x} \in \hat{\Omega}$, $n \in U^+_{new}(\hat{x})$ and $t > 0$,

$$\beta_{\hat{T}^{-t}\hat{x}}(\hat{T}^{-t} n \hat{T}^t) = \beta_{\hat{x}}(n). \tag{6.34}$$

We can write

$$\beta_{\hat{x}}(n) = L_{\hat{x}}(\log n),$$

where $L_{\hat{x}} : \mathrm{Lie}(U^+_{new})(\hat{x}) \to \mathbb{R}$ is a Lie algebra homomorphism (which is in particular a linear map). Let $K \subset \hat{X}$ be a Γ-invariant set with K/Γ of positive measure for which there exists a constant C with $\|L_{\hat{x}}\| \le C$ for all $\hat{x} \in K$. Now for almost all $\hat{x} \in \hat{\Omega}$ and $n \in U^+_{new}(\hat{x})$ there exists a sequence $t_j \to \infty$ so that $T^{-t_j}\hat{x} \in K$ and $T^{-t_j} n T^{t_j} \to e$, where e is the identity element of U^+_{new}. Then, (6.34) applied to the sequence t_j implies that $\beta_{\hat{x}}(n) = 0$ almost everywhere (cf. [BQ1, Proposition 7.4(b)]). This completes the proof of Proposition 6.1. ∎

Proof of Theorem 1.3. This argument follows closely [BQ1, §8]. Let $\mathcal{P}(G/\Gamma)$ denote the space of probability measures on G/Γ. For $\alpha \in \mathcal{P}(G/\Gamma)$, let S_α denote the connected component of the identity of the stabilizer of α with respect to the action of G by left-multiplication on G/Γ. Let

$$\mathcal{F} = \{\alpha \in \mathcal{P}(G/\Gamma) \; : \; S_\alpha \neq \{1\} \text{ and } \alpha \text{ is supported on one } S_\alpha \text{ orbit.}\}$$

The set $\mathcal{F}$ is endowed with the weak-$*$ topology. The group G naturally acts on $\mathcal{F}$. By Ratner's theorems [Ra], $\mathcal{F}$ contains all of the measures invariant and ergodic under a connected non-trivial unipotent subgroup.

Let ν be an ergodic μ-stationary measure on G/Γ. We construct a $\hat{T}^t$ and $\mathcal{U}_1^+$ invariant measure $\hat{\nu}$ on $\hat{\Omega}$ as in §1.

By Proposition 6.1 for almost all $\hat{x} = (x, g\Gamma) \in \hat{\Omega}$, there exists a subgroup $N_1(x) \subset N^+(x)$ such that the conditional measures $\hat{\nu}|_{N_1[\hat{x}]}$ of $\hat{\nu}$ on the $N_1(x)$ orbits on the G/Γ-fiber at x are right-invariant under a non-trivial unipotent subgroup $U^+_{new}(\hat{x})$ of $N_1(x)$. Without loss of generality we may assume that $U^+_{new}(\hat{x})$ is the stabilizer in $N_1(x)$ of $\hat{\nu}|_{N_1[\hat{x}]}$ (otherwise we replace $U^+_{new}(\hat{x})$ by the stabilizer).

Let

$$\Delta(x, g\Gamma) = \{g' \in G/\Gamma \; : \; U^+_{new}(x, g'\Gamma) = U^+_{new}(x, g\Gamma)\}.$$

Let $\hat{\nu}_x$ denote the conditional measure of $\hat{\nu}$ on $\{x\} \times G/\Gamma$. We now disintegrate $\hat{\nu}$ under the map $(x, g\Gamma) \to (x, U^+_{new}(x, g\Gamma))$, or equivalently for $\tilde{\mu}$-almost all $x \in \Omega$ we disintegrate $\hat{\nu}_x$ under the map $g\Gamma \to U^+_{new}(x, g\Gamma)$. We get, for almost all $(x, g\Gamma) \in \hat{\Omega}$, probability measures $\tilde{\nu}_{(x,g\Gamma)}$ on G/Γ supported on $\Delta(x, g\Gamma)$ so that for $\tilde{\mu}$-a.e. $x \in \Omega$,

$$\hat{\nu}_x = \int_{G/\Gamma} \tilde{\nu}_{(x,g\Gamma)} \, d\hat{\nu}_x(g\Gamma).$$

By [EiL3, Corollary 3.4] (cf. [BQ1, Proposition 4.3]), for $\hat{\nu}$-a.e. $(x, g\Gamma) \in \hat{\Omega}$, the measure $\tilde{\nu}_{(x,g\Gamma)}$ is (left) $U^+_{new}(x, g\Gamma)$-invariant.

