Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture.

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1 Introduction

Let Q be an indefinite nondegenerate quadratic form in n variables. Let $\mathcal{L}_Q = Q(\mathbb{Z}^n)$ denote the set of values of Q at integral points. The Oppenheim conjecture, proved by G.A. Margulis (cf. [Mar2, Mar3, Mar4]) states that if $n \geq 3$, and Q is not proportional to a form with rational coefficients, then $\mathcal{L}_Q$ is dense. Since it was conjectured in 1929, the Oppenheim conjecture enjoyed attention and many studies, mostly using analytic number theory methods. See [Mar5], [Bor2] and [Rat6] for references and background.

The proof given in [Mar 2-4] uses a connection, noticed by M.S. Raghunathan, between the Oppenheim conjecture and questions concerning closures of orbits of certain subgroups. Indeed it is not difficult to show that the Oppenheim conjecture (or, more precisely a slightly weaker form of this conjecture) is equivalent to the statement that any relatively compact orbit of $SO(2,1)$ in $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ is compact; in explicit form in the literature this equivalence appears in [Mar 2-4]. (We recently realized that in implicit form this relation already appears in section 10 of an old paper of Cassels and Swinnerton-Dyer [CSD]). In the mid-seventies M.S. Raghunathan noticed that the Oppenheim conjecture would follow from a conjecture about closures of orbits of unipotent subgroups (see [Dan2]). The Raghunathan conjecture states that if G is a connected Lie group, Γ a lattice in G, and U a unipotent subgroup of G then for any

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$x \in G/\Gamma$, there exists a closed subgroup $L = L(x)$ containing U such that the closure of the orbit Ux coincides with Lx. In the literature it was first stated in the paper [Dan2] and in a more general form in [Mar3] (when the subgroup U is not necessarily unipotent but generated by unipotent elements).

Raghunathan's conjecture was eventually proved in full generality by M. Ratner (see [Rat4]). Earlier it was known in the following cases: (a) G is reductive and U is horospherical (see [Dan2]); (b) $G = SL(3, \mathbb{R})$ and $U = \{u(t)\}$ is a one-parameter unipotent subgroup of G such that $u(t) - I$ has rank 2 for all $t \neq 0$, where I is the identity matrix (see [DM2]); (c) G is solvable (see [Sta1] and [Sta2]). We remark that the proof given in [Dan2] is restricted to horospherical U and the proof given in [Sta1] and [Sta2] cannot be applied for nonsolvable G. However the proof in [DM2] together with the methods developed in [Mar 2-4] and [DM1] suggest an approach for proving the Raghunathan conjecture in general by studying the minimal invariant sets, and the limits of orbits of sequences of points tending to a minimal invariant set. This program was being actively pursued at the time Ratner's results were announced (cf. [Sha3]).

Ratner's proof of the Raghunathan conjecture is based on her uniform distribution theorem (see [Rat4]). This theorem states that if $\{u_t\}$ is a one-parameter unipotent subgroup and $x \in G/\Gamma$ then there exist a closed subgroup $L \subset G$ containing U such that the orbit Lx is closed, and an L-invariant Borel probability measure μ on G/Γ supported on Lx such that for any bounded continuous function f on G,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_t x)\, dt = \int_{G/\Gamma} f\, d\mu.$$

Note that the uniform distribution theorem can be considered as the quantitative version of Raghunathan's conjecture for one-parameter unipotent subgroups.

The proof of the uniform distribution theorem is mostly based on another basic result: Ratner's measure classification theorem ("Raghunathan's measure conjecture"). This theorem, conjectured by S.G. Dani in [Dan2], asserts that all finite, ergodic, U-invariant measures on G/Γ are "algebraic". This deep result, which is the hardest part of Ratner's proof of the Raghunathan conjecture, proved to be an important and useful tool for the study of (unipotent) flows on homogeneous spaces. In addition to Ratner's original proof of the measure classification theorem given in [Rat1, Rat2, Rat3], we refer the reader also to a different and shorter proof given by G.A. Margulis and G.M. Tomanov (see [MT1] for the case where G is algebraic and [MT2] for a simple reduction to that case). An alternative approach to the proof of the uniform distribution theorem for the case of a regular unipotent subgroup, also using Ratner's measure classification theorem, is given in [Sha1].

Subsequently, using both Ratner's measure classification theorem, as well as their methods of studying unipotent flows on homogeneous spaces via "linearization", S.G. Dani and G.A. Margulis proved in [DM3] a refined version of the uniform distribution theorem (see Theorem 4.1 below). This theorem is the main ergodic result used in this paper.

It should be noted that Dani's theorem on the behavior of orbits of unipotent flows at infinity (see [Dan3] and [Dan4]) is essentially used in [DM1, DM2, DM3], and [Rat4]. For the case where $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$ this theorem is essentially equivalent to the following assertion: for any lattice Λ in $\mathbb{R}^n$, any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n,\mathbb{R})$ and any $\epsilon > 0$, one can find $\delta > 0$ such that the set

$$\Omega(\Lambda,\delta) = \{t \geq 0 \mid \|u_t z\| > \delta \text{ for all } z \in \Lambda - \{0\}\} \tag{1.1}$$

has density greater then $1-\epsilon$ on every interval $[0,T]$. In a weak form this assertion was proved in [Mar1] (namely it was proved in [Mar1] that $\Omega(\Lambda,\delta)$ is unbounded for some $\delta = \delta(\Lambda) > 0$). Some related questions will be considered in detail below.

Finally, we note that a characteristic property of the unipotent flow, (which distinguishes it from e.g. the geodesic flow) is its close connection to "slow growing" functions like polynomials of bounded degree. In fact most of the results mentioned above for unipotent flows are also true for polynomial maps: see [Sha2]. For further results in the p-adic and S-arithmetic settings see [Rat5] and [MT1].

By "quantitative Oppenheim conjecture" we mean[1] the study of the distribution of values of a quadratic form, e.g. the number of lattice points in a subset Ω of $\mathbb{R}^n$ such that a given quadratic form has values belonging to some given interval, for example determining the asymptotics of the number $V^Q_{(a,b)}(T)$ of integral points v in a ball of radius T with $a < Q(v) < b$. In [DM3] S.G. Dani and G.A. Margulis obtained asymptotically exact lower bounds for $V^Q_{(a,b)}(T)$ and for some more general quantities. In particular they proved that if L is a compact family of indefinite non-degenerate quadratic forms in $n \geq 3$ variables which consists of forms not proportional to rational forms, then

$$\liminf_{T\to\infty} \inf_{Q\in L} \frac{V^Q_{(a,b)}(T)}{\mathrm{Vol}(\{v \in \mathbb{R}^n \mid a < Q(v) < b,\ \|v\| < T\})} \geq 1,$$

where $\mathrm{Vol}(\cdot)$ is the Lebesgue measure on $\mathbb{R}^n$. For a single Q, the fact that the lower limit is positive had been obtained earlier by S.G. Dani and S. Mozes and also

[1]An announcement of the results of this paper appeared in [EMM]

independently by M. Ratner (both unpublished). The proof of the lower bounds in [DM3] is based on the following identity which is immediate from the definitions:

$$\int_T^{\ell T} \int_B \sum_{v \in g\mathbb{Z}^n} \psi(u_t k v)\, d\sigma(k)\, dt = \int_T^{\ell T} \int_B \tilde{\psi}(u_t k g \Gamma)\, d\sigma(k)\, dt, \tag{1.2}$$

where $\{u_t\}$ is a certain one-parameter unipotent subgroup, B is a Borel subset of the maximal compact subgroup K of $SO(p,q)$, σ is the normalized Haar measure on K, ψ is an integrable function on $\mathbb{R}^n - \{0\}$, $\Gamma = SL(n, \mathbb{Z})$ and

$$\tilde{\psi}(g\Gamma) = \sum_{v \in g\mathbb{Z}^n} \psi(v).$$

The left hand side in (1.2) gives a good approximation for $V^Q_{(a,b)}(T)$ for an appropriate choice of ψ and g and the right hand side can be estimated, uniformly when $g\Gamma$ belongs to certain compact subsets of $SL(n,\mathbb{R})/\Gamma$, using the abovementioned version from [DM3] of the uniform distribution theorem.

The function $\tilde{\psi}$ on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is unbounded for any non-negative non-zero continuous function ψ on $\mathbb{R}^n$. Therefore we can not use (1.2) and the uniform distribution theorem to get the asymptotic of $V^Q_{(a,b)}(T)$, because this theorem is proved (and in general true) only for bounded continuous functions. On the other hand, as it was done in [DM3] one can get lower bounds by considering bounded continuous functions $f \leq \tilde{\psi}$ and applying the uniform distribution theorem to f.

It would be possible to obtain the asymptotic of $V^Q_{(a,b)}(T)$ from (1.2) and the uniform distribution theorem if we knew that for some $\delta > 0$ and any continuous nonnegative function ψ on $\mathbb{R}^n$ with compact support,

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \int_K \tilde{\psi}(u_t k g \Gamma)^{1+\delta}\, d\sigma(k)\, dt < \infty. \tag{1.3}$$

Using methods of the papers [Dan1], [Dan4] and [Mar1] it is possible to prove a stronger version of (1.1), i.e. that there exists $\theta = \theta(n) > 0$ (and this θ can be effectively computed) such that

$$\limsup_{T \to \infty} \left\{ \sup_{g \in L} \frac{1}{T} \int_0^T \tilde{\psi}(u_t g \Gamma)^{\theta}\, dt \right\} < \infty$$

for any compact subset $L \subset SL(n,\mathbb{R})$ and any unipotent subgroup $\{u_t\} \subset SL(n,\mathbb{R})$. But these methods are certainly not enough to prove (1.3).

An earlier, technically complex, argument based in part on the ideas of [Dan1], [Dan4] and [Mar1] allowed us to prove (1.3) for the special case $(p,q)=(3,1)$ and δ near zero. Here we present a different argument which uses a new idea for handling the complications arising from the involved structure at infinity of $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. Using this we essentially prove (1.3) for all $0 \leq \delta < 1$ in the case where $p \geq 3$ and $q \geq 1$ (recall that K is a maximal compact subgroup of $SO(p,q)$). If $(p,q)=(2,2)$ or $(p,q)=(2,1)$ then (1.3) is in general not true even for $\delta = 0$ but it becomes true for $\delta = 0$ if we replace in (1.3) the factor $1/T$ by $1/(T\log T)$. Actually, for technical reasons, in the sequel we consider not the integrals from (1.3) but integrals of the form

$$\int_K \tilde{\psi}(a_t k)^{1+\delta}\, d\sigma(k),$$

where $\{a_t\}$ is a certain diagonalizable subgroup of $SO(p,q)$.

2 Statement of Results

Let ρ be a continuous positive function on the sphere $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and let $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \rho(v/\|v\|)\}$. We denote by $T\Omega$ the dilate of Ω by T. Define the following set:

$$V^Q_{(a,b)}(\mathbb{R}) = \{x \in \mathbb{R}^n \mid a < Q(x) < b\}$$

We shall use $V_{(a,b)} = V^Q_{(a,b)}$ when there is no confusion about the form Q. Also let $V_{(a,b)}(\mathbb{Z}) = V^Q_{(a,b)}(\mathbb{Z}) = \{x \in \mathbb{Z}^n \mid a < Q(x) < b\}$. The set $T\Omega \cap \mathbb{Z}^n$ consists of $O(T^n)$ points, $Q(T\Omega \cap \mathbb{Z}^n)$ is contained in an interval of the form $[-\mu T^2, \mu T^2]$, where $\mu > 0$ is a constant depending on Q and Ω. Thus one might expect that for any interval $[a,b]$, as $T \to \infty$,

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \sim c_{Q,\Omega}(b-a)T^{n-2}. \tag{2.1}$$

where $c_{Q,\Omega}$ is a constant depending on Q and Ω. This may be interpreted as "uniform distribution" of the sets $Q(\mathbb{Z}^n \cap T\Omega)$ in the real line. Our main result is that (2.1) holds if Q is not proportional to a rational form, and has signature (p,q) with $p \geq 3$, $q \geq 1$. We also determine the constant $c_{Q,\Omega}$.

If Q is an indefinite quadratic form in n variables, Ω is as above and (a,b) is an interval, we will show that there exists a constant $\lambda = \lambda_{Q,\Omega}$ so that as $T \to \infty$,

$$\mathrm{Vol}(V_{(a,b)}(\mathbb{R}) \cap T\Omega) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}. \tag{2.2}$$

We show that $\lambda_{Q,\Omega}$ as in (2.2) serves as the desired constant $c_{Q,\Omega}$. Thus, our main result is the following:

Theorem 2.1. *Let Q be an indefinite quadratic form of signature (p, q), with $p \geq 3$ and $q \geq 1$. Suppose Q is not proportional to a rational form. Then for any interval (a, b), as $T \to \infty$,*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \sim \lambda_{Q,\Omega}(b-a)T^{n-2},$$

where $n = p + q$, and $\lambda_{Q,\Omega}$ is as in (2.2).

Only the upper bound in this formula is new: the asymptotically exact lower bound was proved in [DM3].

If the signature of Q is $(2, 1)$ or $(2, 2)$ then no universal formula like (2.1) holds. In fact, we have the following theorem:

Theorem 2.2. *Let Ω_0 be the unit ball, and let $q = 1$ or 2. Then for every $\epsilon > 0$ and every interval (a, b) there exists a quadratic form Q of signature $(2, q)$ not proportional to a rational form, and a constant $c > 0$ such that for an infinite sequence $T_j \to \infty$,*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega_0| > cT_j^q(\log T_j)^{1-\epsilon}.$$

The case $q = 1$, $b \leq 0$ of Theorem 2.2 was noticed by P. Sarnak and worked out in detail in [Bre].

While an estimate as in (2.1) does not hold in the case of signatures $(2, 1)$ and $(2, 2)$, we can still establish an upper bound of the form $T^q \log T$. This upper bound is effective, and is uniform over compact sets in the set of quadratic forms. We also give an effective uniform upper bound for the case $p \geq 3$.

Theorem 2.3. *Let $\mathcal{O}(p, q)$ denote the space of quadratic forms of signature (p, q) and discriminant ± 1, let $n = p + q$, (a, b) be an interval, and let $\mathcal{D}$ be a compact subset of $\mathcal{O}(p, q)$. Let ν be a continuous positive function on the unit sphere and let $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \nu(v/\|v\|)\}$. Then, if $p \geq 3$ there exists a constant c depending only on $\mathcal{D}$, (a, b) and Ω such that for any $Q \in \mathcal{D}$ and all $T > 1$,*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| < cT^{n-2}.$$

If $p = 2$ and $q = 1$ or $q = 2$, then there exists a constant $c > 0$ depending only on $\mathcal{D}$, (a, b) and Ω such that for any $Q \in \mathcal{D}$ and all $T > 2$,

$$|V_{(a,b)} \cap T\Omega \cap \mathbb{Z}^n| < cT^{n-2}\log T.$$

Also, for the $(2,1)$ and $(2,2)$ cases, we have the following "almost everywhere" result:

Theorem 2.4. *For almost all quadratic forms Q of signature $(p,q)=(2,1)$ or $(2,2)$*

$$|V_{(a,b)}(\mathbb{Z})\cap T\Omega| \sim \lambda_{Q,\Omega}(b-a)T^{n-2},$$

where $n=p+q$, and $\lambda_{Q,\Omega}$ is as in (2.2).

Theorem 2.4 can be proved using a recent general result of Nevo and Stein [NS]; in §3.6 we will present a self-contained argument which suffices for the application. In [Sar], P. Sarnak proved that an analogous asymptotic formula holds for almost all forms within a specific two-parameter family; this family arises in problems related to Quantum Chaos.

Finally, following [DM3] we give a "uniform" version of Theorem 2.1:

Theorem 2.5. *Let $\mathcal{D}$ be a compact subset of $\mathcal{O}(p,q)$, with $p\geq 3$. Let $n=p+q$, and let Ω be as in Theorem 2.3. Then for every interval $[a,b]$ and every $\theta>0$, there exists a finite subset $\mathcal{P}$ of $\mathcal{D}$ such that each $Q\in\mathcal{P}$ is a scalar multiple of a rational form and for any compact subset $\mathcal{F}$ of $\mathcal{D}-\mathcal{P}$ there exists T_0 such that for all Q in $\mathcal{F}$ and $T\geq T_0$,*

$$(1-\theta)\lambda_{Q,\Omega}(b-a)T^{n-2} \leq |V_{(a,b)}(\mathbb{Z})\cap T\Omega|, \leq (1+\theta)\lambda_{Q,\Omega}(b-a)T^{n-2}$$

where $\lambda_{Q,\Omega}$ is as in (2.2).

As in Theorem 2.1 only the upper bound is new; the asymptotically exact lower bound, which holds even for $SO(2,1)$ and $SO(2,2)$, was proved in [DM3].

Remark 2.6. If we consider $|V_{(a,b)}(\mathbb{R})\cap T\Omega\cap\mathcal{P}(\mathbb{Z}^n)|$ instead of $|V_{(a,b)}(\mathbb{Z})\cap T\Omega|$ (where $\mathcal{P}(\mathbb{Z}^n)$ denotes the set of primitive lattice points), then Theorem 2.1 and Theorem 2.5 hold provided one replaces $\lambda_{Q,\Omega}$ by $\lambda'_{Q,\Omega}=\lambda_{Q,\Omega}/\zeta(n)$, where ζ is the Riemann zeta function.

Remark 2.7. Theorem 2.1 and Theorem 2.5, as well as lower bounds in [DM3] and even the proof of the Oppenheim conjecture in [Mar2-4], are not effective. In the case of Theorem 2.5 it means that we do not know how to determine effectively $\mathcal{P}$ and T_0. In the case of the Oppenheim conjecture, it means that the proof given in [Mar2-4] does not give an algorithm for determining T such that $\{v\in\mathbb{Z}^n \mid a<Q(v)<b, \|v\|<T, v\neq 0\}\neq\emptyset$; this T should be determined in terms of Diophantine approximation properties of the coefficients of Q. The proof in [Mar2-4] is not effective because, in

particular, it implicitly uses the fact that the closure of any relatively compact orbit Fx contains a minimal closed F-invariant subset; this fact has no effective version. The proof of the lower bound in [DM3] and the proofs of Theorem 2.1 and Theorem 2.5 are not effective because these proofs use the uniform distribution theorem and the proof of this theorem uses different facts from ergodic theory which have no effective versions (in particular the fact that the measure $\frac{1}{T}\int_0^T u_t\delta_x\,dt$ converges to the set of invariant measures). It seems to be a very difficult problem to give effective versions of Theorem 2.1, Theorem 2.5 and results from [Mar2-4],[DM1-3],[Rat4].

The paper is organized as follows. In §3 we connect the study of integral values of quadratic forms to estimates of certain integrals over orbits of semisimple groups. We state the results we need in §3.3 and prove the theorems stated in §2 modulo these results in §3.4-§3.6. In §3.7 we discuss "counterexamples", i.e Theorem 2.2.

In the rest of the paper we prove the theorems stated in §3.3. The main result, Theorem 3.5 is naturally a combination of a result for functions with compact support, Theorem 4.4 and an integrability estimate, Theorem 3.2. In §4 we derive Theorem 4.4 from [DM3, Theorem 3] (restated as Theorem 4.1). In §5, which contains most of the new ideas of this paper, we prove Theorem 3.2, and its analogue for the $SO(2,1)$ and $SO(2,2)$ cases, Theorem 3.3. We then complete the proof of Theorem 3.5.

