QUASI-ISOMETRIC RIGIDITY OF NONUNIFORM LATTICES IN HIGHER RANK SYMMETRIC SPACES

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Abstract. We compute the quasi-isometry group of a nonuniform lattice in a semisimple Lie group with no rank one factors, and show that any two such lattices are quasi-isometric if and only if they are commensurable up to conjugation.

We recall the following:

Definition Let X and Y be metric spaces. A (κ, C) *quasi-isometric embedding* is a map $\phi : X \longrightarrow Y$, such that

$$\frac{1}{\kappa} d_X(x_1, x_2) - C \leq d_Y(\phi(x_1), \phi(x_2)) \leq \kappa d_X(x_1, x_2) + C \tag{1}$$

Thus ϕ is bi-Lipschitz on large scales.

If there exists a constant C' such that every point of Y is within a distance C' of a point of $\phi(X)$, then there is a quasi-isometric embedding $\psi : Y \to X$ which is a "coarse inverse of ϕ", i.e. $\sup_{x\in X} d_X(x, \psi(\phi(x))) < \infty$, and $\sup_{y\in Y} d_Y(y, \phi(\psi(y))) < \infty$. In this case ϕ is called a *quasi-isometry* and the spaces X and Y are said to be *quasi-isometric*.

A basic example to keep in mind is that the fundamental group $\pi_1(M)$ (endowed with the word metric) of a compact Riemannian manifold M is quasi-isometric to the universal cover $\widetilde{M}$ of M.

In this paper we prove the following theorem:

Theorem 0.1. *Let Γ be an irreducible nonuniform lattice in a semisimple Lie group G with finite center and no rank 1 factors. Let ϕ be a quasi-isometry of Γ to itself, in the word metric. Then there exists an element g in the commensurator of Γ so that ϕ is a bounded distance away from left multiplication by g. In particular, $QI(\Gamma) \cong Comm(\Gamma)$.*

Corollary 0.2. *Suppose Γ is an irreducible nonuniform lattice in a semisimple Lie group G with finite center and no rank 1 factors, and suppose Λ is a finitely generated group quasi-isometric to Γ. Then there exists a short exact sequence*

$$1 \to F \to \Lambda \to \Gamma' \to 1$$

where F is a finite group, and Γ' is a lattice commensurable to Γ up to conjugation. In particular, two nonuniform irreducible lattices in G are quasi-isometric if and only if they are commensurable up to conjugation.

We note that the rigidity result for non-uniform lattices is much stronger then the analogous result for uniform lattices (in the higher rank case the result for uniform lattices was first proved in [KL]; see also [EF] for another proof). This phenomenon was first noticed by Richard Schwartz

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who did much of the work on the subject, and conjectured Theorem 0.1 and Corollary 0.2 in full generality.

In [S1] Theorem 0.1 and Corollary 0.2 were proved using geometric methods in the case where the rank of G is 1, so that the symmetric space has negative curvature (the case $G = SL(2,\mathbb{R})$ has to be excluded). In [FS] the same result was proved for the case of quadratic Hilbert modular groups, and it was extended to arbitrary Hilbert modular groups in [S2]. The passage to higher rank and non-positive curvature required development of new methods, in particular the "coarse topology" and "coarse separation theorem" of [FS] and [S2]. Even though these methods are quite general, and have proved to be useful in many other questions of coarse geometry (for example [FM]), they do not seem to be suitable for the case when the $\mathbb{Q}$-rank of the lattice is bigger then 1, e.g. $SL(3,\mathbb{Z})$. Our proof is in fact more geometric then topological, and follows a different scheme from the earlier results.

The basic ingredients in our proof are the "quasi-flats with holes" result of [EF], the theorem of Lubotzky-Mozes-Raghunathan establishing the equivalence of the word metric with the induced metric, the Moore ergodicity theorem, and Tits's theory of buildings. The idea of the proof is to prove Theorem 0.1 without extending the map ϕ to the entire symmetric space. Instead (working in the induced metric), we use the quasi-flats with holes result and the Moore ergodicity theorem to construct a map ϕ_0 of the boundary defined almost everywhere. We then show that this map is bi-Hölder on its domain of definition. This allows us to extend ϕ_0 to a homeomorphism of the boundaries. We then show that this extended map induces an order preserving bijection of the associated Tits buildings, which implies by Tits' theorem that ϕ_0 is induced by an isometry. Then we show that ϕ is within a bounded distance of that isometry. A final argument shows that the element $g \in G$ inducing the isometry belongs in fact to the commensurator of Γ.

Most of the argument (through and including the proof of Proposition 6.1) does not use the assumption that G has no rank one factors but only the weaker assumption $\operatorname{rank} G \geq 2$. In the case when $\operatorname{rank} G \geq 2$ but G contains rank one factors, we write down an intermediate result, Proposition 10.1, which follows naturally from the argument in this paper. Proposition 10.1 can then be combined with results from [FS], [S1] and [S2] to prove the analogues of Theorem 0.1 and Corollary 0.2 for this case: see [Fa] for an outline of the argument.

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1. Preliminaries

Notational conventions. We denote by $\mu(\cdot)$ the Haar measure on $\Gamma\backslash G$, normalized so that $\mu(\Gamma\backslash G) = 1$. The Euclidean measure on $\mathbb{R}^n$ is denoted by ν or $|\cdot|$. We denote the measure of the unit ball in $\mathbb{R}^n$ by ω_n. The complement of a set U is denoted U^c.

Let π denote the natural projection from G to the symmetric space $G/K = X$, and let p denote the natural projection from G to $\Gamma\backslash G$. The natural projection from X to $\Gamma\backslash X$ is denoted by $\bar{p}$. Let $\bar{\mu}$ denote the measure on $\Gamma\backslash X$ which is the push-forward of μ; then $\bar{\mu}(\Gamma\backslash X) = 1$.

