Recurrence Properties of Random Walks on Finite Volume Homogeneous Manifolds

Alex Eskin* and Gregory Margulis†

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1 Introduction

Let G be a semisimple Lie group, and Γ a nonuniform irreducible lattice in G. Let π denote the natural projection from G to G/Γ. We recall some results from [Dan1-Dan4], [Mar]. Let $U = \{u_t\}$ be a unipotent one-parameter subgroup of G.

Theorem 1.1. *For* every *point $x \in G/\Gamma$ and every $\epsilon > 0$ there exists a compact set $K \subset G/\Gamma$ such that for all $T > 0$*

$$|\{t \in [0,T] \ : \ u_t x \in K\}| > (1-\epsilon)T \tag{1}$$

More generally, for every compact set $C \in G/\Gamma$ and every $\epsilon > 0$ there exists a compact set K such that (1) holds for all $x \in C$, and all $T > 0$.

A parabolic subgroup P of G is called Γ-*rational* if the unipotent radical of P intersects Γ in a lattice. Note that if Γ is arithmetic then a parabolic subgroup of G is Γ-rational if and only if it is defined over $\mathbb{Q}$.

Theorem 1.2. *There exists a compact set K with the following property: Let $g \in G$ be any element such that gUg^{-1} is not contained in a Γ-rational parabolic subgroup of G. Then $U\pi(g)$ intersects K.*

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Remark 1.3. Essentially the assertion of Theorem 1.2 is that every orbit of U returns to a fixed compact set K. An example that the algebraic condition on gUg^{-1} in Theorem 1.2 is needed can be given as follows: Let G be $SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$. Suppose gUg^{-1} is contained in a $\mathbb{Q}$-parabolic P. Let V be the subspace stabilized by P. Since V is defined over $\mathbb{Q}$, V intersects $\mathbb{Z}^n$ in a lattice L. Note that the volume of the torus $V/V \cap L$ is an invariant of the orbit; if it is sufficiently small, the orbit will not intersect the fixed compact set K.

We emphasize that the key point of Theorem 1.1 and Theorem 1.2 is that Theorem 1.1 holds for every point x, and Theorem 1.2 holds for every element g satisfying an explicit algebraic condition. The fact that the theorems hold almost everywhere is an easy consequence of ergodicity and the Birkhoff ergodic theorem. The proofs rely heavily on the polynomial nature of the unipotent flow, and are based mostly on the techniques of [Mar].

The aim of this paper is to show that similar results hold under some conditions when considering random walks on G/Γ. Let μ be a measure on G satisfying the condition

$$\int_G \|g\|^\delta \, d\mu(g) < \infty \tag{2}$$

for sufficiently small $\delta > 0$, where $\|\cdot\|$ denotes some norm in some finite dimensional representation of G. To state the results we formulate some properties analogous to the conclusions of Theorem 1.2 and Theorem 1.1 which may be possessed by the random walk defined by μ.

Definition 1.4. Let μ be a probability measure on G satisfying (2). Let $\mu^{(m)}$ denote the convolution of μ with itself m times, and for $x \in G$ let δ_x denote the probability measure supported on the point x. The following may hold:

(R1) For every compact set $C \subset G/\Gamma$ and every $\epsilon > 0$ there exists a compact set $K \supset C$ such that for every $x \in C$ and every $m > 0$,

$$(\mu^{(m)} * \delta_x)(K) > (1 - \epsilon)$$

(R2) For every $\epsilon > 0$ there exists a compact set K such that for every $x \in G/\Gamma$, there exists $M = M(x) > 0$ such that for $m > M(x)$,

$$(\mu^{(m)} * \delta_x)(K) > (1 - \epsilon).$$

The constants $M(x)$ can be chosen so that for every compact subset C of G/Γ, $\sup_{x \in C} M(x) < \infty$.

(S) For every $\epsilon > 0$ there exists a compact set K such that for every $g \in G$ either $g(\mathrm{supp}\mu)g^{-1}$ is contained in a Γ-rational parabolic P or there exists $M > 0$ such that for $m > M$, $(\mu^{(m)} * \delta_{\pi(g)})(K) > (1 - \epsilon)$.

The property (R1) is a version of the conclusion of Theorem 1.1 which makes sense in the current context. The property (S) has the same relation to the conclusion of Theorem 1.2. We note that (R2) implies (R1) and (S).

2 Recurrence properties of random walks

To simplify the terminology, we assume that G is a connected algebraic group. All the claims can be easily reduced to this case.

