Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials

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Contents

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1 Introduction

1.1 Moduli spaces of holomorphic differentials

Let Σ be a compact Riemann surface of genus $g > 1$ and ω is a holomorphic 1-form on Σ, i.e. a tensor of the form $\omega(z)dz$ in local coordinates with ω holomorphic. Away from the zeros of ω, we can choose a coordinate ζ so that $\phi = d\zeta$. This determines a Euclidean metric $|d\zeta^2|$ in that chart and the coordinate changes between such charts are of the form $\zeta \to \zeta + c$. Consequently, holomorphic differentials are sometimes referred to as translation surfaces or flat structures with parallel vector fields.

Near a zero of order $k \geq 1$ of ω, we can choose a local coordinate ζ so that ω is given by $\zeta^k \, d\zeta$. The corresponding metric is then $|\zeta^{2k}||d\zeta^2|$. The total angle around the zero is $(2k+2)\pi$, so we say that Σ has a cone singularity with total angle $(2k+2)\pi$.

Definition 1.1. Suppose that $g > 1$ and let μ be a partition of $2g-2$ into $\ell = \ell(\mu)$ parts. We denote by $\mathcal{H}(\mu)$ the moduli space of $(\ell+2)$-tuples

$$(\Sigma, \omega, p_1, \dots, p_\ell)\,,$$

where Σ is a Riemann surface of genus g, and ω is an holomorphic differential on Σ, and

$$(\omega) = \sum_i \mu_i \, [p_i]\,,$$

where (ω) is the divisor of ω, that is, the set of zeros of ω counting multiplicity.

For example if $\mu = (3,1)$, we require that ω has one triple zero p_1 and one simple zero p_2. Similarly, one can consider moduli spaces of pairs (Σ, ω) without ordering of the zeros of ω. This presents no additional difficulties.

One important feature is that these spaces, and the similar spaces of quadratic differentials, admit an ergodic $SL(2,\mathbb{R})$ action. The dynamics of this action is related to billiards in rational polygons and to interval exchange transformations. This circle of ideas has been studied extensively by various authors, e.g. [19, 22, 28, 30, 31, 16, 17, 29, 5, 10].

1.2 Local coordinates and invariant measure on $\mathcal{H}(\mu)$

Consider the relative homology group

$$H_1(\Sigma, \{p_i\}, \mathbb{Z}) \cong \mathbb{Z}^n\,, \quad n = 2g + \ell(\mu) - 1\,,$$

where $\ell(\mu)$ is the number of parts in the partition μ. Choose a basis

$$\{\gamma_1, \dots, \gamma_n\} \subset H_1(\Sigma, \{p_i\}, \mathbb{Z})$$

so that γ_i, $i = 1, \dots, 2g$, form a standard symplectic basis of $H_1(\Sigma, \mathbb{Z})$ and

$$\partial \gamma_{2g+i} = [p_{i+1}] - [p_1]\,, \quad i = 1, \dots, \ell(\mu) - 1\,.$$

The group $Sp(2g,\mathbb{Z})\ltimes\mathbb{Z}^{2g(\ell-1)}$ acts transitively on such bases by changing the basis in $H_1(\Sigma,\mathbb{Z})$ and translating the cycles γ_{2g+i} by elements of $H_1(\Sigma,\mathbb{Z})$. Consider the period map

$$\Phi:\mathcal{H}(\mu)\to\mathbb{C}^n\cong(\mathbb{R}^2)^n$$

defined by

$$\Phi(\Sigma,\omega)=\left(\int_{\gamma_1}\omega,\dots,\int_{\gamma_n}\omega\right)$$

It is known [15] that Φ is a local coordinate system on $\mathcal{H}(\mu)$. In particular,

$$\dim_{\mathbb{C}}\mathcal{H}(\mu)=2g+\ell(\mu)-1\,. \tag{1.1}$$

Let us pull back the Lebesgue measure from $\mathbb{C}^n$ to $\mathcal{H}(\mu)$ using Φ. This is well defined since it is clearly independent of the choice of basis $\{\gamma_i\}$. This measure is infinite, essentially because ω can be multiplied by any complex number. To correct this, we introduce the following

Definition 1.2. Denote by $\mathcal{H}_1(\mu)$ the subset of $\mathcal{H}(\mu)$ defined by the equation $\mathrm{Area}_\omega(\Sigma)=1$, where

$$\mathrm{Area}_\omega(\Sigma)=\frac{\sqrt{-1}}{2}\int_\Sigma\omega\wedge\overline{\omega}$$

is the area of Σ with respect to the metric defined by ω.

In terms of the periods $\phi=\Phi(\Sigma,\omega)$ we have

$$\mathrm{Area}_\omega(\Sigma)=\frac{1}{2}\sum_{i=1}^{g}\left(\phi_i\bar{\phi}_{g+i}-\bar{\phi}_i\phi_{g+i}\right) \tag{1.2}$$

Denote by Q the quadratic form on $\mathbb{R}^{2\dim\mathcal{H}(\mu)}$ defined by (1.2). It follows that the image of $\mathcal{H}_1(\mu)$ under Φ is contained in the hyperboloid $Q(v)=1$ and $\mathcal{H}_1(\mu)$ can be identified with a certain open subset of $Q(v)=1$. We now define a measure ν on $\mathcal{H}_1(\mu)$ as follows.

Definition 1.3. Let a set $E\subset\mathcal{H}_1(\mu)$ lie in the domain of a coordinate chart Φ and let $C\Phi(E)\subset\mathbb{C}^n$ be the cone over $\Phi(E)$ with vertex at the origin $0\in\mathbb{C}^n$. By definition, we set

$$\nu(E)=\mathrm{vol}(C\Phi(E))\,,$$

where the volume on the right is with respect to the Lebesgue measure on $\mathbb{C}^n$.

This measure is invariant under the $SL(2,\mathbb{R})$ action on $\mathcal{H}_1(\mu)$. It is a theorem of Masur [19] and Veech [27] that

$$\nu(\mathcal{H}_1(\mu)) < \infty \,.$$

The main goal of this paper is the computation of these numbers. They arise in particular in problems associated with billiards in rational polygons [31, 6, 7], and also in connection with interval exchanges and the Lyapunov exponents of the Teichmuller geodesic flow [34, 15].

1.3 Volumes and branched coverings

Our approach to the computation of the numbers $\nu(\mathcal{H}_1(\mu))$ is based on the interpretation of $\nu(\mathcal{H}_1(\mu))$ as the asymptotics in a certain enumeration problem, namely, the enumeration of connected branched coverings of a torus as their degree goes to ∞ and the ramification type is fixed. This interpretation was discovered by Kontsevich and Zorich and, independently, by Masur and the first author.

Definition 1.4. Given a partition μ, denote by $\mathcal{C}_d(\mu)$ the weighted the number of connected ramified coverings of the standard torus

$$\sigma : \Sigma \to T \tag{1.3}$$

of degree d, which are ramified over $\ell(\mu)$ fixed points of T, and such that the nontrivial part of the monodromy around the ith point is a cycle of length μ_i. The weight of a covering (1.3) is $|\mathrm{Aut}(\sigma)|^{-1}$, where $\mathrm{Aut}(\sigma)$ is the commutant of the monodromy subgroup of σ inside the symmetric group $S(d)$.

Remark 1.5. Typically, the group $\mathrm{Aut}(\sigma)$ is trivial and, in particular, these weights make no impact on the asymptotics of $\mathcal{C}_d(\mu)$ as $d \to \infty$, see Section 3.1. The purpose of introducing the weights is to make certain exact formulas look better, such as, for example, to make the generating series (1.6) a quasimodular form.

Proposition 1.6. *For any partition μ, we have*

$$\nu(\mathcal{H}_1(\mu)) = \lim_{D\to\infty} D^{-\dim_{\mathbb{C}} \mathcal{H}(\mu)} \sum_{d=1}^{D} \mathcal{C}_d(\mu + \vec{1}) \,, \tag{1.4}$$

where $\mu + \vec{1} = (\mu_1 + 1, \dots, \mu_{\ell(\mu)} + 1)$.

Recall that the dimension of $\mathcal{H}(\mu)$ is given by (1.1). The proof of Proposition 1.6 is elementary and is supplied in Section 3.2 below. The basic idea behind Proposition 1.6 is that to any covering (1.3) we can associate the point

$$(\Sigma, \sigma^*(dz)) \in \mathcal{H}(\mu)\,,$$

where dz is the standard holomorphic differential on T, and counting such points in $\mathcal{H}$ is like counting points of $\mathbb{Z}^{2n}$ inside subsets of $\mathbb{R}^{2n}$, where $n = \dim \mathcal{H}(\mu)$.

Using Proposition 1.6, A. Zorich computed the numbers $\nu(\mathcal{H}_1(\mu))$ for small μ.

1.4 Enumeration of coverings

It will be convenient to introduce the following numbers

$$\mathbf{c}(\mu) = (|\mu|+1) \lim_{D\to\infty} D^{-|\mu|-1} \sum_{d=1}^{D} \mathcal{C}_d(\mu)\,, \tag{1.5}$$

where $|\mu| = \sum \mu_i$. The existence of this limit follows from Proposition 1.6 which states that

$$\operatorname{vol}(\mathcal{H}_1(\mu)) = \frac{\mathbf{c}(\mu+\vec{1})}{\dim \mathcal{H}(\mu)}\,.$$

Heuristically, one should think about (1.5) as saying that

$$\mathcal{C}_d(\mu) \approx \mathbf{c}(\mu)\, d^{|\mu|}$$

for a typical large number d. In this paper, we obtain a general formula for these numbers, and hence for the volumes $\nu(\mathcal{H}_1)$, by developing a systematic approach to the asymptotics of $\mathcal{C}_d(\mu)$ as $d \to \infty$.

Our starting point is an exact result of S. Bloch and the second author [1], who considered certain generating functions, called the n-point functions, which encode the numbers $\mathcal{C}_d(\mu)$. These n-point functions were evaluated in [1] in a closed form as a determinant of ϑ-functions and their derivatives, see also the paper [24] for a simplified approach. This result is reproduced in Theorem 2.17 below. A qualitative conclusion from it is that the following generating function

$$\mathcal{C}(\mu) = \sum_{d=0}^{\infty} q^d\, \mathcal{C}_d(\mu) \tag{1.6}$$

is a *quasimodular form* in the variable q for the full modular groups, that is, a polynomial in the Eisenstein series $G_k(q)$, $k = 2, 4, 6$.

The asymptotics in (1.5) corresponds to the $q \to 1$ asymptotics of (1.6). In principle, using the formula for the n-point function, one can express for any given μ the generating function (1.6) in Eisenstein series. The quasi-modularity of $\mathcal{C}(\mu)$ means that it transforms in a certain way under the transformation

$$q = e^{2\pi i \tau} \mapsto e^{-2\pi i/\tau},$$

which takes $q = 1$ to $q = 0$, thus giving the $q \to 1$ asymptotics of $\mathcal{C}(\mu)$. This quasimodularity is a manifestation of a certain "mirror symmetry" between coverings of very large degree ($q \to 1$) and small degree ($q \to 0$).

In practice, however, the computation of (1.6) becomes very difficult even for relatively small μ.

We therefore pursue a different approach and first investigate the $q \to 1$ asymptotics of the n-point function. Here we find a great simplification, see Theorem 4.7, essentially because the ϑ-functions become trigonometric functions. We then extract from this asymptotics the information about the asymptotics (1.5).

This extraction is still rather nontrivial because, inside of the n-point function, the numbers $\mathcal{C}(\mu)$ are wrapped up in several layers of enciphering, such as going from connected to disconnected coverings. For example, the terms corresponding to connected coverings appear only deep in the asymptotic expansion of the n-point function which requires us to keep track of many orders of asymptotics.

1.5 Summary of results

The answer we obtain for the constants $\mathbf{c}(\mu)$ can be conveniently stated in terms of a certain multilinear form

$$\langle \cdot | \dots | \cdot \rangle_h : \Lambda^* \times \dots \times \Lambda^* \to \mathbb{C}[h^{-1}], \tag{1.7}$$

where Λ^* an algebra closely related to the algebra of symmetric functions. The form (1.7) is such that

$$\langle f_{\mu_1} | \dots | f_{\mu_k} \rangle_h = \mathbf{c}(\mu) \frac{|\mu|!}{h^{|\mu|+1}} + \dots, \tag{1.8}$$

where f_k are certain generators of Λ^*, and dots stand for terms of lower degree in h^{-1}. The evaluation of (1.8) goes in 3 steps.

First, one expresses the generators f_k as polynomials in power-sum generators p_k of Λ. A formula for this expansion is obtained in Theorem 5.5. Then, using an analog of the Wick formula for (1.7) derived in Theorem 6.3, one reduces (1.8) to computations of the constants $\langle\!\langle\, \mu \,\rangle\!\rangle$ defined by

$$\langle p_{\mu_1} | \dots | p_{\mu_k} \rangle_h = \frac{\langle\!\langle\, \mu \,\rangle\!\rangle}{h^{|\mu|+1}} + \dots ,$$

which we call elementary cumulants.

These numbers $\langle\!\langle\, \mu \,\rangle\!\rangle$ are finally computed in Theorem 6.7 in terms of values of the ζ-function at even positive integers, that is, in Bernoulli numbers. In particular, we have

$$\pi^{-2g}\, \nu(\mathcal{H}_1(\mu)) \in \mathbb{Q}$$

for any μ. This rationality was also conjectured by Kontsevich and Zorich.