We denote the identity element of G by 1, and $\pi(1) \in X$ by e.

1.1. Theorems used in the proof.

The proof of Theorem 0.1 is based on the following results:

Theorem 1.1 (Lubotzky-Mozes-Raghunathan). *Let Γ be an irreducible lattice in a semisimple Lie group G, with $\operatorname{rank} G \geq 2$. Then the word metric on Γ is equivalent to the metric induced from the embedding of Γ in the symmetric space G/K.*

Proof. See [LMR]. □

Theorem 1.2 (Moore ergodicity). *Let Γ be an irreducible lattice in a semisimple Lie group G with finite center, $H \subset G$ a non-compact subgroup. Then the H action on $\Gamma\backslash G$ is ergodic.*

Proof. See [Zim, Theorem 2.2.6]. □

Theorem 1.3 (Birkhoff ergodic Theorem for $\mathbb{R}^n$ action). *Let (Y, m) be a finite measure space, and suppose $A \cong \mathbb{R}^n$ acts on Y, preserving m. If the action is ergodic, then for any $f \in L^1(Y)$ and almost all $y \in Y$,*

$$\lim_{r\to\infty} \frac{1}{\omega_n r^n} \int_{B_r} f(a \cdot y)\, d\nu(a) = \int_Y f\, dm$$

where $B_r = B(0, r)$ is the ball of radius r centered at 0.

Maximal flats and Cartan subgroups. The symmetric space X is a quotient G/K, where G is the isometry group of X and K is a maximal compact subgroup. The Killing form $\langle\cdot,\cdot\rangle$ is a nondegenerate quadratic form on the Lie algebra $\mathfrak{g}$ of G defined by $\langle X, Y\rangle = \mathrm{trace}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$. The Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement relative to the Killing form. We fix a maximal Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$, and denote $\exp \mathfrak{a}$ by A. The group A is Abelian and is called a *Cartan subgroup* of G; $\mathfrak{a}$ is a *Cartan subalgebra.* The dimension of $\mathfrak{a}$ is the *rank* of X (or of G), which we denote by n. For any $g \in G$, $\pi(gA) \subset X$ is a maximal flat subspace of X; also every maximal flat can be written as $\pi(gA)$ for some $g \in G$.

As an immediate corollary of Theorem 1.2 and Theorem 1.3 one has the following:

Theorem 1.4 (Birkhoff ergodic theorem for Cartan flow). *Let $G, \mathfrak{a}$ be as above. Then, for any $f \in L^1(\Gamma\backslash G)$ and for μ-a.e. $x \in \Gamma\backslash G$,*

$$\lim_{r\to\infty} \frac{1}{|B(0,r)|} \int_{B(0,r)\subset\mathfrak{a}} f(x \exp v)\, d\nu(v) = \int_{\Gamma\backslash G} f\, d\mu$$

Swiss cheeses. Let U be a subset of $\mathbb{R}^n$. We say that U is "an infinite swiss cheese" if it contains a fraction of the volume, i.e. there exists $\delta < 1$ so that for every $x \in U$,

$$\liminf_{r\to\infty} \frac{|U \cap B(x,r)|}{|B(x,r)|} \geq 1 - \delta \tag{2}$$

We note if the complement of U (i.e. the "holes") contains balls of arbitrarily large radius, the bound in (2) cannot be uniform in the choice of $x \in U$. Given $\epsilon > 0$ and $R > 0$, let

$$U_{(\epsilon,R)} = \{x \in U \;\;:\;\; \text{for all } r > R,\ |U \cap B(x,r)| \geq (1-\epsilon)|B(x,r)|\}$$

Hence, $U_{(\epsilon,R)}$ is a subset of U on which a uniform bound exists.

Theorem 1.5 (Quasi-flats with holes). *Suppose ϕ is a (κ, C) quasi-isometric embedding from $U \subset \mathbb{R}^n$ to X, where U has the induced Euclidean metric. Let $\epsilon > 0$ and $R > 0$ be constants, with ϵ sufficiently small (depending only on κ), and $U_{(\epsilon,R)}$ non-empty. Then $\phi(U_{(\epsilon,R)})$ lies in the N neighborhood of the union of at most M flats in X; here $M = M(\kappa)$ and $N = N(\kappa, \epsilon, R, C)$.*

Proof. See [EF, Theorem 8.1]. □

1.2. **Symmetric spaces and quasi-isometries.** Most of this subsection is standard: we review the results mostly to fix notation. In what follows, we follow the notation of [EF] where possible. Throughout, by a flat we mean a maximal flat.

Weyl chambers, the Weyl group and the longest element: We denote the root system associated to (G, A) by Σ, and fix a positive system Σ^+ and associated simple system Δ for Σ. Let $\mathfrak{a}^+$ denote the subset $\{H \in \mathfrak{a} \ : \ \alpha(H) > 0 \text{ for all } \alpha \in \Delta\}$; the set $\mathfrak{a}^+$ is called the *positive Weyl chamber*. Let $A^+ = \exp \mathfrak{a}^+ \subset A$. For any $g \in G$, the orbit gA^+ is a *Weyl chamber in* G; the set $\pi(gA^+)$ is a *Weyl chamber* in X.

Let M denote the centralizer of A in K. The Weyl group W is the quotient of the normalizer in K of A by M. This is a finite group which acts transitively on the Weyl chambers of A. Let $w_0 \in W$ denote the *longest element*; it takes $\mathfrak{a}^+$ into $-\mathfrak{a}^+$.