Notation: Let H_μ denote the noncompact part of the Zariski closure of the subgroup generated by the support of μ. (By the noncompact part of an algebraic group H we mean the subgroup of H generated by unipotent elements, together with the split part of H).

Our result is the following:

Theorem 2.1. *Suppose H_μ is semisimple, and for all $g \in G$, $gH_\mu g^{-1}$ is not contained in any proper Γ-rational parabolic subgroup of G. Then μ has properties (R1) and (R2). In particular, if $H_\mu = G$, then μ has properties (R1) and (R2).*

Recall that a measure ν on a G-space X is μ-stationary if an only if $\mu * \nu = \nu$. We note the following general lemma (proved in §5):

Lemma 2.2. *Let μ be any measure on G satisfying (R2). Then any μ-stationary locally finite measure on G/Γ is finite.*

As a corollary of Theorem 2.1 and Lemma 2.2 we have the following result conjectured by N. Shah in [Sh]:

Theorem 2.3. *Let G be a semisimple Lie group, and Γ a nonuniform irreducible lattice in G. Let $\Lambda \subset G$ be a countable Zariski dense subgroup (or more generally a countable subgroup with semisimple Zariski closure which is not contained in any conjugate of a proper Γ-rational parabolic subgroup of G). Let $x \in G/\Gamma$ be a point such that the orbit $\Lambda x \subset G/\Gamma$ is discrete. Then Λx is finite.*

Proof of Theorem 2.3. Since Λ is countable, we can find a measure μ such that $\mathrm{supp}\mu = \Lambda$, and (2) holds. Let σ denote counting measure on the discrete set Λx (i.e. $\sigma(C)$ is the cardinality of $C \cap \Lambda x$). Then σ is invariant, hence stationary. Then, in view of Theorem 2.1 and Lemma 2.2, σ is finite. Hence Λx is finite. $\square$

As another corollary, we obtain another proof, in a special case, of the following well known theorem of Borel and Harish-Chandra:

Theorem 2.4. *Let $\mathbf{H}$ be a semisimple algebraic group defined over $\mathbb{Q}$. Then, $\mathbf{H}(\mathbb{Z})$ is a lattice in $\mathbf{H}(\mathbb{R})$ (i.e. $\mathbf{H}(\mathbb{R})/\mathbf{H}(\mathbb{Z})$ has finite $\mathbf{H}(\mathbb{R})$-invariant measure).*

We give a proof of this theorem under the assumption that there exists a faithful representation of $\mathbf{H}$ which is defined over $\mathbb{Q}$ and is irreducible over $\mathbb{R}$.

Proof of Theorem 2.4 (under the assumption). Let $H = \mathbf{H}(\mathbb{R})$, $\Lambda = \mathbf{H}(\mathbb{Z})$. We are assuming that there is a faithful representation $\rho : H \to SL(N, \mathbb{R})$ which is irreducible over $\mathbb{R}$ and defined over $\mathbb{Q}$. Since ρ is defined over $\mathbb{Q}$, we have, after possibly replacing Λ by a subgroup of finite index, $\rho(\Lambda) \subset SL(N, \mathbb{Z})$. Hence, we have $\rho(H)/\rho(\Lambda) \subset SL(N, \mathbb{R})/SL(N, \mathbb{Z})$. Let σ denote the $\rho(H)$-invariant measure on $\rho(H)/\rho(\Lambda)$. Then σ is locally finite. Let μ be any compactly supported absolutely continuous measure on H. Then, by Theorem 2.1, and since ρ is irreducible over $\mathbb{R}$, $\rho(\mu)$ has property (R2). Also since σ is $\rho(H)$-invariant, it is also $\rho(H)$-stationary. Thus by Lemma 2.2, σ is finite. Hence Λ is a lattice in H. $\square$

We also make the following conjecture:

Conjecture 2.5. Suppose H_μ is semisimple (or more generally generated by unipotent elements). Then μ has properties (R1) and (S).

Theorem 2.1 can be generalized somewhat: see Proposition 2.6 below.