We were unable to simplify this answer further in the general case, but in the special case $\mu = (2, \dots, 2)$ which corresponds to differentials with simple zeros (that is, to generic ones), an attractive answer is available. It is given in Theorem 7.1.

1.6 Example of a volume computation

Suppose we want to compute $\nu(\mathcal{H}(3,1))$ or, equivalently, $\mathbf{c}(4,2)$. From Theorem 5.5 we get

$$f_2 = \frac{1}{2}\, p_2 \,, \quad f_4 = \frac{1}{4}\, p_4 - p_2 p_1 + \dots ,$$

where dots stand for lower weight term which make no contribution to the answer.

In general, there exist a very important weight filtration on Λ^* which we discuss in Section 5. It has the property that (1.7) takes it to the filtration of $\mathbb{C}[h^{-1}]$ by degree, which allows us to identify many negligible terms.

By the Wick formula, see Theorem 6.3, we have

$$\begin{aligned}\langle f_4 | f_2 \rangle_h &= \frac{1}{8} \langle p_4 | p_2 \rangle_h - \frac{1}{2} \langle p_2 p_1 | p_2 \rangle_h + \dots = \\ &\frac{h^{-7}}{8} \langle\!\langle\, 4,2 \,\rangle\!\rangle - \frac{h^{-7}}{2} \langle\!\langle\, 2 \,\rangle\!\rangle\, \langle\!\langle\, 2,1 \,\rangle\!\rangle - \frac{h^{-7}}{2} \langle\!\langle\, 1 \,\rangle\!\rangle\, \langle\!\langle\, 2,2 \,\rangle\!\rangle + \dots ,\end{aligned}$$

where dots stand for lower terms. From Theorem 6.7, see also Example 6.9, we conclude that

$$\langle\!\langle\, 1 \,\rangle\!\rangle = \zeta(2) = \frac{\pi^2}{6}\,, \quad \langle\!\langle\, 2 \,\rangle\!\rangle = 0\,,$$

and similarly

$$\langle\!\langle\, 4,2 \,\rangle\!\rangle = \frac{416}{315}\,\pi^6\,, \quad \langle\!\langle\, 2,2 \,\rangle\!\rangle = \frac{16}{45}\,\pi^4\,.$$

Hence

$$\big\langle f_4 \big| f_2 \big\rangle_h = \frac{128}{945}\,\pi^6\, h^{-7} + \dots\,,$$

which means that

$$\mathbf{c}(4,2) = \frac{8}{42525}\,\pi^6\,, \quad \nu(\mathcal{H}(3,1)) = \frac{8}{297675}\,\pi^6\,.$$

This is one of the numbers computed by A. Zorich.

1.7 Connection with random partitions

The quantities (1.7) are, by their construction, certain sums over all partitions λ. The variable h enters this sums as a weight $e^{-h|\lambda|}$ given to a partition λ. The leading term of the $h \to +0$ asymptotics, like in (1.8), describes certain statistical properties of random partitions of a very large size.

Some of our formulas admit a nice probabilistic interpretation, see the Appendix. In particular, one can easily see in our formulas the existence of Vershik's limit shape of a large random partition, see Section A.1 and also the Gaussian correction to this limit shape, see Section A.3.

The point of view of random partitions also provides a very simple explanation why something like (1.6) can never be modular, see Section A.2, which makes the quasimodularity of (1.6) look even more like a miracle.

1.8 Some open problems

The space $\mathcal{H}_1(\mu)$ is sometimes disconnected. The connected components of this space were described in [16]. In particular, there are always at most 3 components and $\mathcal{H}_1(\mu)$ is connected when at least one of the μ_i's is odd. The knowledge if the volumes of connected components is important for applications to ergodic theory. For small genus, volumes of connected components were determined by A. Zorich. Unfortunately, our formulas do not separate the connected components.

Another problem important for applications is to compute the volumes of similarly defined moduli spaces of quadratic differentials.

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2 Counting ramified covering of a torus

2.1 Basics

Let T be a torus and $Z = \{z_1, \dots, z_s\}$ be a collection of distinct points in T. Let $\sigma : \Sigma \to T$ be a ramified covering of T which is unramified outside of Z. All information about σ is encoded in the monodromy action of the fundamental group $\pi_1(T \setminus Z, *)$ on the fiber over the basepoint $* \in T$

$$\pi_1(T \setminus Z, *) \to \mathrm{Aut}(\sigma^{-1}(*)) .$$

If σ is d-fold then any labeling of $\sigma^{-1}(*)$ by $1, \dots, d$ produces an isomorphism

$$\mathrm{Aut}(\sigma^{-1}(*)) \cong S(d) .$$

Therefore, d-fold ramified coverings are in bijection with the orbits of the $S(d)$-action by conjugation on the set of all homomorphisms from $\pi_1(T \setminus Z)$ to $S(d)$

$$\Big\{ d\text{-fold coverings} \Big\} = \mathrm{Hom}(\pi_1(T \setminus Z), S(d)) \Big/ S(d) . \tag{2.1}$$

Introduce the following notation. For any conjugacy classes $C_1, \dots, C_s \subset S(d)$, denote by

$$H_d(C_1, \dots, C_s) \subset \mathrm{Hom}(\pi_1(T \setminus Z), S(d))$$

those homomorphisms that send a small loop around z_i into C_i for $i = 1, \dots, s$. This corresponds to fixing the ramification type (namely C_i) over the points $z_i \in Z$.

A natural way to count the orbits in (2.1) is to weight any orbit σ in (2.1) by $|\,\mathrm{Aut}(\sigma)|^{-1}$ where $\mathrm{Aut}(\sigma)$ is a point stabilizer of σ, that is, the centralizer of the image of $\pi_1(T \setminus Z)$ inside $S(d)$. Introduce the following weighted number of the d-fold coverings with prescribed monodromy $C_1, \dots, C_s$

$$\begin{aligned}\operatorname{Cov}_d(C_1, \dots, C_s) &= \sum_{\sigma \in H_d(C_1,\dots,C_s)/S(d)} \frac{1}{|\operatorname{Aut}(\sigma)|} \\ &= \left|H_d(C_1, \dots, C_s)\right| \Big/ d! \,.\end{aligned}$$

Since the conjugacy classes of $S(d)$ are naturally embedded into conjugacy classes of any bigger symmetric group, it makes sense to introduce the following generating function

$$\operatorname{Cov}(C_1, \dots, C_s) = \sum_{d=0}^{\infty} q^d \operatorname{Cov}_d(C_1, \dots, C_s) \,.$$

Remark 2.1. To avoid possible confusion, we point out that our definition of $\operatorname{Aut}(\sigma)$ does not allow permutations of the marked points $z_1, \dots, z_s$. This will be important in the next subsection where we consider the relation between connected and disconnected coverings. For example, if a covering is a union of two otherwise identical coverings which are ramified over two different points of T then this covering *does not* have an extra $\mathbb{Z}_2$-symmetry.

2.2 Connected and disconnected coverings

The generating function $\operatorname{Cov}(C_1, \dots, C_s)$ counts all, possibly disconnected, coverings with given monodromy $C_1, \dots, C_s$. In particular, $\operatorname{Cov}()$ counts all unramified coverings.

Under the correspondence (2.1), connected components correspond to orbits of the π_1 action on $\{1, \dots, d\}$ and unramified connected components correspond to those orbits on which small loops around the z_i's act trivially. Let

$$H'_d(C_1, \dots, C_s) \subset H_d(C_1, \dots, C_s)$$

be the subset corresponding to coverings without unramified connected components.

Definition 2.2. Let $\operatorname{Cov}'(C_1, \dots, C_s)$ be the generating function for the coverings without unramified connected components. In other words,

$$\operatorname{Cov}'(C_1, \dots, C_s) = \sum_{d=0}^{\infty} q^d \, \frac{|H'_d(C_1, \dots, C_s)|}{d!} \,.$$

Definition 2.3. Similarly, let $\mathcal{C}(C_1,\dots,C_s)$ be the generating function for connected coverings.

Lemma 2.4.

$$\operatorname{Cov}'(C_1,\dots,C_s) = \operatorname{Cov}(C_1,\dots,C_s)\Big/\operatorname{Cov}()\,.$$

Proof. This is equivalent to

$$|H_d(C_1,\dots,C_s)| = \sum_{k=0}^{d}\binom{d}{k}|H'_k(C_1,\dots,C_s)|\,|H_{d-k}()|\,,$$

which is obvious. □

To simplify the exposition, we shall from now on focus on the case when C_i has a single nontrivial cycle of length $m_i\in\{2,3,\dots\}$. The case of more general monodromies presents no extra difficulties but it will be not needed for the application we have in mind. Accordingly, we shall write $\operatorname{Cov}(m)$, where $m=(m_1,\dots,m_s)$, in place of $\operatorname{Cov}(C_1,\dots,C_s)$ and similarly for $\mathcal{C}_d(m)$.

The function $\mathcal{C}$ can be expressed in terms of functions Cov' as follows. Recall that a partition α of a set S is a presentation of the set S as an unordered disjoint union of nonempty subsets

$$S=\alpha_1\sqcup\alpha_2\sqcup\cdots\sqcup\alpha_\ell\,,$$

which are called the blocks of α. The number $\ell=\ell(\alpha)$ is the length of the partition α. We denote by Π_s the set of all partitions of $\{1,\dots,s\}$. Any covering $\sigma\in H_d(m)$ produces a partition $\alpha=\alpha(\sigma)\in\Pi_s$ as follows. Two numbers i and j belong to the same block of α if and only if the corresponding ramifications occur on the same connected component.

It is clear that the same argument that establishes Lemma 2.4 shows that

$$\operatorname{Cov}'(m)=\sum_{\alpha\in\Pi_s}\prod_{k=1}^{\ell(\alpha)}\operatorname{Cov}^\circ\left(m_{\alpha_k}\right)\,, \tag{2.2}$$

where $m_{\alpha_k}=\{m_i\}_{i\in\alpha_k}$.

Remark 2.5. Recall that the set Π_n of partitions of an n-element set is partially ordered: if $\alpha,\beta\in\Pi_n$ we say that $\alpha<\beta$ if α is a refinement of β, that is, if the blocks of β consist of whole blocks of α. The maximal element of this poset is the partition $\widehat{n}$ into one block. The Möbius function of the partially ordered set Π_n is well known to satisfy

$$\text{Möbius}\,(\widehat{n},\alpha) = (-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)!\,,$$

see, for example, Section 3.10.4 of [26].

Applying the Möbius inversion to (2.2) results in the following:

Lemma 2.6. *We have*

$$\mathcal{C}(m) = \sum_{\alpha\in\Pi_s} (-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)! \prod_{k=1}^{\ell(\alpha)} \mathrm{Cov}'\left(m_{\alpha_k}\right)\,,$$

where $m_{\alpha_k} = \{m_i\}_{i\in\alpha_k}$.

2.3 Coverings and sums over partitions

Definition 2.7. Let C be a conjugacy class in $S(d)$. Let f_C be the following function of a partition λ

$$f_C(\lambda) = \#C\,\frac{\chi^\lambda(C)}{\dim\lambda}\,,$$

where χ^λ is the character of the irreducible representation of $S(d)$ corresponding to the partition λ, $\chi^\lambda(C)$ is its value on any element of C, and $\dim\lambda = \chi^\lambda(1)$ is the dimension of representation λ.

If C is the class of an m-cycle we shall write f_m instead of f_C. Also, note the difference between partitions and partitions of a set. In the above definition we have simply partitions whereas in the previous section we used partitions of a set.

For the number of ramified coverings, there exists the following expression in terms of the function f_C which goes back essentially to Burnside, see Exercise 7 in §238 of [3]. In exactly this form it is presented, for example, in [4].

Proposition 2.8. *We have*

$$\mathrm{Cov}_d(C_1,\dots,C_s) = \sum_{|\lambda|=d}\prod_{i=1}^{s} f_{C_i}(\lambda)\,,$$

where the sum is over all partitions λ *of the number* d.

It is known that for any conjugacy class C the function $f_C(\lambda)$ is a polynomial function of a partition λ in the following sense.

Let $\Lambda^*(n)$ be the algebra of polynomials in $\lambda_1, \dots, \lambda_n$ which are symmetric in the variables $\lambda_i - i$. This algebra is filtered by the degree of a polynomial. Let the algebra Λ^* be the projective limit of these algebras $\Lambda^* := \varprojlim \Lambda^*(n)$ as filtered algebras with respect to homomorphisms that set the last variable to 0. This is the algebra of *shifted symmetric functions*, see [13, 25]. By construction, any $f \in \Lambda^*$ has a well defined degree and can be evaluated at any partition λ. There is the following result, see [13] and also [25]

Proposition 2.9 ([13]). *We have $f_C \in \Lambda^*$ and the degree of f_C is the number of non-fixed points of any permutation from C.*

Various expressions are known for this polynomial; for example, its expression in the shifted Schur functions is given by the formula (15.21) in [25].

It is clear that we have

$$\mathrm{Cov}(m) = \sum_\lambda q^{|\lambda|} \prod_i f_{m_i}(\lambda)\,,$$

where the sum is over all partitions λ. In particular, the generating function for the unramified coverings is

$$\mathrm{Cov}() = \sum_\lambda q^{|\lambda|} = (q)_\infty^{-1}\,,$$

where $(q)_\infty = \prod_{n\geq 1}(1-q^n)$.