Subflats, hyperplanes, polyhedra and orbits of subgroups of A**:** For a subset σ of Σ, let

$$A_\sigma = \{a \in A \mid \alpha(\log a) = 0, \quad \forall \alpha \in \sigma\}$$

If $\sigma \subset \Delta$, let

$$A_\sigma^+ = \{a \in A \ : \ \alpha(\log a) = 0, \quad \forall \alpha \in \sigma, \qquad \text{and } \alpha(\log a) > 0, \quad \forall \alpha \in \Delta - \sigma.\}$$

We call the orbits of A_σ "σ-subflats in G" (or just subflats) and the orbits of A_σ^+ "σ-walls" (or just walls) in G. If $\sigma = \emptyset$ then the orbit of $A_\emptyset^+$ (denoted A^+) is a *Weyl chamber* in G; if $\sigma = \{\alpha\}$ we write A_α instead of $A_{\{\alpha\}}$ and call the orbit a *hyperplane* in G. By a subflat, wall, Weyl chamber, or hyperplane in X we mean the projection to X of the corresponding object in G via π.

It is easy to check that the intersection of two flats is either empty, or a point or a subflat. Conversely, for every subflat $S \subset A$ passing through the identity 1 there exists $k_S \in K$ such that $k_S A k_S^{-1} \cap A = S$.

By a *polyhedron* we mean a (possibly unbounded) convex subset of a flat bounded by hyperplanes. A flat is also considered a polyhedron. If U is a polyhedron in X, we denote by ∂U its boundary as a subset of $\mathbb{R}^n$, not as a subset of X (otherwise we will always have $\partial U = U$).

The Furstenberg boundary $\hat{X}$ **and the Tits building** $\mathcal{T}(X)$**:** The Furstenberg boundary $\hat{X}$ is defined as the equivalence classes of Weyl chambers in X, with two chambers considered equivalent if the Hausdorff distance between them is finite. Each Weyl chamber is equivalent to a Weyl chamber with origin at the point e. Since every Weyl chamber with origin at e is of the form $\pi(kA^+)$ for some $k \in K$, this identifies $\hat{X}$ with K/M.

If $\sigma \subset \Delta$, we denote by P_σ the standard parabolic subgroup associated to σ. We have the Langlands decomposition $P_\sigma = M_\sigma A_\sigma N_\sigma$, where A_σ is as above, M_σ is semisimple and N_σ is unipotent. Write $K_\sigma = M_\sigma \cap K$. $P_\emptyset$ is a *Borel* subgroup of G, denoted by B.

Any parabolic subgroup is of the form $gP_\sigma g^{-1}$ for some $\sigma \subset \Sigma$ and some $g \in G$. The *Tits building* $\mathcal{T}(X)$ is the set of parabolic subgroups of G, partially ordered by reverse inclusion. Hence the maximal dimension simplices of $\mathcal{T}(X)$ correspond to conjugates of B. Since B is its own normalizer, the set of conjugates of B is $G/B \cong K/M$, where the last identification used the Iwasawa decomposition $G = KB$, and the fact that $M = K \cap B$.

Thus every maximal dimension simplex of $\mathcal{T}(X)$ corresponds to a point in K/M. The same argument shows that every simplex D in $\mathcal{T}(X)$ corresponds to a point in K/K_σ, with $\dim(D) = n - |\sigma|$.

This description of the $\mathcal{T}(X)$ is convenient for the argument in this paper. As in [EF] we denote the K-invariant distance on K/M by d_K.

Galleries, combinatorial distance and the singular subset. Each pair of n-simplices in the Tits building lies in an apartment. A *gallery* is a sequence of n-simplices, with successive simplices sharing a codimension 1 face. By the Tits *combinatorial distance* $d_c(k_1, k_2)$ we mean the length of a minimal gallery in the Tits building connecting the n-simplices $k_1 M$ and $k_2 M$. This implies that $\sup_{k_1,k_2} d_c(k_1,k_2) < \infty$. Let $\ell = \max_{k_1,k_2} d_c(k_1,k_2)$, and as in [EF], let $\mathcal{S} = \{k \in K \ : \ d_c(1,k) < \ell\}$. The *singular subset* $\mathcal{S} \subset K$ is closed, and $\mathcal{S} = K \cap \bigcup_{w \in W - \{w_0\}} BwB$, where B is the Borel subgroup. This exhibits $\mathcal{S}$ as a union of cells, of varying dimensions.

Graded quasi-isometric embeddings. If X_1 and X_2 are metric spaces, and $z_0 \in X_1$, we say that a map $q : X_1 \to X_2$ is a (κ,ϵ,ρ) *graded quasi-isometric embedding centered at* z_0 if for all $x, y \in X_1$ with $d(y,z_0) \geq d(x,z_0)$,

$$\frac{1}{\kappa} d(x,y) - \max(\rho, \epsilon d(x,z_0)) \leq d(q(x),q(y)) \leq \kappa d(x,y) + \max(\rho, \epsilon d(x,z_0)) \tag{3}$$

Thus any (κ,ρ) quasi-isometric embedding is also a (κ,ϵ,ρ) graded quasi-isometric embedding. Graded quasi-isometric embeddings are more general in the sense that the "coarseness constant" is allowed to grow linearly with distance to the origin z_0.

Graded q.i. embeddings (c.f [EF, §3.1]) are a useful technical device both for the proof of Theorem 1.5 and for some of the arguments below. Two (κ,ϵ,ρ) graded q.i. embeddings q_1 and q_2 centered at the same point z_0 are *equivalent* if for all $x \in X_1$, $d(q_1(x), q_2(x)) \leq \max(\rho, \epsilon d(x,z_0))$.

Neighborhoods, Conical neighborhoods.

We denote the r-tubular neighborhood of a set U by $\mathrm{Nbhd}(U,r)$. The *metric interior* $\mathrm{Int}(U,r)$ is defined to be $U \setminus \mathrm{Nbhd}(\partial U, r)$.