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3 Integral Values of quadratic forms

In this section we relate the counting problems of §2 to certain integral expressions involving the orthogonal group of the quadratic form and the space of lattices $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. Roughly this is done as follows. Let f be a bounded function on $\mathbb{R}^n - \{0\}$ vanishing outside a compact subset. For $g \in SL(n,\mathbb{R})$ let

$$\tilde{f}(g) = \sum_{v\in\mathbb{Z}^n} f(gv). \tag{3.1}$$

The proof is based on the identity of the form

$$\int_K \tilde{f}(a_t k g)\,dk = \sum_{v\in\mathbb{Z}^n}\int_K f(a_t k g v)\,dk, \tag{3.2}$$

obtained by integrating (3.1). In (3.2) $\{a_t\}$ is a certain diagonal subgroup of the orthogonal group of Q, and K is a maximal compact subgroup of the orthogonal group of Q. Then for an appropriate function f and an appropriate element $g \in G$, the right hand side is then related to the number of lattice points $v \in [e^t/2, e^t]\partial\Omega$ with $a < Q(v) < b$. For a precise statement see Proposition 3.7 below. We then establish the asymptotics of the left-hand side using the ergodic theory of unipotent flows and some other techniques.

3.1 Lattices and the function α.

Let Δ be a lattice in $\mathbb{R}^n$. We say that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the volume of $L/(L \cap \Delta)$. Let us note that $d(L)$ is equal to the norm of $u_1 \wedge \cdots \wedge u_\ell$ in the exterior power $\bigwedge^\ell(\mathbb{R}^n)$ where $\ell = \dim L$, $(u_1, \cdots, u_\ell)$ is a basis over $\mathbb{Z}$ of $L \cap \Delta$, and the norm on $\bigwedge(\mathbb{R}^n)$ is induced from the Euclidean norm on $\mathbb{R}^n$. If $L = \{0\}$ we write $d(L) = 1$. A lattice is Δ *unimodular* if $d_\Delta(\mathbb{R}^n) = 1$. The space of unimodular lattices is canonically identified with $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

Let us introduce the following notation:

$$\alpha_i(\Delta) = \sup\left\{ \frac{1}{d(L)} \,\middle|\, L \text{ is a } \Delta\text{-rational subspace of dimension } i \right\}, \quad 0 \le i \le n,$$
$$\alpha(\Delta) = \max_{0 \le i \le n} \alpha_i(\Delta). \tag{3.3}$$

Lemma 3.1 ([Sch, Lemma 2]). *Let f be a bounded function on $\mathbb{R}^n$ vanishing outside a compact subset. Then there exists a positive constant $c = c(f)$ such that*

$$\tilde{f}(\Delta) < c\alpha(\Delta),$$

for any lattice Δ in $\mathbb{R}^n$. Here $\tilde{f}$ is the function on the space of lattices defined in (3.1).

3.2 Quadratic Forms

Let $n \ge 3$, and let $p \ge 2$. We denote $n - p$ by q, and assume $q > 0$. Let $\{e_1, e_2, \dots e_n\}$ be the standard basis of $\mathbb{R}^n$. Let Q_0 be the quadratic form defined by

$$Q_0\left(\sum_{i=1}^n v_i e_i\right) = 2v_1 v_n + \sum_{i=2}^p v_i^2 - \sum_{i=p+1}^{n-1} v_i^2 \quad \text{for all } v_1, \dots, v_n \in \mathbb{R}. \tag{3.4}$$

It is straightforward to verify that Q_0 has signature (p, q). Let $G = SL(n, \mathbb{R})$, the group of $n \times n$ matrices of determinant 1. For each quadratic form Q and $g \in G$, let Q^g denote the quadratic form defined by $Q^g(v) = Q(gv)$ for all $v \in \mathbb{R}^n$. Recall that $\mathcal{O}(p, q)$ denotes the set of quadratic forms of signature (p, q) and discriminant ± 1. By the well known classification of quadratic forms over $\mathbb{R}$, for each $Q \in \mathcal{O}(p, q)$ there exists $g \in G$ such that $Q = Q_0^g$. For any quadratic form Q let $SO(Q)$ denote the special orthogonal group corresponding to Q; namely $\{g \in G \mid Q^g = Q\}$. Let $H = SO(Q_0)$. Then the map $H \backslash G \to \mathcal{O}(p, q)$ given by $Hg \to Q_0^g$ is a homeomorphism.

For $t \in \mathbb{R}$, let a_t be the linear map so that $a_t e_1 = e^{-t} e_1$, $a_t e_n = e^t e_n$, and $a_t e_i = e_i$, $2 \leq i \leq n-1$. Then the one-parameter group $\{a_t\}$ is contained in H. Let $\hat{K}$ be the subgroup of G consisting of orthogonal matrices, and let $K = H \cap \hat{K}$. It is easy to check that K is a maximal compact subgroup of H, and consists of all $h \in H$ leaving invariant the subspace spanned by $\{e_1 + e_n, e_2, \ldots, e_p\}$. We denote by m the normalized Haar measure on K.

3.3 The main theorems

In this subsection we state without proof the main theorems used to prove the results stated in §2. The proofs of the theorems in this subsection will be given in §4 and §5. We keep the notation of the previous two subsections.

The main ergodic result which we use, namely Theorem 4.1 (i.e. [DM3, Theorem 3]) holds for bounded continuous functions, but not for arbitrary continuous functions from $L^1(G/\Gamma)$, (see the remark following Theorem 4.1). However, for a non-negative bounded continuous function f on $\mathbb{R}^n$, the function $\tilde{f}$ defined in (3.1) is non-negative, continuous, and L^1 but unbounded (it is in $L^s(G/\Gamma)$ for $1 \leq s < n$, where $G = SL(n, \mathbb{R})$, and $\Gamma = SL(n, \mathbb{Z})$). Thus in order to carry out the integrals in (3.2) and prove the upper bounds in the theorems of §2 we need to examine carefully the situation at the "cusp" of G/Γ, i.e outside of compact sets.

By Lemma 3.1, the function $\tilde{f}(g)$ on the space of unimodular lattices G/Γ is majorized by the function $\alpha(g)$. The function α is more convenient since it is invariant under the left action of the maximal compact subgroup $\hat{K}$ of G, and its growth rate at infinity is known explicitly. The main integrability estimate we will use is the following theorem, proved in §5:

Theorem 3.2. *If $p \geq 3$, $q \geq 1$ and $0 < s < 2$, or if $p = 2$, $q \geq 1$ and $0 < s < 1$, then for any lattice Δ in $\mathbb{R}^n$*

$$\sup_{t>0} \int_K \alpha(a_t k \Delta)^s \, dm(k) < \infty.$$

The upper bound is uniform as Δ varies over compact sets in the space of lattices.

This result can be interpreted as follows. For a lattice Δ in G/Γ and for $h \in H$, let $f(h) = \alpha(h\Delta)$. Since α is left $\hat{K}$-invariant, f is a function on the symmetric space $X = K\backslash H$. Theorem 3.2 is the statement that if $p \geq 3$, then the averages of f^s, $0 < s < 2$ over the sets Ka_tK in X remain bounded as $t \to \infty$, and the bound is uniform as one varies Δ over compact sets. We remark that in the case $q = 1$, the rank of X is 1, and the sets Ka_tK are metric spheres of radius t, centered at the origin.

In the case $p = 2$, the analogue of the assertion in Theorem 3.2 fails even for $s = 1$, and the best possible result is the following theorem which states that the average of α over large spheres in X grows at most linearly with the radius of the sphere. This yields the cases $p = 2$, $q = 1$, and $p = 2$, $q = 2$ in Theorem 2.3.

Theorem 3.3. *If $p = 2$ and $q = 1$ or 2, then for any lattice Δ in $\mathbb{R}^n$,*

$$\sup_{t>1} \frac{1}{t} \int_K \alpha(a_t k\Delta)\, dm(k) < \infty. \tag{3.5}$$

The upper bound is uniform as Δ varies over compact sets in the space of lattices.

To prove Theorem 2.1 one may use the following theorem:

Theorem 3.4. *Suppose $p \geq 3$, $q \geq 1$. Let ϕ be a continuous function on G/Γ (the space of lattices in $\mathbb{R}^n$ with determinant 1). Assume that for some s, $0 < s < 2$ and some $C > 0$,*

$$|\phi(\Delta)| < C\alpha(\Delta)^s, \qquad \text{for all } \Delta \in G/\Gamma.$$

Let $x_0 \in G/\Gamma$ be a unimodular lattice such that Hx_0 is not closed. Let ν be any continuous function on K. Then

$$\lim_{t\to+\infty} \int_K \phi(a_t k x_0)\nu(k)\, dm(k) = \int_K \nu\, dm \int_{G/\Gamma} \phi(y)\, d\mu(y).$$

Theorem 3.4 is proved by combining Theorem 3.2 with the results of §4. To prove Theorem 2.5 we will need the following generalization of Theorem 3.4:

Theorem 3.5. *Suppose $p \geq 3$, $q \geq 1$. Let ϕ, ν be as in Theorem 3.4, and let $\mathcal{C}$ be any compact set in G/Γ. Then for any $\epsilon > 0$ there exist finitely many points $x_1, \ldots, x_\ell \in G/\Gamma$ such that*

(i) *The orbits $Hx_1, \dots, Hx_\ell$ are closed and have finite H-invariant measure.*

(ii) *For any compact subset F of $\mathcal{C} - \bigcup_{1\le i\le \ell} Hx_i$, there exists $t_0 > 0$, so that for all $x \in F$ and $t > t_0$,*

$$\left| \int\limits_K \phi(a_t k x)\nu(k)\, dm(k) - \int\limits_{G/\Gamma} \phi\, d\mu \int\limits_K \nu\, dm \right| \le \epsilon. \tag{3.6}$$

Theorem 3.5 will be proved in §5 by combining Theorem 4.4 with Theorem 3.2.

3.4 A Lemma about vectors in $\mathbb{R}^n$.

We denote the standard inner product on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. Let $\mathbb{R}^n_+$ denote the set $\{x \in \mathbb{R}^n \ : \ \langle x, e_1 \rangle > 0\}$. Let $c_{p,q} = 2^{(n-2)/2}/(\gamma_{p-1}\gamma_{q-1})$, where for $m \in \mathbb{N}$, γ_m denotes the volume of S^m. We use the following lemma about vectors in $\mathbb{R}^n$:

Lemma 3.6. *Let f be a continuous function of compact support on $\mathbb{R}^n_+$ and let*

$$J_f(r, \zeta) = \frac{1}{r^{n-2}} \int_{\mathbb{R}^{n-2}} f\left(r, x_2, \dots, x_{n-1}, x_n\right) dx_2 \dots dx_{n-1},$$

where in the integral $x_n = (\zeta - Q_0(0, x_2, \dots, x_{n-1}, 0))/(2r)$, so that $Q_0(r, x_2, \dots, x_n) = \zeta$. Let ν be a non-negative continuous function on S^{n-1}. Then for every $\epsilon > 0$ there exists $T_0 > 0$ such that for every t with $e^t > T_0$, and every $v \in \mathbb{R}^n$ with $\|v\| > T_0$,

$$\left| c_{p,q} e^{(n-2)t} \int_K f(a_t k v)\nu(k^{-1}e_1)\, dm(k) - J_f\left(\|v\| e^{-t}, Q_0(v)\right) \nu\left(\frac{v}{\|v\|}\right) \right| < \epsilon, \tag{3.7}$$

Proof. Let E denote the support of f. Let $c_1 = \inf_{v\in E}\langle v, e_1\rangle$, $c_2 = \sup_{v \in E}\langle v, e_1 \rangle$. From the definition of a_t and the fact that $a_t \in H = SO(Q_0)$, it follows that $f(a_t w) = 0$, unless

$$|\langle w, e_1\rangle \langle w, e_n \rangle| \le \beta \tag{3.8}$$

$$c_1 \le \langle w, e_1 \rangle e^{-t} \le c_2 \tag{3.9}$$

$$\pi'(w) \in \pi'(E), \tag{3.10}$$

where β depends only on E and π' denotes the projection onto the span of $e_2, \dots, e_{n-1}$. For w satisfying (3.8) and (3.9) we have $\langle w, e_n \rangle = O(e^{-t})$. This, together with (3.10) and (3.9) imply that if $f(a_t w) \neq 0$ and t is large, then

$$\|w\| = \langle w, e_1 \rangle + O(e^{-t}). \tag{3.11}$$

Hence if $f(a_t kv) \neq 0$ and t is large, then $kv/\|v\|$ must be near e_1. Now the uniform continuity of ν implies that for every $\delta > 0$ there exists $t_0 > 0$ so that for every v with $\|v\| > t_0$ and every k in the support of the function $k \mapsto f(a_t kv)$,

$$\left|\nu(k^{-1}e_1) - \nu(v/\|v|)\right| < \delta. \tag{3.12}$$

Note that if (3.8), (3.9) and (3.10) hold, then

$$\langle a_t w, e_1\rangle = \langle w, e_1\rangle e^{-t} = e^{-t}\|w\| + O(e^{-2t}), \tag{3.13}$$

by (3.11). Also

$$\langle a_t w, e_i\rangle = \langle w, e_i\rangle, \qquad 2 \le i \le n-1. \tag{3.14}$$

Finally,

$$\begin{aligned}\langle a_t w, e_n\rangle &= \Big(Q_0(w) - Q_0(0, \langle w, e_2\rangle, \dots, \langle w, e_{n-1}\rangle, 0)\Big)/(2\langle a_t w, e_1\rangle) \\ &= \Big(Q_0(w) - Q_0(0, \langle w, e_2\rangle, \dots, \langle w, e_{n-1}\rangle, 0)\Big)/(2e^{-t}\|w\|) + O(e^{-t}).\end{aligned} \tag{3.15}$$

Hence, using (3.13), (3.14) and (3.15) applied with $w = kv$, and the uniform continuity of f, we see that for any $\delta > 0$ there exists $t_0 > 0$ so that if $t > t_0$ then on the support of the function $k \mapsto f(a_t kv)$,

$$\left|f(a_t kv) - f\left(\|v\|e^{-t}, \langle kv, e_2\rangle, \dots, \langle kv, e_{n-1}\rangle, v_n\right)\right| < \delta, \tag{3.16}$$

where v_n is determined by $Q_0(\|v\|e^{-t}, \langle kv, e_2\rangle, \dots, \langle kv, e_{n-1}\rangle, v_n) = Q_0(v)$.

It is convenient to change basis by letting $f_1 = (e_1 + e_n)/\sqrt{2}$, $f_n = (e_1 - e_n)/\sqrt{2}$, $f_j = e_j$, $2 \le j \le n-1$. In this basis $K \cong SO(p) \times SO(q)$ consists of orthogonal matrices preserving the subspace L_1 spanned by $f_1, \dots f_p$, and the subspace L_2 spanned by $f_{p+1}, \dots f_n$. For $i = 1, 2$, let π_i denote orthogonal projection onto L_i. Write $r_i = \|\pi_i(v)\|$; then the orbit Kv is the product of two spheres $r_1 S^{p-1} \times r_2 S^{q-1}$, where S^{p-1} denotes the unit sphere in L_1 and S^{q-1} denotes the unit sphere in L_2. We claim that $f(a_t kv) = 0$ for all $k \in K$ unless both r_1 and r_2 are comparable to $\|v\|$. Indeed, suppose $w \in Kv$ is such that $f(a_t w) \neq 0$. Then from (3.8) and (3.9) it follows that $\langle w, e_n\rangle = O(e^{-t})$. Now if we write $w = \sum w_i f_i$, then $w_1 = 2^{-1/2}\langle w, e_1\rangle + O(e^{-t})$, $w_n = 2^{-1/2}\langle w, e_1\rangle + O(e^{-t})$, and for $2 \le i \le n-1$, $w_i = O(1)$. Hence $\|\pi_1(w)\| = w_1 + O(e^{-t})$, and $\|\pi_2(w)\| = w_n + O(e^{-t})$, It follows that for $i = 1, 2$,

$$r_i = \|\pi_i(w)\| = 2^{-1/2}\langle w, e_1\rangle + O(e^{-t}) = 2^{-1/2}\|w\| + O(e^{-t}), \tag{3.17}$$

where the last estimate follows from (3.11).

The equation (3.10) implies that if $f(a_t kv) \neq 0$, then kv is within a bounded distance from $r_1 f_1 + r_2 f_n$. As $\|v\|$ increases, so do the r_i, and the the normalized Haar measure on $r_1 S^{p-1}$ near $r_1 f_1$ (resp. $r_2 S^{q-1}$ near $r_2 f_2$) tends to $(1/\gamma_{p-1}) r_1^{1-p} dw_2 \dots dw_p$ (resp. $(1/\gamma_{q-1}) r_2^{1-q} dw_{p+1} \dots dw_{n-1}$). Now the lemma follows from combining this fact with (3.16), (3.12) and (3.17). ☐

3.5 Passage to the space of lattices.

Proof of Theorem 2.3. Choose a compact set $C \subset G$ so that every quadratic form in $\mathcal{D}$ is a scalar multiple of Q_0^x for some $x \in C$. There exists a constant β so that for every $x \in C$ and every $v \in \mathbb{R}^n$, $\beta^{-1}\|v\| \leq \|xv\| \leq \beta\|v\|$. Let $\epsilon > 0$ be an arbitrarly small number, and let $x \in C$ be arbitrary. Choose a continuous non-negative function f on $\mathbb{R}_+^n$ of compact support so that $J_f \geq 1 + \epsilon$ on $[\beta^{-1}, 2\beta] \times [a, b]$. Then if $v \in \mathbb{R}^n$ satisfies $e^t \leq \|v\| \leq 2e^t$, $a \leq Q_0(xv) \leq b$, then $J_f(\|xv\|e^{-t}, Q_0(xv)) \geq 1 + \epsilon$. Then by Lemma 3.6, for sufficiently large t,

$$c_{p,q} e^{(n-2)t} \int_K f(a_t kxv)\, dm(k) \geq 1 \qquad \text{if } e^t \leq \|v\| \leq 2e^t \quad \text{and } a \leq Q_0(xv) \leq b. \tag{3.18}$$

Then summing over $v \in \mathbb{Z}^n$ satisfying the condition of (3.18), we get

$$\begin{aligned} |V_{(a,b)}(\mathbb{Z}) \cap [e^t, 2e^t] S^{n-1}| &\leq \sum_{v \in \mathbb{Z}^n} c_{p,q} e^{(n-2)t} \int_K f(a_t kxv)\, dm(k) \\ &= c_{p,q} e^{(n-2)t} \int_K \tilde{f}(a_t kx)\, dm(k), \end{aligned}$$

where $\tilde{f}$ is defined in §3.1. By Lemma 3.1, $\tilde{f}(g) \leq c_1 \alpha(g)$ for some constant c_1. Hence if $p \geq 3$ then by Theorem 3.2, $\int_K \tilde{f}(a_t kx)\, dm(k)$ is bounded for all t, uniformly for $x \in C$. Thus if $p \geq 3$,

$$|V_{(a,b)}(\mathbb{Z}) \cap [T, 2T] S^{n-1}| \leq c_2 T^{n-2}$$

for some constant $c_2 > 0$, and we have let $T = e^t$. If $(p, q) = (2, 1)$ or $(p, q) = (2, 2)$, then by Theorem 3.3,

$$|V_{(a,b)}(\mathbb{Z}) \cap [T, 2T] S^{n-1}| \leq c_3 T^{n-2} \log T,$$

for some constant $c_3 > 0$. Now the theorem follows from the compactness of Ω and an easy geometric series argument. ☐

Proposition 3.7. *Let f be a non-negative continuous function of compact support on $\mathbb{R}^n_+$, and ν a positive continuous function on S^{n-1}.*

(i) Suppose $p \geq 3$. Then for every $\epsilon > 0$ and every $x \in G$ there exists $t_0 > 0$ so that for $t > t_0$,

$$\left| e^{-(n-2)t} \sum_{v \in \mathbb{Z}^n} (J_f)(\|xv\|e^{-t}, Q_0(xv))\nu(xv/\|xv\|) - c_{p,q} \int_K \tilde{f}(a_t kx)\nu(k^{-1}e_1)\, dm(k) \right| < \epsilon. \tag{3.19}$$

The constant t_0 can be chosen uniformly as x varies over a compact subset of G.