Introduce the following linear functional on the algebra Λ^*

Definition 2.10. For any $F \in \Lambda^*$, set

$$\langle F \rangle_q = (q)_\infty \sum_\lambda q^{|\lambda|}\, F(\lambda)\,.$$

In particular, $\langle 1 \rangle_q = 1$. More generally, for $s = 1, 2, \dots$ consider the following multilinear functional on $(\Lambda^*)^{\times s}$

$$\left\langle F_1 \middle| F_2 \middle| \dots \middle| F_s \right\rangle_q = \sum_{\alpha \in \Pi_s} (-1)^{\ell(\alpha)-1} (\ell(\alpha)-1)! \prod_{k=1}^{\ell(\alpha)} \left\langle \prod_{i \in \alpha_k} F_i \right\rangle_q$$

In other words, $\langle f\rangle_q$ is the expected value of f if the probability of a partition λ is proportional to $q^{|\lambda|}$. In the physical language, $\langle f\rangle_q$ is the Gibbsian average of f with respect to the "energy" function $\lambda \mapsto |\lambda|$. The functional $\left\langle F_1\middle|F_2\middle|\dots\middle|F_s\right\rangle_q$ in the physical language would correspond to the "connected" part of $\langle F_1 F_2 \cdots F_s\rangle_q$. It is no coincidence that it counts quite precisely the connected coverings.

Indeed, the following is an immediate corollary of Lemmas 2.4 and 2.6.

Proposition 2.11. *We have*

$$\operatorname{Cov}'(m) = \langle f_{m_1} f_{m_2} \cdots f_{m_s}\rangle_q \,, \tag{2.3}$$

$$\mathcal{C}(m) = \left\langle f_{m_1}\middle|f_{m_2}\middle|\dots\middle|f_{m_s}\right\rangle_q \,. \tag{2.4}$$

2.4 Formula for n-point functions

Our strategy for evaluation of the quantities (2.4) is the following. By multilinearity, it suffices to compute $\left\langle F_1\middle|F_2\middle|\dots\middle|F_s\right\rangle_q$, where the $F_i's$ range over any linear basis of the algebra Λ^* and then expand the functions f_m in this linear basis.

Remark 2.12. One fact supporting such a roundabout approach, aside of the fact that it appears to be very difficult to evaluate (2.4) directly, is the following. As our choice of the parameter q for the generating function suggests, the averages $\langle\,\cdot\,\rangle_q$ have some modular properties. More concretely, they are quasi-modular, see [1] and below. It turns out, however, that (2.4) are linear combinations of quasi-modular forms of different weights or, in other words, they are inhomogeneous elements of the algebra of quasi-modular forms. The other basis of Λ^*, which will be introduced momentarily, does have the property that $\langle\,\cdot\,\rangle_q$ takes basis vectors to homogeneous quasi-modular forms.

A very convenient linear basis of the algebra Λ^* is formed by monomials in the following generators

$$p_k(\lambda) = \sum_{i=0}^{\infty} \left[(\lambda_i - i + 1/2)^k - (-i+1/2)^k\right] + (1-2^{-k})\zeta(-k)\,. \tag{2.5}$$

This peculiar expression is in fact a natural ζ-function regularization of the divergent sum $\sum_{i=0}^{\infty}(\lambda_i - i + 1/2)^k$. More precisely, since $\lambda_i = 0$ for all but finitely many i the first sum in (2.5) is finite while the second term in (2.5) is the natural regularization for $\sum_{i=0}^{\infty}(-i+1/2)^k$.

Remark 2.13. It is an experimental fact that the somewhat annoying $\frac{1}{2}$'s in the definition of p_k are actually very useful, see [1, 2, 13, 24, 25]. In other words, it turns out that the so-called *modified Frobenius coordinates*, which are the usual Frobenius coordinates plus $\frac{1}{2}$ for the half of a diagonal square, are the most convenient coordinates on partitions. For example, these $\frac{1}{2}$'s make the p_k behave well under the involution ω in the algebra Λ^*, see Section 5.4.

It is also convenient to introduce the following generating function

$$\mathbf{e}^\lambda(x) = \sum_i e^{(\lambda_i - i + 1/2)x} .$$

This sum converges provided $\Re x > 0$ and has a simple pole at $x = 0$ with residue 1. We have (see the formula (0.18) in [1])

$$p_k(\lambda) = k! \left[x^k\right] \mathbf{e}^\lambda(x) , \tag{2.6}$$

where $[x^k]$ denotes the coefficient of x^k in the Laurent series expansion about $x = 0$. Therefore, all averages of the form $\left\langle \prod p_{k_i} \right\rangle_q$ are encoded in the following generating function.

Definition 2.14. We call the following generating function

$$F(x_1, \dots, x_n) = \left\langle \prod \mathbf{e}^\lambda(x_i) \right\rangle_q$$

the *n-point function.* Similarly, we also consider more general generating functions

$$F(x_1, \dots, x_i \,\big|\, x_{i+1}, \dots, x_j \,\big|\, x_{j+1}, \dots \,\big|\, \dots, x_n) = \\ \left\langle \mathbf{e}^\lambda(x_1) \cdots \mathbf{e}^\lambda(x_i) \,\big|\, \mathbf{e}^\lambda(x_{i+1}) \cdots \mathbf{e}^\lambda(x_j) \,\big|\, \mathbf{e}^\lambda(x_{j+1}) \cdots \,\big|\, \cdots \mathbf{e}^\lambda(x_n) \right\rangle_q$$

which we call the *connected functions.*

It is clear that the connected functions are, by Definition 2.10, polynomials in the n-point functions.

The following claim follows immediately from (2.6)

Proposition 2.15. *Let μ be a multi-index $\mu = (\mu_1, \dots, \mu_n)$. We have*

$$\left\langle p_\mu \right\rangle_q = \mu! \left[x^\mu\right] F(x) ,$$

where $x = (x_1, \dots, x_n)$ *and, as usual,*

$$p_\mu = \prod_i p_{\mu_i}\,, \quad \mu! = \prod_i \mu_i\,, \quad x^\mu = \prod_i x^{\mu_i}\,.$$

Similarly,

$$\left\langle p_\mu \,|\, p_\nu \,|\, p_\eta \,|\, \dots \right\rangle_q = \mu!\,\nu!\,\eta!\cdots \left[x^\mu y^\nu z^\eta \cdots\right] F(x\,|\,y\,|\,z\,|\,\dots)\,.$$

Definition 2.16. The quantities

$$\left\langle p_\mu \,|\, p_\nu \,|\, p_\eta \,|\, \dots \right\rangle_q \,,$$

which appear in the above proposition and which will be of primary interest to us in this paper, will be called *cumulants*.

Proposition 2.15 is, of course, only useful if one can compute the n-point functions. The n-point functions were computed in [1] (see also [24]) as certain determinants involving theta functions and their derivatives. Introduce the following odd genus 1 theta function

$$\vartheta(x) = \vartheta_{\frac{1}{2},\frac{1}{2}}(x;q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+\frac{1}{2})^2}{2}} e^{(n+\frac{1}{2})x}\,.$$

This is the only odd genus 1 theta functions and its precise normalization is not really important because the formulas will be homogeneous in ϑ. The formula for the n-point functions is the following

Theorem 2.17 ([1]). *We have*

$$F(x_1,\dots,x_n) = \sum_{\substack{\text{all } n! \text{ permutations}\\ \text{of } x_1,\dots,x_n}} \frac{\det\left[\dfrac{\vartheta^{(j-i+1)}(x_1+\dots+x_{n-j})}{(j-i+1)!}\right]_{i,j=1}^n}{\vartheta(x_1)\,\vartheta(x_1+x_2)\cdots\vartheta(x_1+\dots+x_n)}\,, \quad (2.7)$$

where in the $n!$ *summands the* x_i*'s have to be permuted in all possible ways,* $\vartheta^{(k)}$ *stands for the* k*-th derivative of* ϑ*, and by the usual convention that* $1/k! = 0$ *if* $k < 0$ *we do not have negative derivatives.*

In principle, one can use this formula to give a formula for the connected functions but it appears to be difficult to simplify the answer in any attractive manner. However, in the $q \to 1$ limit, which corresponds to the limit of coverings of very large degree, the situation simplifies and useful formulas for the connected functions become available.

3 Coverings of large degree and volumes of moduli spaces

3.1 Coverings with automorphisms

Suppose that a d-fold connected covering

$$\sigma : \Sigma \to T$$

has a nontrivial automorphism, that is, suppose that there exists a permutation $h \in S(d)$ which commutes with the monodromy subgroup $G \subset S(d)$ of σ.

Since the sets of fixed points of h^k, $k = 1, 2, \dots$, are G-stable and G is transitive, they must be either empty or all of $\{1, \dots, d\}$ for any k. It follows that the cycle type of h is of the form

$$(\underbrace{d_1, \dots, d_1}_{d_2 \text{ times}}),$$

for some factorization $d = d_1 d_2$.

Let $\mathbb{Z}_{d_1}$ be the cyclic group generated by h. We have the following factorization $\sigma = \sigma'' \circ \sigma'$

$$\sigma : \Sigma \xrightarrow[d_1\text{-fold}]{\sigma'} \Sigma/\mathbb{Z}_{d_1} \xrightarrow[d_2\text{-fold}]{\sigma''} T\,. \tag{3.1}$$

Because the group G commutes with h, the size of d_1 is bounded in terms of the ramification type μ of σ.

On the other hand, the genus of $\Sigma/\mathbb{Z}_{d_1}$ is strictly less than the genus of Σ by the Riemann-Hurwitz formula and the number of ramification points of σ'' is at most the number of ramification points of σ.

We will see in the next Section that the number of connected genus g coverings of degree $\leq D$ with ℓ ramification points grows like $D^{2g+\ell-1}$ as $D \to \infty$. Hence the number of coverings admitting a factorization of the form (3.1) grows slower than the number of all coverings.

In particular, the proportion of those coverings of degree $\leq D$ which have nontrivial automorphisms becomes negligible as $D \to \infty$.

3.2 Proof of Proposition 1.6

Recall that $p_1, \ldots, p_\ell$ denote the zeros of ω and μ_i's are the corresponding multiplicities. Also recall that we choose the basis

$$\{\gamma_1, \ldots, \gamma_n\} \subset H_1(\Sigma, \{p_i\}, \mathbb{Z})$$

so that that γ_i, $i = 1, \ldots, 2g$, form a standard symplectic basis of $H_1(\Sigma, \mathbb{Z})$ and

$$\partial\gamma_{2g+i} = [p_{i+1}] - [p_1]\,, \quad i = 1, \ldots, \ell(\mu) - 1\,.$$

We have the following elementary

Lemma 3.1. (cf. [29]) *Consider $\phi = \Phi(\Sigma, \omega) \in \mathbb{C}^{\dim \mathcal{H}(\mu)}$. We have $\phi_i \in \mathbb{Z}^2$, $i = 1, \ldots, 2g$, if and only if the following holds:*

(a) *there exists a holomorphic map σ from Σ to the standard torus $T = [0, 1]^2$,*

(b) $\omega = \sigma^{-1}(dz)$,

(c) *$\{p_i\}$ is the set of critical points of σ,*

(d) *the ramification of σ at p_i is of the form $z \mapsto z^{\mu_i+1}$,*

(e) $\sigma(p_{i+1}) - \sigma(p_1) = \phi_{2g+i} \mod \mathbb{Z}^2$,

(f) *the degree of σ is equal to* $\mathrm{Area}_w(\Sigma) = \frac{\sqrt{-1}}{2}\int_\Sigma \omega \wedge \overline{\omega}$.

Proof. The sufficiency of the conditions in the lemma is clear. To prove necessity, define the map σ by

$$\sigma(z) = \int_p^z \omega \qquad \mod \mathbb{Z}^2\,,$$

where $p \in \Sigma$ is arbitrary. This map σ is well defined because $\int_\gamma \omega \in \mathbb{Z}^2$ for any closed path $\gamma \subset \Sigma$. The required properties of σ follow easily from the definitions. $\square$

We note the map σ depends only on (M, ω) and not on the choice of homology basis. Now we finish the proof of Proposition 1.6 as follows.

Choose a vector $\beta \in \mathbb{C}^{\dim \mathcal{H}(\mu)}$ such that

$$\beta_i \in \mathbb{Z}^2, \quad i = 1, \ldots, 2g\,, \quad \beta_i \neq \beta_j \mod \mathbb{Z}^2\,, \quad i, j > 2g, i \neq j\,.$$

Let a set $E \subset \mathcal{H}_1(\mu)$ lie in the domain of a coordinate chart Φ and denote by C_D the cone

$$C_D = \left\{ t\Phi(\Sigma,\omega), (\Sigma,\omega) \in E, t \in [0,\sqrt{D}] \right\} \subset \mathbb{C}^{\dim \mathcal{H}(\mu)} .$$

By definition of ν we have

$$D^{-\dim \mathcal{H}(\mu)} \left| C_D \cap (\mathbb{Z}^{2\dim \mathcal{H}(\mu)} + \beta) \right| \to \operatorname{vol}(C_1) = \nu(E) , \quad D \to \infty .$$

On the other hand, by Lemma 3.1, every point of the intersection $C_D \cap (\mathbb{Z}^{2\dim \mathcal{H}(\mu)} + \beta)$ corresponds to a covering σ of degree $\leq D$ with ramification type μ. Thus, $\nu(E)$ is the asymptotics of the number of those covering which correspond to the subset E of the moduli space.