Let U be a subset of a metric space Y, and $y_0 \in Y$ a point. We let $U[c;y_0]$ be a neighborhood of U whose size is proportional to the distance from y_0, namely:

$$U[c;y_0] = \{y \in Y \ : \ \exists u \in U \text{ with } d(y,u) < c d(y,y_0)\}$$

As in [EF], if $U \subset \mathbb{R}^n$, we write $U[c]$ as shorthand for $U[c;0]$, and if $U \subset X$, we write $U[c]$ as shorthand for $U[c;e]$.

Hausdorff distance and ϵ-equivalence: If U, V are subsets of a metric space, we denote the Hausdorff distance between their closures by $hd(U,V)$. If $hd(U,V) < \infty$, we say that U is *equivalent* to V and write $U \sim V$.

Let x_0 be a point and let $A_R(x_0)$ denote the annulus $\{x \ : \ R < d(x,x_0) < 2R\}$. We say that two subsets U and V are *ϵ-equivalent* and write $U \sim_\epsilon V$ if for all sufficiently large R,

$$hd(U \cap A_R(x_0), V \cap A_R(x_0)) \leq \epsilon R. \tag{4}$$

(clearly this is meaningful only if ϵ is small).

Because of the triangle inequality, the notion of ϵ-equivalence is (almost) independent of the point x_0. Indeed, if $U \sim_\epsilon V$ with origin x_0, and z_0 is any other point, then $U \sim_\delta V$ with origin z_0 for any $\delta > \epsilon$.

We will occasionally need a more quantitative version. We write $U \sim_{\epsilon,x_0,\rho} V$ if for all $R > \rho$, (4) holds. It is easy to check that if $U \sim_{\epsilon,x,\rho} V$ then outside the ball $B(x,\rho)$

$$U \subset V[2\epsilon;x] \text{ and } V \subset U[2\epsilon;x]. \tag{5}$$

Conversely, if (5) holds outside the ball $B(x,\rho)$, then $U \sim_{4\epsilon,x,\rho} V$.

Multiplicative constants depending only κ and X: As in [EF], the argument produces a proliferation of constants depending only on κ and X. (Essentially there is an extra factor of κ each time (1) or (3) is used). Thus we use the $\gg$, $\ll$, $\prec$, $\succ$ and $O(\cdot)$ notations in the same way as defined in [EF, §3.1].

1.3. **Structure of the proof.** The proof of Theorem 0.1 is a succession of steps; in each step we gain more control over the map ϕ. Each step is proved in its own section; only the results stated in the introductory subsection of each section are used in subsequent sections.

We now begin the proof of Theorem 0.1. In view of Theorem 1.1, we may assume that ϕ is a (κ, C) quasi-isometry $\Gamma \to \Gamma$ in the *induced* metric.

Step 1 (Extending ϕ to the "neutered space")**.** For any compact subset $\mathcal{C}$ of $\Gamma\backslash X$, ϕ can be extended to a (κ, C') quasi-isometry of $\bar{p}^{-1}(\mathcal{C})$ to $\bar{p}^{-1}(\mathcal{C})$. Here C' depends on κ, C and $\mathcal{C}$.

Proof. This construction is now standard; see e.g. [S1], [FS], [S2]. Since for every point y of $p^{-1}(\mathcal{C})$ is within a uniformly bounded distance of a point in Γ, one may just define $\phi(y)$ to be $\phi(\gamma)$ where γ is the point in Γ closest to y, (one chooses randomly if there are several possibilities for γ). The resulting map is easily verified to be a quasi-isometry. $\square$

For notational convenience, we define ϕ on $\pi^{-1}(\bar{p}^{-1}(\mathcal{C})) \subset G$ by making ϕ constant along the K fibers.

Choice of $\mathcal{C}$ and ϵ. We choose $\epsilon > 0$ so that Theorem 1.5 holds, and also so that $\epsilon \ll 1$, i.e. for any multiplicative constant $\lambda = \lambda(\kappa)$ which arises in §§2-§7 $\lambda\epsilon < 1/100$. Note that ϵ depends only on κ.

We now choose $\mathcal{C}$ so that $\bar{\mu}(\mathcal{C}) > 1 - \epsilon/4$. The set $\mathcal{C}$ and the constant ϵ are fixed throughout the argument.

2. Quasi-flats with holes and the Moore ergodicity theorem

Definition 2.1 (Γ-invariant family of flats)**.** A family of $\mathcal{F}$ of flats is Γ-*invariant* if for every $\gamma \in \Gamma$ and every flat $F \in \mathcal{F}$, $\gamma F \in \mathcal{F}$. A Γ-invariant family $\mathcal{F}$ of flats is said to have *full measure* if there exists a set $E \subset \Gamma\backslash G$ with $\mu(E) = 1$ such that for any $y \in p^{-1}(E)$, $\pi(yA) \in \mathcal{F}$.

The approach in this paper is based on the following lemma, which follows trivially from the ergodic theorem Theorem 1.4:

Lemma 2.2. *There exists constants $R > 0$, $\rho > 0$, a family $\mathcal{F}$ of flats in X and for each $F \in \mathcal{F}$ subsets $\Omega_F \subset \Omega'_F \subset F$ with the following properties:*

(a) *$\mathcal{F}$ is a Γ-invariant family of full measure.*
(b) *For every $F \in \mathcal{F}$, $U_F = F \cap \bar{p}^{-1}(\mathcal{C})$ is not empty. In fact, for any $x \in F$,*

$$\liminf_{r\to\infty} \frac{|U_F \cap B(x,r)|}{|B(x,r)|} \geq 1 - \frac{\epsilon}{4}$$

The quasi-isometry ϕ is defined on U_F.