The set Δ. We now define a certain finite collection Δ of maximal parabolic subgroups of G. If G has real rank 1, then since Γ is non-uniform there exists a Γ-rational minimal parabolic subgroup P_0 of G, and we define $\Delta = \{P_0\}$. If the real rank of G is at least 2 then Γ is arithmetic, hence there exists a $\mathbb{Q}$-structure on G such that Γ is commensurable with the set $G(\mathbb{Z})$ of integer points (with respect to this $\mathbb{Q}$-structure). We fix a maximal $\mathbb{Q}$-split torus A_0 for G, and we let $\Delta = \{P_1, \dots, P_r\}$ where the P_k are the standard parabolic subgroups with respect to A_0. For every $P_i \in \Delta$, there exists a representation $\rho_i : G \to GL(V_i)$ and vectors $w_i \in V_i$ such that the stabilizer of $\mathbb{R}w_i$ is P_i.

Condition A. We say that μ satisfies *Condition A* if it satisfies (2) and for all sufficiently small $\delta > 0$ there exist $c < 1$ and $n > 0$ such that for all i, $1 \le i \le r$ and all $v \in Gw_i$,

$$\int_G \frac{1}{\|\rho_i(g)v\|^\delta}\, d\mu^{(n)}(g) \le \frac{c}{\|v\|^\delta} \tag{3}$$

Theorem 2.1 follows immediately from the following two propositions:

Proposition 2.6. *Suppose μ satisfies Condition A. Then μ has properties (R1) and (R2).*

Proposition 2.7. *Suppose H_μ is semisimple and for any $g \in G$, $gH_\mu g^{-1}$ is not contained in any Γ-rational parabolic subgroup of G. Then μ satisfies Condition A.*

Proposition 2.6 will be proved in §3 and Proposition 2.7 will be proved in §4. Lemma 2.2 will be proved in §5.

3 Systems of inequalities

In this section we prove Proposition 2.6. The proof is based on the following:

Lemma 3.1. *Suppose that there exists a positive continuous function $u : G/\Gamma \to \mathbb{R}$ with the following properties:*

(i) *$u(x) \to \infty$ as $x \to \infty$ in G/Γ.*

(ii) *There exist constants $c_1 < 1$ and $b > 0$ and $n > 0$ such that for any $x \in G/\Gamma$,*

$$\int_G u(gx)\, d\mu^{(n)}(g) \le c_1 u(x) + b. \tag{4}$$

Then μ has property (R2).

Proof. We note that it is enough to prove (R2) for m in an arithmetic progression. After iterating (4) and summing the geometric series, we obtain for any multiple m of n,

$$\int_G u(gx)\, d\mu^{(m)}(g) \le c_1^{m/n} u(x) + b_1, \tag{5}$$

where $b_1 = b/(1-c_1)$. Note that for any $R > 0$ the set $\{y \in G/\Gamma \ : \ u(y) < R\}$ is compact. We choose $K = \{y \in G/\Gamma \ : \ u(y) < 2b_1/\epsilon\}$; then, for m large enough so that $c_1^{m/n} u(x) < b_1$,

$$\frac{(2b_1)}{\epsilon}(\mu^{(m)} * \delta_x)(K^c) \leq \int_G u(hx)\, d\mu^{(m)}(h) \leq c_1^{m/n} u(x) + b_1 \leq 2b_1,$$

hence $(\mu^{(m)} * \delta_x)(K^c) < \epsilon$ as required. $\square$

3.1 Construction of the function u in the $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$ case.

In this case the construction is more transparent, and follows closely that of [EMM]. The representation ρ_i of condition A is the representation of $G = SL(d,\mathbb{R})$ on the i'th exterior power of $\mathbb{R}^d$, and we can take $w_i = e_1 \wedge \cdots \wedge e_i$, where $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{R}^n$. We now recall some notation and results from [EMM]:

Let Δ be a lattice in $\mathbb{R}^d$. We say that a subspace L of $\mathbb{R}^d$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the volume of $L/(L \cap \Delta)$. Let us note that $d(L)$ is equal to the norm of $u_1 \wedge \cdots \wedge u_\ell$ in the exterior power $\bigwedge^\ell(\mathbb{R}^d)$ where $\ell = \dim L$, $(u_1, \cdots, u_\ell)$ is a basis over $\mathbb{Z}$ of $L \cap \Delta$, and the norm on $\bigwedge(\mathbb{R}^d)$ is induced from the Euclidean norm on $\mathbb{R}^d$. If $L = \{0\}$ we write $d(L) = 1$. A lattice is Δ *unimodular* if $d_\Delta(\mathbb{R}^d) = 1$. The space of unimodular lattices is canonically identified with $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$.