Now for the whole moduli space $\mathcal{H}_1(\mu)$, it follows from the proof of the finiteness of the volume in [19, 27] that for every $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \mathcal{H}_1(\mu)$ such that $\nu(K_\epsilon) \geq \nu(\mathcal{H}_1(\mu)) - \epsilon$ and it is easy to show that $\mathcal{H}_1(\mu)$ has a rectifiable boundary. Hence

$$\nu(\mathcal{H}_1(\mu)) = \lim_{D\to\infty} D^{-\dim_{\mathbb{C}} \mathcal{H}(\mu)} \sum_{d=1}^{D} \mathcal{C}_d(\mu + \vec{1}) , \tag{3.2}$$

as was to be shown.

3.3 Large degree coverings and $q \to 1$ asymptotics

Recall that we introduced in (1.5) the constants $\mathbf{c}(\mu)$ such that

$$\sum_{d=0}^{D} \mathcal{C}_d(m) \sim \mathbf{c}(m) \frac{D^{|m|+1}}{|m|+1} , \quad D \to \infty .$$

where $|m| = \sum m_i$. We now observe that $\mathbf{c}(m)$ is determined by the leading order asymptotics of

$$\mathcal{C}(m) = \left\langle f_{m_1} \middle| f_{m_2} \middle| \dots \middle| f_{m_s} \right\rangle_q$$

as $q \to 1$. Namely, we have the following proposition which follows from the elementary power series identity

$$\frac{1}{1-q} \sum_{d=0}^{\infty} q^d \, a_d = \sum_{d=0}^{\infty} q^d \sum_{k=0}^{d} a_k .$$

Proposition 3.2.

$$\left\langle f_{m_1} \middle| f_{m_2} \middle| \dots \middle| f_{m_s} \right\rangle_q = \mathbf{c}(m) \, \frac{|m|!}{(1-q)^{|m|+1}} + O\left((1-q)^{-|m|}\right) , \quad q \to 1 \, .$$

The $q \to 1$ asymptotics of the n-point functions and of the connected functions will be considered in the next section.

4 Asymptotics of connected functions.

4.1 Asymptotics of n-point functions.

It will be convenient to replace the parameter q, $|q| < 1$, by a new parameter h, $\Re h > 0$, related to q by

$$q = e^{-h} \, .$$

The $q \to 1$ limit corresponds to the $h \to +0$ limit and $\dfrac{1}{1-q} \sim \dfrac{1}{h}$. The following proposition describes the behavior of the ϑ-function in this limit:

Proposition 4.1. *We have*

$$\frac{\vartheta(hx, e^{-h})}{\vartheta'(0, e^{-h})} = h \, \frac{\sin(\pi x)}{\pi} \, \exp\left(\frac{hx^2}{2}\right) \, \left(1 + O\left(e^{-\frac{4\pi^2}{h}}\right)\right) \qquad (4.1)$$

as $h \to +0$ uniformly in x. This asymptotic relation can be differentiated any number of times.

Proof. The Jacobi imaginary transformation (see e.g. Section 1.9 in [23]) yields

$$\vartheta(hx, e^{-h}) = \sqrt{\frac{2\pi}{h}} \, \exp\left(\frac{hx^2}{2}\right) \, \vartheta\left(-2\pi i x, e^{-\frac{4\pi^2}{h}}\right) \, .$$

We have

$$\vartheta\left(-2\pi i x, e^{-\frac{4\pi^2}{h}}\right) = \sum_{n \in Z} (-1)^n \, \exp\left(-\frac{2\pi^2(n+\frac{1}{2})^2}{h}\right) \, e^{-2\pi i (n+\frac{1}{2})x} \, .$$

It is obvious that this series, together with all derivatives, is dominated in the $h \to +0$ limit by only two terms, namely the terms with $n = 0$ and $n = -1$ which combine into a multiple of $\sin(\pi x)$. All other terms differ by a factor of at least $O\left(e^{-\frac{4\pi^2}{h}}\right)$. $\square$

Remark 4.2. As we will see below, all Laurent coefficients of all connected functions behave asymptotically like powers of h as $h \to +0$. Therefore, error terms of the form $\exp\big(-\mathrm{const}/h\big)$ are completely negligible.

We want to introduce an operation $\mathbf{A}$ of "taking the asymptotics" which replaces all ϑ-functions and their derivatives by their asymptotics as $h \to +0$. Since the n-point functions (2.7) and all connected functions are homogeneous in ϑ, we can ignore the constant factor $h\,\vartheta'(0, e^{-h})/\pi$. Let us, therefore, make the following:

Definition 4.3. Introduce the following substitution operator $\mathbf{A}$

$$\mathbf{A}(g) = g\Big|_{\vartheta(hx,\, e^{-h}) \,\mapsto\, \sin(\pi x)\ \exp(hx^2/2)}\,,$$

where g is any expression containing ϑ-functions and their derivatives.

In particular, we have

$$\mathbf{A}\left(\vartheta^{(k)}\,(hx)\right) = \frac{1}{h^k}\,\frac{d^k}{dx^k}\,\sin(\pi x)\,\exp\left(\frac{hx^2}{2}\right)\,, \tag{4.2}$$

where $k = 0, 1, 2, \dots$.

Definition 4.4. Introduce the following *asymptotic n-point function*

$$A(x_1, \dots, x_n) = \mathbf{A}\left(F(hx_1, \dots, hx_n)\right)\,.$$

In other words, this is the result of substituting (4.1) into the formula for the n-point functions and discarding the error terms. Similarly, define the *asymptotic connected functions*

$$A(x_1, \dots \mid \dots \mid \dots, x_n) = \mathbf{A}\left(F(hx_1, \dots \mid \dots \mid \dots, hx_n)\right)\,.$$

Our next goal is to derive a formula for the asymptotic n-point function. We will see that it is considerably more simple than the n-point functions (2.7).

Definition 4.5. Introduce the following function

$$\mathcal{S}(x_1, \dots, x_n) = \frac{\pi(x_1 + \dots + x_n)^{n-1}}{\sin(\pi(x_1 + \dots + x_n))}\,.$$

More generally, given any partition $\alpha \in \Pi_n$

$$\{1,\dots,n\} = \alpha_1 \sqcup \dots \alpha_{\ell(\alpha)}$$

set, by definition,

$$\mathcal{S}_\alpha(x_1,\dots,x_n) = \prod_{k=1}^{\ell(\alpha)} \mathcal{S}(x_{\alpha_k})\,,$$

where $x_{\alpha_k} = \{x_i\}_{i\in\alpha_k}$.

Remark 4.6. Because, eventually, we will be expanding the functions $\mathcal{S}$ into Laurent series we recall the following Taylor series

$$\frac{\pi x}{\sin(\pi x)} = \sum_{k=0}^{\infty} (2 - 2^{-2k+2})\,\zeta(2k)\,x^{2k}\,.$$

Theorem 4.7. *We have*

$$A(x_1,\dots,x_n) = e^{-\frac{h}{2}(\sum x_i)^2} \sum_{\alpha\in\Pi_n} h^{-\ell(\alpha)}\,\mathcal{S}_\alpha(x_1,\dots,x_n)\,, \tag{4.3}$$

where the summation is over all partitions α of the set $\{1,\dots,n\}$ and the functions S_α were defined in Definition 4.5.

Remark 4.8. It is clear that

$$A(x_1) = \exp\left(-\frac{hx_1^2}{2}\right)\,\frac{\pi}{h\,\sin(\pi x_1)}\,,$$

and, thus, (4.3) is satisfied if $n = 1$.

Remark 4.9. Recall that the n-point functions are, by their definition, certain averages over the set of all partitions. As $q \to 1$, larger and larger partitions play an important role in these averages, so the $q \to 1$ asymptotics of the n-point functions is, in a sense, the study of a very large random partition, see the Appendix. In particular, the factorization of the leading order asymptotics

$$A(x_1,\dots,x_n) = h^{-n}\prod_{i=1}^{n}\frac{\pi}{\sin(\pi x_i)} + O(h^{-n+1})\,,\quad h\to +0\,,$$

corresponds to the existence of Vershik's limit shape of a typical large partition.

The proof of Theorem 4.7 will be based on a sequence of lemmas. First, note that the denominators of all summands in (2.7) have a factor of $\vartheta(x_1 + \cdots + x_n)$. It is convenient to set, by definition,

$$\widetilde{F}(x_1, \dots, x_n) = \vartheta(x_1 + \cdots + x_n)\, F(x_1, \dots, x_n)\,.$$

Similarly, set

$$\widetilde{A}(x_1, \dots, x_n) = \mathbf{A}\left(\widetilde{F}(hx_1, \dots, hx_n)\right) = \sin\left(\pi\left(\sum x_i\right)\right) e^{\frac{h}{2}(\sum x_i)^2} A(x_1, \dots, x_n)\,.$$

We have the following

Lemma 4.10. *The function $\widetilde{A}(x_1, \dots, x_n)$ is a polynomial expression in h^{-1}, the variables x_i, and cotangents of the form $\cot\left(\pi \sum_{i\in S} x_i\right)$, where S is a subset of $\{1, \dots, n\}$. The degree of $\widetilde{A}$ in h^{-1} equals n.*

Proof. Observe that all ϑ-functions appear in $\widetilde{F}(hx_1, \dots, hx_n)$ in the following combinations: either they appear in pairs of the form

$$\frac{\vartheta^{(k)}(hy)}{\vartheta(hy)}\,, \quad y = \sum_{i\in S} x_i\,,$$

where S is a subset of $\{1, \dots, n\}$, or else they appear as the nullwerts

$$\vartheta^{(k)}(0)\,.$$

It is clear from (4.2) that the asymptotics in the either case is a polynomial in h^{-1} of degree k with coefficients involving y and $\cot(y)$. It remains to observe that in all monomials which appear in the expansion of the determinant in (2.7) the orders of the derivatives sum up to n. $\square$

It is clear that $A(x_1, \dots, x_n)$ is meromorphic with at most first order poles at the divisors

$$\mathcal{D}_{S,m} = \left\{\sum_{i\in S} x_i = m\right\}\,, \quad S \subset \{1, \dots, n\}\,, \quad m \in \mathbb{Z}\,,$$

and no other singularities.

Remark 4.11. For any nonsingular point $x = (x_1, \dots, x_n)$, the asymptotic n-point function $A(x)$ describes the polynomial in h terms in the asymptotics of $F(hx)$ as $h \to +0$. More generally, since the asymptotics (4.1) is uniform in x, any nonsingular contour integral of A represents the asymptotics of the corresponding integral for F. In particular, the residues of A at the divisors $\mathcal{D}_{S,m}$ are determined by the corresponding residues of F.

Lemma 4.12. *The function $A(x_1, \dots, x_n)$ is regular at the divisors $\mathcal{D}_{S,0}$ provided $|S| > 1$. At $\mathcal{D}_{\{1\},0} = \{x_1 = 0\}$ we have*

$$A(x_1, \dots, x_n) = \frac{1}{hx_1} A(x_2, \dots, x_n) + \dots$$

where dots stand for regular terms.

Proof. Follows, as explained in Remark 4.11, from the corresponding facts for F, see Section 9 in [1] or Section 3 of [24]. □

Lemma 4.13. *The function $A(x_1, \dots, x_n)$ satisfies the following difference equation*

$$A(x_1 - 1, \dots, x_n) = -e^{h\left(\sum x_i - \frac{1}{2}\right)} \times \\ \sum_{S=\{i_1, i_2, \dots\} \subset \{2,\dots,n\}} (-1)^{|S|} A(x_1 + x_{i_1} + x_{i_2} + \cdots, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_2}}, \dots), \quad (4.4)$$

where the sum is over all subsets S of $\{2, \dots, n\}$ and hats mean that the corresponding terms should be omitted.

Proof. Follows from the difference equation satisfied by F, see Section 8 in [1] or Section 3 of [24]. □

Definition 4.14. Given a partition $\alpha \in \Pi_n$ and a subset $S \subset \{1, \dots, n\}$, write $S \subset \alpha$ if S is a subset of one of the blocks of α.

Lemma 4.15. *The right-hand side of (4.3) satisfies the same difference equation (4.4) as A does.*

Proof. Observe that the binomial theorem and the definition of the function $\mathcal{S}$ imply that

$$\mathcal{S}(x_1 - 1, \dots, x_k) = \\ \sum_{S=\{i_1, i_2, \dots\} \subset \{2,\dots,k\}} (-1)^{|S|+1} \mathcal{S}(x_1 + x_{i_1} + x_{i_2} + \cdots, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_2}}, \dots),$$

where the sum is over all subsets S of $\{2,\ldots,k\}$ and hats mean that the corresponding terms should be omitted. Interchanging the order of summation in the partition α and in the subset S one obtains

$$\sum_{\alpha\in\Pi_n} h^{-\ell(\alpha)}\,\mathcal{S}_\alpha(x_1-1,\ldots,x_n) = \\ \sum_{\alpha\in\Pi_n} h^{-\ell(\alpha)} \sum_{\{1\}\cup S\subset\alpha} (-1)^{|S|+1}\mathcal{S}_\alpha(x_1+x_{i_1}+x_{i_2}+\cdots,\ldots,\widehat{x_{i_1}},\ldots) = \\ \sum_{S\subset\{2,\ldots,n\}} (-1)^{|S|+1} \sum_{\alpha'\in\Pi_{n-|S|}} h^{-\ell(\alpha')}\,\mathcal{S}_{\alpha'}(x_1+x_{i_1}+x_{i_2}+\cdots,\ldots)\,, \quad (4.5)$$

where α' is a partition of the set with $n-|S|$ elements which is obtained from the partition α by mapping $\{1\}\cup S$ to a point. Note that $\{1\}\cup S\subset\alpha$, which according to Definition 4.14 means that 1 and S belong to the same block of α, implies $\ell(\alpha)=\ell(\alpha')$.