(c) *For every $F \in \mathcal{F}$, $\Omega'_F \subset (U_F)_{(\epsilon,R)}$ is not empty. In fact, for any $x \in F$,*

$$\liminf_{r\to\infty} \frac{|\Omega'_F \cap B(x,r)|}{|B(x,r)|} \geq 1 - \frac{\epsilon}{2} \tag{6}$$

where the notation $(U_F)_{(\epsilon,R)}$ is as in Theorem 1.5.

(d) *For every $F \in \mathcal{F}$, $\Omega_F \subset (\Omega'_F)_{(\epsilon,\rho)}$ is not empty. In fact for any $x \in F$,*

$$\liminf_{r\to\infty} \frac{|\Omega_F \cap B(x,r)|}{|B(x,r)|} \geq 1 - \frac{\epsilon}{2}$$

Remark. From (c) and Theorem 1.5, it follows that for all $F \in \mathcal{F}$, $\phi(\Omega'_F)$ is close to a finite union of flats. Thus, in view of (6) we have information about the image under ϕ of "most of the volume in almost all flats".

We need (d) for a technical reason which will become apparent shortly.

Proof of Lemma 2.2. Let χ denote the characteristic function of the set $\pi^{-1}(\mathcal{C}) \subset \Gamma\backslash G$. Then, in our normalizations, $\int_{\Gamma\backslash G} \chi \, d\mu \geq 1 - \epsilon/4$. Then by Theorem 1.4, there exists a set $E' \subset \Gamma\backslash G$ such that $\mu(E') = 1$, and for all $x \in E'$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \chi(x \exp v) \, d\nu(v) \geq 1 - \epsilon/4.$$

where B_r denotes the ball $B(0,r) \subset \mathfrak{a}$.

Suppose $y \in p^{-1}(E')$, and let $F = \pi(yA)$. Note that for $v \in \mathfrak{a}$,

$$\pi(y \exp v) \in U_F = F \cap \bar{p}^{-1}(\mathcal{C}) \quad \text{precisely when} \quad \chi(y \exp v) = 1. \tag{7}$$

Thus if we let $\mathcal{F}_1 = \{\pi(yA) \ : \ y \in p^{-1}(E')\}$, (a)-(b) of Lemma 2.2 hold for $\mathcal{F}_1$ instead of $\mathcal{F}$.

To get (c) we may argue as follows: For any $j > 0$, let

$$E'_j = \left\{ x \in E' \ : \ \text{for all } r > j, \ \frac{1}{|B_r|} \int_{B_r} \chi(x \exp v) \, d\nu(v) \geq 1 - \epsilon/2 \right\} \tag{8}$$

Then E'_j is an increasing family of sets and $\bigcup_{j=1}^{\infty} E'_j = E'$. Thus, there exists an integer R such that $\mu(E'_R) \geq 1 - \epsilon/4$. Let $E_R = E'_R \cap \pi^{-1}(\mathcal{C})$; then $\mu(E_R) \geq 1 - \epsilon/2$.

Note that if $y \in p^{-1}(E')$ and $F = \pi(yA)$, then in view of (7) and (8), $\pi(yA \cap p^{-1}(E_R)) \subset (U_F)_{(\epsilon,R)}$. Thus we let $\Omega'_F = \pi(yA \cap p^{-1}(E_R))$.

Let χ_R denote the characteristic function of E_R. Then by the ergodic theorem Theorem 1.4, there exists a set $E'' \subset E'$ with $\mu(E'') = 1$ such that for $x \in E''$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \chi_R(x \exp v) \, d\nu(v) = \mu(E_R) \geq 1 - \epsilon/2 \tag{9}$$

Thus we can let $\mathcal{F}_2 = \{\pi(yA) \ : \ y \in p^{-1}(E'')\}$. Since $\mu(E'') = 1$, part (a) holds for $\mathcal{F}_2$. Since $\mathcal{F}_2 \subset \mathcal{F}_1$, part (b) also holds. Finally part (c) holds for $\mathcal{F}_2$ in view of (9).

To get (d) we repeat the argument for (c). Let

$$E''_j = \left\{ x \in E'' \ : \ \text{for all } r > j, \ \frac{1}{|B_r|} \int_{B_r} \chi_R(x \exp v) \, d\nu(v) \geq 1 - \epsilon \right\} \tag{10}$$

Then E''_j is an increasing family, with $\bigcup_{j=1}^{\infty} E''_j = E''$. Thus there exists an integer ρ such that $\mu(E''_\rho) \geq 1 - \epsilon/4$. Let $\Omega = E''_\rho \cap \pi^{-1}(\mathcal{C})$, then $\mu(\Omega) \geq 1 - \epsilon/2$. For $F = \pi(yA) \in \mathcal{F}_2$, we set $\Omega_F = \pi(yA \cap \Omega)$. Then in view of (10), $\Omega_F \subset (\Omega'_F)_{(\epsilon,\rho)}$. Let χ_Ω denote the characteristic function of Ω. Thus $\pi(y \exp v) \in \Omega_F$ precisely when $p(y \exp v) \in \Omega$.