(ii) If $(p, q) = (2, 1)$ or $(2, 2)$ then the equation (3.19) still holds for a fixed $x \in G$, any $\epsilon > 0$ and t is sufficiently large, under the following additional assumption: there exists a non-negative continuous function f_+ of compact support on $\mathbb{R}^n_+$ with the support of f contained in the interior of the support of f_+, and

$$\sup_{t>0} \int_K \tilde{f}_+(a_t kx)\, dm(k) < \infty.$$

Proof. Suppose $p \geq 3$. Since J_f has compact support, the number of non-zero terms in the sum on the left hand side of (3.19) is bounded by $ce^{(n-2)t}$ because of Theorem 2.3. Then applying Lemma 3.6 with xv instead of v and ϵ/c instead of ϵ and summing over $v \in \mathbb{Z}^n$ we get (3.19). This proves (i).

In view of the proof of (i), to prove (ii) we need only show that the number of non-zero terms in the sum on the left hand side of (3.19) is $O(e^{(n-2)t})$. Note that the support of J_f is contained in the interior of the support of J_{f_+}; hence there exists $\rho > 0$ such that on the support of J_f, $J_{f_+} > \rho > 0$.

We apply Lemma 3.6 with $\nu = 1$, xv instead of v, f_+ instead of f, and $\epsilon < \rho/3$, and sum over the set $S = \{v \in \mathbb{Z}^n \ : \ J_f(\|xv\|e^{-t}, Q_0(xv)) \neq 0\}$. We get

$$\sum_{v \in S} J_{f_+}(\|xv\|e^{-t}, Q_0(xv)) \leq c_{p,q} e^{(n-2)t} \int_K \tilde{f}_+(a_t kx)\, dm(k) + \epsilon|S|, \tag{3.20}$$

where $|S|$ denotes the cardinality of S. By the assumption on f_+ the integral in (3.20) is bounded, hence the right hand side of (3.20) is bounded by $\epsilon|S| + O(e^{(n-2)t})$. Since each term in the sum in (3.20) is of size at least ρ, we get $\rho|S| < \epsilon|S| + O(e^{(n-2)t})$. Since $\rho > 2\epsilon$, this implies that $|S|$ is indeed $O(e^{(n-2)t})$. This completes the proof of (ii). $\square$

Lemma 3.8.

(i) Suppose h is a continuous function of compact support on $\mathbb{R}^n - \{0\} \times \mathbb{R}$. Then

$$\lim_{T\to\infty} \frac{1}{T^{n-2}} \int_{\mathbb{R}^n} h(v/T, Q_0(v))\, dv =$$
$$= c_{p,q} \int_K \int_{-\infty}^{\infty} \int_0^{\infty} h(rk^{-1}e_1, \zeta) r^{n-2} (dr/2r)\, d\zeta\, dm(k). \quad (3.21)$$

The convergence is uniform as h varies over equicontinuous families of functions vanishing outside a given compact set.

(ii) Let ρ be a continuous positive function on the sphere, and let $\Omega = \{v \in \mathbb{R}^n \mid \|v\| < \rho(v/\|v\|)\}$. Then there exists a constant $\lambda = \lambda_{Q,\Omega}$ so that as $T \to \infty$,

$$\mathrm{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap T\Omega) \sim \lambda_{Q,\Omega}(b-a)T^{n-2}.$$

Proof. Write $v = \sum_{i=1}^n v_i f_i$, where the $\{f_i\}$ is the basis defined in the proof of Lemma 3.6. As in Lemma 3.6, let $r_1 = (\sum_{i=1}^p v_i^2)^{1/2}$, $r_2 = \left(\sum_{i=p+1}^n v_i^2\right)^{1/2}$. Then $Q_0(v) = r_1^2 - r_2^2$, and $\|v\| = (r_1^2 + r_2^2)^{1/2}$. Using polar coordinates (r_1, ω_1) and (r_2, ω_2) on the spaces $\mathrm{span}(f_1, \dots, f_p)$ and $\mathrm{span}(f_{p+1}, \dots, f_n)$ respectively, we get, for any function h of compact support on $\mathbb{R}^n - \{0\} \times \mathbb{R}$,

$$\int_{\mathbb{R}^n} h(v/T, Q_0(v))\, dv =$$
$$\int_0^{\infty} \int_0^{\infty} \int_{S^{p-1}\times S^{q-1}} h\left(\frac{r_1\omega_1 + r_2\omega_2}{T}, r_1^2 - r_2^2\right) r_1^{p-1} r_2^{q-1}\, d\omega_1\, d\omega_2\, dr_1\, dr_2.$$

Since h has compact support in $\mathbb{R}^n - \{0\}$, for large T the integrand is zero unless r_1 and r_2 are comparable to T. Using the compactness of the support of h in the second variable we get that the integrand is zero unless $r_2 = r_1 + O(1/T)$. Letting $\zeta = r_1^2 - r_2^2$, we see that $dr_1\, dr_2 = (1 + O(1/T))(dr_1/2r_1)d\zeta$ and, $(r_1\omega_1 + r_2\omega_2)/T = r_1(\omega_1 + \omega_2)/T + O(1/T)$. Hence

$$\lim_{T\to\infty} \frac{1}{T^{n-2}} \int_{\mathbb{R}^n} h\left(v/T, Q_0(v)\right)\, dv =$$
$$= \lim_{T\to\infty} \frac{1}{T^{n-2}} \int_{S^{p-1}\times S^{q-1}} \int_{-\infty}^{\infty} \int_0^{\infty} h\left(\frac{r_1(\omega_1+\omega_2)}{T}, \zeta\right) r_1^{n-2} \frac{dr_1}{2r_1}\, d\zeta d\omega_1\, d\omega_2$$
$$= c_{p,q} \int_K \int_{-\infty}^{\infty} \int_0^{\infty} h(rk^{-1}e_1, \zeta) r^{n-2} (dr/2r)\, d\zeta\, dm(k),$$

where in the last line we have written $(\omega_1+\omega_2)/2^{1/2}=k^{-1}e_1$ and made the change of variable $r=2^{1/2}r_1/T$. This proves (i).

We now prove (ii). Since we may write $Q=Q_0^x$ and replace Ω by $x^{-1}\Omega$, we may assume $Q=Q_0$. Let $\chi(v,\zeta)$ denote the characteristic function of $[1/2,1]\partial\Omega\times[a,b]$. We claim that (3.21) holds for $h=\chi$. Indeed, for any $m>1$, we may choose continuous compactly supported functions h_m^+ and h_m^-, such that $0\le h_m^-\le\chi\le h_m^+\le 1$, h_m^+ vanishes outside $[1/2-1/m,1+1/m]\partial\Omega\times[a-1/m,b+1/m]$ and $h_m^-=\chi$ on $[1/2+1/m,1-1/m]\partial\Omega\times[a+1/m,b-1/m]$. Let $L(h)$ denote the right hand side of (3.21). By part (i),

$$L(h_m^-)\le\liminf_{T\to\infty}\frac{1}{T^{n-2}}\int_{\mathbb{R}^n}\chi(v/T,Q_0(v))\,dv\le\limsup_{T\to\infty}\frac{1}{T^{n-2}}\int_{\mathbb{R}^n}\chi(v/T,Q_0(v))\,dv\le L(h_m^+).$$

From the definition of $L(\cdot)$ it is clear that $\lim_{m\to\infty}L(h_m^+)=\lim_{m\to\infty}L(h_m^-)=L(\chi)$. Hence (3.21) indeed holds for $h=\chi$. Hence,

$$\lim_{T\to\infty}\frac{1}{T^{n-2}}\operatorname{Vol}(V_{(a,b)}^{Q_0}(\mathbb{R})\cap[T/2,T]\partial\Omega)=L(\chi)=c(b-a).$$

Now (ii) follows from an easy geometric series argument. $\square$

Remark. One can verify that:

$$\lambda_{Q,\Omega}=\int\limits_{L\cap\Omega}\frac{dA}{\|\nabla Q\|},$$

where L is the lightcone $Q=0$ and dA is the area element on L.

Lemma 3.9. *Suppose f is a continuous function of compact support on $\mathbb{R}_+^n$, ν a non-negative continuous function on S^{n-1}. Let $h(v,\zeta)=J_f(\|v\|,\zeta)\nu(v/\|v\|)$. Then*

$$\lim_{T\to\infty}\frac{1}{T^{n-2}}\int_{\mathbb{R}^n}h(v/T,Q_0(v))\,dv=c_{p,q}\int_{G/\Gamma}\tilde{f}\int_K\nu(k^{-1}e_1)\,dm(k).$$

The convergence is uniform as ν varies over equicontinuous families of functions and f varies over equicontinuous families of functions vanishing outside a given compact set in $\mathbb{R}_+^n$.

Proof. The Siegel integral formula is the identity:

$$\int_{G/\Gamma}\tilde{f}(g)\,dg=\int_{\mathbb{R}^n}f(v)\,dv. \tag{3.22}$$

Also using the change of variable $(x_1, \ldots, x_n) \to (x'_1, x'_2, \ldots, x'_{n-1}, \zeta)$, where $x'_j = x_j$, $1 \le j \le n-1$ and $\zeta = Q_0(x_1, \ldots, x_n)$, we get (writing r for x'_1):

$$\int_{\mathbb{R}^n} f(v)\, dv = \int_{-\infty}^{\infty} \int_0^{\infty} J_f(r, \zeta) r^{n-2} (dr/2r)\, d\zeta. \tag{3.23}$$

Now the lemma follows from (3.22), (3.23) and Lemma 3.8 part (i), with $h(v, \zeta) = J_f(\|v\|, \zeta)\nu(v/\|v\|)$. $\square$

Proof of Theorems 2.1 and 2.5. We note that Theorem 2.1 is a special case of Theorem 2.5, and so we only prove the latter. By rescaling, we may assume that all the forms in $\mathcal{D}$ have discriminant 1. Let C be a compact subset of G such that every $Q \in \mathcal{D}$ is of the form Q_0^x for some x in C.

Let $\mathcal{C}$ denote the space of continuous functions on $\mathbb{R}^n - \{0\} \times \mathbb{R}$ vanishing outside a fixed compact set, with the topology of uniform convergence. Then by Lemma 3.8 part (i), the functional L on $\mathcal{C}$ given by

$$L(h) = \lim_{T \to \infty} \frac{1}{T^{n-2}} \int_{\mathbb{R}^n} h(v/T, Q_0(v))\, dv$$

is continuous. From the proof of part (ii) of Lemma 3.8 it follows that if χ is the characteristic function of $[1/2, 1]\partial\Omega \times [a, b]$, then for every $\epsilon > 0$ and every $x \in C$ there exist functions $h_{x,+}$ and $h_{x,-}$ in $\mathcal{C}$ such that for all v, ζ

$$h_{x,-}(xv, \zeta) \le \chi(v, \zeta) \le h_{x,+}(xv, \zeta), \tag{3.24}$$

and

$$|L(h_{x,+}) - L(h_{x,-})| < \epsilon. \tag{3.25}$$

Let $\mathcal{L}$ denote the space of finite linear combinations of functions on $\mathbb{R}^n \times \mathbb{R}$ of the form $J_f(\|v\|, \zeta)\nu(v/\|v\|)$ where f is a continuous compactly supported function on $\mathbb{R}^n_+$, and ν a non-negative continuous function on S^{n-1}. By the definition it easily follows that every continuous compactly supported function on $\mathbb{R}^+ \times \mathbb{R}$ vanishing outside a fixed compact set is of the form J_f for f a continuous compactly supported function on $\mathbb{R}^n_+$; indeed the mapping $f \mapsto J_f$ is essentially integration along the leaves of the foliation whose leaves are the hyperboloids $Q_0 = const$. Then by the Stone-Weierstrass Theorem (cf. [Rud]) $\mathcal{L}$ is dense in the space $\mathcal{C}$. Hence, using the compactness of C, we may assume that all $h_{x,+}, h_{x,-}$ belong to some finite collection $\mathcal{H}$ of functions in $\mathcal{L}$. By Proposition 3.7, Theorem 3.2 and Lemma 3.9 there exist points $x_1, \dots x_\ell$ in G/Γ so that Hx_i is closed for each i, and for each compact subset F of $C - \bigcup_{i=1}^{\ell} Hx_i$ there exists $t_0 > 0$ so that for all $h \in \mathcal{H}$, and all $x \in F$

$$\left| e^{-(n-2)t} \sum_{v \in \mathbb{Z}^n} h(xve^{-t}, Q_0(xv)) - L(h) \right| < \epsilon, \tag{3.26}$$

and

$$\left| e^{-(n-2)t} \int_{\mathbb{R}^n} h(xve^{-t}, Q_0(xv)) - L(h) \right| \leq \epsilon. \tag{3.27}$$

Combining (3.24), (3.25), (3.26) and (3.27) with a suitable ϵ, we see that for every $\theta > 0$ there exist finitely many points $x_1, \dots, x_m$ with Hx_i closed for each i, so that for every compact subset F of $C - \bigcup_{i=1}^{m} Hx_i$ there exists $t_0 > 0$ such that for every $x \in F$ and every $t > t_0$,

$$(1-\theta) \int_{\mathbb{R}^n} \chi(ve^{-t}, Q_0(xv))\, dv \leq \sum_{v \in \mathbb{Z}^n} \chi(ve^{-t}, Q_0(xv)) \leq (1+\theta) \int_{\mathbb{R}^n} \chi(ve^{-t}, Q_0(xv))\, dv,$$

i.e.

$$(1-\theta)\operatorname{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap \Omega_t) \leq |V^Q_{(a,b)}(\mathbb{Z}) \cap \Omega_t| \leq (1+\theta)\operatorname{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap \Omega_t),$$

where $\Omega_t = [e^t/2, e^t]\partial\Omega$.

Let $\mathcal{P} = \mathcal{D} \cap \{\mathbb{R}Q_0^{x_1}, \dots, \mathbb{R}Q_0^{x_m}\}$ (for a quadratic form Q, $\mathbb{R}Q$ means the multiples of Q). By [DM3, Proposition 9.3] all the quadratic forms in $\mathcal{P}$ are multiples of rational forms, and for every compact subset $\mathcal{F}$ of $\mathcal{D} - \mathcal{P}$ there exists a compact subset F of $C - \bigcup_{1 \leq i \leq m} Hx_i$ such that every quadratic form $Q \in \mathcal{F}$ is a multiple of Q_0^x for some $x \in F$.

This implies that for sufficiently large T, for every $Q \in \mathcal{F}$,

$$(1-\theta)\operatorname{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap T\Omega) \leq |V^Q_{(a,b)}(\mathbb{Z}) \cap T\Omega| \leq (1+\theta)\operatorname{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap T\Omega).$$

To finish the proof of the theorem it only remains to remark that the asymptotics of the volume $\operatorname{Vol}(V^Q_{(a,b)}(\mathbb{R}) \cap T\Omega)$ are of the form (2.2) by Lemma 3.8 (ii). $\square$

Remark. To prove Theorem 2.1 and Theorem 2.5 for the counting function of primitive lattice points defined in Remark 2.6, one may argue as above but instead of $\tilde{f}$ one may use

$$\hat{f}(g) = \sum_{\substack{v \in \mathbb{Z}^n - 0 \\ v \text{ primitive}}} f(gv).$$

With this definition the analogue of (3.22) is

$$\int_{G/\Gamma} \hat{f}(g)\,dg = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f(v)\,dv.$$

3.6 Convergence a.e. in the $(2,1)$ and $(2,2)$ cases

Lemma 3.10. *The function α on the space of unimodular lattices in $\mathbb{R}^m$, $SL(m,\mathbb{R})/SL(m,\mathbb{Z})$, belongs to every L^μ, $1 \le \mu < m$.*

Proof. Let N denote the set of upper triangular matrices, and K the set of orthogonal matrices. For $\beta \in \mathbb{R}$, let $A_\beta = \{\mathrm{diag}(a_1, \ldots, a_m) \ : \ a_i/a_{i+1} < \beta\}$. One can choose a compact set $\Omega \subset N$ and a number $\beta > 0$ so that the *Siegel set*: $\mathfrak{S} = K A_\beta \Omega$ contains a fundamental domain for the action of Γ on G (see [Bor1, p. 34] for a proof). Hence it is enough to show that the integral of α^μ over $\mathfrak{S}$ is finite. Note that for $g = kan \in \mathfrak{S}$, $\alpha(g) \le C\alpha(a)$. By its definition, $\alpha(a) = \max_{1 \le j \le m-1} \alpha_j(a)$, where $\alpha_j(a) = \prod_{1 \le i \le j} a_i^{-1}$. It is convenient to pass to the Lie algebra by writing for $1 \le i \le m-1$, $a_i/a_{i+1} = e^{-\beta_i}$. Then on A_β, $\beta_i \ge -\log\beta$ for all i. A straightforward calculation (or [BMP]) shows that $\alpha_j = \exp \sum_{r=1}^{m-1} c_{jr}\beta_r$, where the coefficients c_{jr} satisfy $c_{jr} = c_{rj}$, $c_{jj} = j(n-j)/m$, and $c_{ij} \le c_{ii}$ for all i, j. By [Kna, §V.6], the measure of integration in KAN coordinates is $e^{-2\rho}\left(\prod_r d\beta_r\right) dn\,dk$, where $2\rho = \sum_{r=1}^{m-1} r(m-r)\beta_r$. Hence for each j, $1 \le j \le m-1$, and $1 \le \mu < m$,

$$\begin{aligned}\int_{\mathfrak{S}} \alpha_j^\mu &= \int_K \int_{\log A_\beta} \int_\Omega \alpha_j(kan)^\mu\, e^{-2\rho}\, dn\, d\beta_1 \ldots d\beta_{m-1}\, dk \\ &\le C' \prod_{r=1}^{m-1} \int_{-\log\beta}^{\infty} e^{(c_{jr}\mu - r(m-r))\beta_r}\, d\beta_r < \infty.\end{aligned}$$

Since $\alpha = \max_j \alpha_j$, the lemma follows. $\square$

In view of Lemma 3.10 and the results subsection §3.5 Theorem 2.4 can also be derived from the general pointwise ergodic theorem of [NS]. We choose to present a self-contained argument which uses additional properties of the function α.

We keep the notation of the previous subsections. For $g \in G$, we denote by $d(g,1)$ the distance in the symmetric space $\hat{K}\backslash G$ between the point $\hat{K}g$ and the origin $\hat{K}$. For $1 \le r \le \infty$, we denote the L^r norm of a function f by $\|f\|_r$.