Now the obvious identity

$$e^{-\frac{h}{2}(\sum x_i-1)^2} = e^{h\left(\sum x_i-\frac{1}{2}\right)}\,e^{-\frac{h}{2}(\sum x_i)^2} \quad (4.6)$$

completes the proof. □

Now we can complete the proof of Theorem 4.7

Proof of Theorem 4.7. By induction on n. The case $n=1$ is clear, see Remark 4.8.

Suppose $n>2$. Denote by $A^{[?]}(x_1,\ldots,x_n)$ the right-hand side of (4.3). We know that $A^{[?]}$ satisfies the same difference equation as $A(x_1,\ldots,x_n)$ does. We claim that it also has the same singularities as A does.

Indeed, $A^{[?]}$ is regular at the divisors $\mathcal{D}_{S,0}$, $|S|>0$, because $\mathcal{S}(x_1,\ldots,x_k)$ is regular at $\{x_1+\cdots+x_k=0\}$ provided $k>0$. It is also clear that on $\{x_1=0\}$ we have

$$A^{[?]}(x_1,\ldots,x_n) = \frac{1}{hx_1}A^{[?]}(x_2,\ldots,x_n)+\ldots$$

and so, by induction hypothesis, A and $A^{[?]}$ have identical singularities at all divisors $\mathcal{D}_{S,0}$. Since they also satisfy the same difference equation, all of their singularities are identical.

It follows that the function

$$\sin\left(\pi\left(\sum x_i\right)\right)\, e^{\frac{h}{2}(\sum x_i)^2}\left[A(x)-A^{[?]}(x)\right] \tag{4.7}$$

is regular everywhere. By the difference equation, the induction hypothesis, and (4.6) this function is also periodic in all x_i's with period 1. From Lemma 4.10 we conclude that (4.7) grows at most polynomially as $\Im x_i \to \infty$ and, therefore, it is a constant. Since both A and $A^{[?]}$ are regular at $\{x_1+\dots+x_n = 0\}$, the function (4.7) vanishes there. It follows that it is identically zero. This completes the proof. $\square$

We conclude this subsection by the following asymptotic version of Proposition 2.15. It is clear from Theorem 4.7 that the asymptotic n-point function $A(x_1,\dots,x_n)$ can be expanded into a Laurent series in $x_1,\dots,x_n$ in the neighborhood of the origin. Same is true about the asymptotic connected functions since they are polynomials in the n-point functions. The Laurent coefficients of these connected functions are responsible for the $h \to +0$ asymptotics of the cumulants:

Proposition 4.16. *We have*

$$\begin{aligned}\left\langle p_\mu \,|\, p_\nu \,|\, p_\eta \,|\, \dots \right\rangle_q &= \\ &h^{-|\mu|-|\nu|-|\eta|-\cdots}\,\mu!\,\nu!\,\eta!\cdots\left[x^\mu y^\nu z^\eta\cdots\right] A(x\,|\,y\,|\,z\,|\,\dots)+O(\dots)\,,\end{aligned} \tag{4.8}$$

where A is the asymptotic connected function, $|\mu|=\sum_i \mu_i$, and $O(\dots)$ stands for an error term of the following type

$$O(\dots)=O\left(\frac{e^{-\mathrm{const}/h}}{h^{\mathrm{const}}}\right).$$

Proof. The Laurent coefficients of A are certain contour integrals and hence by Remark 4.11 they represent the asymptotics of the corresponding coefficients of F. $\square$

Definition 4.17. Let $\langle\,\cdot\,\rangle_h$ denote the polynomial in h^{-1} part of the asymptotics of $\langle\,\cdot\,\rangle_q$ as $q=e^{-h}\to 1$, that is, the asymptotics of $\langle\,\cdot\,\rangle_q$ without the exponentially small terms.

For example, Proposition 3.2 can be restated as

$$\left\langle f_{m_1}\middle| f_{m_2}\middle|\dots\middle| f_{m_s}\right\rangle_h = \mathbf{c}(m)\,\frac{|m|!}{h^{|m|+1}}+\dots \tag{4.9}$$

where dots stand for terms of smaller degree in h^{-1}.

4.2 Asymptotics of the connected functions

The following notation will be useful in manipulation the connected functions. Recall that in Remark 2.5 we introduced a partial ordering on the set Π_n of partitions of an n-element set.

Definition 4.18. Let Q be a sequence of functions $Q(x_1, \dots, x_n)$, where $n = 1, 2, \dots$. For any partition $\alpha \in \Pi_n$ set, by definition

$$Q_\alpha(x_1, \dots, x_n) = \prod_{\text{blocks } \alpha_k} Q(x_{\alpha_k}),$$

where $x_{\alpha_k} = \{x_i\}_{i \in \alpha_k}$. Similarly, for any $\alpha \in \Pi_n$ introduce the corresponding connected function

$$Q\left(\big|_\alpha x\right) = Q\left(x_{\alpha_1} \,\big|\, x_{\alpha_2} \,\big|\, \dots\right) = \sum_{\beta \geq \alpha} (-1)^{\ell(\beta)-1} (\ell(\beta)-1)!\, Q_\beta(x).$$

It is clear that definition is consistent with Definitions 2.10, 4.5.

Definition 4.19. Given two partitions α and β, denote by $\alpha \wedge \beta$ the *meet* of α and β, that is, the minimal partition consisting of whole blocks of both α and β. We say that α and β are *transversal* and write $\alpha \perp \beta$ if

$$\ell(\alpha) + \ell(\beta) - \ell(\alpha \wedge \beta) = n.$$

Remark 4.20. Transversal pairs of partitions are extremal in the sense that for any $\alpha, \beta \in \Pi_n$ we have

$$\ell(\alpha) + \ell(\beta) - \ell(\alpha \wedge \beta) \leq n.$$

Indeed, any block β_k of β can intersect at most $|\beta_k|$ blocks of α and therefore

$$\ell(\alpha) - \ell(\alpha \wedge \beta) \leq \sum_{k=1}^{\ell(\beta)} (|\beta_k| - 1) = n - \ell(\beta).$$

In other words, $\alpha \perp \beta$ if β bonds the blocks of α as effectively as possible.

Our goal in this section is to prove a formula for the leading order asymptotics of connected functions as $h \to 0$. In other words, we want to compute the term with the minimal exponent of h in the asymptotic connected functions

$$A\left(\big|_{\rho}x\right) = \sum_{\alpha\geq\rho}(-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)!\,A_\alpha(x)\,,$$

where $x=(x_1,\dots,x_n)$, $\rho\in\Pi_n$, and the $A_\alpha(x)$'s are products of the asymptotic n-point functions. This leading order asymptotics is described in the following

Theorem 4.21. *As $h\to+0$ we have*

$$A\left(\big|_{\rho}x\right) = h^{-n+\ell(\rho)-1}\sum_{\alpha\perp\rho}\mathcal{S}_\alpha(x)\,\mathcal{T}_{\alpha\wedge\rho}(x) + O\left(h^{-n+\ell(\rho)}\right)\,,$$

where

$$\mathcal{T}_\beta(x) = (-1)^{\ell(\beta)-1}\left(\sum x_i\right)^{\ell(\beta)-2}\prod_{\text{blocks }\beta_k}\left(\sum_{i\in\beta_k}x_i\right)\,.$$

Remark 4.22. Observe that if $\ell(\beta)=1$ then $\mathcal{T}_\beta(x)=1$.

In preparation for the proof of Theorem 4.21 we introduce the following function

$$E(x_1,\dots,x_n) = \exp\left(-\frac{h}{2}\left(\sum x_i\right)^2\right)\,.$$

It is clear that

$$E(x) = 1+O(h)\,,\quad h\to 0\,.$$

The next proposition describes the $h\to 0$ asymptotics of the connected versions of E

Proposition 4.23. *Let $\rho\in\Pi_n$ be a partition. As $h\to 0$ we have*

$$E\left(\big|_{\rho}x\right) = h^{\ell(\rho)-1}\,\mathcal{T}_\rho(x) + O(h^{\ell(\rho)})\,.$$

Proof. Recall that, by definition,

$$E\left(\big|_{\rho}x\right) = \sum_{\alpha\geq\rho}(-1)^{\ell(\alpha)-1}(\ell(\alpha)-1)!\,E_\alpha(x)\,.$$

We have

$$E_\alpha(x) = \exp\left(-\frac{h}{2}\sum x_i^2\right)\,\exp\left(-h\sum_{\{i\neq j\}\subset\alpha}x_ix_j\right)\,, \tag{4.10}$$

where, we recall Definition 4.14, $\{i,j\} \subset \alpha$ means that $\{i,j\}$ is a subset of a block of α.

The first factor in (4.10) is a common factor for all α. The Taylor series expansion of the second factor in (4.10) can be interpreted as summation over certain graphs Γ with multiple edges

$$\exp\left(-h \sum_{\{i\neq j\}\subset\alpha} x_i x_j\right) = \sum_{\Gamma\subset\alpha} (-h)^{\sum m(e)} \prod_{\text{edges } e=\{i,j\}} \frac{(x_i x_j)^{m(e)}}{m(e)!}\,,$$

where $\Gamma \subset \alpha$ means that $e = \{i,j\} \subset \alpha$ for any egde e of Γ, no edges from a vertex to itself are allowed, and $m(e)$ is a nonnegative integer, called multiplicity, which is assigned to any edge e.

The Möbius inversion in the partially ordered set Π_n, see Remark 2.5, implies that

$$E\left(\big|_\rho x\right) = e^{-\frac{h}{2}\sum x_i^2} \sum_{\rho\text{-connected } \Gamma} (-h)^{\sum m(e)} \prod_{\text{edges } e=\{i,j\}} \frac{(x_i x_j)^{m(e)}}{m(e)!}\,,$$

where ρ-connected means that Γ becomes connected after collapsing all blocks of ρ to points, that is, after passing to the quotient

$$\{1,\dots,n\} \to \{1,\dots,n\}/\rho \cong \{1,\dots,\ell(\rho)\}\,.$$

It is now clear that the minimal possible exponent of h is $\ell(\rho)-1$ and it is achieved by those graphs Γ which have no multiple edges and project onto spanning trees of $\{1,\dots,\ell(\rho)\}$. That is,

$$E\left(\big|_\rho x\right) = (-h)^{\ell(\rho)-1} \sum_{\substack{\text{spanning trees}\\ \text{on } \{1,\dots,\ell(\rho)\}}} \prod_{\text{edges } e=\{k,l\}} y_k y_l + O(h^{\ell(\rho)})\,,$$

where $y_k = \sum_{i\in\rho_k} x_i$. It is known, see Problem 3.3.44 in [8], that this sum over spanning trees equals

$$\sum_{\text{spanning trees}} = \left(\sum y_k\right)^{\ell(\rho)-2} \prod_k y_k\,,$$

which concludes the proof. $\square$

Remark 4.24. Call a forest with vertices $\{1,\dots,n\}$ a *ρ-spanning forest* if it has $\ell(\rho)-1$ edges and connects all blocks of ρ. Equivalently, a forest is ρ-spanning if it projects onto a spanning tree on the quotient $\{1,\dots,n\}/\rho$.

It is clear from the proof of the above proposition that

$$\mathcal{T}_\rho = (-1)^{\ell(\rho)-1} \sum_{\substack{\rho\text{-spanning}\\ \text{forests}}} \prod_{\text{edges } e=\{i,j\}} x_i x_j \,.$$

Proof of Theorem 4.21. By definition, we have

$$A\left(\big|_\rho x\right) = \sum_{\beta\geq\rho} (-1)^{\ell(\beta)-1}(\ell(\beta)-1)!\, A_\beta(x)\,.$$

Substituting Theorem 4.7 in this sum yields

$$A\left(\big|_\rho x\right) = \sum_{\beta\geq\rho} (-1)^{\ell(\beta)-1}(\ell(\beta)-1)!\, E_\beta(x) \sum_{\alpha\leq\beta} h^{-\ell(\alpha)}\, \mathcal{S}_\alpha(x)\,.$$

Interchanging the order of summation we obtain

$$\begin{aligned} A\left(\big|_\rho x\right) &= \sum_{\alpha} h^{-\ell(\alpha)}\, \mathcal{S}_\alpha(x) \sum_{\beta\geq\alpha\wedge\rho} (-1)^{\ell(\beta)-1}(\ell(\beta)-1)!\, E_\beta(x) \\ &= \sum_{\alpha} h^{-\ell(\alpha)}\, \mathcal{S}_\alpha(x)\, E\left(\big|_{\alpha\wedge\rho} x\right)\,. \end{aligned}$$

Using Proposition 4.23 we conclude that

$$A\left(\big|_\rho x\right) = \sum_{\alpha} h^{-\ell(\alpha)+\ell(\alpha\wedge\rho)-1}\, \mathcal{S}_\alpha(x)\, \left(\mathcal{T}_{\alpha\wedge\rho} + \dots\right)\,,$$

where dots stand for lower order terms. We know from Remark 4.20 that the exponent $-\ell(\alpha)+\ell(\alpha\wedge\rho)-1$ takes its minimal value $-n+\ell(\rho)-1$ precisely when $\alpha\perp\rho$. This concludes the proof. $\square$

Definition 4.25. Introduce the following notation for the coefficient of h in the leading asymptotics of the connected functions

$$A_{\text{lead}}\left(\big|_\rho x\right) = \sum_{\alpha\perp\rho} \mathcal{S}_\alpha(x)\, \mathcal{T}_{\alpha\wedge\rho}(x)\,. \tag{4.11}$$

It is clear that Proposition 4.16 and Theorem 4.21 imply the formula for the leading asymptotics of the cumulants as $h \to +0$

Definition 4.26. We call the number $\mathrm{wt}(\mu) = |\mu| + \ell(\mu)$ the weight of a partition μ.