By Theorem 1.4, there exists a subset $E''' \subset E''$ with $\mu(E''') = 1$ such that for any $x \in E'''$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \chi_\Omega(x \exp v) \, d\nu(v) = \mu(\Omega) \geq 1 - \epsilon/2 \tag{11}$$

Thus we can let $\mathcal{F} = \{\pi(yA) \ : \ y \in p^{-1}(E''')\}$. Part (a) holds since $\mu(E''') = 1$, parts (b)-(c) hold since $\mathcal{F} \subset \mathcal{F}_2$, and part (d) holds because of (11). □

Combining Lemma 2.2 with Theorem 1.5 we get the following:

Step 2 (ϕ maps almost all flats to finite unions of flats)**.** There exist constants $M = M(\kappa)$, $\rho = \rho(\kappa, \epsilon, C)$ and $N = N(\kappa, \epsilon, C, \rho)$, a family of flats $\mathcal{F}$, and for each $F \in \mathcal{F}$ nonempty subsets $\Omega_F \subset \Omega'_F \subset F$, such that the following holds:

(a) $\mathcal{F}$ is a Γ-invariant family of full measure.
(b) For every $F \in \mathcal{F}$, there exist at most M flats $F'_1, \dots, F'_M$ such that $\phi(\Omega'_F)$ lies within a distance N of $\bigcup_{j=1}^M F'_j$.
(c) For every $F \in \mathcal{F}$, every $x \in \Omega_F$, and every $r > \rho$,

$$\frac{|B(x,r) \cap \Omega'_F|}{|B(x,r)|} \geq 1 - \epsilon \tag{12}$$

Hence there exists a map $\tilde{\phi} : F \to \mathrm{Nbhd}(\bigcup_{j=1}^M F'_j, N)$, which agrees with ϕ on Ω'_F, and is a $(\kappa, 2\kappa\epsilon, \rho)$ graded quasi-isometric embedding centered at x for any $x \in \Omega_F$.
(d) For every $F \in \mathcal{F}$, and any $x \in F$,

$$\liminf_{r\to\infty} \frac{|\Omega_F \cap B(x,r)|}{|B(x,r)|} \geq 1 - \epsilon \tag{13}$$

(e) There exists a subset $\Omega \subset \Gamma\backslash G$ with $\mu(\Omega) \geq 1 - \epsilon/2$ such that for any $x \in p^{-1}(\Omega)$, $F = \pi(xA) \in \mathcal{F}$ and $\pi(x) \in \Omega_F$.

The map $\tilde{\phi}$ of part (c) is just the map obtained by $\tilde{\phi}(u) = \phi(v)$ where v a point of Ω'_F nearest u (c.f. Step 1). In view of (12), $\tilde{\phi}$ is indeed a $(\kappa, 2\kappa\epsilon, \rho)$ graded q.i. embedding.

The set Ω of part (e) is the same as in the proof of Lemma 2.2. Part (e) will be used in §7 to "return from the boundary".

3. Mapping flats to flats

Step 3 (ϕ maps most flats to single flats)**.** There exists a full-measure Γ-invariant family of flats $\mathcal{F}^* \subset \mathcal{F}$ such that for every $F \in \mathcal{F}^*$ the following hold:

(i) Parts (b) and (c) of Step 2 hold with $M = 1$; i.e. there exists a flat F' such that $\phi(\Omega'_F) \subset \mathrm{Nbhd}(F', N)$. (the sets Ω_F, Ω'_F and the constant N are the same as in Step 2).
(ii) There exists a constant $\lambda = \lambda(\kappa)$ such that for every Weyl chamber $\mathfrak{C} \subset F$ there exists a chamber $\mathfrak{C}' \subset F'$ such that $\phi(\mathfrak{C} \cap \Omega'_F) \sim_{\lambda\epsilon} \mathfrak{C}'$.

Remark. The proof of this step is somewhat involved.

Transverse Flats. For each $\alpha \in \Sigma$, we pick $k_\alpha \in K$ such that:

$$k_\alpha A k_\alpha^{-1} \cap A = A_\alpha = \{a \in A \ : \ \alpha(a) = 0\}.$$

We fix the k_α for the remainder of the paper. Now if $F = \pi(zA)$ is a flat, and $x = \pi(za) \in F$, then we define

$$\mathrm{trans}_\alpha(x; z) = \pi(zak_\alpha A)$$

Then $F_\alpha = \mathrm{trans}_\alpha(x; z)$ is a flat, which intersects F in the hyperplane $L_\alpha = \pi(zaA_\alpha)$ passing through x. Let

$$\mathrm{Trans}_F(x) = \{\mathrm{trans}_\alpha(x; z) \ : \ \alpha \in \Sigma, \text{ and } F = \pi(zA)\}$$

We call any flat $F_0 \in \mathrm{Trans}_F(x)$ a *transverse* flat to F at x. The name is motivated by the following lemma:

Lemma 3.1 (Transverse flats)**.** *Suppose F is a flat, $x \in F$, and $F_0 \in \mathrm{Trans}_F(x)$. Then*

(a) *There exists a hyperplane $L \subset F$ with $x \in L$ and $F \cap F_0 = L$.*
(b) *There exists constants λ_0, λ_1 depending only on X (and the choice of the k_α) such that for $r > \lambda_0$,*

$$hd\big(F \cap \mathrm{Nbhd}(F_0, r), L\big) \leq \lambda_1 r$$

This lemma expresses in a quantitative way the fact that the flats F and F_0 come close only near L. Part (a) follows from the definition of k_α. Part (b) is essentially the fact that geodesics in F_1 and F_2 emanating from a point of L and orthogonal to L diverge; cf. the discussion at the beginning of §5. □

The sets Ω_z^* and Ω_F^*: Suppose $F = \pi(zA) \in \mathcal{F}$. Let

$$\Omega_z^* = \{x \in \Omega_F \ : \ \mathrm{trans}_\alpha(x; z) \in \mathcal{F} \text{ and } x \in \Omega_{\mathrm{trans}_\alpha(x;z)} \text{ for all } \alpha \in \Sigma\} \tag{14}$$

Hence if $x \in \Omega_z^*$, then for every hyperplane L passing through x there exists a flat $F_L \in \mathrm{Trans}_F(x)$ such that $x \in \Omega_{F_L}$.