We introduce the following notation:

$$\begin{aligned} \alpha_i(\Delta) &= \sup \left\{ \frac{1}{d(L)} \,\middle|\, L \text{ is a } \Delta\text{-rational subspace of dimension } i \right\}, \quad 0 \leq i \leq d, \\ \alpha(\Delta) &= \max_{0 \leq i \leq d} \alpha_i(\Delta). \end{aligned} \tag{6}$$

Lemma 3.2 ([EMM], Lemma 5.6). *For any two Δ-rational subspaces L and M*

$$d(L)d(M) \geq d(L \cap M)d(L + M). \tag{7}$$

In view of (2) we may write $\mu^{(n)} = \mu_1 + \mu_2$, where μ_1 has compact support and for any $1 \leq i \leq d-1$,

$$\int_G \|\rho_i(g)\|^\delta \, d\mu_2(g) \leq \frac{1-c}{3} \tag{8}$$

Lemma 3.3. *Suppose μ satisfies condition A, and let n, δ and c be as in (3). Let $\mu_1 < \mu^{(n)}$ be any compactly supported measure. Then there exists a constant $\omega > 1$ such that for any lattice Λ in $\mathbb{R}^d$, and any $1 \le i \le d-1$,*

$$\int_G \alpha_i(g\Lambda)^\delta \, d\mu_1(g) < c\, \alpha_i(\Lambda)^\delta + \omega^{2\delta} \max_{0<j\le\min\{d-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\, \right)^\delta. \tag{9}$$

Proof. Let Λ be a lattice in $\mathbb{R}^d$, and let M be a Λ-rational subspace of $\mathbb{R}^d$. Then, for any $g \in G$, gM is also a $g\Lambda$-rational subspace. By (3) applied to $v_1 \wedge \cdots \wedge v_l$ where $(v_1, \cdots, v_\ell)$ is a basis over $\mathbb{Z}$ of $M \cap \Lambda$, we have

$$\int_G \frac{1}{d_{g\Lambda}(gM)^\delta} \, d\mu_1(g) \le c \frac{1}{d(M)^\delta}. \tag{10}$$

There exists a Λ-rational subspace L_i of dimension i such that

$$\frac{1}{d_\Lambda(L_i)} = \alpha_i(\Lambda). \tag{11}$$

The inequality (10) implies

$$\int_G \frac{d\mu_1(g)}{d_{g\Lambda}(gL_i)^\delta} < c \frac{1}{d_\Lambda(L_i)^\delta}. \tag{12}$$

Let $\omega = \max_{\substack{g\in\mathrm{supp}\mu_1 \\ 0<j<d}} \|\rho_j(g)^{\pm 1}\|$. We have that for any Λ-rational subspace L, and any $g \in \mathrm{supp}\mu_1$,

$$\omega^{-1} \le \frac{d_{g\Lambda}(gL)}{d_\Lambda(L)} \le \omega. \tag{13}$$

Let us denote the set of Λ-rational subspaces L of dimension i with $d_\Lambda(L) < \omega^2 d_\Lambda(L_i)$ by Ψ_i. We get from (13) that for a Λ-rational i-dimensional subspace $L \notin \Psi_i$

$$d_{g\Lambda}(gL) > d_{g\Lambda}(gL_i), \qquad g \in \mathrm{supp}\, \mu_1. \tag{14}$$

It follows from (12), (14) and the definition of α_i that

$$\int_G \alpha_i(g\Lambda)^\delta \, d\mu(g) < c\alpha_i(\Lambda)^\delta \qquad \text{if } \Psi_i = \{L_i\}. \tag{15}$$

Assume now that $\Psi_i \neq \{L_i\}$. Let $M \in \Psi_i,\ M \neq L_i$. Then $\dim(M+L_i) = i+j,\ j > 0$. Now using (11), (13) and Lemma 3.2 we get that for any $g \in \mathrm{supp}\mu_1$,

$$\begin{aligned} \alpha_i(g\Lambda) < \omega\alpha_i(\Lambda) = \frac{\omega}{d_\Lambda(L_i)} &< \frac{\omega^2}{\sqrt{d_\Lambda(L_i)d_\Lambda(M)}} \\ &\leq \frac{\omega^2}{\sqrt{d_\Lambda(L_i \cap M)d_\Lambda(L_i + M)}} \\ &\leq \omega^2\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}. \end{aligned} \tag{16}$$

Hence if $\Psi_i \neq \{L_i\}$

$$\int_G \alpha_i(g\Lambda)^\delta \, d\mu_1(g) \leq \omega^{2\delta} \max_{0<j\leq\min\{d-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\,\right)^\delta. \tag{17}$$

Combining (15) and (17) we get that for any lattice $\Lambda \subset \mathbb{R}^n$, (9) holds. $\square$

Lemma 3.4. *For sufficietly small $\epsilon > 0$ and $\delta > 0$ the function*

$$u(\Lambda) = \sum_{i=0}^{d} \epsilon^{i(d-i)}\alpha_i(\Lambda)^\delta, \tag{18}$$

satisfies the conditions of Lemma 3.1.