Theorem 4.27. *Let $\mu, \dots, \eta$ be a collection of s partitions. Then*

$$\langle p_\mu \,|\, \dots \,|\, p_\eta \rangle_h = \frac{\mu! \cdots \eta! \left[x^\mu \cdots z^\eta\right] A_{\mathrm{lead}}(x \,|\, \dots \,|\, z)}{h^{\mathrm{wt}(\mu)+\dots+\mathrm{wt}(\eta)-s+1}} + \dots, \tag{4.12}$$

where $\mathrm{wt}(\mu) = |\mu| + \ell(\mu)$ *and dots stand for terms of smaller degree in* h^{-1}.

We will address the task of actually picking the Laurent coefficients of $A_{\mathrm{lead}}\left(\big|_\rho x\right)$ below in Sections 6 and 7. First, we take a small detour and consider the properties of the weight function $\mathrm{wt}(\mu)$ which was introduced in Theorem 4.27

5 Weight filtration in Λ^*

5.1 Weight grading and weight filtration

The weight function $\mathrm{wt}(\mu) = |\mu| + \ell(\mu)$ introduced in Definition 4.26 has the following interpretation.

It is known, see [1], and can be seen from the formula (2.7) for the n-point functions, that for any partition μ

$$\langle p_\mu \rangle_q \in QM_{\mathrm{wt}(\mu)},$$

where QM_* is the graded algebra of the quasi-modular form which is the polynomial algebra in the Eisenstein series $G_k(q)$, $k = 2, 4, 6$. Therefore, the *weight grading* of Λ^* which is defined by assigning the generators $\{p_k\}$ the weights

$$\mathrm{wt}(p_k) = k + 1, \quad k = 1, 2, \dots,$$

is very natural in the sense that the linear map $\langle \cdot \rangle_q : \Lambda^* \to QM_*$ preserves it. It is clear that

$$\langle p_\mu \,|\, \dots \,|\, p_\eta \rangle_q \in QM_{\mathrm{wt}(\mu)+\dots+\mathrm{wt}(\eta)}.$$

Proposition 4.27 says that

$$\left\langle p_\mu \,|\, \dots \,|\, p_\eta \right\rangle_h = \text{const}\, h^{-\,\mathrm{wt}(\mu)-\cdots-\mathrm{wt}(\eta)+\#\text{ of partitions}-1} + \dots .$$

Since we are interested in the coefficient of the lowest power of h which is not identically zero by weight considerations, we introduce the following

Definition 5.1. We call the filtration of Λ^* associated to the weight grading the *weight filtration.*

It is clear that for any $g_1, \dots, g_s \in \Lambda^*$ the constant in the expansion

$$\left\langle g_1 \,|\, \dots \,|\, g_s \right\rangle_h = \text{const}\, h^{-\sum \mathrm{wt}(g_i)+s-1} + \dots$$

depends only on the top weight terms of $g_1, \dots, g_s$.

5.2 Elementary description of the weight filtration

In contrast to the weight grading, the weight filtration is very easy to describe in completely elementary terms.

By construction, the algebra Λ^* is a projective limit of the algebras $\Lambda^*(n)$ of shifted symmetric functions in n variables. The algebra $\Lambda^*(n)$ is isomorphic to the algebra of symmetric polynomials in

$$\xi_i = \lambda_i - i + \text{const}\,, \quad i = 1, \dots, n\,,$$

where any constant will do.

It is easy to see that the induced filtration of $\Lambda^*(n)$ is the same as the one obtained by assigning weight $(k+1)$ to the polynomial

$$\bar{p}_k = \sum \xi_i^k\,, \quad k = 1, 2, \dots .$$

Let $\bar{m}_\mu \in \Lambda^*(n)$ be the monomial symmetric function in the ξ_i's, that is, the sum of all monomials which can be obtained from ξ^μ by permuting the ξ_i's. Recall that the notation $\mu = 1^{\varkappa_1} 2^{\varkappa_2} 3^{\varkappa_3} \dots$ means that μ has $\varkappa_k$ parts equal to k. The following lemma is immediate

Lemma 5.2. *For any partition* $\mu = 1^{\varkappa_1} 2^{\varkappa_2} 3^{\varkappa_3} \dots$ *we have*

$$\bar{p}_\mu = \prod \bar{p}_{\mu_i} = \varkappa!\, m_\mu + \dots\,,$$

where dots stand for lower weight terms.

Definition 5.3. Define the weight of a monomial ξ^μ by $\mathrm{wt}(\xi^\mu) = \mathrm{wt}(\mu) = |\mu| + \ell(\mu)$ or, in other words,

$$\text{weight} = \text{degree} + \#\text{ of variables}\,.$$

It is clear that the k-th subspace of the weight filtration is spanned by monomials of weight $\leq k$. In other words, we have the following

Proposition 5.4. *The weight of any shifted symmetric function g is the maximum of the weights of all monomials in g.*

5.3 Top weight term of f_k

The purpose of this subsection is to prove the following formula for the top weight term of f_k

Theorem 5.5. *We have*

$$f_k = k^{-1} \sum_{\mathrm{wt}(\lambda)=k+1} \frac{(-k)^{\ell(\lambda)-1}}{\varkappa!}\, p_\lambda + \dots\,,$$

where the sum is over all partitions $\lambda = 1^{\varkappa_1} 2^{\varkappa_2} 3^{\varkappa_3} \dots$ of weight $k+1$ and dots stand for lower weight terms.

Remark 5.6. In fact, the dots in the above formula stand for terms of weight at most $k-1$ as will be shown in the next subsection. In particular, since there are no partitions of weight 1 we have

$$f_2 = \tfrac{1}{2}\, p_2 \tag{5.1}$$

Proof. We can assume that the number of variables λ_i is finite and equal to $n \gg 0$ and switch to the variables $\xi_i = \lambda_i + n - i$. It is known, see Example I.7.7 in [18] , that

$$f_k = \frac{1}{k} \sum_{i=1}^{n} (\xi_i \downarrow k) \prod_{j \neq i} \left(1 - \frac{k}{\xi_i - \xi_j}\right)\,,$$

where $(\xi_i \downarrow k) = \xi_i(\xi_i - 1)\cdots(\xi_i - k + 1)$. Expand all fractions in geometric series assuming that

$$|\xi_1| > |\xi_2| > \dots > |\xi_n|\,.$$

We have

$$f_k = \frac{1}{k}\sum_{i=1}^{n} (\xi_i \downarrow k) \prod_{j=1}^{i-1} \left(1 + k\sum_{l=0}^{\infty} \frac{\xi_i^l}{\xi_j^{l+1}}\right) \prod_{j=i+1}^{n} \left(1 - k\sum_{l=0}^{\infty} \frac{\xi_j^l}{\xi_i^{l+1}}\right)$$

Now let μ is a partition of weight $k+1$ and let us compute the coefficient of ξ^μ in the above expression.

Observe that only the first summand produces positive powers of ξ_1 and, moreover, the monomials of maximal weight come from the expansion of

$$\xi_1^k \prod_{j=2}^{n} \left(1 - k\sum_{l=0}^{\infty} \frac{\xi_j^l}{\xi_1^{l+1}}\right) \tag{5.2}$$

Clearly, the coefficient of ξ^μ in the expansion of (5.2) equals $(-k)^{\ell(\mu)-1}$. By Lemma 5.2 this concludes the proof. $\square$

The statement of Theorem 5.5 can be rewritten as follows

$$f_k = -\frac{1}{k^2} \sum_{\sum_i (i+1)\varkappa_i = k+1} \prod \frac{(-k\, p_i)^{\varkappa_i}}{\varkappa_i!} + \dots\,,$$

where the summation is over all $(\varkappa_1, \varkappa_2, \dots) \in \mathbb{Z}_{\geq 0}^\infty$ satisfying the condition $\sum_i (i+1)\varkappa_i = k+1$. This can be restated as follows.

Proposition 5.7. *We have*

$$f_k = -k^{-2}\, \left[z^{k+1}\right]\, P(z)^k + \dots\,,$$

where $\left[z^{k+1}\right]$ *stands for the coefficient of* z^{k+1}*, the dots stand for the lower order terms, and* $P(z)$ *is the following generating function*

$$P(z) = \exp\left(-\sum_{i\geq 1} z^{i+1}\, p_i\right)\,.$$

5.4 Involution and parity in Λ^*

The algebra Λ^* has a natural involutive automorphism ω which acts as follows

$$[\omega \cdot f](\lambda) = f(\lambda')\,,$$

where λ is a partition and λ' the dual partition (that is, the result of flipping the diagram of λ along the diagonal), see Section 4 in [25].

For any permutation g, we have

$$\chi^{\lambda'}(g) = \operatorname{sgn}(g)\, \chi^{\lambda}(g)$$

and, therefore,

$$\omega \cdot f_k = (-1)^{k+1} f_k \,.$$

Similarly it can be shown (for example, by expanding the statement of Lemma 5.1 in [1] into a series) that

$$\omega \cdot p_k = (-1)^{k+1} p_k \,.$$

It follows that the expansion of f_k in p_μ contains only terms of weight

$$\operatorname{wt}(\mu) \equiv k+1 \mod 2 \,,$$

which justifies Remark 5.6.

Remark 5.8. Note that since $|\lambda| = |\lambda'|$ we have

$$\left\langle f \right\rangle_q = \left\langle \omega \cdot f \right\rangle_q$$

for any $f \in \Lambda^*$. In particular,

$$\left\langle p_\mu \right\rangle_q = \left\langle f_\mu \right\rangle_q = 0 \,, \quad \operatorname{wt}(\mu) \equiv 1 \mod 2 \,,$$

which, of course, makes sense since there are no quasimodular forms of odd weight. In terms of coverings, this parity condition just means that the product of monodromies of all ramifications has to be an even permutation.

6 Asymptotics of cumulants

6.1 Analog of Wick's formula for cumulants

Given a multi-index $m = (m_1, \dots, m_n)$ and a partition $\rho \in \Pi_n$, we write

$$\left\langle \big|_\rho\, p_m \right\rangle_h = \left\langle \prod_{i \in \rho_1} p_{m_i} \,\middle|\, \dots \,\middle|\, \prod_{i \in \rho_{\ell(\rho)}} p_{m_i} \right\rangle_h \,.$$

Recall that the Wick formula is a rule to compute expectations of any polynomial in Gaussian normal variables η_i given means $\langle \eta_i \rangle$ and covariances $\langle \eta_i \,|\, \eta_j \rangle$ of these variables. Our purpose in this section is to prove a similar rule which reduces the computation of any cumulants $\left\langle \big|_\rho \, p_m \right\rangle_h$ to computations of the following elementary ones:

Definition 6.1. We call the coefficients $\langle\!\langle\, m \,\rangle\!\rangle = \langle\!\langle\, m_1, \dots, m_n \,\rangle\!\rangle$ in the expansion

$$\langle p_{m_1} \,|\, \dots \,|\, p_{m_n} \rangle_h = \frac{\langle\!\langle\, m \,\rangle\!\rangle}{h^{|m|+1}} + \dots \,,$$

the *elementary cumulants.*

To state the analog of the Wick rule we need the following:

Definition 6.2. Given two partitions $\alpha, \beta \in \Pi_n$ we say that they are *complementary* and write $\alpha \top \beta$ if $\alpha \perp \beta$ and $\alpha \wedge \beta = \widehat{n}$, where $\widehat{n} \in \Pi_n$ is the partition into one block. In other words, $\alpha \top \beta$ if β bonds all parts of α and does so using the minimal number of bonds.