As we defined it, the set Ω_z^* depends on z, and not only on $F = \pi(zA)$. Even though it is not essential for the argument, to simplify some notation we set

$$\Omega_F^* = \bigcup_{\{z \text{ with } \pi(zA)=F\}} \Omega_z^*$$

Thus $\Omega_F^* \subset \Omega_F \subset \Omega_F' \subset F$.

Lemma 3.2 (the family $\mathcal{F}^*$). *There exists a full-measure Γ-invariant family $\mathcal{F}^* \subset \mathcal{F}$ such that for any $F \in \mathcal{F}^*$ and any $x \in F$,*

$$\liminf_{r\to\infty} \frac{|\Omega_F^* \cap B(x,r)|}{|B(x,r)|} \geq 1 - |\Sigma|\epsilon$$

Proof. Let

$$\Omega^* = \bigcap_{\alpha\in\Sigma} \Omega k_\alpha^{-1}$$

where Ω is as in the proof of Lemma 2.2. Then $\mu(\Omega^*) \geq 1 - |\Sigma|\epsilon$. Suppose $z \in G$. Note that if $p(za) \in \Omega^*$ for some $a \in A$, then $p(zak_\alpha) \in \Omega$ for all $\alpha \in \Sigma$, hence $x = \pi(za) = \pi(zak_\alpha) \in \Omega_{\pi(zak_\alpha A)} = \Omega_{\mathrm{trans}_\alpha(x;z)}$ for all $\alpha \in \Sigma$. Hence:

$$x = \pi(za) \in \Omega_z^* \text{ if } p(za) \in \Omega^* \tag{15}$$

Let χ_{Ω^*} be the characteristic function of Ω^*. By the ergodic theorem Theorem 1.4, there exists a set $E^* \subset E''' \subset \Gamma\backslash G$ with $\mu(E^*) = 1$ (where E''' is as in the proof of Lemma 2.2) such that for all $x \in E^*$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \chi_{\Omega^*}(x \exp v)\, d\nu(v) = \mu(\Omega^*) \geq 1 - |\Sigma|\epsilon \tag{16}$$

Let $\mathcal{F}^* = \{\pi(zA) \ : \ z \in p^{-1}(E^*)\}$. Then for $F \in \mathcal{F}^*$ Lemma 3.2 holds in view of (15) and (16). □

Step 3 will follow from the following:

Proposition 3.3. *Let $\mathcal{F}^*$ be the family defined in Lemma 3.2. Then for every flat $F \in \mathcal{F}^*$, (i) and (ii) of Step 3 hold.*

The proof of Proposition 3.3, which involves no ergodic theory, occupies the rest of this section. Along the way, we will prove the following result, which will be used in §5:

Proposition 3.4. *There exist constants $\lambda = \lambda(\kappa)$ and $\rho' = \rho'(\kappa, C, \epsilon, X)$ such that the following holds: Suppose $F \in \mathcal{F}^*$, and $x \in \Omega_F^*$. Then:*

(a) *For every hyperplane $L \subset F$ passing through x there exists a hyperplane L' such that $\tilde\phi(L) \sim_{\lambda\epsilon, \tilde\phi(x), \rho'} L'$.*
(b) *For every chamber $\mathfrak{C} \subset F$ based at x there exists a chamber $\mathfrak{C}'$ such that $\tilde\phi(\mathfrak{C}) \sim_{\lambda\epsilon, \tilde\phi(x), \rho'} \mathfrak{C}'$.*

3.1. **Intersections of quasi-flats.** Our aim in this subsection is to prove the following:

Lemma 3.5 (Hyperplanes map to finite unions of hyperplanes). *Suppose $F \in \mathcal{F}^*$, $x_0 \in F$ and R is sufficiently large (depending on F). Let $A_R = A(R, 2R; \phi(x_0)) \subset X$, Then there exists $M' > 0$ (depending only on κ, X) such that for any hyperplane L in F, there exist at most M' hyperplanes $L'_1 \cdots L'_{M'}$ such that*

$$\tilde{\phi}(L) \cap A_R \subset \mathrm{Nbhd}\left(\bigcup_{j=1}^{M'} L'_j, O(\epsilon R)\right) \cap A_R$$

The proof uses the following lemma:

Lemma 3.6 (Quasi-flats are unions of polyhedra). *Suppose $F \in \mathcal{F}$, $x \in \Omega_F$ and $R > \lambda' \rho$ (where ρ is as in Step 2 and $\lambda' = \lambda'(\kappa)$). Let B_R denote $B(\phi(x), R)$. Then there exist $m = m(\kappa)$ and $\lambda = \lambda(\kappa)$ and convex polyhedra $\mathcal{P}_1, \ldots, \mathcal{P}_m$ such that*

$$hd\left(\tilde{\phi}(F) \cap B_R, \bigcup_{j=1}^{m} \mathcal{P}_j \cap B_R\right) < \lambda \epsilon R$$

Remark. If instead of the assumption $R \gg \rho$ we made the assumption "R sufficiently large depending on F", Lemma 3.6 would follow from [EF, Lemma 8.3] (restated as Lemma 3.11 below). For the case of non-graded quasi-isometries, intersecting with a ball is not necessary. Thus Lemma 3.6 can be viewed as a strengthening of [EF, Lemma 8.3] in the sense that a the Hausdorff distance from a quasi-flat to some explicitly given set is bounded only in terms of the quasi-isometry constants.

Proof of Lemma 3.6. By Step 2 we have a $(\kappa, 2\kappa\epsilon, \rho)$ graded quasi-isometry centered at x, $\tilde{\phi} : F \to \mathrm{Nbhd}(\bigcup_{j=1}^{M} F'_j, N)$.

Fix $\eta'' \gg \eta \gg \eta' \gg 1$. In the proof of this lemma, all implicit constants depend only on κ and X, and the dependence on η'', η and η' is shown explicitly.