Proof. Note that $u(\Lambda)$ satisfies (i) of Lemma 3.1. Let $\delta_0 > 0$ be some choice of δ satisfying (2) and Condition A, and let $\delta = \delta_0/d$. Since for any $1 \leq j \leq d-1$ and any $g \in G$, $\|\rho_j(g)\| \leq \|g\|^d$, we have for some constant C,

$$u(g\Lambda) \leq C\|g\|^{d\delta}u(\Lambda).$$

Hence, in view of (2), we may decompose $\mu^{(n)} = \mu_1 + \mu_2$ such that μ_1 is compactly supported and

$$\int_G u(g\Lambda)\, d\mu_2(g) \leq \frac{1-c}{3}u(\Lambda), \tag{19}$$

where $c < 1$ and $n > 0$ are as in (3). Let A_1 denote the averaging operator on G/Γ given by

$$(A_1 f)(\Lambda) = \int_G f(g\Lambda)\, d\mu_1(g). \tag{20}$$

Let $q(i) = i(d-i)$. Then by direct computations $2q(i) - q(i+j) - q(i-j) = 2j^2$. Therefore we get from Lemma 3.3 that for any $i, 0 < i < d$, and any positive $\epsilon < 1$

$$
\begin{aligned}
A_1(\epsilon^{q(i)}\alpha_i^\delta) &< c\,\epsilon^{q(i)}\alpha_i^\delta + \omega^2 \max_{0<j\le\min\{d-i,i\}} \epsilon^{q(i)-\frac{q(i+j)+q(i-j)}{2}} \sqrt{\epsilon^{q(i+j)}\alpha_{i+j}^\delta \epsilon^{q(i-j)}\alpha_{i-j}^\delta} \\
&\le c\,\epsilon^{q(i)}\alpha_i^\delta + \epsilon\,\omega^2 \max_{0<j\le\min\{d-i,i\}} \sqrt{\epsilon^{q(i+j)}\alpha_{i+j}^\delta \epsilon^{q(i-j)}\alpha_{i-j}^\delta}.
\end{aligned}
\tag{21}
$$

Since $\epsilon^{q(i)}\alpha_i^\delta < u, \alpha_0 = 1$ and $\alpha_d = 1/d(\Delta) = 1$, the inequalities (21) imply the following inequality:

$$
(A_1 u)(\Lambda) < 1 + 1 + c\,u(\Lambda) + d\epsilon\omega^2 u(\Lambda).
$$

Taking $\epsilon = \frac{1-c}{3d\omega^2}$ we see that

$$
(A_1 u)(\Lambda) = \int_G u(gx)\,d\mu_1(\Lambda) < \frac{1+2c}{3}u(\Lambda) + 2 \tag{22}
$$

Now, in view of (22) and (19), (ii) of Lemma 3.1 also holds. This completes the proof of Lemma 3.4, and hence the proof of Proposition 2.6 in the $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$ case. $\square$

3.2 Construction of the function u in the general case.

Let P_0 denote the minimal Γ-rational parabolic subgroup of G. Then we have the Langlands decomposition $P_0 = M_0A_0N_0$, where M_0 is semisimple, N_0 is the unipotent radical of P_0, and A_0 is as in the definition of Δ. From the general theory, $\Gamma \cap M_0$ is a cocompact lattice in M_0, and $\Gamma \cap N_0$ is a cocompact lattice in N_0. Let $\mathfrak{a}$ denote the Lie algebra of A_0. We identify $\mathfrak{a}$ with its dual using the Killing form.