Theorem 3.11 (Howe-Moore). *Let G be a connected simple Lie group with finite center, and let Γ be a lattice in G. Let $\hat{K}$ denote the maximal compact subgroup of G. Then there exists a constant $\lambda > 0$, so that the following holds: Let $f_1, f_2 \in L^2(G/\Gamma)$ be orthogonal to the constant function. Suppose either both f_1 and f_2 are smooth of compact support, or both f_1 and f_2 are left $\hat{K}$ invariant. Then there exists a constant $C > 0$, depending on f_1, f_2 so that*

$$\int_{G/\Gamma} f_1(gx) f_2(x)\, dx \leq C e^{-\lambda d(g,1)}.$$

Proof. See [How],[KM, Theorem A.5], [KS]. □

Lemma 3.12. *Suppose $t > 0$, and let x, y, e be points in $\mathbb{H}^2$ with $d(x,e) = d(y,e) = t$. Let $\theta = \theta_e(x,y)$ denote the visual angle between x and y as viewed from e. Then for any β with $0 < \beta < 1$ there exists $t_0 > 0$ such that if $\theta \geq e^{-\beta t}$ and $t > t_0$, then $d(x,y) \geq \beta t$.*

Proof. The hyperbolic law of cosines [Bea] yields

$$\begin{aligned} \cosh(d(x,y)) &= 1 + \sinh^2 t \cdot (1 - \cos\theta) \\ &= 1 + 2(\sinh^2 t)\sin^2(\theta/2). \end{aligned}$$

Hence, (assuming $e^{-\beta t} < \pi$),

$$d(x,y) \geq \cosh^{-1}\left(1 + 2(\sinh^2 t)\sin^2(e^{-\beta t}/2)\right). \tag{3.28}$$

Now the desired estimate easily follows from the fact that the right hand side of (3.28) is asymptotic to $(2-\beta)t$ as $t \to \infty$. □

Proposition 3.13. *Suppose $(p,q) = (2,1)$ or $(2,2)$. Let ϕ be a continuous function on $G/\Gamma \approx$ the space of lattices in $\mathbb{R}^n$ with determinant 1. Assume that for some s, $0 < s < n/2$ and some $C > 0$,*

$$|\phi(\Delta)| < C\alpha(\Delta)^s, \qquad \text{for all } \Delta \in G/\Gamma.$$

Let ν be any bounded function on K. Then for almost all $x_0 \in G/\Gamma$,

$$\lim_{t\to+\infty} \int_K \phi(a_t k x_0)\nu(k)\, dm(k) = \int_K \nu\, dm \int_{G/\Gamma} \phi(y)\, d\mu(y).$$

Proof. Any continuous function on G/Γ can be approximated in $C(G/\Gamma)$ from above and below by smooth functions. Hence we may assume the function ϕ is smooth. We may also assume ν is positive.

Note that for any $g \in G$ and any lattice Δ, $\frac{\alpha_i(g\Delta)}{\alpha_i(\Delta)} \leq \| \bigwedge^i(g)\|$. Thus $\log\alpha$ is Lipschitz, hence uniformly continuous.

Let $\epsilon > 0$ be arbitrary. Let $A(r) = \{x \in G/\Gamma \ : \ \alpha(x) > r\}$. Choose a smooth nonnegative function g_r on G/Γ such that $g_r(x) = 1$ if $x \in A(r+1)$, $g_r(x) = 0$ if $x \notin A(r)$ and $0 \leq g_r(x) \leq 1$ if $x \in A(r) - A(r+1)$. Let $\phi_1 = \phi g_r$ and $\phi_2 = \phi - \phi g_r$ so that $\phi = \phi_1 + \phi_2$. By Lemma 3.10 we can choose $\theta > 0$ so that $\alpha^{s+\theta} \in L^2(G/\Gamma)$. Then $\phi_1(x) \leq C\alpha(x)^s g_r(x) = C\alpha(x)^{s+\theta}g_r(x)\alpha(x)^{-\theta} \leq Cr^{-\theta}\alpha(x)^{s+\theta}$. Let $\psi(x) = Cr^{-\theta}\alpha(x)^{s+\theta}$, so that $|\phi_1| \leq \psi$ and $\psi \in L^2(G/\Gamma)$. The function $\log\alpha^{s+\theta}$ is uniformly continuous, hence there exists a constant $M > 0$ so that for all $x \in G/\Gamma$ and all $\tau \in [0,1]$, $\alpha^{s+\theta}(a_\tau x) \leq M\alpha^{s+\theta}(x)$, hence $\psi(a_\tau x) \leq M\psi(x)$. By choosing r sufficiently large we may make $\|\psi\|_1 \leq \epsilon/6M$.

For any function f on G/Γ, let $(A_t f)(x) = \int_K f(a_t kx)\nu(k)\,dm(k)$. Suppose $f \in L^2$ is orthogonal to the constants, and f is either smooth of compact support or left K-invariant. Then by Theorem 3.11, there exists $\lambda > 0$ and $C = C_f > 0$, so that for any $g \in G$,

$$\left| \int_{G/\Gamma} f(gx)f(x)\,dx \right| \leq C_f e^{-\lambda d(g,1)}. \tag{3.29}$$

Now

$$\begin{aligned}\|A_t f\|_2^2 &= \int_{G/\Gamma}\int_{K\times K} f(a_t k_1 x)f(a_t k_2 x)\nu(k_2)\nu(k_1)\,dm(k_2)\,dm(k_1)\,dx = \\ &= \int_K \left(\int_{G/\Gamma} f(a_t k a_t^{-1}x)f(x)\,dx\right)(\nu * \check{\nu})(k)\,dm(k),\end{aligned} \tag{3.30}$$

where $\check{\nu}$ is the "inverted" function $\check{\nu}(k) = \nu(k^{-1})$, and $\nu * \check{\nu}$ is the convolution $(\nu * \check{\nu})(k) = \int_K \nu(kk')\nu(k')\,dm(k')$.

Fix β with $0 < \beta < 1$, and let $\mathcal{U} = \{k \in K \ : \ d(a_t k a_t^{-1}, 1) \leq \beta t\}$. By Lemma 3.12, $m(\mathcal{U}) \leq e^{-\beta t}$, Hence, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\int_{\mathcal{U}}\int_{G/\Gamma} f(a_t k a_t^{-1}x)f(x)\,dx\ (\nu * \check{\nu})(k)\,dm(k)\right| &\leq m(\mathcal{U})\max(\nu * \check{\nu})\|f\|_2^2 \\ &\leq e^{-\beta t}\max(\nu * \check{\nu})\|f\|_2^2.\end{aligned} \tag{3.31}$$

If $k \notin \mathcal{U}$, then $d(a_t k a_t^{-1}, 1) \geq \beta t$, and thus, by (3.29),

$$\left| \int_{K-\mathcal{U}} \int_{G/\Gamma} f(a_t k a_t^{-1} x) f(x) \, dx (\nu * \check{\nu})(k) \, dm(k) \right| \leq C_f e^{-\lambda \beta t} \max(\nu * \check{\nu}). \tag{3.32}$$

Combining (3.30) (3.31) and (3.32) we get for sufficiently large t,

$$\|A_t f\|_2^2 \leq (e^{-\beta t} \|f\|_2^2 + C_f e^{-\lambda \beta t}) \max(\nu * \check{\nu}) < e^{-\lambda' t}, \tag{3.33}$$

where λ' is independent of t. Hence

$$\begin{aligned} \left| \left\{ x \in G/\Gamma \ : \ A_t \psi(x) > \frac{\epsilon}{3M} \right\} \right| &\leq \left| \left\{ x \in G/\Gamma \ : \ \left| A_t \psi - \int \psi \int \nu \right| > \frac{\epsilon}{6M} \right\} \right| \\ &\leq \left(\frac{6M}{\epsilon} \right)^2 e^{-\lambda' t}, \end{aligned} \tag{3.34}$$

where $|\cdot|$ denotes Haar measure on G/Γ. Note that $A_{t+\tau}\psi < M A_t \psi$, for all $0 \leq \tau \leq 1$. Thus, since $|\phi_1| \leq \psi$,

$$|\{ x \in G/\Gamma \ : \ |A_{t+\tau}\phi_1(x)| > \frac{\epsilon}{3} \quad \text{for some } \tau \in [0,1] \}| \leq \left(\frac{6M}{\epsilon} \right)^2 e^{-\lambda' t}. \tag{3.35}$$

Since ϕ_2 is uniformly continuous, there exists $\delta > 0$ such that $|A_{t+\tau}\phi_2(x) - A_t \phi_2(x)| < \epsilon/3$ for all $t > 0$, $x \in G/\Gamma$ and $\tau \in [0, \delta]$. Hence using (3.33),

$$\left| \left\{ x \in G/\Gamma \ : \ \left| A_{t+\tau}\phi_2 - \int \phi_2 \int \nu \right| > \frac{2\epsilon}{3} \quad \text{for some } \tau \in [0, \delta] \right\} \right| < \left(\frac{3}{\epsilon} \right)^2 e^{-\lambda' t}. \tag{3.36}$$

Let

$$E_t = \{ x \in G/\Gamma \ : \ |(A_{t+\tau}\phi)(x) - \int \phi \int \nu| > \epsilon \quad \text{for some } \tau \in [0, \delta] \}.$$

Then, combining (3.35) and (3.36) we get

$$|E_t| \leq C e^{-\lambda' t},$$

where C is independent of t. Hence,

$$\left| \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_{n\delta} \right| = 0,$$

which is equivalent to the statement of the proposition. □

Remark. One can easily modify the proof to handle signatures other than $(2,1)$ and $(2,2)$, and s with $0 \le s < n$.

Proof of Theorem 2.4. Theorem 2.4 follows from Proposition 3.13, Proposition 3.7 part (ii) and Lemma 3.9; the argument is similar to the argument used in the proof of Theorem 2.1 on page 18. □

3.7 Counterexamples.

In this subsection we prove Theorem 2.2. For $\alpha > 0$, let $Q_\alpha(x_1, x_2, x_3)$ denote the quadratic form $x_1^2 + x_2^2 - \alpha^2 x_3^2$, and let $\hat{Q}_\alpha(x_1, x_2, x_3, x_4)$ denote the quadratic form $x_1^2 + x_2^2 - \alpha^2(x_3^2 + x_4^2)$. The quadratic forms we will construct to prove Theorem 2.2 will be of the form Q_α if $n = 3$, and $\hat{Q}_\alpha$ if $n = 4$.

The following lemma is well known, and can be deduced from the properties of the divisor function:

Lemma 3.14. *Suppose α is rational. Then*

$$|\{x \in \mathbb{Z}^3 \mid Q_\alpha(x) = 0, \quad \|x\| \le T\}| \sim c_\alpha T \log T, \tag{3.37}$$

and

$$|\{x \in \mathbb{Z}^4 \mid \hat{Q}_\alpha(x) = 0, \quad \|x\| \le T\}| \sim c_\alpha T^2 \log T. \tag{3.38}$$

Lemma 3.15. *Suppose $[a, b] = [1/4, 1]$ or $[a, b] = [-1, -1/4]$. Then for every $\epsilon > 0$ and $S > 0$, the set of $\beta \in \mathbb{R}$ for which there exists a $T > S$ such that*

$$|V_{(a,b)}^{Q_\beta}(\mathbb{Z}) \cap T\Omega_0)| \ge T\,(\log T)^{1-\epsilon}$$

is dense. Similarly, the set of $\beta \in \mathbb{R}$ for which there exists a $T > S$ such that

$$|V_{(a,b)}^{\hat{Q}_\beta}(\mathbb{Z}) \cap T\Omega_0)| \ge T^2\,(\log T)^{1-\epsilon}$$

is dense.

Proof of Lemma 3.15. For α rational and $T > 0$, let $L(\alpha, T)$ denote the set of $x \in \mathbb{Z}^3$ such that $Q_\alpha(x) = 0$ and $T/2 \le \|x\| \le T$. Then by (3.37) for sufficiently large

T (depending on α), the cardinality of $L(\alpha, T)$ is at least $(c_\alpha/4) T \log T > T(\log T)^{1-\epsilon}$. Note that if $(x_1, x_2, x_3) \in L(\alpha, T)$, then

$$\frac{T^2}{4(1+\alpha^2)} \le x_3^2 \le \frac{T^2}{1+\alpha^2}, \tag{3.39}$$

so that if

$$\beta^2 = \alpha^2 - \frac{1+\alpha^2}{T^2}, \tag{3.40}$$

then for $x \in L(\alpha, T)$,

$$Q_\beta(x) = Q_\alpha(x) + \frac{1+\alpha^2}{T^2} x_3^2$$

satisfies $(1/4) \le Q_\beta(x) \le 1$ by (3.39). Hence $L(\alpha, T) \subset V^{Q_\beta}_{(1/4,1)}(\mathbb{Z}) \cap T\Omega_0$, and hence the cardinality of the latter is at least $T(\log T)^{1-\epsilon}$. Similarly, if we choose

$$\beta^2 = \alpha^2 + \frac{1+\alpha^2}{T^2}, \tag{3.41}$$

then the cardinality of $V^{Q_\beta}_{(-1,-1/4)}(\mathbb{Z}) \cap T\Omega_0)$ is at least $T(\log T)^{1-\epsilon}$. The set of numbers β satisfying (3.40) or (3.41) for rational α and sufficiently large T is clearly dense. The proof for the $(2,2)$ case is identical. $\square$

Remark. The above proof can be interpreted geometrically as follows: The light cone of the form Q_β is slightly wider (resp. narrower) then the light cone of Q_α, so that the integral points $L(\alpha, T)$ which are on the light cone of Q_α are inside (resp. outside) the light cone of Q_β.

Proof of Theorem 2.2. Since the interval $[a, b]$ must intersect either the positive reals or the negative reals, we may assume after passing to a subset and rescaling that e.g. $[a, b] = [1/8, 2]$ or $[a, b] = [-2, -1/8]$. We assume the former for notational simplicity; the latter case is proved identically. For a given $S > 0$ let $\mathcal{U}_S$ denote the set of $\gamma \in \mathbb{R}$ for which there exists $\beta \in \mathbb{R}$ and $T > S$ with

$$|V^{Q_\beta}_{(1/4,1)}(\mathbb{Z}) \cap T\Omega_0| \ge T(\log T)^{1-\epsilon}, \tag{3.42}$$

and

$$|\beta - \gamma| < T^{-3}. \tag{3.43}$$

Then $\mathcal{U}_S$ is open and dense by Lemma 3.15. By the Baire category theorem, the intersection $\cap_{j=1}^{\infty}\mathcal{U}_{2^j}$ is not empty; in fact it is of second category so uncountable. Let γ be an irrational point in this intersection. Then there exist infinite sequences β_j, T_j satisfying (3.42) and (3.43), with β replaced by β_j and T replaced by T_j. The equation (3.43) implies that for $\|x\| < T_j$, $|Q_{\beta_j}(x) - Q_\gamma(x)| < c/T_j < 1/8$, so that $V_{(1/8,2)}^{Q_\gamma}(\mathbb{Z}) \cap T_j\Omega_0$ contains $V_{(1/4,1)}^{Q_{\beta_j}}(\mathbb{Z}) \cap T_j\Omega_0$. Thus the cardinality of the former is at least $T_j(\log T_j)^{1-\epsilon}$ by (3.42). $\square$

Remarks. We have actually proved that the set of $\gamma \in \mathbb{R}$ for which Theorem 2.2 holds for Q_γ is of second category in the Baire sense. Also from the proof it is clear that in the statement of Theorem 2.2, $(\log T)^{1-\epsilon}$ can be replaced by $\log T/\nu(T)$ where $\nu(T)$ is any unbounded increasing function. One can also obtain results analogous to Theorem 2.2 for the function $|\mathcal{L}'_Q(T\Omega_0, [a,b])|$ counting primitive lattice points, but the construction for the $SO(2,1)$ case must be modified: instead of (3.37) one can use the fact that for rational α, the number of integral points on the one-sheeted hyperboloid $x^2 + y^2 - \alpha^2 z^2 = 1$ with $x^2 + y^2 + z^2 < T^2$ grows like $T \log T$. (However the number of integral points x on any rational two-sheeted hyperboloid satisfying $\|x\| < T$ is asymptotic to a constant times T). We omit the details.

4 Some results on the equidistibution of translates

Consider the following situation: Let G be a connected Lie group, Γ a lattice in G, H a connected semisimple subgroup of G, and K a maximal compact subgroup of H. In this section we study the question of whether large spheres in H (and more generally, sets of the form $a_t K x$ where $x \in G$ and $A = \{a_s \mid s \in \mathbb{R}\}$ is a suitable one-parameter subgroup of H) are equidistributed in the quotient G/Γ. In our application, $G = SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$, $H \cong SO(p,q)$ where $p \geq q \geq 1$, $p+q = n$, and $K \subset H$ is isomorphic to $SO(p) \times SO(q)$. Theorem 4.4 below states that if one restricts to bounded continuous test functions ϕ, then under certain conditions on H and K and A (which are satisfied for the application) the answer is yes, as long Hx is not closed. (the case where Hx is closed corresponds to a quadratic form which is a multiple of a rational form). Furthermore, Theorem 4.4 implies that if one fixes a test function ϕ, then the convergence is uniform on compact sets away from a finite number of closed H orbits.

The following theorem of Dani and Margulis answers the analogous question for averages along orbits of one-parameter unipotent groups, instead of over spheres in a semisimple subgroup H. The "uniformity" of the convergence as one varies the base point x is crucial for the application.

For any closed subgroups H and W of G let $X(H,W) = \{g \in G \mid Wg \subset gH\}$.

Theorem 4.1 ([DM3], Theorem 3). *Let G be a connected Lie group and let Γ be a lattice in G. Let μ be the G-invariant probability measure on G/Γ. Let $U = \{u_t\}$ be an Ad-unipotent one-parameter subgroup of G and let ϕ be a bounded continuous function on G/Γ. Let $\mathcal{D}$ be a compact subset of G/Γ and let $\epsilon > 0$ be given. Then there exist finitely many proper closed subgroups $H_1 = H_1(\phi, \mathcal{D}, \epsilon), \cdots, H_k = H_k(\phi, \mathcal{D}, \epsilon)$ such that $H_i \cap \Gamma$ is a lattice in H_i for all i, and compact subsets $C_1 = C_1(\phi, \mathcal{D}, \epsilon), \cdots, C_k = C_k(\phi, \mathcal{D}, \epsilon)$ of $X(H_1, U), \cdots, X(H_k, U)$ respectively, for which the following holds: For any compact subset F of $\mathcal{D} - \bigcup_{1 \le i \le k} C_i\Gamma/\Gamma$ there exists a $T_0 \ge 0$ such that for all $x \in F$ and $T > T_0$*

$$\Big|\frac{1}{T}\int_0^T \phi(u_t x)\, dt - \int_{G/\Gamma} \phi \, d\mu\Big| < \epsilon.$$

Remark. It can be easily checked that in Theorem 4.1 the assumption that ϕ is bounded can not be weakened to $\phi \in L^1$, even if we assume that $\log \phi$ is uniformly continuous. For example if $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$, and $U = \{u_t\} = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ the function α of §3.1 belongs to $L^{2-\epsilon}$ for every $\epsilon > 0$; yet there exists a point $x \in G$ such that the orbit Ux is dense in G/Γ, but the averages

$$\frac{1}{T}\int_0^T \alpha(u_t x)\, dt$$

are unbounded. Since in our application we need to integrate an unbounded function in L^1, this presents a difficulty, which is handled in §5.

In the rest of this section, we deduce Theorem 4.4, from Theorem 4.1. This relies on the the well known fact that in the hyperbolic plane large circles are well approximated by horocyles. Theorem 4.4 immediately implies Theorem 3.5 for functions of compact support.