Now we have the following Wick-type formula:

Theorem 6.3. *We have*

$$\left\langle \big|_\rho \, p_m \right\rangle_h = h^{-\operatorname{wt}(m) + \ell(\rho) - 1} \sum_{\alpha \top \rho} \prod_{k=1}^{\ell(\alpha)} \langle\!\langle\, m_{\alpha_k} \,\rangle\!\rangle + \dots \,,$$

where $m_{\alpha_k} = \{m_i\}_{i \in \alpha_k}$, $\operatorname{wt}(m) = \sum (m_i + 1)$, *and dots stand for lower order terms.*

Example 6.4. We have

$$\begin{aligned} h^{a+b+c+d+2} \, \langle p_a \,|\, p_b \,|\, p_c \, p_d \rangle_h = \langle\!\langle\, a, b, c \,\rangle\!\rangle \, \langle\!\langle\, d \,\rangle\!\rangle + \langle\!\langle\, a, b, d \,\rangle\!\rangle \, \langle\!\langle\, c \,\rangle\!\rangle \\ + \langle\!\langle\, a, c \,\rangle\!\rangle \, \langle\!\langle\, b, d \,\rangle\!\rangle + \langle\!\langle\, a, d \,\rangle\!\rangle \, \langle\!\langle\, b, c \,\rangle\!\rangle + \dots \,. \end{aligned}$$

Example 6.5. Note, in particular, that if $\rho = \widehat{n}$ then the only partition complementary to ρ is the partition into 1-element blocks. It follows that

$$\langle p_m \rangle_h = h^{-\operatorname{wt}(m)} \prod_i \langle\!\langle\, m_i \,\rangle\!\rangle + \dots \,.$$

This factorization of the leading order asymptotics corresponds to the limit shape for uniform measure on partitions, see Section A.1.

Similarly, we have

$$\left\langle p_\mu \middle| p_\nu \right\rangle_h = h^{-\operatorname{wt}(\mu)-\operatorname{wt}(\nu)+1} \sum_{k,l} \langle\!\langle\, \mu_k, \nu_l \,\rangle\!\rangle \prod_{i\neq k} \langle\!\langle\, \mu_i \,\rangle\!\rangle \prod_{j\neq l} \langle\!\langle\, \nu_j \,\rangle\!\rangle + \dots \,,$$

which is the usual Wick's rule for the Gaussian correction to the limit shape, see Section A.3. The covariance matrix of this Gaussian correction is

$$\operatorname{Covar}(p_k, p_l) = h^{-k-l-1} \; \langle\!\langle\, k, l \,\rangle\!\rangle \;.$$

Proof. By definition of the cumulants and of the elementary cumulants, we have

$$\langle p_m \rangle_h = h^{-|m|-\ell(\alpha)} \sum_{\alpha} \prod_{k=1}^{\ell(\alpha)} \langle\!\langle\, m_{\alpha_k} \,\rangle\!\rangle + \dots \,,$$

and therefore

$$\left\langle \big|_\rho \, p_m \right\rangle_h = h^{-|m|-\ell(\alpha)} \sum_{\alpha\wedge\rho=\widehat{n}} \prod_{k=1}^{\ell(\alpha)} \langle\!\langle\, m_{\alpha_k} \,\rangle\!\rangle + \dots \,.$$

By Remark 4.20, for any α such that $\alpha \wedge \rho = \widehat{n}$, we have

$$|m| + \ell(\alpha) \leq |m| - \ell(\rho) + n + 1 = \operatorname{wt}(m) - \ell(\rho) + 1 \,,$$

with the equality if and only if $\alpha \top \rho$. $\square$

Theorem 6.3 reduces the computation of the asymptotics of cumulants to the asymptotics $\langle\!\langle\, m \,\rangle\!\rangle$ of the elementary cumulants. These numbers will be considered in the following subsection.

6.2 Asymptotics of elementary cumulants

Definition 6.6. We set

$$\mathfrak{z}(k) = \begin{cases} (2-2^{2-k})\,\zeta(k) & k \text{ even}\,, \\ 0 & k \text{ odd}\,. \end{cases}$$

By Remark 4.6 this means that

$$\frac{\pi x}{\sin(\pi x)} = \sum_{k=0}^{\infty} \mathfrak{z}(k)\, x^{2k} \,. \tag{6.1}$$

If ρ is a partition into 1-element blocks then any α is transversal to it and $\rho \wedge \alpha = \alpha$. Therefore, from Theorem 4.27 and (6.1) we get

$$\begin{aligned}\langle\!\langle\, m \,\rangle\!\rangle &= m!\, \left[x^m\right] \sum_{\alpha} \mathcal{S}_\alpha(x)\, \mathcal{T}_\alpha(x) \\ &= m!\, \left[x^m\right] \sum_{\alpha} (-1)^{\ell(\alpha)-1} \left(\sum x_i\right)^{\ell(\alpha)-2} \prod_{k=1}^{\ell(\alpha)} \sum_{j=0}^{\infty} \mathfrak{z}(j) \left(\sum_{i\in\alpha_k} x_i\right)^{j+|\alpha_k|-1} ,\end{aligned}$$

where $m = (m_1, \ldots, m_n)$ is a multi-index, the sum is over all partitions $\alpha \in \Pi_n$.

Using the expansion

$$\left(\sum x_i\right)^{\ell(\alpha)-2} = \sum_{d_1,\ldots,d_{\ell(\alpha)}} \binom{\ell(\alpha)-2}{d_1,\ldots,d_{\ell(\alpha)}} \prod_k \left(\sum_{i\in\alpha_k} x_i\right)^{d_k}$$

we obtain the following

Theorem 6.7. *For any $m = (m_1, \ldots, m_n)$ we have*

$$\langle\!\langle\, m \,\rangle\!\rangle = \sum_{\alpha\in\Pi_n} (-1)^{\ell(\alpha)-1} (\ell(\alpha)-2)! \times \sum_{d} (d!)^{-1} \prod_{k=1}^{\ell(\alpha)} |m_{\alpha_k}|!\; \mathfrak{z}\left(|m_{\alpha_k}| - |\alpha_k| - d_k + 1\right) , \tag{6.2}$$

where $|m_{\alpha_k}| = \sum_{\alpha_k} m_i$ and the summation is over all $\ell(\alpha)$-tuples

$$d = (d_1, \ldots, d_{\ell(\alpha)})$$

of nonnegative integers such that $\sum d_k = \ell(\alpha) - 2$ and

$$d_k \equiv 1 + |m_{\alpha_k}| - |\alpha_k| \mod 2 \,, \quad k = 1, \ldots, \ell(\alpha) \,.$$

The $\alpha = \widehat{n}$ term in (6.2) should be understood as $|m|!\, \mathfrak{z}(|m| - n + 2)$.

Remark 6.8. Given a partition μ with even parts, write

$$\mathfrak{z}_\mu = \prod_i \mathfrak{z}(\mu_i)\,.$$

Observe that all $\mathfrak{z}_\mu$ which appear in (6.2) satisfy

$$|\mu| = |m| - n + 2\,.$$

For any k, we have $\mathfrak{z}(k)/\pi^k \in \mathbb{Q}$, and hence

$$\langle\!\langle\, m_1, \dots, m_n \,\rangle\!\rangle \,/\pi^{|m|-n+2} \in \mathbb{Q}\,.$$

Example 6.9. In particular, we have

$$\langle\!\langle\, k \,\rangle\!\rangle = k!\, \mathfrak{z}(k+1)\,, \tag{6.3}$$

$$\langle\!\langle\, k, l \,\rangle\!\rangle = (k+l)!\, \mathfrak{z}(k+l) - k!\, l!\, \mathfrak{z}(k)\, \mathfrak{z}(l)\,, \tag{6.4}$$

As already mentioned in Example 6.5, these formulas describe the limit shape of a large random partitions and the covariance matrix for the central limit correction to it.

Remark 6.10. In general, all $\mathfrak{z}_\mu$ which appear in (6.2) are distinct. However, for small m there may be many like terms and collecting them may lead to substantial simplifications. In the next section, we will consider the most interesting of such special cases, namely, the case of $m = (2, \dots, 2)$ which corresponds to the case of simple branched coverings.

7 The case of simple branched coverings

Consider the case when all ramifications are simple, that is, their monodromies only transpose a pair of sheets of the covering. Such coverings are enumerated by $\left\langle f_2 \,|\, \dots \,|\, f_2 \right\rangle_q$. By virtue of (5.1), the asymptotics $\mathbf{c}(2, \dots, 2)$ of the number of simple coverings is the following

$$\mathbf{c}(\underbrace{2, \dots, 2}_{n \text{ times}}) = \frac{1}{(2n)!\, 2^n} \langle\!\langle\, 2, \dots, 2 \,\rangle\!\rangle\,.$$

The aim of this section is to prove the following

Theorem 7.1. *We have*

$$\frac{\mathbf{c}(2,\dots,2)}{n!} = \sum_{\substack{\text{even } \mu \\ |\mu|=n+2}} \frac{(-1)^{\ell(\mu)-1}}{\varkappa!\,(2n-\ell(\mu)+2)!} \left(\prod_i (2\mu_i-3)!!\right) \mathfrak{z}_\mu \,, \tag{7.1}$$

where n is the number of 2's, the summation is over all even partitions $\mu = 2^{\varkappa_2}4^{\varkappa_4}6^{\varkappa_6}\dots$ of the number $n+2$, and $\varkappa! = \varkappa_2!\,\varkappa_4!\cdots$.

The following lemma is well known and elementary to prove

Lemma 7.2. *For any function h and any $L = 1,2,\dots$ we have*

$$\frac{1}{L!}\left(\sum_{k=1}^{\infty} h(k)\,\frac{t^k}{k!}\right)^L = \sum_{n=l}^{\infty} \frac{t^n}{n!} \sum_{\alpha\in\Pi_n,\,\ell(\alpha)=L} \prod_1^L h\left(|\alpha_k|\right) \,,$$

where the summation is over all partitions $\alpha \in \Pi_n$ which have exactly L parts.

Proof. Follows by extracting terms of degree L in h from the formula

$$\exp\left(\sum_{k=1}^{\infty} h(k)\,\frac{t^k}{k!}\right) = \prod_{k=1}^{\infty} \exp\left(h(k)\,\frac{t^k}{k!}\right) = \sum_{n=l}^{\infty} \frac{t^n}{n!} \sum_{\alpha\in\Pi_n} \prod h\left(|\alpha_k|\right) \,.$$

□

Proof of Theorem 7.1. The formula (6.2) specializes to

$$\langle\!\langle\, 2,\dots,2 \,\rangle\!\rangle = \sum_{\alpha\in\Pi_n} (-1)^{\ell(\alpha)-1}(\ell(\alpha)-2)! \times$$

$$\sum_d \prod_{k=1}^{\ell(\alpha)} \frac{(2\,|\alpha_k|)!}{d_k!}\, \mathfrak{z}\left(|\alpha_k| - d_k + 1\right) \,, \tag{7.2}$$

where the summation is over $d = (d_1,\dots,d_{\ell(\alpha)})$ satisfying the conditions described above. In particular, $\sum d_i = \ell(\alpha) - 2$.

Recall that the $\alpha = \widehat{n}$ term in (7.2) is to be understood as $(2n)!\,\mathfrak{z}(n+2)$. This is in agreement with the coefficient of $\mathfrak{z}(n+2)$ in (7.1). Therefore, in what follows we can assume that $\ell(\alpha) \geq 2$.

We know from Remark 6.8 that all $\mathfrak{z}_\mu$ appearing in (7.2) satisfy

$$|\mu| = n + 2\,.$$

Let $\mu = 2^{\varkappa_2} 4^{\varkappa_4} 6^{\varkappa_6}$ be one such partition and pick the coefficient of $\mathfrak{z}_\mu$ in (7.2). This means that of $\ell(\alpha)$ blocks of α we have to chose $\varkappa_2$ blocks for which we take $d_k = |\alpha_k| - 1$ so that to produce the factor of $\mathfrak{z}(2)^{\varkappa_2}$. After that, we select $\varkappa_4$ blocks of α for which we take $d_k = |\alpha_k| - 3$, and so on. In the remaining

$$\varkappa_0 = \ell(\alpha) - \ell(\mu)$$

parts of α we take $d_k = |\alpha_k| + 1$ which results in the factor $\mathfrak{z}(0) = 1$. This can be imagined as painting the parts of α into different colors which we call "0", "2", "4" etc.

Observe that the summands in (7.2) depend not on the actual partition α but rather on the sizes of blocks of a given color. For any color $s = 0, 2, 4, \dots$ we can use Lemma 7.2 with $h_s(k) = \frac{(2k)!}{(k-s+1)!}$ and this yields the following formula

$$\left[\mathfrak{z}_\mu\right] \langle\!\langle\, 2, \dots, 2 \,\rangle\!\rangle = n!\, \left[t^n\right] \sum_{l=2}^{\infty} (-1)^{l-1} (l-2)! \times \\ \prod_{s=0,2,\dots} \frac{1}{\varkappa_s!} \left(\sum_{k=1}^{\infty} \frac{(2k)!}{(k-s+1)!\, k!}\, t^k \right)^{\varkappa_s}, \quad (7.3)$$

where $\varkappa_0 = l - \ell(\mu)$ is the only one of the $\varkappa_i$'s that depends on l.