Let $\alpha_1, \dots, \alpha_r$ denote the roots, which we view as elements of the dual of $\mathfrak{a}$. A *Siegel set* is a set $\mathfrak{S} = K\mathcal{M}\mathcal{A}\mathcal{N}$, where K is a maximal compact subgroup of G, $\mathcal{M} \in M_0$ and $\mathcal{N} \in N_0$ are compact, and $\mathcal{A} = \{a \in A \ :\ \alpha_k(\log a) < C,\ 1 \le k \le r\}$, where $C > 0$ is some positive constant. It follows from reduction theory that for appropriate choices of $\mathcal{M}$, $\mathcal{N}$ and C there exists a finite set $J \in G$ such that for every $g \in G$, the intersection $\mathfrak{S} \cap g\Gamma J$ is not empty.

Since $G = KP_0 = KM_0A_0N_0$ we may decompose $g_1 = k(g_1)m_0(g_1)\exp H(g_1)n_0(g_1)$, where $k(g_1) \in K$, $m_0(g_1) \in M_0$, $\exp H(g_1) \in A_0$, and $n_0(g_1) \in N_0$. Hence,

$$
g_1 = k(g_1)m_0(g_1)\exp H(g_1)n_0(g_1) \tag{23}
$$

Then, for $g_1 \in \mathfrak{S}$, we have for any root α_j,

$$\alpha_j(-H(g_1)) > C \tag{24}$$

where $C < 0$ is an absolute constant. (Hence $-H(g_1)$ is a finite distance from the positive Weyl chamber).

Let ρ_k, w_k be as in Condition A. Let $P_k = M_k A_k N_k$ be the Langlands decomposition of P_k, where as above $N_k \subset N_0$ is the unipotent radical of P_k, $M_k \supset M_0$ is semisimple and $A_k \subset A_0$ is one dimensional. Let $d_k(g) = \|\rho_k(gw_k)\|$. Then, by construction, if $g = kman$ where k is in the maximal compact K, $m \in M_k$, $a \in a_k$ and $n \in N_k$, then $d_k(g) = d_k(a)$. It follows from structure theory that $|\log d_k(a) - c_k \omega_k(\log a)| < C_1$, where $C_1 < 0$ and $c_k > 0$ are absolute constants, and ω_k is the coroot corresponding to the root α_k; i.e. $\omega_k(\alpha_k) = 1$, and $\omega_k(\alpha_j) = 0$ if $j \neq k$. Hence, if $g = km_0 \exp(H(g)) n_0$, where $k \in K$, $m_0 \in M_0$, $a_0 = \exp(H(g)) \in A_0$, $n_0 \in N_0$, then $\log d_k(g) = \omega_k(\log a_0) = \omega_k(H(g))$.

Let

$$\beta_k(g) = \max_{\gamma \in \Gamma} \frac{1}{d_k(g\gamma)^{1/c_k}} \tag{25}$$

It follows from reduction theory that up to an absolute constant, $\beta_k(g) = \beta_k(g_1)$, where g_1 is any element of $g\Gamma J \cap \mathfrak{S}$. Hence,

$$|\log \beta_k(g) - \omega_k(-H(g_1))| < C \tag{26}$$

where $H(g_1)$ is as in (23), and C is an absolute constant. In particular, it follows that the function $\beta_k : G \to \mathbb{R}$ is bounded from below away from 0.

We may apply the identity

$$\alpha_k = \sum_j \langle \alpha_k, \alpha_j \rangle \omega_j, \tag{27}$$

to $H(g_1)$ and combine with (24) and (26) to obtain that for all $g \in G$ and all k,

$$\sum_j \langle \alpha_k, \alpha_j \rangle \log \beta_j(g) > C'. \tag{28}$$

where C' is an absolute constant.

Furthermore, for every constant C there exists a constant C' such that if for some k and some $g \in G$ and $g_1 \in g\Gamma J \cap \mathfrak{S}$ there exists $g_2 \in g\Gamma J$ such that $\frac{1}{d_k(g_2)} > \frac{1}{C d_k(g_1)}$, $d_k(g_2) \neq d_k(g_1)$ then it follows from reduction theory that

$$\alpha_k(-H(g_1))) < C' \tag{29}$$

(i.e. $-H(g_1)$ is "near the k'th wall" of the positive Weyl chamber). Hence, in this case

$$\sum_j \langle \alpha_k, \alpha_j \rangle \log \beta_j(g) < C''. \tag{30}$$

Choose $q_j > 0$ such that $\sum_j q_j \omega_j$ belongs to the positive Weyl chamber. Then for any k,