Lemma 4.2. *Let K be a compact group with the normalized Haar measure m acting continuously on a locally compact second countable space X. Let $A \subset X$ be a closed subset and $F \subset X$ a compact subset. Assume that*

$$m(\{k \in K \mid kx \in A\}) = 0$$

for any $x \in F$. Then for any $\epsilon > 0$ one can find an open subset $U_\epsilon \subset X$ containing A such that

$$m(\{k \in K \mid kx \in U_\epsilon\}) < \epsilon,$$

for any $x \in F$.

Proof. Since $m(\{k \in K \mid kx \in A\}) = 0,\ x \in F$, there exist a continuous function f_x on X, an open subset $W_x,\ A \subset W_x \subset X$, and a neighborhood V_x of x such that $0 \le f_x(y) \le 1$ for any $y \in X,\ f_x(w) = 0$ for any $w \in W_x$ and

$$\int_K f_x(kv)\, dm(k) > 1 - \epsilon$$

for any $v \in V_x$. The subset F is compact. Therefore one can find a finite number of points $x_1, \cdots, x_k \in F$ such that $F \subset \bigcup_{1 \le i \le k} V_{x_i}$. Let us denote the maximum of functions $f_i,\ 1 \le i \le k$, by f. Then $0 \le f(y) \le 1$ for any $y \in X,\ f(w) = 0$ for any $w \in U_\epsilon \stackrel{\text{def}}{=} \bigcap_{1 \le i \le k} W_{x_i}$ and

$$\int_K f(kx) dm(k) > 1 - \epsilon$$

for any $x \in F$. It remains to notice now that

$$m(\{k \in K \mid kx \in U_\epsilon\}) \le 1 - \int_K f(kx)\, dm(k),$$

because $\chi_{U_\epsilon} \le 1 - f$ where χ_{U_ϵ} denotes the characteristic function of U_ϵ.

Theorem 4.3. *(cf. [DM3, Proposition 9.4]) Let $G, \Gamma, \mu, U = \{u_t\},\ \phi,\ \mathcal{D}$ and $\epsilon > 0$ be the same as in Theorem 4.1. Let H be a connected semisimple subgroup of G and let K be a maximal compact subgroup of H with the normalized Haar measure m. Assume that the following conditions are satisfied:*

(a) *H has finite center and has no compact factors;*

(b) *$U \subset H$ and U is not contained in any proper normal subgroup of H;*

(c) *H is a maximal proper connected subgroup of G and H has finite index in the normalizer $N_G(H)$.*

Then there exist finitely many points $x_1, \cdots, x_\ell \in G/\Gamma$ such that

(I) *the orbits $Hx_1, \cdots, Hx_\ell$ are closed and have finite H-invariant measure;*

(II) *for any compact subset F of $\mathcal{D} - \bigcup_{1\le i\le \ell} Hx_i$ there exists T_0 such that for all $x \in F$ and $T > T_0$*

$$m\Big\{k \in K \Big| \quad \Big|\frac{1}{T}\int\limits_0^T \phi(u_t kx)\,dt - \int\limits_{G/\Gamma} \phi\,d\mu\Big| > \epsilon\Big\} < \epsilon, \tag{4.1}$$

and, consequently,

$$\begin{aligned}
&\Big|\frac{1}{T}\int\limits_0^T\int\limits_K \phi(u_t kx)q(k)\,dt\,dm(k) - \int\limits_{G/\Gamma}\phi\,d\mu\int\limits_K q\,dm\Big| \le \\
&\le \int\limits_K |q(k)|\,\Big|\frac{1}{T}\int\limits_0^T \phi(u_t kx)\,dt - \int\limits_{G/\Gamma}\phi\,d\mu\Big|\,dm(k) \le \\
&\le \epsilon \sup_{k\in K}|q(k)|\,(1 + 2\sup_{y\in G/\Gamma}|\phi(y)|),
\end{aligned} \tag{4.2}$$

for any bounded measurable function q on K.

Proof. Let us consider subgroups $H_1 = H_1(\phi,\ K\mathcal{D}, \epsilon), \cdots, H_k = H_k(\phi, K\mathcal{D}, \epsilon)$ and subsets $C_1 = C_1(\phi, K\mathcal{D}, \epsilon), \cdots, C_k = C_k(\phi, K\mathcal{D}, \epsilon)$ defined in Theorem 4.1, and let us define subsets

$$Y_i = \{g \in G \mid Kg \subset X(H_i, U)\}, \qquad 1 \le i \le k.$$

The subgroup S generated by $\bigcup_{k\in K} k^{-1}Uk$ is normalized by $U \cup K$. Since H is semisimple with finite center, any connected subgroup L of H containing K can be represented as $L = H'K$ where H' is a connected normal subgroup of H. Therefore conditions (a) and (b) imply that $S = H$. Let $y \in Y_i$. Then $Uky \subset kyH_i$, or equivalently, $k^{-1}Uk \subset yH_iy^{-1}$ for any $k \in K$. This and the equality $S = H$ imply that $H \subset yH_iy^{-1}$. Now using condition (c) we get

$$H = yH_i^\circ y^{-1} \qquad \text{for any } y \in Y_i, \qquad 1 \le i \le k, \tag{4.3}$$

where H_i^o denotes the connected component of identity of H_i. We deduce from (4.3) that $y_1^{-1}y_2 \in N_G(H)$ for any $y_1, y_2 \in Y_i$. Now using (4.3) and condition (c) we get that Y_i can be covered by a finite number of right cosets of H. Hence there exist finitely many points $x_1, \cdots, x_\ell \in G/\Gamma$ such that

$$\bigcup_{1 \le i \le k} Y_i\Gamma/\Gamma \subset \bigcup_{1 \le j \le \ell} Hx_j. \tag{4.4}$$

Since $H_i \cap \Gamma$ is a lattice in H_i we get from (4.3) that Hx_j, $1 \le j \le \ell$, carries finite H-invariant measure. Thus it remains to check property (II).

Since $X(H_i, U)$ is a real analytic submanifold of G and the group K is connected we have that $m(\{k \in K \mid kg \in X(H_i, U)\gamma\}) = 0$ for any $g \in G - Y_i\gamma$, $\gamma \in \Gamma$. Therefore,

$$m(\{k \in K \mid kg \in \bigcup_{1 \le i \le k} X(H_i,\ U)\Gamma\}) = 0$$

for any $g \in G - \bigcup_{1 \le i \le k} Y_i\Gamma$. But $C_i \subset X(H_i,\ U)$ and $F \subset \mathcal{D} - \bigcup_{1 \le j \le \ell} Hx_j$. This and (4.4) imply that

$$m(\{k \in K \mid kx \in \bigcup_{1 \le i \le k} C_i\Gamma/\Gamma\}) = 0$$

for any $x \in F$. Now applying Lemma 4.2 we get that there exists an open subset $W \subset G/\Gamma$, $W \supset \bigcup_{1 \le i \le k} C_i\Gamma/\Gamma$ such that $m(\{k \in K \mid kx \in W\}) < \epsilon$ for any $x \in F$. It remains to notice that since $C_i = C_i(\phi, K\mathcal{D},\ \epsilon)$ there exists $T_0 \ge 0$ such that

$$\Big|\frac{1}{T}\int_0^T \phi(u_t y)\, dt - \int_{G/\Gamma} \phi\, d\mu\Big| < \epsilon$$

for all $y \in K\mathcal{D} - W$ and $T > T_0$. $\square$

Theorem 4.4. *Let G be a connected Lie group, Γ a lattice in G, μ the G-invariant probability measure on G/Γ, ϕ a bounded continuous function on G/Γ, $\mathcal{D}$ a compact subset of G/Γ, H a connected semisimple subgroup of G, K a maximal compact subgroup of H with the normalized Haar measure m, and $A = \{a_s \mid s \in \mathbb{R}\}$ a one-parameter subgroup of H. Assume that H satisfies conditions (a) and (c) of Theorem 4.3. Let us assume also that*

(a) *A is not contained in any proper normal subgroup of H;*

(b) *there exists a continuous homomorphism with finite kernel* $\rho : SL_2(\mathbb{R}) \to H$ *such that* $\rho(D) = A$ *and* $\rho(SO(2)) \subset K$ *where* $D = \{d_t = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) | t > 0\}$.

Then for any $\epsilon > 0$ *and any bounded measurable function* q *on* K *there exist finitely many points* $x_1, \cdots, x_\ell \in G/\Gamma$ *such that*

(I) *the orbits* $Hx_1, \cdots, Hx_\ell$ *are closed and have finite* H*-invariant measure;*

(II) *for any compact subset* F *of* $\mathcal{D} - \bigcup_{1\le i\le \ell} Hx_i$ *there exists* $S_0 > 0$ *such that for all* $x \in F$ *and* $s > S_0$

$$\Big|\int\limits_K \phi(a_s kx)q(k)\, dm(k) - \int\limits_{G/\Gamma} \phi\, d\mu \int\limits_K q\, dm\Big| \le \epsilon.$$

Proof. Since ϕ is bounded and continuous and q is bounded, there exist bounded uniformly continuous functions ϕ_n' and ϕ_n'' on G/Γ and q_n on K such that $\phi_n' \le \phi \le \phi_n''$, $\lim\limits_{n\to\infty} \int\limits_{G/\Gamma} (\phi_n'' - \phi_n')\, d\mu = 0$ and $\lim\limits_{n\to\infty} \int\limits_K |q_n - q|\, dm(k) = 0$. Hence we can assume that ϕ and q are uniformly continuous (with respect to the metric induced by a left invariant metric on G).

Let us denote $\left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right)$ by u_t and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ by w. Then simple calculations show that

$$d_t = b_t u_t k_t w,$$

where

$$b_t = \frac{1}{\sqrt{1+t^{-2}}}\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1+t^{-2}\end{pmatrix} \qquad k_t = \frac{1}{\sqrt{1+t^{-2}}}\begin{pmatrix} 1 & t^{-1} \\ -t^{-1} & 1\end{pmatrix}.$$

Then $k_t \in SO(2)$ and

$$\lim_{t\to+\infty} b_t = \lim_{t\to+\infty} k_t = e. \tag{4.5}$$

Let us denote $\rho(d_t)$, $\rho(b_t)$, $\rho(u_t)$, $\rho(k_t)$ and $\rho(w)$ by d_t', b_t', u_t', k_t' and w' respectively. Then for any $t > 0$ and $x \in G/\Gamma$

$$\begin{aligned}
&\int\limits_K \phi(d_t' kx)q(k)\, dm(k) = \int\limits_K \phi(b_t'u_t'k_t'w'kx)q(k)\, dm(k) = \\
&= \int\limits_K \phi(b_t'u_t'k_t'w'kx)q(k)\, dm(k_t'w'k) = \\
&= \int\limits_K \phi(b_t'u_t'kx)q(w'^{-1}k_t'^{-1}k)\, dm(k).
\end{aligned} \tag{4.6}$$

It follows from (4.6) that for any positive r and t

$$
\begin{aligned}
&\Big|\int_K \phi(d'_t kx)q(k)\,dm(k) - \int_K \phi(u'_{rt}\,kx)q(w'^{-1}k)dm(k)\Big| \leq \\
\leq&\Big|\int_K (\phi(d'_{rt}kx) - \phi(d'_t kx))q(k)\,dm(k)\Big| + \\
+&\Big|\int_K (\phi(d'_{rt}kx) - \phi(u'_{rt}kx))q(w'^{-1}k)\,dm(k)\Big| = \qquad (4.7)\\
=&\Big|\int_K (\phi(d'_r d'_t kx) - \phi(d'_t kx))q(k)\,dm(k)\Big| + \\
+&\Big|\int_K \phi(b'_{rt}u'_{rt}kx)q(w'^{-1}k'^{-1}_{rt}k) - \phi(u'_{rt}kx)q(w'^{-1}k)\,dm(k)\Big|.
\end{aligned}
$$

The functions ϕ and q are uniformly continuous. Therefore we can deduce from (4.5) and (4.7) the existence of $T_1 > 0$ and $\alpha > 0$ such that

$$\Big|\int_K \phi(d'_t kx)q(k)\,dm(k) - \int_K \phi(u'_{rt}kx)q(w'^{-1}k)\,dm(k)\Big| \leq \frac{\epsilon}{2},$$

whenever $t > T_1$ and $|r-1| < \alpha$. Hence

$$\Big|\int_K \phi(d'_t kx)q(k)\,dm(k) - \frac{1}{\alpha T}\int_T^{(1+\alpha)T}\int_K \phi(u'_t kx)q(w'^{-1}k)\,dm(k)\,dt\Big| \leq \frac{\epsilon}{2}, \qquad (4.8)$$

if $T > T_1$. Let us denote the one-parameter subgroup $\{u'_t | t \in \mathbb{R}\}$ by U. Any proper normal subgroup of $\rho(SL_2(\mathbb{R}))$ is finite. But A is not contained in any proper normal subgroup of H and $U \subset \rho(SL(2,\mathbb{R}))$. Hence U is not contained in any proper normal subgroup of H. Thus conditions (a) - (c) of Theorem 4.3 are satisfied. Let $x_1, \cdots, x_\ell \in G/\Gamma$ be points defined as in Theorem 4.3. Then property (I) is true and for any compact subset F of $\mathcal{D} - \bigcup_{1\leq i\leq \ell} Hx_i$ there exists $T_2 > 0$ such that for all $x \in F$ and $T > T_2$

$$\Big|\frac{1}{\alpha T}\int_T^{(1+\alpha)T}\int_K \phi(u'_t kx)q(w'^{-1}k)\,dm(k) - \int_{G/\Gamma}\phi\,d\mu\int_K q\,dm\Big| \leq \frac{\epsilon}{2} \qquad (4.9)$$

(we use that $\int\limits_K q(w'^{-1}k)\, dm(k) = \int\limits_K q(k)\, dm(k)$). Combining (4.8) and (4.9) we get for all $x \in F$ and $t > \max\{T_1,\ T_2\}$

$$\Big| \int\limits_K \phi(\rho(d_t)kx)q(k)\, dm(k) - \int\limits_{G/\Gamma} \phi\, d\mu \int\limits_K q\, dm \Big| \leq \epsilon. \tag{4.10}$$

Let $\rho'(y) = \rho(wyw^{-1})$, $y \in SL_2(\mathbb{R})$. Then $\rho'(d_t) = a_{-s}$ if $\rho(d_t) = a_s$. Now it remains to recall that $\rho(D) = A$ and to apply (4.10) to the homomorphism ρ and ρ'. $\square$

The following theorem is the specialization of Theorem 4.4 to the case considered in §3.

Theorem 4.5. *Let $\{a_s | s \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $SO(2,1)$. Let $p \geq 2$ and $q \geq 1$. Denote $p+q$ by n, $SO(p,q)$ by H, $SL(n,\mathbb{R})$ by G, and $SO(p) \times SO(q)$ by K. Let Γ be a lattice in G and let ϕ be any continuous function f on G/Γ vanishing outside a compact set. Then for any $\epsilon > 0$ and any bounded measurable function q on K and every compact subset $\mathcal{D}$ of G/Γ there exist finitely many points $x_1, \cdots, x_\ell \in G/\Gamma$ such that*

(I) *the orbits $Hx_1, \cdots, Hx_\ell$ are closed and have finite H-invariant measure;*

(II) *for any compact subset F of $\mathcal{D} - \bigcup_{1 \leq i \leq \ell} Hx_i$ there exists $S_0 > 0$ such that for all $x \in F$ and $s > S_0$*

$$\Big| \int\limits_K \phi(a_s kx)q(k)\, dm(k) - \int\limits_{G/\Gamma} \phi\, d\mu \int\limits_K q\, dm \Big| \leq \epsilon.$$

Proof. Note that $H^0 \cong SO(p,q)^0$ is a maximal proper connected subgroup of $SL(n,\mathbb{R})$. Also if $H_1 \subset H^0$ is the subgroup of H^0 fixing $\{e_3, \ldots, e_{n-1}\}$, then $H_1 \cong SO(2,1)$, $\{a_t\} \subset H_1$, and there is a homomorphism ρ from $SL(2,\mathbb{R})$ to $H_1 \subset H^0$ so that $\rho\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = a_t$. Hence the conditions (a) and (b) of Theorem 4.4 are satisfied, with H^0 in place of H. Thus the conclusions of Theorem 4.4 hold for H^0, hence for H. $\square$

5 The upper bounds

In this section we prove Theorem 3.2 and Theorem 3.3. We recall the notation from §3: G is $SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$, $\hat{K} \cong SO(n)$ is a maximal compact subgroup of G,

$H \cong SO(p,q) \subset G$, $K = H \cap \hat{K}$ is a maximal compact subgroup of H, and X is the symmetric space $K \backslash H$. From its definition (3.3), the function $\alpha(\Delta)$ is the maximum over $1 \leq i \leq n$ of $\hat{K}$ invariant functions $\alpha_i(\Delta)$. The main idea of the proof is to show that the α_i^s satisfy a certain system of integral inequalities which imply the desired bounds.

If $p \geq 3$ and $0 < s < 2$, or if $(p,q) = (2,1)$ or $(2,2)$ and $0 < s < 1$, we show that for any $c > 0$ there exist $t > 0$, and $\omega > 1$ so that the the functions α_i^s satisfy the following system of integral inequalities in the space of lattices:

$$A_t \alpha_i^s \leq c_i \alpha_i^s + \omega^2 \max_{0 < j \leq \min(n-i,i)} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}, \tag{5.1}$$

where A_t is the averaging operator $(A_t f)(\Delta) = \int_K f(a_t k \Delta)$, and $c_i \leq c$ (Lemma 5.7). If $(p,q) = (2,1)$ or $(2,2)$ and $s = 1$, then (5.1) also holds (for suitably modified functions α_i), but some of the constants c_i cannot be made smaller than 1.

Let $f_i(h) = \alpha_i(h\Delta)$, so that each f_i is a function on the symmetric space X. When one restricts to an orbit of H, (5.1) becomes:

$$A_t f_i^s \leq c_i f_i^s + \omega^2 \max_{0 < j \leq \min(n-i,i)} \sqrt{f_{i+j}^s f_{i-j}^s}. \tag{5.2}$$

If $\operatorname{rank} X = 1$, then $(A_t f)(h)$ can be interpreted as the average of f over the sphere of radius $2t$ in X, centered at h. In §5.4 we show that if the f_i satisfy (5.2) then for any $\epsilon > 0$, the function $f = f_{\epsilon,s} = \sum_{0 \leq i \leq n} \epsilon^{i(n-i)} f_i^s$ satisfies the scalar inequality:

$$A_t f \leq cf + b, \tag{5.3}$$

where t, c and b are constants. This inequality is studied in §5.3. We show that if c is sufficiently small, then (5.3) for a fixed t together with the uniform continuity of $\log f$ imply that $(A_r f)(1)$ is bounded as a function of r, which is the conclusion of Theorem 3.2. If $c = 1$, which will occur in the $SO(2,1)$ and $SO(2,2)$ cases, then (5.3) implies that $(A_r f)(1)$ is growing at most linearly with the radius. In §5.4, we complete the proof of Theorem 3.2, and also prove Theorems 3.3 and 3.5.

Throughout the proof we consider the functions $\alpha(g)^s$ for $0 < s < 2$ even though for the application to quadratic forms we only need $s = 1 + \delta$. This yields a better integrability result, and is also necessary for the proof of Theorem 3.4 and Theorem 3.5.

5.1 Averages of the functions $1/d_i^s$ over spheres.

Recall that the function d_i is the norm of a certain vector in the exterior power $\bigwedge^i(\mathbb{R}^n)$. In this subsection it is convenient to work with norms of vectors in arbitrary finite dimensional representations.