We can do the simultaneous $U^+_{new}(x, g\Gamma)$-ergodic decomposition of all the measures $\tilde{\nu}_{(x,g\Gamma)}$ for almost all $(x, g\Gamma) \in \hat{\Omega}$ to get

$$\tilde{\nu}_{(x,g\Gamma)} = \int_{G/\Gamma} \zeta(x, g'\Gamma) \, d\tilde{\nu}_{(x,g\Gamma)}(g'\Gamma), \tag{6.35}$$

where $\zeta : \hat{\Omega} \to \mathcal{F}$ is a $\hat{\nu}$-measurable map such that for almost all $(x, g\Gamma) \in \hat{\Omega}$, ζ is constant along the fiber $\Delta(x, g\Gamma)$. (In fact, for any $\beta \in \mathcal{F}$, $\zeta(x, g\Gamma) = \beta$ if and only if $g\Gamma$ is β-generic for the action of $U^+_{new}(x, g\Gamma)$ on $\Delta(x, g\Gamma)$). Integrating (6.35) over $g\Gamma \in G/\Gamma$ we obtain for almost all $x \in \Omega$,

$$\hat{\nu}_x = \int_{G/\Gamma} \zeta(x, g\Gamma) \, d\hat{\nu}_x(g\Gamma). \tag{6.36}$$

The uniqueness of the ergodic decomposition and the $\hat{T}$ and $\mathcal{U}_1^+$-equivariance of the subgroups $N_1(x)$ and $U^+_{new}(\hat{x})$ shows that

$$\zeta(x, g\Gamma) = (T^t_x)_* \zeta(\hat{T}^t(x, g\Gamma)) \tag{6.37}$$

and for $u \in \mathcal{U}_1^+$,

$$\zeta(ux, g\Gamma) = \zeta(x, g\Gamma). \tag{6.38}$$

Therefore (see [BQ1, Lemma 3.2(e)]), the push-forward $\eta = \zeta_* \hat{\nu}$ is a μ-stationary probability measure on $\mathcal{F}$.

By [Ra, Theorem 1.1] the set $\mathcal{G}$ of G-orbits on $\mathcal{F}$ is countable. Let $\bar{\eta}$ denote the push-forward of η to $\mathcal{G}$. Then by [BQ1, Lemma 8.3], $\bar{\eta}$ is invariant under the support of μ.

Since $\hat{\nu}$ is ergodic, so is η. Thus $\bar{\eta}$ is supported at one point. Then, η is supported on $G\nu_0$, where $\nu_0 \in \mathcal{F}$. Let H denote the stabilizer of ν_0. By the definition of $\mathcal{F}$, ν_0 is supported on a single H-orbit.

We can now write $\zeta(x, g\Gamma) = \theta(x, g\Gamma)\nu_0$, where $\theta : \hat{\Omega} \to G/H$. Then θ satisfies (6.37) and (6.38) and then, again by [BQ1, Lemma 3.2(e)], the pushforward $\lambda = \theta^* \hat{\nu}$ is a μ-stationary measure on G/H. Then, by [BQ1, Proposition 6.7], $H = G$. Therefore, ν is Haar measure. □

7. Case II

In this section, we will prove Theorem 1.4.

7.1. Initial reductions.

Recall that $\mathcal{S}$ denotes the support of μ.

Proposition 7.1. *Let ν be a μ-stationary measure on G/Γ, and suppose that Case II holds, (see §1). Then ν is $G_{\mathcal{S}}$ invariant, and $\hat{\nu} = \mu^{\mathbb{Z}} \times \nu$.*

Proof. This is essentially contained in [B-RH], see also [EsL, §11.1-§11.3].

Standing assumptions and notation. In view of Proposition 7.1 in the rest of §7 we will assume that ν is $G_{\mathcal{S}}$-invariant. Also, we may replace $\mathcal{S}$ by a finite subset $\mathcal{S}'$ such that $G_{\mathcal{S}'}$ is still Zariski dense in G. Thus, in the rest of §7, we will assume that $\mathcal{S}$ is finite.

7.2. Dimensions of invariant measures.

For $g \in G$ and $r > 0$, let

$$B(r) = \{\exp(\mathbf{v}) \in G \ : \ \mathbf{v} \in \mathfrak{g} \text{ and } \|\mathbf{v}\| \leq r\}.$$

We define, for $g\Gamma \in G/\Gamma$, the "lower local dimension"

$$\underline{\dim}(\nu, g\Gamma) = \liminf_{r \to 0} \frac{\log \nu(B(r)g\Gamma)}{\log r}.$$

By the ergodicity of $\hat{T}$, for ν-a.e. $g \in G$, $\underline{\dim}(\nu, g)$ is independent of g. We denote the common value by $\underline{\dim}(\nu)$.