We have

$$\sum_{k=0}^{\infty} \binom{2k}{k} t^k = \frac{1}{\sqrt{1-4t}}$$

and, therefore, for $s = 2, 4, 6, \dots$ we obtain

$$\sum_{k=1}^{\infty} \frac{(2k)!}{(k-s+1)!\, k!}\, t^k = t^{s-1} \frac{d^{s-1}}{dt^{s-1}} \frac{1}{\sqrt{1-4t}} \\ = 2^{s-1}\, (2s-3)!!\, \frac{t^{s-1}}{(1-4t)^{s-1/2}} \quad (7.4)$$

For $s = 0$, introduce the following notation

$$H = \sum_{k=1}^{\infty} \frac{(2k)!}{(k+1)!\, k!}\, t^k = \frac{1 - \sqrt{1-4t}}{2t} - 1\,.$$

We compute

$$\sum_{l=2}^{\infty}(-1)^{l-1}\frac{(l-2)!}{(l-\ell(\mu))!}\,H^{l-\ell(\mu)} = (-1)^{\ell(\mu)-1}\,(\ell(\mu)-2)!\,(1+H)^{1-\ell(\mu)}$$
$$= (-1)^{\ell(\mu)-1}\,(\ell(\mu)-2)!\frac{2^{\ell(\mu)-1}\,t^{\ell(\mu)-1}}{\left(1-\sqrt{1-4t}\right)^{\ell(\mu)-1}}\,.$$

Putting it all together using the equalities

$$\sum s\,\varkappa_s = |\mu| = n+2\,,\quad \sum \varkappa_s = \ell(\mu)$$

we obtain

$$\left[\mathfrak{z}_\mu\right]\langle\!\langle\, 2,\dots,2\,\rangle\!\rangle = (-1)^{\ell(\mu)-1}\,n!\,\frac{2^{n+1}\,(\ell(\mu)-2)!}{\prod_{s\geq 2}\varkappa_s!}\left(\prod_i(2\mu_i-3)!!\right)\times$$
$$\left[t^n\right]\frac{t^{n+1}}{(1-4t)^{n+2-\ell(\mu)/2}\,\left(1-\sqrt{1-4t}\right)^{\ell(\mu)-1}}\,,$$

It remains to show that

$$\left[t^{-1}\right]\frac{1}{(1-4t)^{n+2-l/2}\,\left(1-\sqrt{1-4t}\right)^{l-1}} = \frac{1}{2}\binom{2n}{l-2}\,.$$

Recall that the residue of a differential form is independent on the choice of coordinates. Using the change of variables $z = 1-\sqrt{1-4t}$, which implies that $dt = \frac{1}{2}(1-z)\,dz$, we compute

$$\left[t^{-1}\right]\frac{1}{(1-4t)^{n+2-l/2}\,\left(1-\sqrt{1-4t}\right)^{l-1}} =$$
$$\frac{1}{2}\left[z^{-1}\right]\frac{1}{(1-z)^{2n+3-l}\,z^{l-1}} = \frac{1}{2}\binom{2n}{l-2}\,.$$

This concludes proof. $\square$

A Large random partitions

A.1 Leading asymptotics and Vershik's limit shape of a typical random partition

The computations we do in this paper can be interpreted probabilistically as follows. We consider the following probability measure $\mathfrak{P}$ on partitions

$$\mathfrak{P}(\lambda) = \frac{e^{-h|\lambda|}}{Z},$$

where Z is the partition function

$$Z = \sum_\lambda q^{|\lambda|} = \prod_{n\geq 1} (1-q^n)^{-1}, \quad q = e^{-h}.$$

It is a Gibbsian measure with the energy function $\lambda \mapsto |\lambda|$ and inverse temperature h.

Our algebra Λ^* is naturally an algebra of functions on partitions and what we can compute is the polynomial terms in the asymptotics of the corresponding expectations $\langle \cdot \rangle_h$ as $h \to +0$. The limit $h \to +0$ describes the behavior of random partitions of N as $N \to \infty$ and some properties of $\langle \cdot \rangle_h$ have a nice interpretation in these terms.

In particular, we have the factorization of the leading order asymptotics, see Example 6.5,

$$\langle p_m \rangle_h = \prod_i h^{-m_i - 1} \, \langle\!\langle\, m_i \,\rangle\!\rangle + \dots ,$$

where the numbers $\langle\!\langle\, m_i \,\rangle\!\rangle$ are given by

$$\langle\!\langle\, k \,\rangle\!\rangle = \begin{cases} k!(2-2^{-k+1})\zeta(k+1), & k \text{ odd}, \\ 0, & k \text{ even}, \end{cases}$$

see Example 6.9.

It is a general principle that if for some probability $\mathfrak{M}$ measure the map

$$g \xrightarrow{\text{Expectation}} \langle g \rangle = \int g \, d\mathfrak{M}$$

is multiplicative, then $\mathfrak{M}$ is a δ-measure. Indeed, the multiplicativity implies $\mathrm{Var}(g) = \langle g^2 \rangle - \langle g \rangle^2 = 0$ for any g.

Thus, the multiplicativity of the leading order of the asymptotics reveals the existence of the limit shape of a typical large partition.

This limit shape is, of course, the Vershik's limit shape of a typical large random partition [32] as we shall see momentarily. Consider the uniform measure on the set of all partitions of N. Then, as $N \to \infty$, the diagrams of typical partitions are concentrated, see [32] for more precise statements, in the neighborhood of the shape bounded by the following curve

$$\exp\left(-\sqrt{\frac{\zeta(2)}{N}}\,x\right) + \exp\left(-\sqrt{\frac{\zeta(2)}{N}}\,y\right) = 1\,.$$

The expected size $|\lambda|$ of a partition λ with respect to the measure $\mathfrak{P}$ is

$$\langle p_1 \rangle_h = \frac{\zeta(2)}{h^2} + \dots\,, \qquad h \to +0\,,$$

and its variance is

$$\operatorname{Var}(|\lambda|) = \langle p_1 \,|\, p_1 \rangle_h = \frac{\langle\!\langle\, 1,1 \,\rangle\!\rangle}{h^3} + \dots = \frac{\pi^2}{3h^3} + \dots = o\left(\langle p_1 \rangle_h^2\right)\,.$$

Therefore, $\mathfrak{P}$-typical partitions have size $\approx \zeta(2)/h^2$ and hence are concentrated in the neighborhood of the shape bounded by the following curve

$$\Upsilon = \left\{ e^{-hx} + e^{-hy} = 1 \right\}.$$

We will now check that if a partition λ is close to Υ then the $p_k(\lambda)$ is close to $\langle p_k \rangle_h$. Informally, this can be stated as follows

$$\langle p_k \rangle_h = p_k\left(\Upsilon\right) + \dots\,.$$

By performing the summation along the rows of λ, one easily checks that

$$p_k(\lambda) = k \sum_{(i,j)\in\lambda} (j-i)^{k-1} + \dots\,,$$

where the summation is over all squares (i,j) in the diagram of λ and dots stand for a linear combination of $p_i(\lambda)$ with $i < k$. If λ is close to Υ then

$$\left|\{(i,j) \in \lambda, j - i = m\}\right| \approx h^{-1} \ln\left(1 + e^{-h|m|}\right)\,,$$

and therefore

$$p_k(\lambda) \sim \frac{k}{h} \sum_{m \in \mathbb{Z}} m^{k-1} \ln\left(1 + e^{-h|m|}\right) \sim \frac{k}{h^{k+1}} \int_{-\infty}^{\infty} u^{k-1} \ln\left(1 + e^{-|u|}\right) \, du \,.$$

The last integral obviously vanishes if k is even and for k odd it is twice the value of the following Mellin transform

$$\int_0^\infty u^{s-1} \ln\left(1 + e^{-u}\right) \, du = \left(1 - 2^{-s}\right) \Gamma(s) \, \zeta(s+1) \,.$$

Thus, we see that indeed $\langle p_k \rangle_h = p_k\,(\Upsilon) + \dots$.

A.2 Quasimodularity and limit shape fluctuations

We recall that averages $\langle p_\mu \rangle_q$ are quasimodular forms in q of weight $\operatorname{wt}(\mu) = |\mu| + \ell(\mu)$. They are only quasi-modular and *not modular*. This may look like an unfortunate circumstance from some other points of view, but is actually very natural from the point of view of random partitions. For if $\langle p_\mu \rangle_q$ *were* modular that would mean that the limit shape Υ is incredibly rigid in the sense that fluctuations of random partitions around Υ would be very, very small.

For example, if $\langle p_\mu \rangle_q$ were modular then, because there is only one empty partition, the variance

$$\operatorname{Var}(p_\mu) = \langle p_\mu \,|\, p_\mu \rangle_q$$

would have no q^0 term and hence would be a *modular cusp form*. Consequently, we would have

$$\operatorname{Var}(p_\mu) = O\left(\frac{e^{-4\pi^2/h}}{h^{\text{const}}}\right) \,, \quad h \to +0 \,.$$

In other words, not only this variance would not grow (compare this to $\langle p_\mu \rangle_q \propto h^{-\operatorname{wt}(\mu)}$), but it actually would decay to 0 faster than any power of the parameter h.

In real life, of course, we have

$$\operatorname{Var}(p_\mu) \propto h^{-2\operatorname{wt}(\mu)+1} \,,$$

and we have similar power laws for all other cumulants, see Theorem 6.3. They describe global fluctuation of a random partition about the limit shape Υ to all orders in h.

It is curious to notice that, on the level of formulas, these fluctuations are there ultimately because the expectations $\langle p_\mu \rangle_q$ involve the weight 2 Eisenstein series

$$G_2(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d \right) q^n \,,$$

which has a two-term modular transformation law

$$G_2\left(e^{-h}\right) = -\frac{4\pi^2}{h^2}\, G_2\left(e^{-4\pi^2/h}\right) - \frac{1}{2h}$$

and, hence, a two-term polynomial asymptotics as $h \to +0$

$$G_2\left(e^{-h}\right) = -\frac{\pi^2}{6h^2} - \frac{1}{2h} + O\left(\frac{e^{-4\pi^2/h}}{h^2}\right) .$$

A.3 Central limit theorem

Let us center and scale the random variables p_k, that is, introduce the variables

$$\widetilde{p}_k = h^{k+1/2} \left(p_k - \langle p_k \rangle_h\right) .$$

We have $\langle \widetilde{p}_k \rangle_h = 0$ and

$$\operatorname{Covar}\left(\widetilde{p}_k, \widetilde{p}_l\right) = \langle\!\langle\, k, l \,\rangle\!\rangle + O(h) \,, \tag{A.1}$$

as $h \to +0$, where, see Example 6.9,

$$\langle\!\langle\, k, l \,\rangle\!\rangle = (k+l)!\, \mathfrak{z}(k+l) - k!\, l!\, \mathfrak{z}(k)\, \mathfrak{z}(l) \,.$$

Here

$$\mathfrak{z}(k) = \begin{cases} \left(2 - 2^{2-k}\right) \zeta(k) & k \text{ even}\,, \\ 0 & k \text{ odd}\,. \end{cases}$$

It follows from our Wick formula, see Theorem 6.3, that the leading asymptotics of averages of the form

$$\langle \widetilde{p}_{m_1} \cdots \widetilde{p}_{m_n} \rangle_h$$

is given by the usual Wick rule with the covariance matrix (A.1). Hence the variables $\widetilde{p}_k$ are asymptotically Gaussian normal with mean zero and covariance (A.1). They describe the Gaussian fluctuation of a typical partition of N around its limit shape.

Similar central limit theorems are known in the literature for partitions into distinct parts [33] and for the Plancherel measure on partitions [12].

Further terms in the asymptotics of $\langle \cdot \rangle_h$ may not have such a transparent probabilistic interpretation.

A.4 Correlation functions and n-point functions

For $y \in \mathbb{Z} + \frac{1}{2}$, consider the following function of a partition λ

$$\delta_y(\lambda) = \begin{cases} 1\,, & y \in \{\lambda_i - i + \frac{1}{2}\}\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then the averages of the form

$$\left\langle \delta_{y_1}(\lambda) \cdots \delta_{y_k}(\lambda) \right\rangle_q$$

represent the probability to find the numbers $y_1, \dots, y_k$ among the numbers $\{\lambda_i - i + \frac{1}{2}\}$. In other words, they are the *correlation functions* for the 0/1 random process defined on $\mathbb{Z} + \frac{1}{2}$ by $y \mapsto \delta_y(\lambda)$.

We can write the function $\mathbf{e}^\lambda$ considered in Section 2.4 in the following form

$$\mathbf{e}^\lambda(\xi) = \sum_i e^{(\lambda_i - i + 1/2)\xi} = \sum_y e^{\xi y}\, \delta_y(\lambda)\,.$$

Therefore, we have, for example,

$$F(\xi_1|\xi_2) = \left\langle \mathbf{e}^\lambda(\xi_1) \middle| \mathbf{e}^\lambda(\xi_2) \right\rangle_q = \sum_{y_1, y_2} e^{\xi_1 y_1 + \xi_2 y_2} \left\langle \delta_{y_1}(\lambda) \middle| \delta_{y_2}(\lambda) \right\rangle_q\,.$$

In other words, the n-point functions and the connected functions are Laplace transforms of the correlation functions and their connected analogs.

The correlation functions have a nice integral representation, see [24], from which using the Laplace method one can, in principle, derive their asymptotics. This is another possible approach to the asymptotics of the n-point functions.

In particular, in the leading order of the asymptotics, the correlation functions factorize, which means that the local shape of a large random partitions is a random walk. This factorization is also reflected in the leading order factorization of the correlation functions.

This triviality of the local shape represents a striking contrast to the situation with the Plancherel measure [2], where the local properties are nontrivial and interesting. Conversely, the global properties of the Plancherel measure are quite simple, whereas in our situation their behavior is rather involved.

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