Lemma 5.1. *Let V be a finite dimensional real inner product space, A a self-adjoint linear transformation of V, K a closed connected subgroup of the orthogonal group $O(V)$, and Q a closed subset of the unit sphere $\{v \in V \mid \|v\| = 1\}$. Assume that the only eigenvalues of A are $-1, 0$ and 1 and denote by W^-, W^0 and W^+ the eigenspaces corresponding respectively to $-1, 0$ and 1. Let ℓ be a positive integer, and assume that the following conditions are satisfied:*

(a) *$Kv \not\subset W^- + W^0$ for any $v \in Q$;*

(b) *For any nonzero $v \in W^-$ the codimension of the subspace $\{x \in \text{Lie}(K) \mid xv \in W^-\}$ in the Lie algebra Lie (K) of K is at least ℓ or, equivalently, there exist a ℓ-dimensional subspace $L_v \subset \text{Lie}(K)$ such that $xv \notin W^-$ for any nonzero $x \in L_v$.*

Then for any s, $0 < s < \ell$,

$$\lim_{t \to +\infty} \sup_{v \in Q} \int\limits_K \frac{dm(k)}{\|\exp(tA)kv\|^s} = 0, \tag{5.4}$$

where m is the normalized Haar measure on K.

Proof. Let $p : V \to W^+ \oplus W^0$ and $p^+ : V \to W^+$ denote the natural orthogonal projections. For any $v \in V$ and $r \geq 0$, let us consider the following subsets of K:

$$D(v, r) = \{k \in K \mid \|p(kv)\| \leq r\} \qquad \text{and}$$
$$D^+(v, r) = \{k \in K \mid \|p^+(kv)\| \leq r\}.$$

It follows from the condition (b) that there exists $\epsilon > 0$ so that the differential of function $f_v(k) = p(kv)$ has rank at least ℓ at every point $k \in K$ with $\|p(kv)\| < \epsilon\|v\|$. It then follows from the implicit function theorem (or e.g. [Spi, Vol I, Theorem II.9]) that around every point k with $\|p(kv)\| < \epsilon\|v\|$ there exist coordinate systems for K and $W^+ \oplus W^0$ such that in these coordinates the function f_v has the form $f_v(u_1, \ldots, u_m) = (u_1, u_2, \ldots, u_\ell, \psi_{\ell+1}(u), \ldots, \psi_m(u))$ where the ψ_i are smooth functions, and that this coordinate system depends smoothly on v. Then by compactness,

$$C \stackrel{\text{def}}{=} \sup_{\|v\|=1,\ r>0} \frac{m(D(v, r))}{r^\ell} < \infty. \tag{5.5}$$

Since $D^+(v,0)$ is an algebraic subvariety in K and K is connected we get from the condition (a) that $m(D^+(v,0)) = 0$ for every $v \in Q$. This and the compactness of Q easily imply

$$\lim_{r\to 0} \sup_{v\in Q} m(D^+(v,r)) = 0. \tag{5.6}$$

It is clear that

$$\|\exp(tA)kv\|^s \geq \max\{\|p(kv)\|^s, e^{ts}\|p^+(kv)\|^s\}. \tag{5.7}$$

In view of (5.5) and (5.7) we have

$$\int\limits_{D(v,r)-D(v,r/2)} \frac{dm(k)}{\|\exp(tA)kv\|^s} \leq \frac{Cr^\ell}{(r/2)^s} = (2^{-s}C)r^{\ell-s}, \tag{5.8}$$

for any $v \in V, \|v\| = 1$, and any $r > 0$. But $m(D(v,0)) = 0$. Therefore for any $v \in V, \|v\| = 1$, and any $r > 0$

$$\begin{aligned} &\int\limits_{D(v,r)} \frac{dm(k)}{\|\exp(tA)kv\|^s} = \\ &= \int\limits_{D(v,0)} \frac{dm(k)}{\|\exp(tA)kv\|^s} + \sum_{n=0}^{\infty} \int\limits_{D(v,2^{-n}r)-D(v,2^{-n-1}r)} \frac{dm(k)}{\|\exp(tA)kv\|^s} \\ &\leq \frac{C}{2^{-s}-2^{-\ell}} r^{\ell-s}. \end{aligned} \tag{5.9}$$

Using (5.7) we get for any $v \in V,\ r > 0$ and $r_1 > 0$

$$\begin{aligned} &\int\limits_{K-D(v,r)} \frac{dm(k)}{\|\exp(tA)kv\|^s} \\ &\leq \int\limits_{K-D^+(v,r_1)} \frac{dm(k)}{\|\exp(tA)kv\|^s} + \int\limits_{D^+(v,r_1)-D(v,r)} \frac{dm(k)}{\|\exp(tA)kv\|^s} \leq \\ &\leq e^{-ts}r_1^{-s} + m(D^+(v,r_1))r^{-s}. \end{aligned} \tag{5.10}$$

Combining (5.6), (5.9) and (5.10) we easily get (5.4). $\square$

Lemma 5.2. *Let V, A, K, Q, W^-, W^0 and W^+ be the same as in Lemma 5.1. Assume that $Kv \not\subset W^0$ for any $v \in Q$ and there exists a closed self-adjoint subgroup H_1 of $GL(V)$ with the following properties:*

(a) *the Lie algebra Lie (H_1) of H_1 contains A;*

(b) *H_1 is locally isomorphic to $SL(2,\mathbb{C})$ or, equivalently, to $SO(3,1)$;*

(c) *$H_1 \cap K$ is a maximal compact subgroup of H_1.*

Then (5.4) *holds for any s satisfying* $0 < s < 2$.

Proof. After a suitable normalization, we may assume that the only eigenvalues of $\operatorname{Ad} A$ are $-\lambda, 0$ and λ, where $\lambda = 2$ if $H_1 \cong SL(2,\mathbb{C})$ and $\lambda = 1$ if $H_1 \cong SO(3,1)$. Then in view of (a) and (b)

$$\mathrm{Lie}(H_1) = \mathfrak{h}^- \oplus \mathfrak{h}^0 \oplus \mathfrak{h}^+,$$

where

$$\mathfrak{h}^- = \{x \in \mathrm{Lie}(H_1) \mid [A,x] = -\lambda x\}, \tag{5.11}$$
$$\mathfrak{h}^0 = \{x \in \mathrm{Lie}(H_1) \mid [A,x] = 0\}, \tag{5.12}$$
$$\mathfrak{h}^+ = \{x \in \mathrm{Lie}(H_1) \mid [A,x] = \lambda x\}. \tag{5.13}$$

We also have that $\mathfrak{h}^-, \mathfrak{h}^0$ and $\mathfrak{h}^+$ are subalgebras of Lie (H_1) and that for any nonzero $x \in \mathfrak{h}^+$ there exists a unique $\alpha(x) \in \mathfrak{h}^-$ with $[x, \alpha(x)] = A$. Note that x, A and $\alpha(x)$ span a Lie algebra isomorphic to $s\ell_2(\mathbb{R}) = \mathrm{Lie}(SL(2,\mathbb{R}))$. Now using standard arguments about representations of $s\ell_2(\mathbb{R})$ we get that for any $v \in W^- + W^0, v \notin W^-$,

$$\{x \in \mathrm{Lie}(H_1) \mid xv \in W^- + W^0\} = \mathfrak{h}^- \oplus \mathfrak{h}^0. \tag{5.14}$$

The intersection of the Lie algebra of any compact self-adjoint subgroup of H_1 with $\mathfrak{h}^- \oplus \mathfrak{h}^0$ belongs to $\mathfrak{h}^0$. Therefore

$$\mathrm{Lie}(K) \cap (\mathfrak{h}^- \oplus \mathfrak{h}^0) \subset \mathfrak{h}^0. \tag{5.15}$$

Since $Kv \not\subset W^0$ for any $v \in Q$ we get from (5.14) and (5.15) that $Kv \not\subset W^- + W^0$ for any $v \in Q$. Thus the condition (a) of Lemma 5.1 is satisfied. On the other hand since $\dim(K \cap H_1) = 3$ and $\dim(\mathrm{Lie}(K) \cap \mathfrak{h}^0) = 1$ we can deduce from (5.14) and (5.15) that the condition (b) of Lemma 5.1 is also satisfied with $\ell = 2$. Now we apply Lemma 5.1 and get (5.4). $\square$

Lemma 5.3. *Let H^0 denote the connected component of the identity of the group $SO(p,q)$ and let $K = SO(p)\times SO(q)$ be the maximal compact subgroup of H^0. Assume that $(p,q) \neq (2,2)$. Then for any nonzero proper subspace L of $\mathbb{R}^{p+q}$ the subgroup*

$$H_L \stackrel{\text{def}}{=} \{h \in H^0 \mid h(kL) = kL \text{ for any } k \in K\}$$

is contained in K.

Proof. Since the set KL is invariant under K we have that K normalizes H_L. Thus a subgroup F generated by $K \cup H_L$ normalizes H_L. But K is a maximal proper subgroup of H^0 (because $(p,q) \neq (2,2)$, H^0 is simple). Therefore if $H_L \not\subset K$ then $F = H^0$ that is H_L is a normal subgroup of H^0. Since H^0 is simple, either $H_L = H^0$ or $H_L \subset \{\pm E\}$. But since H^0 acts irreducibly on $\mathbb{R}^{p+q}$ the first case is not possible. Thus $H_L \subset K$. □

As in §3.2, we realize $SO(p,q)$ as $SO(Q_0)$ where Q_0 is the quadratic form defined by (3.4). Let $H_1 \cong SO(2,1)$ denote the subgroup of $H = SO(Q_0)$ fixing $\{e_3, \dots, e_{n-1}\}$. We refer to the embedding $H_1 \hookrightarrow H$ as the *standard embedding* of $SO(2,1)$ into $SO(p,q)$.

Proposition 5.4. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $SO(2,1)$. Let p and q be positive integers and let $0 < i < p+q$. Let*

$$F(i) = \{x_1 \wedge x_2 \wedge \dots \wedge x_i \mid x_1, x_2, \cdots, x_i \in \mathbb{R}^{p+q}\} \subset \bigwedge^i(\mathbb{R}^{p+q}).$$

Then, if $p \geq 3$, or if $p = 2$, $q = 2$ and $i \neq 2$, then for any s, $0 < s < 2$,

$$\lim_{t\to\infty} \sup_{v\in F(i),\ \|v\|=1} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0. \tag{5.16}$$

where $K = SO(p) \times SO(q)$ and $SO(2,1)$ is embedded into $SO(p,q)$. If $p = 2$ and $q = 1$, or if $p = 2$, $q = 2$ and $i = 2$, then (5.16) holds for any s, $0 < s < 1$.

Proof. Let ρ_i denote the i-th exterior representation of $SO(p,q)$ in $\bigwedge^i(\mathbb{R}^{p+q})$. Since $\{a_t\} \subset SO(2,1)$ we can assume that $a_t = \exp tA$ where A is a self-adjoint linear transformation with the only eigenvalues $-1, 0$ and 1 and such that the multiplicities of eigenvalues -1 and 1 is equal to 1. Then the only eigenvalues of $\rho_i(A)$ are also $-1, 0$ and 1. Let W_i^-, W_i^0 and W_i^+ denote the corresponding eigenspaces of $\rho_i(A)$. It is clear that for any $t \neq 0$

$$W_i^0 = \{v \in \bigwedge^i(\mathbb{R}^{p+q}) \mid a_t v = v\}. \tag{5.17}$$

Suppose $(p,q) \neq (2,2)$. It follows from Lemma 5.3 that for any $t \neq 0$ and any $v \in F(i),\ v \neq 0$,

$$a_t \notin \{h \in H \mid h(kv) = kv \text{ for any } k \in K\}. \tag{5.18}$$

Combining (5.17) and (5.18) we get that $Kv \not\subset W_i^0$ for any $v \notin F(i), v \neq 0$. If $p \geq 3$ it remains now to apply Lemma 5.2. If $p = 2$ and $q = 1$, then one can apply Lemma 5.1 directly with $\ell = 1$.

For the $SO(2,2)$ assertions, recall that $SO(2,2)$ is locally isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, via the action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ on $M_2(\mathbb{R}) \cong \mathbb{R}^4$ given by $(h_1,h_2) \cdot v = h_1^t v h_2$. This action preserves the determinant which is a quadratic form on $M_2(\mathbb{R})$ of signature $(2,2)$. If we choose a basis so that the quadratic form det has the form (3.4), the standard embedded $SO(2,1)$ becomes locally isomorphic to the diagonal subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and a_t becomes $\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right) \times \left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right)$. Hence if $i = 1$, then the spaces W^- and W^0 of Lemma 5.1 consist of matrices of the forms $\left(\begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$, respectively. Thus (a) and (b) of Lemma 5.1 are satisfied, with $\ell = 2$. The assertion for $i = 3$ follows by duality. To obtain the assertion for $i = 2$, recall that the the representation ρ_2 of $SO(2,2)$ splits into two three dimensional representations. Each invariant subspace is fixed by one of the $SL(2,\mathbb{R})$ factors, while the other acts via the three dimensional representation of $SL(2,\mathbb{R})$, hence fixes a quadratic form of signature $(2,1)$. If we write $K = K_1 \times K_2$ and denote the projections onto the irreducible subspaces by π_1 and π_2, then for any $c > 0$, $0 < s < 1$, and for sufficiently large t,

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} \leq \int_K \min\left(\frac{1}{\|\pi_1(a_t k v)\|^s}, \frac{1}{\|\pi_2(a_t k v)\|^s}\right) dm(k) \tag{5.19}$$

$$\leq \min\left(\int_{K_1} \frac{dm(k_1)}{\|\pi_1(a_t k_1 v)\|^s}, \int_{K_2} \frac{dm(k_2)}{\|\pi_2(a_t k_2 v)\|^s}\right) \tag{5.20}$$

$$\leq \min\left(\frac{c}{\|\pi_1(v)\|^s}, \frac{c}{\|\pi_2(v)\|^s}\right) \text{ by the assertion for } SO(2,1) \tag{5.21}$$

$$\leq \frac{2^s c}{\|v\|^s}. \tag{5.22}$$

Since c is arbitrary, the assertion follows. □

In the next lemma we obtain an analogous result for the case $(p,q) = (2,1)$, $s = 1$.

Lemma 5.5. *Let $H \cong SO(2,1)$ be the orthogonal group of the quadratic form $x^2 + y^2 - z^2$. Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of H, and let*

$K = H \cap O(3)$ *denote the maximal compact of* H. *We define another norm* $\|\cdot\|^*$ *on* $\mathbb{R}^3$ *by*

$$\|(x, y, z)\|^* = \max(\sqrt{x^2 + y^2}, |z|). \tag{5.23}$$

Then, for any $v \in \mathbb{R}^3$, $v \neq 0$, *and any* $t > 0$,

$$\int_K \frac{dm(k)}{\|a_t k v\|^*} \leq \frac{1}{\|v\|^*}. \tag{5.24}$$

Proof. The group $K \cong SO(2)$ acts by rotations in the xy plane and fixes the z axis. The norm $\|\cdot\|^*$ is equivalent to the standard Euclidean norm $\|\cdot\|$, and is K-invariant. Thus the function $f_v(h) = 1/\|hv\|^*$ is well defined on $K \backslash H \cong \mathbb{H}^2$. Note that the cosets $K(a_t K) \subset \mathbb{H}^2$ are metric circles centered at the origin. The Laplace operator (i.e the Casimir of H) is given by

$$\Delta = -(x^2 + y^2 - z^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) + E^2 + E,$$

where E is the Euler operator

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

(see [HT, Chapter III, Eq. (2.3.4)]. A direct calculation shows that if $x^2+y^2-z^2 > 0$, then

$$\Delta f_v = \Delta\left(\frac{1}{\sqrt{x^2 + y^2}}\right) = -\frac{x^2 + y^2 + z^2}{(x^2 + y^2)^{3/2}} < 0. \tag{5.25}$$

The equation (5.25) also implies that if $x^2 + y^2 - z^2 = 0$, then

$$\Delta f_v = \Delta\left(\frac{1}{\sqrt{x^2 + y^2}}\right) = 0.$$

Finally, if $x^2 + y^2 - z^2 < 0$, then $|z| > 0$, and

$$\Delta f_v = \Delta\left(\frac{1}{|z|}\right) = (x^2 + y^2 - z^2)\frac{2}{|z|^3} < 0.$$

Hence f_v is always superharmonic with respect to the hyperbolic metric on $\mathbb{H}^2$. Thus, the averages of f_v over circles are bounded by the value at the center, which implies (5.24), (c.f. [Hel, Ch. IV, Proposition 2.5]. $\square$

5.2 A system of inequalities

Lemma 5.6. *For any two Δ-rational subspaces L and M*

$$d(L)d(M) \geq d(L \cap M)d(L+M). \tag{5.26}$$

Proof. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/(L \cap M)$ denote the natural projection. Then $d(L) = d(\pi(L))d(L\cap M)$, $d(M) = d(\pi(M))d(L\cap M)$ and $d(L+M) = d(\pi(L+M))d(L\cap M)$. On the other hand the inequality (5.26) is equivalent to the inequality

$$\frac{d(L)}{d(L\cap M)}\frac{d(M)}{d(L\cap M)} \geq \frac{d(L+M)}{d(L\cap M)}.$$

Therefore replacing L, M and $L+M$ by $\pi(L)$, $\pi(M)$ and $\pi(L+M)$ we can assume that $L \cap M = \{0\}$. Let $(e_1, \cdots, e_\ell)$, $\ell = \dim L$, and $(e_{\ell+1}, \cdots, e_{\ell+m})$, $m = \dim M$, be bases in L and M respectively. Then

$$\begin{aligned} d(L)d(M) &= \|e_1 \wedge \cdots \wedge e_\ell\| \, \|e_{\ell+1} \wedge \cdots \wedge e_{\ell+m}\| \\ &\geq \|e_1 \wedge \cdots \wedge e_\ell \wedge e_{\ell+1} \wedge \cdots \wedge e_{\ell+m}\| \geq d(L+M) \end{aligned} \tag{5.27}$$

that proves (5.26) (the second inequality in (5.27) is true because $(L\cap\Delta)+(M\cap\Delta) \subset (L+M)\cap\Delta$. □

Lemma 5.7. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $SO(2,1)$. Let p and q be positive integers, and denote $p+q$ by n. Denote $SO(p)\times SO(q)$ by K. Suppose $p \geq 3$, $q \geq 1$ and $0 < i < n$, or $p=2$, $q=2$ and $i = 1$ or 3. Then for any s, $0 < s < 2$, and any $c > 0$ there exist $t > 0$ and $\omega > 1$ such that for any lattice Λ in $\mathbb{R}^n$*

$$\int_K \alpha_i(a_t k\Lambda)^s \, dm(k) < \frac{c}{2}\alpha_i(\Lambda)^s + \omega^2 \max_{0<j\leq\min\{n-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\,\right)^s. \tag{5.28}$$

If $p=2$, $q=1$ and $i=1,2$, or if $p=2$, $q=2$ and $i=2$, then for any s, $0<s<1$, and any $c>0$ there exist $t>0$ and $\omega>1$ such that (5.28) holds.