Proposition 7.2. *Under the assumptions of Theorem 1.4,* $\underline{\dim}(\nu) = 0$.

Remark. If there are no zero Lyapunov exponents, this follows from [LX] (which is based on [BPS]). We will give a proof of the trivial special case we need below (allowing for zero exponents).

Let
$$B_0(\epsilon) = \{\exp(\mathbf{v}) \;\;:\;\; \mathbf{v} \in \mathfrak{g},\, \|\mathbf{v}\| \le \epsilon\}.$$
For $x \in \mathcal{S}^{\mathbb{Z}}$ and $n \in \mathbb{N}$ let $B^n(x)$ denote the "Bowen ball" centered at the identity 1 of G, i.e.
$$B^n(x) = \{h \in G \;\;:\;\; \text{for all } 0 \le m \le n,\, (x_m \dots x_0)h(x_m \dots x_0)^{-1} \in B_0(\epsilon)\}.$$

Lemma 7.3. *For any unit* $\mathbf{v} \in \mathfrak{g}$*, for* $\mu^{\mathbb{Z}}$*-a.e.* $x \in \mathcal{S}^{\mathbb{Z}}$*, for all sufficiently large* n*,*
$$|\{t \;\;:\;\; \exp t\mathbf{v} \in B^n(x)\}| \le e^{-\alpha n},$$
where $\alpha > 0$ *depends only on the Lyapunov spectrum.*

Proof. See [BQ1, Lemma 7.3]. (IS THIS THE RIGHT REFERENCE?) □

The fiber entropy. Let ξ be a finite measurable partition of G/Γ. Then the limit
$$\lim_{n\to\infty} \frac{1}{n} H_{\nu_{x^-}}\Big(\bigvee_{i=0}^{n-1} (T_x^i)^{-1}\xi\Big) \equiv \lim_{n\to\infty} -\frac{1}{n} \sum_{A \in \bigvee_{i=0}^{n-1}(T_x^i)^{-1}\xi} \nu_{x^-}(A) \log \nu_{x^-}(A)$$
exists and is constant for $\mu^{\mathbb{Z}}$-a.e. x. We denote its value by $h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T}, \xi)$. Then, we define the fiber entropy $h^{G/\Gamma}_{\hat{\nu}}(\hat{T})$ to be the supremum over all finite measurable partitions ξ of $h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T}, \xi)$.

We recall the following:

Lemma 7.4. *For* $\epsilon > 0$*,* $\epsilon'' > 0$*,* $n \in \mathbb{N}$ *and* $x \in \mathcal{S}^{\mathbb{Z}}$*, let* $N(n, x, \epsilon, \epsilon'')$ *denote the smallest number of Bowen balls* $B^n(x, \epsilon)g\Gamma \subset G/\Gamma$ *needed to cover a set of* ν*-measure at least* $1 - \epsilon''$*. Then, for* $\mu^{\mathbb{Z}}$*-a.e.* $x \in \mathcal{S}^{\mathbb{Z}}$ *and any* $0 < \epsilon'' < 1$*,*
$$\lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(n, x, \epsilon, \epsilon'') = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, x, \epsilon, \epsilon'') = h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T}).$$

Proof. The analogous formula for the case of a single measure preserving trasformation is due to Katok [Ka, Theorem I.I]. The precise statement we need is given as [Zhu, Theorem 3.1] □

Corollary 7.5. *Let* $N(n, x, \epsilon, \epsilon'')$ *be as in Lemma 7.4. Then for any* $\epsilon > 0$*, any* $0 < \epsilon'' < 1$ *and* $\mu^{\mathbb{Z}}$*-a.e.* $x \in \mathcal{S}^{\mathbb{Z}}$*,*
$$\lim_{n\to\infty} \frac{1}{n} \log N(n, x, \epsilon, \epsilon'') = 0.$$

Proof. In our setting the fiber entropy $h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T})$ is zero. Now the statement follows immediately from the fact that for fixed n, x, ϵ'', $N(n, x, \epsilon, \epsilon'')$ is decreasing as a function of ϵ. $\square$