$$\sum_j q_j \langle \alpha_j, \alpha_k \rangle > 0. \tag{31}$$

Let $u_j(g) = \beta_j(g)^{1/q_j}$, so that $\log \beta_j(g) = q_j \log u_j(g)$. Hence, if (30) holds,

$$\sum_j \langle \alpha_k, \alpha_j \rangle q_j \log u_j(g) < C'' \tag{32}$$

Since $\langle \alpha_k, \alpha_k \rangle > 0$ and $\langle \alpha_k, \alpha_j \rangle \leq 0$ for $j \neq k$, (32) may be rewritten as

$$u_k \leq C \prod_{j \neq k} u_j^{\lambda_{jk}}, \tag{33}$$

where

$$\lambda_{jk} = \frac{q_j |\langle \alpha_j, \alpha_k \rangle|}{q_k \langle \alpha_k, \alpha_k \rangle} \tag{34}$$

Note that $\lambda_{jk} \geq 0$ and in view of (31),

$$\sum_{j \neq k} \lambda_{jk} < 1. \tag{35}$$

We note that the functions u_k satisfy the estimate

$$u_k(g'g) \leq \|g'\|_k^{1/q_k} u_k(g) \tag{36}$$

where $\|\cdot\|_k$ is the operator norm in the representation ρ_k.

Now choose $\delta > 0$ such that (2) and Condition A hold for δ/q_k instead of δ, $1 \leq k \leq r$. Let c and n be as in Condition A (with δ/q_k instead of δ). In view of (2) we may write $\mu^{(n)} = \mu_1 + \mu_2$, where μ_1 has compact support and for any $1 \leq k \leq r$,

$$\int_G \|g\|_k^\delta \, d\mu_2(g) < \frac{1-c}{3}. \tag{37}$$

Let A_1 denote the averaging operator

$$(A_1 f)(g) = \int_G f(g'g) \, d\mu_1(g').$$

Let $n > 0$ be as in Condition A, and for sufficiently small $\delta > 0$ and $g \in G$ consider the average $(A_1 u_k^\delta)(g)$. If for every h in the support of μ_1 the maximum in the definition of β_k (hence of u_k) is achieved by the same $\gamma \in \Gamma$, then, by Condition A we have, $(A_1 u_k)(g) \leq c u_k(g)$, with $c < 1$. If the maximum is achieved by different γ depending on the choice of g' in the (compact) support of μ_1, then (29) holds. Hence, in that case (33) also holds. Thus in all cases, we obtain the system of inequalities:

$$A_1 u_k^\delta \leq c u_k^\delta + C' \prod_{j \neq k} u_j^{\delta \lambda_{jk}} + B' \tag{38}$$

where $c < 1$, $C' < \infty$, $B' < \infty$ and we have used again the compactness of the support of μ_1. The additive constant B' arises because we have assumed that the u_i are bounded from below.

For $\epsilon > 0$, (38) may be rewritten as

$$A_1 (\epsilon u_k)^\delta \leq c (\epsilon u_k)^\delta + \epsilon_1 \prod_{j \neq k} (\epsilon u_j)^{\delta \lambda_{jk}} + B''$$

and in view of (35), $\epsilon_1 > 0$ can be made arbitrarily small by choosing $\epsilon > 0$ small enough. Note that by (35), Jensen's inequality and the fact that the u_j are bounded from below we have

$$\prod_{j \neq k} (\epsilon u_j)^{\delta \lambda_{jk}} \leq C_1 \sum_{j \neq k} (\epsilon u_j)^\delta.$$

Hence $u = \sum_k (\epsilon u_k)^\delta$ satisfies the inequality

$$A_1 u \leq c' u + b,$$

where $c' < 1 - \frac{(1-c)}{3}$ and $b > 0$. Now in view of (37), we have

$$A u \leq c_1 u + b \tag{39}$$

where $c_1 < 1$ and $(Af)(g) = \int_G f(g'g)\, d\mu^{(n)}(g')$.

□

4 Averaging operators

In this section we prove Proposition 2.7.