Proof. Fix $c > 0$. In view of Proposition 5.4 one can find $t > 0$ such that

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} < \frac{c}{2},$$

whenever $v \in F(i), \|v\| = 1$. It follows that

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} < \frac{c}{2} \cdot \frac{1}{\|v\|^s}, \tag{5.29}$$

for any $v \in F(i), v \neq 0$. Let Λ be a lattice in $\mathbb{R}^n$. There exists a Λ-rational subspace L_i of dimension i such that

$$\frac{1}{d_\Lambda(L_i)} = \alpha_i(\Lambda). \tag{5.30}$$

The inequality (5.29) implies

$$\int_K \frac{dm(k)}{d_{a_t k\Lambda}(a_t k L_i)^s} < \frac{c}{2} \frac{1}{d_\Lambda(L_i)^s}. \tag{5.31}$$

Let $\omega = \max_{0<j<n} \| \bigwedge^j(a_t)\|$. (In fact $\omega = e^t$). We have that

$$\omega^{-1} \leq \frac{\|a_t v\|}{\|v\|} \leq \omega,\ 0 < j < n,\ v \in F(j). \tag{5.32}$$

Let us denote the set of Λ-rational subspaces L of dimension i with $d_\Lambda(L) < \omega^2 d_\Lambda(L_i)$ by Ψ_i. We get from (5.32) that for a Λ-rational i-dimensional subspace $L \notin \Psi_i$

$$d_{a_t k\Lambda}(a_t k L) > d_{a_t k\Lambda}(a_t k L_i), \qquad k \in K. \tag{5.33}$$

It follows from (5.31), (5.33) and the definition of α_i that

$$\int_K \alpha_i(a_t k\Lambda)^s \, dm(k) < \frac{c}{2}\alpha_i(\Lambda)^s \text{ if } \Psi_i = \{L_i\}. \tag{5.34}$$

Assume now that $\Psi_i \neq \{L_i\}$. Let $M \in \Psi_i,\ M \neq L_i$. Then $\dim(M+L_i) = i+j,\ j > 0$. Now using (5.30), (5.32) and Lemma 5.6 we get that for any $k \in K$

$$\begin{aligned} \alpha_i(a_t k\Lambda) < \omega\alpha_i(\Lambda) = \frac{\omega}{d_\Lambda(L_i)} &< \frac{\omega^2}{\sqrt{d_\Lambda(L_i) d_\Lambda(M)}} \\ &\leq \frac{\omega^2}{\sqrt{d_\Lambda(L_i \cap M) d_\Lambda(L_i + M)}} \\ &\leq \omega^2 \sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}. \end{aligned} \tag{5.35}$$

Hence if $\Psi_i \neq \{L_i\}$

$$\int_K \alpha_i(a_t k\Lambda)^s \, dm(k) \leq \omega^2 \max_{0<j\leq \min\{n-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\right)^s. \tag{5.36}$$

Combining (5.34) and (5.36) we get that for any lattice $\Lambda \subset \mathbb{R}^n$, (5.28) holds. $\square$

In the rest of this subsection we obtain similar systems of inequalities for the $SO(2,1)$ and $SO(2,2)$ cases, with $s = 1$. For $H = SO(2,1)$, Δ a lattice in $\mathbb{R}^3$, and L a Δ-rational subspace of $\mathbb{R}^3$, let $d^*_\Delta(L) = \|e_1 \wedge \ldots e_\ell\|^*$ where $(e_1, \ldots e_l)$ is a basis for $\Delta \cap L$. (The norm $\|\cdot\|^*$ defined in (5.23) on $\mathbb{R}^3 = \bigwedge^1(\mathbb{R}^3)$ can be extended to $\bigwedge^2(\mathbb{R}^3)$ by duality.) For $1 \leq i \leq 2$, let

$$\alpha^*_i(\Delta) = \sup\left\{ \frac{1}{d^*_\Delta(L)} \middle| L \text{ is a } \Delta\text{-rational subspace of dimension } i \right\}. \tag{5.37}$$

Clearly for any Δ,

$$(1/2)\alpha_i(\Delta) < \alpha^*_i(\Delta) < 2\alpha_i(\Delta). \tag{5.38}$$

Lemma 5.8. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $H = SO(2,1)$, and denote $SO(2)$ by K. Then there exist $t_0 > 0$ and $\omega > 1$, such that for any $t < t_0$, for any unimodular lattice Λ in $\mathbb{R}^3$, and $1 \leq i \leq 2$,*

$$\int_K \alpha^*_i(a_t k\Lambda) \, dm(k) < \alpha^*_i(\Lambda) + \omega^2 \sqrt{\alpha_{3-i}(\Lambda)}. \tag{5.39}$$

Proof. The argument is identical to the proof of Lemma 5.7 except that one uses Lemma 5.5 instead of Proposition 5.4. $\square$

Now let $H = SO(2,2)$. The space $V = \bigwedge^2(\mathbb{R}^4)$ splits as a direct sum $V_1 \oplus V_2$ of two invariant subspaces, where on each V_i, H preserves a quadratic form Q_i of signature $(2,1)$. We define on each V_i a Euclidean norm $\|\cdot\|^*_i$ by (5.23) (adapted to Q_i). Let π_i denote the orthogonal projections from V to V_i. Now let Δ be a lattice in $\mathbb{R}^4$, and let L be a two-dimensional Δ-rational subspace of $\mathbb{R}^4$. For $1 \leq i \leq 2$, let

$$d^{i,\#}_\Delta(L) = \|\pi_i(e_1 \wedge e_2)\|^*_i, \tag{5.40}$$

where $\{e_1, e_2\}$ is a basis over $\mathbb{Z}$ for $\Delta \cap L$. Then let

$$\alpha^\#_2(\Delta) = \sup_L \left\{ \min\left(\frac{1}{d^{1,\#}_\Delta(L)}, \frac{1}{d^{2,\#}_\Delta(L)} \right) \right\}. \tag{5.41}$$

The supremum is taken over Δ-rational two dimensional subspaces L. By construction, for any Δ,

$$C^{-1}\alpha_2^{\#}(\Delta) < \alpha_2(\Delta) < C\alpha_2^{\#}(\Delta), \tag{5.42}$$

where C is an absolute constant.

Lemma 5.9. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $SO(2,1)$, where $SO(2,1)$ is diagonally embedded in $H = SO(2,2)$, under its local identification with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. Denote $SO(2) \times SO(2)$ by K, and the maximal compact of $SO(2,1)$ by $\tilde{K}$. Then there exist $t_0 > 0$ and $\omega > 1$, such that for any $t < t_0$ and for any unimodular lattice Λ in $\mathbb{R}^4$,*

$$\int_{\tilde{K}} \alpha_2^{\#}(a_t\tilde{k}\Lambda)\, dm(\tilde{k}) < \alpha_2^{\#}(\Lambda) + \omega^2\sqrt{\alpha_1(\Lambda)\alpha_3(\Lambda)}. \tag{5.43}$$

Proof. The group $\tilde{K}$ is diagonally embedded in K. Recall that each $SO(2,2)$ invariant subspace V_i of $\bigwedge^2(\mathbb{R}^4)$ is fixed pointwise by one of the $SL(2,\mathbb{R})$ factors, while the other fixes a quadratic form of signature $(2,1)$. Thus, for $1 \le i \le 2$, the inequalities:

$$\int_{\tilde{K}} \frac{dm(\tilde{k})}{\|\pi_i(a_t\tilde{k}v)\|_i^*} \le \frac{1}{\|\pi_i(v)\|_i^*} \tag{5.44}$$

follow immediately from Lemma 5.5. Hence,

$$\begin{aligned}\int_{\tilde{K}} \min\left(\frac{1}{\|\pi_1(a_t\tilde{k}v)\|_1^*}, \frac{1}{\|\pi_2(a_t\tilde{k}v)\|_2^*}\right) dm(k) &\le \min\left(\int_{\tilde{K}} \frac{dm(\tilde{k})}{\|\pi_1(a_t\tilde{k}v)\|_1^*}, \int_{\tilde{K}} \frac{dm(\tilde{k})}{\|\pi_2(a_t\tilde{k}v)\|_2^*}\right) \\ &\le \min\left(\frac{1}{\|\pi_1(v)\|_1^*}, \frac{1}{\|\pi_2(v)\|_2^*}\right).\end{aligned} \tag{5.45}$$

The rest of the proof is identical to that of Lemma 5.7 except that (5.45) is used in place of Proposition 5.4. $\square$

5.3 Coarsely Superharmonic Functions.

Let $n \in \mathbb{N}^+$ and let D_n^+ denote the set of diagonal matrices $d(\lambda_1, \cdots, \lambda_n) \in GL_n(\mathbb{R})$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. For any $g \in GL_n(\mathbb{R})$, consider the Cartan decomposition $g = k_1(g)d(g)k_2(g), k_1(g), k_2(g) \in K = O_n(\mathbb{R})$, $d(g) \in D_n^+$ and denote by $\lambda_1(g) \ge \lambda_2(g) \ge \cdots \ge \lambda_n(g)$ the eigenvalues of $d(g)$.

Lemma 5.10. *For every $\epsilon > 0$ there exists a neighborhood U of e in $O_n(\mathbb{R})$ such that*

$$\left|\frac{\lambda_i(d_1 k d_2)}{\lambda_i(d_1)\lambda_i(d_2)} - 1\right| < \epsilon \tag{5.46}$$

for any $d_1, d_2 \in D_n^+$, $k \in U$ and $1 \le i \le n$.

Proof. Let $(e_1, \cdots, e_n)$ be the standard orthonormal basis in $\mathbb{R}^n$. If $k \in O_n(\mathbb{R})$ and $\langle ke_1, e_1\rangle > 1 - \epsilon$ then

$$\|d_1 k d_2 e_1\| > (1-\epsilon)\lambda_1(d_1)\lambda_1(d_2). \tag{5.47}$$

On the other hand, for any $g \in GL_n(\mathbb{R})$.

$$\lambda_1(g) = \|g\| \ge \|ge_1\|. \tag{5.48}$$

Since $\|d_1 k d_2\| \le \|d_1\|\,\|d_2\|$ it follows from (5.47) and (5.48) that

$$1 \ge \frac{\lambda_1(d_1 k d_2)}{\lambda_1(d_1)\lambda_1(d_2)} > 1 - \epsilon, \tag{5.49}$$

if $\langle ke_1, e_1\rangle > 1-\epsilon$. Analogously considering the representation of $GL_n(\mathbb{R})$ in the i-th exterior product $\bigwedge^i(\mathbb{R}^n)$ of $\mathbb{R}^n$ we get that

$$1 \ge \frac{(\lambda_1\lambda_2\cdots\lambda_i)(d_1 k d_2)}{(\lambda_1\lambda_2\cdots\lambda_i)(d_1 d_2)} > 1 - \epsilon, \tag{5.50}$$

if $k \in O_n(\mathbb{R})$ and $\langle \bigwedge^i(k)(e_1\wedge\cdots\wedge e_i), e_1\wedge\cdots\wedge e_i\rangle > 1-\epsilon$. It is clear that there exists a neighborhood U of identity in $O_n(\mathbb{R})$ such that $\langle \bigwedge^i(k)(e_1\wedge\cdots\wedge e_i),\ e_1\wedge\cdots\wedge e_i\rangle > \sqrt{1-\epsilon}$ for every $k \in U$ and $1 \le i \le n$. But

$$\lambda_i(g) = \frac{(\lambda_1\lambda_2\cdots\lambda_i)(g)}{(\lambda_1\lambda_2\cdots\lambda_{i-1})(g)}.$$

Therefore (5.46) follows from (5.50). □

Lemma 5.11. *(cf. the "wavefront lemma" [EM, Theorem 3.1]) Let H be a self-adjoint connected reductive subgroup of $GL_n(\mathbb{R})$, let $K = O_n(\mathbb{R}) \cap H$ be a maximal compact subgroup of H and let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of H. Then for every neighborhood V of e in H there exists a neighborhood U of e in K such that*

$$a_t U a_s \subset K V a_t a_s K \tag{5.51}$$

for any $t \ge 0$ and $s \ge 0$.

Proof. Conjugating a_t by an element of K we can assume that $\{a_t \mid t \geq 0\} \subset D_n^+$. It is easy to see that there exists $\epsilon > 0$ such that $h_1 \in V h_2$ whenever $h_1, h_2 \in D_n^+$ and $\left|\frac{\lambda_i(h_1)}{\lambda_i(h_2)} - 1\right| < \epsilon$ for every $1 \leq i \leq n$. Take a neighborhood U such that (5.46) is satisfied. Then (5.51) is true for this U. $\square$

Proposition 5.12. *Let H be a self-adjoint reductive subgroup of $GL_n(\mathbb{R})$, let $K = O_n(\mathbb{R}) \cap H$, let m denote the normalized measure on K, and let $A = \{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of H. Let $\mathcal{F}$ be a family of strictly positive functions on H having the following properties:*

(a) *The logarithms $\log f$ for $f \in \mathcal{F}$ are equicontinuous with respect to a left-invariant uniform structure on H or, equivalently, for any $\epsilon > 0$ there exists a neighborhood $V(\epsilon)$ of 1 in H such that for any $f \in \mathcal{F}$,*

$$(1 - \epsilon) f(h) < f(uh) < (1 + \epsilon) f(h)$$

for any $h \in H$ and $u \in V(\epsilon)$;

(b) *The functions $f \in \mathcal{F}$ are left K-invariant, that is $f(Kh) = f(h)$, $h \in H$,*

(c) $\sup_{f \in \mathcal{F}} f(1) < \infty$.

Then there exists $0 < c = c(\mathcal{F}) < 1$ such that for any $t > 0$ and $b > 0$ there exists $B = B(t, b) < \infty$ with the following property: If $f \in \mathcal{F}$ and

$$\int_K f(a_t k h)\, dm(k) < c f(h) + b \tag{5.52}$$

for any $h \in KAK \subset H$, then

$$\int_K f(a_\tau k)\, dm(k) < B$$

for any $\tau > 0$.

Proof. Fix $f \in \mathcal{F}$, and let

$$\tilde{f}(h) = \int_K f(hk)\, dm(k).$$

Properties (a), (b), (c) of the function f imply that $\tilde{f}$ has the same properties. Hence it suffices to show that the conclusion of the proposition holds for $\tilde{f}$. Therefore we can assume that

$$f(KhK) = f(h),\ h \in H, \tag{5.53}$$

and we have to prove that

$$\sup_{\tau > 0} f(a_\tau) < B < \infty. \tag{5.54}$$

It follows from property (a) that

$$\frac{1}{2} f(h) < f(uh) < 2f(h),\ h \in H,\ u \in V = V(\frac{1}{2}). \tag{5.55}$$

According to Lemma 5.11 there exists a neighborhood U of 1 in H such that $a_t U a_\tau \in KVa_t a_\tau K$ for any $t \geq 0$ and $\tau \geq 0$. Then we get from (5.53) and (5.55) that

$$\int\limits_K f(a_t k a_\tau)\, dm(k) \geq \int\limits_{U \cap K} f(a_t k a_\tau)\, dm(k) > \frac{1}{2} m(U \cap K) f(a_t a_\tau). \tag{5.56}$$

Suppose for some $t > 0$ and $b > 0$

$$\int\limits_K f(a_t k h)\, dm(k) < \frac{1}{4} m(U \cap K) f(h) + b,\ h \in H. \tag{5.57}$$

It follows from (5.56) and (5.57) that for some $b' > 0$,

$$f(a_t a_\tau) < \frac{1}{2} f(a_\tau) + b', \text{ for all } \tau > 0. \tag{5.58}$$

Using induction on ℓ we get from (5.58) that

$$f(a_{\ell t}) < 2 \max\{f(1), b'\},\ ; \ell \in \mathbb{N}^+. \tag{5.59}$$

Since $\{a_r \mid 0 \leq r \leq t\}$ belongs to V^i for some i where $V^1 = V,\ V^i = V\ V^{i-1}$, it follows that $\sup\limits_{h \in H,\, 0 \leq r \leq t} \frac{f(a_r h)}{f(h)} < \infty$. Therefore (5.59) and property (c) imply (5.54). $\square$

Lemma 5.13. *Let $H = SL(2, \mathbb{R})$, $K = O(2) \cap H$, and let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of H. Suppose f is a strictly positive continuous function on H with the following properties:*

(a) *The logarithm of f is uniformly continuous with respect to a left-invariant uniform structure on H, or equivalently, for any $\epsilon > 0$ there exists a neighborhood $V(\epsilon)$ of 1 in H such that*

$$(1-\epsilon)f(h) < f(uh) < (1+\epsilon)f(h)$$

for any $h \in H$ and $u \in V(\epsilon)$;

(b) *f is left K-invariant, i.e $f(Kh) = f(h)$*

(c) *There exist $t_0 > 0$ and $b > 0$, such that for $0 \le t < t_0$ and any $h \in H$,*

$$\int_K f(a_t kh)\, dm(k) < f(h) + b. \tag{5.60}$$

(d) *$f(1) < \beta$.*

Then there exists $B > 0$ depending only on b, β and t_0 and the modulus of continuity of $\log f$, so that for all $t > 1$,

$$\int_K f(a_t k)\, dm(k) < Bt. \tag{5.61}$$

Proof. Let

$$\hat{f}(h) = \int_K f(hk)\, dm(k).$$

Then $\hat{f}$ also has properties (a), (b), (c) and (d). Thus we may assume $f(KhK) = f(h)$. Let $\rho(h)$ be defined by $\rho(Ka_tK) = \log\cosh\frac{t}{2}$, where $t > 0$. Then f and ρ may be interpreted as radially symmetric functions on $K\backslash H = \mathbb{H}^2$. Then $\Delta\rho = c_0$, where Δ is the Laplace operator on $\mathbb{H}^2$, and c_0 is an absolute constant (depending on the normalizations). Hence, for a fixed $\tau \in [0, t_0]$,

$$\int_K \rho(a_\tau kh)\, dm(k) = \rho(h) + c_1, \tag{5.62}$$

where c_1 depends only on τ. Let $u(h) = f(h) - (b/c_1)\rho(h)$. Then (5.60) and (5.62) imply that

$$\int_K u(a_\tau kh)\, dm(k) \le u(h). \tag{5.63}$$

The estimate (5.63) implies that the minimum of $u(Kh)$ on the disk of radius t centered at the origin is attained within a distance τ of the boundary. Since $f(1) = \beta$, $u(1) < \beta - (b/c_1)\log\cosh 0 \leq \beta$, hence $u(Kh) < \beta$ for some point Kh on the circle of radius t_1, where $t - \tau < t_1 < t$. But u is radially symmetric, hence $u(Kh) < \beta$ on the entire circle of radius t_1. Then

$$\int_K f(a_{t_1}k)\, dm(k) < \beta + \frac{b}{c_1}t_1.$$

To get (5.61) for arbitrary t it remains to use again property (a). □

5.4 Averages over large spheres.

In this subsection we complete the proofs of Theorem 3.2, Theorem 3.3 and Theorem 3.5.