Proof of Proposition 7.2. By Lemma 7.3 there exist $x^1, \dots, x^M \in \mathcal{S}^{\mathbb{Z}}$ and $\alpha > 0$ such that for any $g_1, \dots, g_M \in G$,

$$\bigcap_{m=1}^{M} B^n(x^m) g_m \subset B(e^{-\alpha n}) g' \qquad \text{for some } g' \in G. \tag{7.1}$$

Then, by Corollary 7.5, for every $\epsilon > 0$ and $\epsilon' > 0$ there exists $M > 0$, and $x^1, \dots x^M \in \mathcal{S}^{\mathbb{Z}}$ such that (7.1) holds, and for all sufficiently large n, for each $1 \le m \le M$, there exists $Q_m^{(n)} \subset G/\Gamma$ of measure at least $1 - \epsilon/M$ such that $Q_m^{(n)}$ can be covered by $e^{\epsilon' n}$ Bowen balls of the form $B^n(x^m)g'\Gamma$. Then, $Q^{(n)} = \bigcap_{m=1}^M Q_m^{(n)}$ satisfies $\nu(Q^{(n)}) > 1-\epsilon$, and also $Q^{(n)}$ can be covered by at most $e^{M\epsilon' n}$ sets of the form

$$\bigcap_{m=1}^{M} B^n(x^m) g_m \Gamma.$$

Therefore, by (7.1), there exists a finite set $\Delta \subset G/\Gamma$ of cardinality at most $e^{M\epsilon' n}$ such that

$$Q^{(n)} \subset \bigcup_{g'\Gamma \in \Delta} B(e^{-\alpha n}) g' \Gamma.$$

Let

$$\Delta' = \{ g'\Gamma \in \Delta \ : \ \nu(B(e^{-\alpha n}) g'\Gamma) \le \epsilon |\Delta|^{-1} \}.$$

Then,

$$\nu\left(\bigcup_{g'\Gamma \in \Delta'} B(e^{-\alpha n}) g'\Gamma \right) \le \sum_{g'\Gamma \in \Delta'} \nu(B(e^{-\alpha n}) g'\Gamma) \le |\Delta| \left(\epsilon |\Delta|^{-1} \right) = \epsilon.$$

Let $\hat{Q}^{(n)} = \bigcup_{g'\Gamma \in \Delta \setminus \Delta'} B(e^{-\alpha n}) g'\Gamma$. Then, $\nu(\hat{Q}^{(n)}) \ge (1 - 2\epsilon)$, and each $g\Gamma \in \hat{Q}^{(n)}$ is contained in a set of the form $B(e^{-\alpha n}) g'\Gamma$ with $\nu(B(e^{-\alpha n}) g'\Gamma) > \epsilon |\Delta|^{-1}$. Therefore, for each $g\Gamma \in \hat{Q}^{(n)}$,

$$\nu(B(3e^{-\alpha n}) g\Gamma) \ge \epsilon |\Delta|^{-1} \ge \epsilon\, e^{-M\epsilon' n}.$$

Let Q_∞ denote the set of $g\Gamma \in G/\Gamma$ such that $g\Gamma \in \hat{Q}^{(n)}$ for infinitely many n. Then, $\nu(Q_\infty) \ge 1 - 2\epsilon$ and for each $g\Gamma \in Q_\infty$ there exists a sequence $r_k = 3e^{-\alpha n_k}$ with $r_k \to 0$ such that

$$\nu(B(r_k) g\Gamma) \ge \epsilon\, r_k^{(M/\alpha)\epsilon'},$$

i.e.

$$\frac{\log \nu(B(r_k) g\Gamma)}{\log r_k} \le (M/\alpha)\epsilon' + \frac{|\log \epsilon|}{|\log r_k|}.$$

Since ϵ and ϵ' are arbitrary and $|\log r_k| \to \infty$ as $r_k \to 0$, this implies $\underline{\dim}(\nu, g\Gamma) = 0$. $\square$

7.3. **Margulis Functions.** Recall that μ is a probability measure on G, supported on a finite set $\mathcal{S}$. Let r be small enough so that the exponential map restricted to the set $\{\mathbf{v} \in \mathfrak{g} \ : \ \|\mathbf{v}\| \leq r\}$ is a diffeomorphism onto its image.

Let $d : G \times G \to \mathbb{R}$ be defined by

$$d(g, g') = \begin{cases} \|\mathbf{v}\| & \text{if } g' = \exp(\mathbf{v})g, \ \mathbf{v} \in \mathfrak{g} \text{ and } \|\mathbf{v}\| < r, \\ r & \text{otherwise.} \end{cases}$$

Let $\delta > 0$ be a small parameter to be chosen later (idependently of ϵ) and let $f : G \times G \to \mathbb{R}$ be defined by

$$f(g_1, g_2) = \sup_{\gamma \in \Gamma} d_\epsilon(g_1, g_2\gamma)^{-\delta}.$$

Then, f descends to a function $G/\Gamma \times G/\Gamma \to \mathbb{R}$ which we also denote by f.