Lemma 4.1. *Suppose H is a semisimple algebraic subgroup of $GL(V)$ without compact factors, such that V does not have any H-invariant vectors. Suppose μ is a Zariski dense measure on H satisfying (2). Then, there exist $N > 0$ and $c > 0$ such that for all $n > N$ and all nonzero $v \in V$,*

$$\frac{1}{n}\int_H \log\frac{\|hv\|}{\|v\|}\, d\mu^{(n)}(h) > c$$

Proof. See [Bou, Chapter III, Corollary 3.4 p. 53-54]. See also [Furm] and the original paper [Furs] for closely related statements. $\square$

Lemma 4.2. *Let H, V and μ be as in Lemma 4.1, Then, for all sufficiently small $\delta > 0$ there exist c_1, $0 < c_1 < 1$ and $N > 0$ such that for all $n > N$ and all $v \in V$,*

$$\int_H \|hv\|^{-\delta}\, d\mu^{(n)}(h) < c_1\|v\|^{-\delta}. \tag{40}$$

Proof. Without loss of generality, we assume that $\|v\| = 1$. We first show that there exists $n > 0$ for which(40) holds. Let n, c be as in Lemma 4.1. Since H is semisimple, $H \subset SL(V)$. Hence there exists $k > 0$ depending only on the dimension of V, such that for $h \in H$, $\|h^{-1}\| \leq \|h\|^k$. Then, for all $\delta_0 > 0$,

$$\|hv\|^{-\delta_0} \leq \|h^{-1}\|^{\delta_0} \leq \|h\|^{k\delta_0}. \tag{41}$$

Now, using Taylor's theorem, for some $0 < \delta_0 < \delta$,

$$\|hv\|^{-\delta} = 1 - \delta\log\|hv\| + \frac{1}{2}\delta^2(\log\|hv\|)^2\|hv\|^{-\delta_0}$$

Then, using (41),

$$\|hv\|^{-\delta} \leq 1 - \delta\log\|hv\| + \frac{1}{2}\delta^2(\log\|h\|)^2\|h\|^{k\delta}$$

Hence, for sufficiently small $\delta > 0$, using Lemma 4.1,

$$\begin{aligned}\int_H \|hv\|^{-\delta}\, d\mu^{(n)}(h) &\leq 1 - \delta nc + \frac{1}{2}\delta^2\int_H (\log\|h\|)^2\|h\|^{k\delta}\, d\mu^{(n)}(h)\\ &\leq 1 - \delta nc + \delta^2 C_n\end{aligned}$$

where C_n is independent of δ, and in the last line we have used the condition (2). Now choose $\delta < C_n/(2c)$. Then (40) holds for n. To see that it holds for $m > n$, note that $\mu^{(m)} = \mu^{(n)} * \mu^{(m-n)}$ and $\mu^{(m-n)}$ is a probability measure. $\square$

Proof of Proposition 2.7. Let P_k, ρ_k, V_k, w_k be as in condition A. Let $V_k^{H_\mu}$ denote the H_μ-invariant subspace of V_k on which the action of H_μ is trivial, and let V_k' denote the complementary H_μ-invariant subspace of V_k. The assumption of Proposition 2.7 implies that Gw_k does not intersect $V_k^{H_\mu}$. Let π_k' denote the projection onto V_k'. Let K denote the maximal compact subgroup of G. Since $G = KP_k$, and P_k stabilizes the span of w_k, Gw_k is projectively compact. Hence there exists a constant $c_0 > 0$ such that for all $v \in Gw_k$, $\|\pi_k'(v)\| > c_0\|v\|$. Now the proposition follows from Lemma 4.2 applied to $\pi_k'(v)$. $\square$

5 Proof of Lemma 2.2

Let $X = G/\Gamma$, and let σ be a locally finite μ-stationary measure on X. Consider the space $G \times X$ with the measure $\mu \times \sigma$. Then, since σ is μ-stationary, for any (compact) $K \subset X$,

$$(\mu \times \sigma)\{(g, x) \;:\; gx \in K\} = \sigma(K).$$

Hence,

$$\int_X \mu\{\, g \;:\; gx \in K\}\, d\sigma(x) = \sigma(K)$$

Hence, for any compact subset C of X,

$$\int_C \mu\{\, g \;:\; gx \in K\}\, d\sigma(x) \leq \sigma(K)$$

Now after convolving, we see that for any $n > 0$,

$$\int_C \mu^{(n)}\{\, g \;:\; gx \in K\}\, d\sigma(x) \leq \sigma(K)$$

But by (R2), for n sufficiently large, for any $x \in C$, $\mu^{(n)}\{\, g \;:\; gx \in K\} \geq (1-\epsilon)$. Then,

$$\sigma(C) \leq \frac{\sigma(K)}{1-\epsilon}$$

Since C is arbitrary, we see that $\sigma(X) \leq \frac{\sigma(K)}{1-\epsilon}$. $\square$

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