Proof of Theorem 3.2. Define functions $f_0, f_1, \cdots, f_n$ on $H = SO(p,q)$ by the following equalities

$$f_i(h) = \alpha_i(h\Delta),\ h \in H,\ 0 \leq i \leq n.$$

Since $\alpha(a_t k\Delta)^s = \max\limits_{0\leq i\leq n} f_i(a_t k)^s < \sum\limits_{0\leq i\leq n} f_i(a_t k)^s$ it is enough to show that

$$\sup_{t>0,\ 0\leq i\leq n} \int\limits_K f_i^s(a_t k)\, dm(k) < \infty. \tag{5.64}$$

Let A_t denote the averaging operator defined by

$$(A_t f)(h) = \int_K f(a_t k h)\, dm(k), \qquad h \in H.$$

As in Proposition 5.4, let

$$F(i) = \{x_1 \wedge x_2 \wedge \cdots \wedge x_i \mid x_1, x_2, \cdots, x_i \in \mathbb{R}^n\} \subset \textstyle\bigwedge^i(\mathbb{R}^n).$$

Since $\|Kv\| = \|v\|$ and $\frac{\|hv\|}{\|v\|} \leq \|\bigwedge^i(h)\|$, for any $v \in F(i)$ and $h \in H$, each f_i has properties (a) and (b) of Proposition 5.12. Applying Lemma 5.7 to $\Lambda = h\Delta$ we see that for any $i, 0 < i < n$, and $h \in H$

$$A_t f_i^s < \frac{c}{2} f_i^s + \omega^2 \max_{0<j\leq \min\{n-i,i\}} \sqrt{f_{i+j}^s f_{i-j}^s}. \tag{5.65}$$

Let us denote $q(i) = i(n-i)$. Then by direct computations $2q(i) - q(i+j) - q(i-j) = 2j^2$. Therefore we get from (5.65) that for any $i, 0 < i < n$, and any positive $\epsilon < 1$

$$\begin{aligned} A_t(\epsilon^{q(i)} f_i^s) &< \frac{c}{2}\epsilon^{q(i)} f_i^s + \omega^2 \max_{0<j\leq \min\{n-i,i\}} \epsilon^{q(i)-\frac{q(i+j)+q(i-j)}{2}} \sqrt{\epsilon^{q(i+j)} f_{i+j}^s \epsilon^{q(i-j)} f_{i-j}^s} \\ &\leq \frac{c}{2}\epsilon^{q(i)} f_i^s + \epsilon\omega^2 \max_{0<j\leq \min\{n-i,i\}} \sqrt{\epsilon^{q(i+j)} f_{i+j}^s \epsilon^{q(i-j)} f_{i-j}^s}. \end{aligned} \tag{5.66}$$

Consider the linear combination

$$f_{\epsilon,s} = \sum_{0\leq i\leq n} \epsilon^{q(i)} f_i^s.$$

The function $f_{\epsilon,s}$ then also has properties (a) and (b) of Proposition 5.12. Since $\epsilon^{q(i)} f_i^s < f_{\epsilon,s}, f_0 = 1$ and $f_n = 1/d(\Delta)$, the inequalities (5.66) imply the following inequality:

$$A_t f_{\epsilon,s} < 1 + d(\Delta)^{-s} + \frac{c}{2} f_{\epsilon,s} + n\epsilon\omega^2 f_{\epsilon,s}. \tag{5.67}$$

Taking $\epsilon = \frac{c}{2n\omega^2}$ we see that (5.52) from Proposition 5.12 also holds. Furthermore property (a) and (5.52) of Proposition 5.12 hold with the same constants for any unimodular lattice $\Delta \in \mathbb{R}^n$. Since $f_{\epsilon,s}(1) \leq n\alpha(\Delta)^s$, $f_{\epsilon,s}(1)$ is uniformly bounded as Δ varies over a compact set $\mathcal{C}$ of unimodular lattices. Hence the family $\mathcal{F}$ of functions $f_{\epsilon,s}$ obtained as Δ varies over $\mathcal{C}$ satisfies all the conditions of Proposition 5.12. Since $\alpha_i(h\Delta)^s = f_i(h)^s \leq \epsilon^{-q(i)} f_{\epsilon,s}(h)$, Proposition 5.12 implies that there exists a constant $B > 0$ so that for each i, all $t > 0$, and all $\Delta \in \mathcal{C}$,

$$\int_K \alpha_i(a_t k\Delta)^s \, dm(k) < B.$$

From this the theorem follows. $\square$

Proof of Theorem 3.3. Suppose $p = 2$ and $q = 1$. Let $\mathcal{C}$ be a compact set of lattices in $\mathbb{R}^3$; by rescaling we may assume that $\mathcal{C}$ consists of unimodular lattices. Applying Theorem 3.2 with $s = 1/2$, we see that there is a constant $B_1 > 0$, so that for any $t \geq 0$ and $i = 1, 2$, and any $\Delta \in \mathcal{C}$,

$$\int_K \alpha_i(a_t k\Delta)^{1/2} \, dm(k) < B_1. \tag{5.68}$$

For $i=1,2$, let $f_i(h) = \alpha_i^*(h\Delta)$ (where α_i^* is defined in (5.37)). By Lemma 5.8 there exist $t_0 > 0$ and $\omega > 1$ such that for $0 \le t < t_0$, any $h \in H$ and any unimodular lattice Δ,

$$\int_K f_i(a_t k h)\, dm(k) < f_i(h) + \omega^2 \alpha_{3-i}(h\Delta)^{1/2}. \tag{5.69}$$

Let $\tilde{f}_i(h) = \int_K f_i(hk)\, dm(k)$. Then writing (5.69) with h replaced by hk' and integrating over $k' \in K$ we get

$$\int_K \tilde{f}_i(a_t k h)\, dm(k) < \tilde{f}_i(h) + \omega^2 \int_K \alpha_{3-i}(hk'\Delta)^{1/2}\, dm(k'). \tag{5.70}$$

Since we may write $h = k a_s k''$ for $k, k'' \in K$ and the α_i are left K-invariant, the integral on the right hand side of (5.70) is bounded by (5.68). Thus, for $0 \le t < t_0$ and any $h \in H$,

$$\int_K \tilde{f}_i(a_t k h)\, dm(k) < \tilde{f}_i(h) + b.$$

where $b = \omega^2 B_1$. Thus for $i = 1,2$, $\tilde{f}_i$ satisfies all the conditions of Lemma 5.13, with constants independent of the choice of $\Delta \in \mathcal{C}$. Hence for $i=1,2$ and $t > 1$,

$$\int_K f_i(a_t k)\, dm(k) = \int_K \tilde{f}_i(a_t k)\, dm(k) < B_2 t, \tag{5.71}$$

where B_2 is independent of t or the choice of $\Delta \in \mathcal{C}$. Now (3.5) follows from (5.71) and (5.38).

Now suppose $p = 2$ and $q = 2$. Let $\mathcal{C}$ be a compact set of unimodular lattices in $\mathbb{R}^4$. Applying Theorem 3.2 with $s = 1/2$, we see that there is a constant $B_4 > 0$, so that for any $t \ge 0$ and $i = 1,2,3$,

$$(A_t \alpha_i^{1/2})(\Delta) < B_4. \tag{5.72}$$

(recall that for any function f on G/Γ, $(A_t f)(\Delta) = \int_K f(a_t k \Delta)\, dm(k)$). For $i=1$, and $i = 3$, let $f_i(h) = \alpha_i(h\Delta)$. Then Lemma 5.7 implies that for any $c > 0$ there exist $t > 0$ and $\omega > 0$ such that for any $h \in H$ and $i = 1,3$,

$$(A_t f_i)(h) < \frac{c}{2} f_i(h) + \omega^2 \alpha_2(h\Delta)^{1/2}. \tag{5.73}$$

For $i = 1, 3$, let $\tilde{f}_i(h) = \int_K f_i(hk)\, dm(k)$. Writing (5.73) with hk' instead of h and integrating over $k' \in K$ we get that for all $h \in H$ and $i = 1, 3$,

$$(A_t \tilde{f}_i)(h) < \frac{c}{2}\tilde{f}_i(h) + \omega^2 \int_K \alpha_2(hk'\Delta)^{1/2}\, dm(k'). \tag{5.74}$$

Let $A = \{a_\tau \mid \tau \in \mathbb{R}\}$, and suppose $h \in KAK$. Then, since α_2 is left-K invariant the integral on the right hand side of (5.74) is bounded by (5.72). Hence for any $c > 0$ there exist $t > 0$ and $b' > 0$ so that for all $h \in KAK$,

$$(A_t \tilde{f}_i)(h) < \frac{c}{2}\tilde{f}_i(h) + b',$$

where $b' = \omega^2 B_1$. Then by Proposition 5.12, there exists $B_5 > 0$ so that for all $t > 0$, and $i = 1, 3$,

$$\tilde{f}_i(K a_t K) = \int_K f_i(a_t k)\, dm(k) < B_5,$$

i.e for all $t > 0$ and $i = 1, 3$,

$$\int_K \alpha_i(a_t k \Delta)\, dm(k) < B_5. \tag{5.75}$$

Since the constants in conditions (a)-(c) and equation (5.52) of Proposition 5.12 can be chosen uniformly as Δ varies over $\mathcal{C}$, the constant B_5 does not depend on the choice of $\Delta \in \mathcal{C}$.

Now let $f_2(h) = \alpha_2^{\#}(h\Delta)$. By Lemma 5.9, there exist $t_0 > 0$ and $\omega > 0$ so that for any $0 < t < t_0$ and any $h \in H$,

$$(\tilde{A}_t f_2)(h) < f_2(h) + \omega^2 \sqrt{\alpha_1(h\Delta)\alpha_3(h\Delta)} \le f_2(h) + \omega^2(\alpha_1(h\Delta) + \alpha_3(h\Delta)). \tag{5.76}$$

where $(\tilde{A}_t f)(h) = \int_{\tilde{K}} f(a_t \tilde{k} h)\, dm(k)$ and $\tilde{K}$ denotes the maximal compact of $\tilde{H} = SO(2,1)$, which is naturally (i.e diagonally) embedded in $SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. If we let

$$\tilde{f}_2(h) = \int_K f_2(hk)\, dm(k),$$

then using (5.76) with hk in place of h and integrating over $k \in K$ we get

$$(\tilde{A}_t \tilde{f}_2)(h) < \tilde{f}_2(h) + \omega^2 \int_K [\alpha_1(hk\Delta) + \alpha_3(hk\Delta)]\, dm(k). \tag{5.77}$$

Since α_1 and α_3 are left K-invariant, and $\tilde{H} \subset KAK$, (5.77) and (5.75) imply that for $h \in \tilde{H}$,

$$(\tilde{A}_t \tilde{f}_2)(h) < \tilde{f}_2(h) + b'',$$

where $b'' = 2\omega^2 B_5$. Thus $\tilde{f}_2$ restricted to $\tilde{H} \cong SO(2,1)$ satisfies all the conditions of Lemma 5.13, hence there exists $B_6 > 0$ so that for all $t > 1$ and all $\Delta \in \mathcal{C}$,

$$\int_K \alpha_2^{\#}(a_t k \Delta)\, dm(k) = \int_K \tilde{f}_2(a_t k)\, dm(k) < B_6 t. \tag{5.78}$$

Now (3.5) follows from (5.78), (5.42) and (5.75). $\square$

Proof of Theorem 3.5. We can assume that ϕ is nonnegative. Let $A(r) = \{x \in G/\Gamma \; : \; \alpha(x) > r\}$. Choose a continuous nonnegative function g_r on G/Γ such that $g_r(x) = 1$ if $x \in A(r+1)$, $g_r(x) = 0$ if $x \notin A(r)$ and $0 \leq g_r(x) \leq 1$ if $x \in A(r) - A(r+1)$. Then

$$\begin{aligned} &\int_K \phi(a_t k x)\nu(k)\, dm(k) = \\ &= \int_K (\phi g_r)(a_t k x)\nu(k)\, dm(k) + \int_K (\phi - \phi g_r)(a_t k x)\nu(k)\, dm(k). \end{aligned} \tag{5.79}$$

But (letting $\beta = 2 - s$), $(\phi g_r)(y) \leq B_1 \alpha(y)^{2-\beta} g_r(y) = B_1 \alpha(y)^{2-\frac{\beta}{2}} g_r(y) \alpha(y)^{-\frac{\beta}{2}} \leq B_1 r^{-\frac{\beta}{2}} \alpha(y)^{2-\frac{\beta}{2}}$ (the last inequality is true because $g_r(y) = 0$ if $\alpha(y) \leq r$). Therefore

$$\int_K (\phi g_r)(a_t k x)\nu(k)\, dm(k) \leq B_1 r^{-\frac{\beta}{2}} \int_K \alpha(a_t k x)^{2-\frac{\beta}{2}} \nu(k)\, dm(k). \tag{5.80}$$

According to Theorem 3.2 there exists B such that

$$\int_K \alpha(a_t k x)^{2-\frac{\beta}{2}}\, dm(k) < B$$

for any $t \geq 0$ and uniformly over $x \in \mathcal{C}$. Then (5.80) implies that

$$\int_K (\phi g_r)(a_t k x)\nu(k)\, dm(k) \leq B B_1 (\sup \nu) r^{-\frac{\beta}{2}}. \tag{5.81}$$

Since the function $\phi - \phi g_r$ is continuous and has a compact support, Theorem 4.5 implies that for every $\epsilon > 0$ there exists a finite set of points $x_1, \ldots, x_\ell$ with Hx_i closed for each i so that for every compact subset F of $\mathcal{C} - \bigcup_{i=1}^{\ell} Hx_i$ there exists $t_0 > 0$ such that for every $t > t_0$ and every $x \in F$,

$$\left| \int_K (\phi - \phi g_r)(a_t k x)\nu(k)\, dm(k) - \int_{G/\Gamma} (\phi - \phi g_r)(y)\, d\mu(y) \int_K \nu(k)\, dm(k) \right| < \frac{\epsilon}{2}. \quad (5.82)$$

It is easy to see that (5.79), (5.81) and (5.82) imply (3.6) if r is sufficiently large. $\square$

References

[Bea] A. Beardon. *The Geometry of Discrete Groups.* Springer-Verlag, 1983.

[Bor1] A. Borel. *Introduction aux groupes arithmetiques.* Publ. de l'Inst. Math. del l'Univ. de strassbourg XV. Hermann, Paris, 1969.

[Bor2] A. Borel. Values of indefinite quadratic forms at integral points and flows on spaces of lattices. *Bull. Amer. Math. Soc. (N.S.)* **32**(1995), 184–204.

[Bre] T. Brennan. . Princeton University undergraduate thesis, 1994.

[BMP] M.R. Brenner, R.V Moody, and J. Patera. *Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras.* M. Dekker, New York, 1985.

[CSD] J.W.S. Cassels and H.P.F. Swinnerton-Dyer. On the product of three homogeneous linear forms and the indefinite ternary quadratic forms. *Philos. Trans. Roy. Soc. London Ser. A.* **248**(1955), 73–96.

[Dan1] S.G. Dani. On invariant measures, minimal sets and a lemma of Margulis. *Invent. Math* **51**(1979), 239–260.

[Dan2] S.G. Dani. Invariant measures and minimal sets of horoshperical flows. *Invent. Math* **64**(1981), 357–385.

[Dan3] S.G. Dani. On orbits of unipotent flows on homogeneous spaces. *Ergod. Theor. Dynam. Syst.* **4**(1984), 25–34.

[Dan4] S.G. Dani. On orbits of unipotent flows on homogenous spaces II. *Ergod. Theor. Dynam. Syst* **6**(1986), 167–182.

[DM1] S.G. Dani and G.A. Margulis. Values of quadratic forms at primitive integral points. *Invent. Math.* **98**(1989), 405–424.

[DM2] S.G. Dani and G.A. Margulis. Orbit closures of generic unipotent flows on homogeneous spaces of $SL(3,\mathbb{R})$. *Math. Ann.* **286**(1990), 101–128.

[DM3] S.G. Dani and G.A. Margulis. Limit distributions of orbits of unipotent flows and values of quadratic forms. *Advances in Soviet Math.* **16**(1993), 91–137.

[EMM] A. Eskin, G. Margulis, and S. Mozes. On a quantitative version of the Oppenheim conjecture. *ERA. Amer. Math. Soc.* **01**(1995), 124–130.

[EM] A. Eskin and C. McMullen. Mixing, counting and equidistribution in Lie groups. *Duke Math. J.* **71**(1993), 143–180.

[Hel] S. Helgason. *Groups and Geometric Analysis.* Academic Press, New York, 1984.

[How] R. Howe. A notion of rank for unitary representations of the classical groups. In A. Figa Talàmianca, editor, *Harmonic analysis and group representaions.* CIME, 1980.

[HT] R. Howe and E.C. Tan. *Non-abelian harmonic analysis: applications of $SL(2,\mathbb{R})$.* Springer-Verlag, New York, 1992.

[KS] A. Katok and R. Spatzier. Differential rigidity of hyperbolic abelian actions. *Publ. Math. IHES* **79**(1994), 131–156.

[KM] D. Kleinbok and G.A. Margulis. Bounded orbits of non-quasiunipotent flows on homogeneous spaces. *AMS. Math. Soc. Trans. (2)* **171**(1996), 141–172.

[Kna] A. W. Knapp. *Representation Theory of Semisimple Groups.* Princeton University Press, 1986.

[Mar1] G. A. Margulis. On the action of unipotent groups in the space of lattices. In *Lie Groups and their representations, (Proc. of Summer School in Group Representations, Bolyai Janos Math. Soc., Akademai Kiado, Budapest, 1971)*, pages 365–370. Halsted, New York, 1975.

[Mar2] G. A. Margulis. Formes quadratiques ind‘efinies et flots unipotents sur les spaces homog‘enes. *C. R. Acad. Sci. Paris Ser. I* **304**(1987), 247–253.

[Mar3] G. A. Margulis. Discrete Subgroups and Ergodic Theory. In *Number theory, trace formulas and discrete subgroups (a symposium in honor of A. Selberg)*, pages 377–398. Academic Press, Boston, MA, 1989.

[Mar4] G. A. Margulis. Indefinite quadatic forms and unipotent flows on homogeneous spaces. In *Dynamical systems and ergodic theory*, volume 23, pages 399–409. Banach Center Publ., PWN — Polish Scientific Publ., Warsaw, 1989.

[Mar5] G. A. Margulis. Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory. In *Proc. of ICM (Kyoto, 1990)*, pages 193–215. Math. Soc. of Japan and Springer, 1991.

[MT1] G.A. Margulis and G.M. Tomanov. Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. *Invent. Math.* **116**(1994), 347–392.

[MT2] G.A. Margulis and G.M. Tomanov. Measure rigidity for almost linear groups and its applications. *J. d'Anal. Math.* **69**(1996), 25–54.

[NS] A. Nevo and E. Stein. A generalization of Wiener's pointwise ergodic theorem. Preprint.

[Rat1] M. Ratner. Strict measure rigidity for unipotent subgroups of solvable groups. *Invent. Math.* **101**(1990), 449–482.

[Rat2] M. Ratner. On measure rigidity of unipotent subgroups of semisimple groups. *Acta. Math.* **165**(1990), 229–309.

[Rat3] M. Ratner. On Raghunathan's measure conjecture. *Annals of Math.* **134**(1991), 545–607.

[Rat4] M. Ratner. Raghunathan's topological conjecture and distributions of unipotent flows. *Duke Math. J.* **63**(1991), 235–290.

[Rat5] M. Ratner. Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups. *Duke Math. J.* **77**(1995), 275–382.

[Rat6] M. Ratner. Interactions between Lie groups, ergodic theory and number theory. In *Proc. of ICM (Zurich, 1994)*.

[Rud] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 2d edition, 1974.

[Sar] P. Sarnak. Values at integers of binary quadratic forms. In preparation.

[Sch] W. Schmidt. Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height. *Duke Math. J.* **35**(1968), 327–339.

[Sha1] N. A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.* **289**(1991), 315–334.

[Sha2] N. A. Shah. Limit distributions of polynomial trajectories on homogeneous spaces. *Duke Math. J.* **75**(1994), 711–732.

[Sha3] N. A. Shah. PhD thesis, Tata Institute for Fundamental Research.

[Spi] M. Spivak. *A Comprehensive Introduction to Differential Geometry*. Publish or Perish, Inc., 1970.

[Sta1] A.N. Starkov. Solvable homogeneous flows. *Math. Sbornik* **176**(1987), 242–259. In Russian.

[Sta2] A.N. Starkov. The ergodic decomposition of flows on homogenous spaces of finite volume. *Math. Sbornik* **180**(1989), 1614–1633. In Russian.

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