In the next Lemma and Proposition, we assume that Γ is cocompact. (The general argument is done in [EsL, Lemma 11.12 and Proposition 11.13]). We present the cocompact case here since it is much shorter and easier to follow.

After replacing r by a smaller number, we may assume that for any $g\Gamma$ in G/Γ, the injectivity radius is at least r. The proof of Theorem 1.4 is based on the following Margulis inequality:

Lemma 7.6. *Suppose Γ is cocompact. There exists $n \in \mathbb{N}$ sufficiently large (depending only on μ), and $\delta > 0$ sufficiently small (depending only on n and μ), and constants $c = c(\mu, n, \delta) < 1$, and $b = b(\mu, n, \delta, r) > 0$ such that for all $g_1\Gamma$, $g_2\Gamma \in G/\Gamma$,*

$$\int_G f(gg_1\Gamma, gg_2\Gamma)\, d\mu^{(n)}(g) \leq c f(g_1\Gamma, g_2\Gamma) + b. \tag{7.2}$$

Proof. This is essentially [EMar, Lemma 4.2]. The constant b is needed in case for some g in the support of $\mu^{(n)}$, $d(gg_1, gg_2) > r$. $\square$

Proposition 7.7. *Suppose Γ is cocompact, and ν is non-atomic. Then, for any $\eta > 0$ there exists $K'' \subset G/\Gamma$ with $\nu(K'') > 1 - c(\eta)$ where $c(\eta) \to 0$ as $\eta \to 0$ and a constant $C = C(\eta, \epsilon)$ such that for any $g\Gamma \in K''$,*

$$\int_{G/\Gamma} f(g\Gamma, g'\Gamma)\, d\nu(g'\Gamma) < C. \tag{7.3}$$

Remark. This proposition is true even in the case where Γ is not cocompact: see [EsL, Proposition 11.13].

Proof. By iterating (7.2), for any $g_1\Gamma, g_2\Gamma \in G/\Gamma$,

$$\limsup_{k\to\infty} \int_G f(g' g_1\Gamma, g' g_2\Gamma)\, d\mu^{(kn)}(g') \le \frac{b}{1-c}. \tag{7.4}$$

By the random ergodic theorem [Kif, Corollary I.2.2], there exists a function $\phi : G/\Gamma \times G/\Gamma \to \mathbb{R}$ such that

$$\int_{G/\Gamma\times G/\Gamma} \phi\, d(\nu\times\nu) = \int_{G/\Gamma\times G/\Gamma} f\, d(\nu\times\nu),$$

and for $\mu^{\mathbb{Z}}$ a.e. $x \in \mathcal{S}^{\mathbb{Z}}$ and $\nu\times\nu$ a.e. $g_1\Gamma,\ g_2\Gamma \in G/\Gamma$,

$$\phi(g_1\Gamma, g_2\Gamma) = \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} f(x_{jn}\dots x_1 g_1\Gamma, x_{jn}\dots x_1 g_2\Gamma). \tag{7.5}$$

Then, integrating both sides of (7.5) over $S^{\mathbb{Z}} \times G/\Gamma \times G/\Gamma$, using Fatou's lemma to take the limsup outside the integral, and then using (7.4), we get

$$\int_{G/\Gamma\times G/\Gamma} f\, d(\nu\times\nu) \le \frac{b}{1-c}.$$

This immediately implies the lemma. □

Proof of Theorem 1.4. Choose $\eta > 0$ and let K'', C be as in Proposition 7.7. Then it follows from (7.3) that for all $r > \epsilon > 0$ and all $g\Gamma \in K''$,

$$\nu(B(\epsilon)g\Gamma) \le C(\eta)\epsilon^{\delta},$$

hence

$$\frac{\log \nu(B(\epsilon)g\Gamma)}{\log\epsilon} \ge \delta - \frac{|\log C(\eta)|}{|\log\epsilon|}.$$

This implies $\underline{\dim}(\nu, g\Gamma) \ge \delta$, contradicting Proposition 7.2.

Thus, ν has an atomic part. Then by the ergodicity of $\hat{T}$, ν is atomic, and all atoms have the same measure. Therefore ν is finitely supported. □

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