Let λ_0 be as in Lemma B.1 part (i). For each pair (i, j) there is an interval $I_{ij} = [d(F'_i, F'_j), \lambda_0(1 + d(F'_i, F'_j))]$ for which Lemma B.1 cannot be applied. Since the number of flats M is $O(1)$ (i.e depends only κ and X), we may choose $r_0 = O(\eta'' \epsilon R)$ such that $r_0 \gg \epsilon R$ and $[r_0, \eta'' r_0]$ does not intersect any I_{ij}. For example, $r_0 = r_0(\eta'' \epsilon R)$ can be constructed as follows: Denote by I the union of the half-open intervals $(\frac{1}{\eta''} d(F'_i, F'_j), \lambda_0(1 + d(F'_i, F'_j))]$. Then let $r_0(t)$, $t \in (0, \infty)$ be a piecewise linear function (with slope 1) continuous from the right, and with jumps such that the range of $r_0(t)$ is $(0, \infty) \setminus I$. Note that $t \le r_0(t) \le \lambda_1 (\eta'')^{M(M-1)/2} t$ where λ_1 depends only on λ_0 and M.

For $r_0 < r < \eta'' r_0$, let $\mathcal{P}_{ij}(r)$ be as in Lemma B.1 part (i) with $F_1 = F'_i$, $F_2 = F'_j$ if $r_0 > \lambda_0(1 + d(F'_i, F'_j))$ and let $\mathcal{P}_{ij}(r) = \emptyset$ if $d(F'_i, F'_j) > \eta'' r_0$. Note that $\mathcal{P}_{ii}(r) = F'_i$.

For $r \in [r_0, \eta r_0]$, $s \in [r_0, \eta r_0]$, $s < r$ and $\sigma \subset \{1, \ldots, M\}$ let

$$\mathcal{P}_{i,\sigma}(s, r) = \bigcap_{j \in \sigma} \mathcal{P}_{ij}(s) \bigcap_{j \notin \sigma} \mathcal{P}_{ij}(r)^c$$

Note that

$$F'_i = \bigcup_{\sigma \subset \{1, \ldots, M\}} \mathcal{P}_{i,\sigma}(s, r) \setminus \bigcup_{j=1}^{M} (\mathcal{P}_{ij}(r) \setminus \mathcal{P}_{ij}(s))$$

From its definition $\mathcal{P}_{ij}(r) \setminus \mathcal{P}_{ij}(s)$ is contained in a union of neighborhoods of hyperplanes, each of width $O(\eta r_0)$. Hence, for each i,

$$hd\left(F'_i, \bigcup_{\sigma\subset\{1,\ldots,M\}} \mathcal{P}_{i,\sigma}(s,r)\right) = O(\eta r_0) \tag{17}$$

For any $a < b$, let

$$F'_{i,\sigma}(a,b) = \{x \in F'_i \ : \ d(x, F'_j) \le a, \quad \forall j \in \sigma \qquad \text{and} \qquad d(x, F_j) > b, \quad \forall j \notin \sigma\} \tag{18}$$

Note that $F'_{i,\sigma} = \emptyset$ if $i \notin \sigma$. Let λ_1, λ_2 be as in Lemma B.1 part (i). Then, by Lemma B.1,

$$\bigcap_{j\in\sigma} \mathcal{P}_{ij}(\lambda_1 a) \bigcap_{j\notin\sigma} \mathcal{P}_{ij}(\lambda_2 b)^c \subset F'_{i,\sigma}(a,b) \subset \bigcap_{j\in\sigma} \mathcal{P}_{ij}(\lambda_2 a) \bigcap_{j\notin\sigma} \mathcal{P}_{ij}(\lambda_1 b)^c$$

Hence,

$$\mathcal{P}_{i,\sigma}(\lambda_1 a, \lambda_2 b) \subset F'_{i,\sigma}(a,b) \subset \mathcal{P}_{i,\sigma}(\lambda_2 a, \lambda_1 b) \tag{19}$$

The hyperplanes bounding $\mathcal{P}_{i,\sigma}(s,r)$ subdivide F'_i into convex polyhedra; for each pair (s,r) there is a certain combinatorial pattern (i.e which hyperplanes bound which convex polyhedra). The possible number of patterns is bounded depending only on X, and it is easy to see that the set of (s,r) which give rise to a fixed pattern is a convex subset of $\mathbb{R}^2$. Thus we have a decomposition of the square $[r_0, \eta r_0]^2$ into a finite number of disjoint convex sets. Hence, if $\eta \gg \eta'$ is sufficiently large, one of the sets must contain a box of side length $\eta r_0/\eta'$ i.e. there exists $(s_1, r_1) \in [r_0, \eta r_0]^2$ such that the combinatorial pattern is the same for all $(s,r) \in [s_1/\eta', s_1] \times [r_1/\eta', r_1]$. Without loss of generality, we may assume that $r_1 > s_1$. Hence,

$$hd(\mathcal{P}_{i,\sigma}(s,r), \mathcal{P}_{i,\sigma}(s_1,r_1)) = O(r_1) \qquad \text{for all } (s,r) \in [s_1/\eta', s_1] \times [r_1/\eta', r_1]. \tag{20}$$

Suppose $a \in [2s_1/(\lambda_1\eta'), s_1/(2\lambda_2)]$ and $b \in [2r_1/(\lambda_1\eta'), r_1/(2\lambda_2)]$ Then by (20) and (19)

$$hd(F'_{i,\sigma}(a,b), \mathcal{P}_{i,\sigma}(s_1,r_1)) = O(r_1) = O(\eta r_0) \tag{21}$$

By (18), for any $i, j \in \sigma$, $F'_{i,\sigma}(a,b) \subset \mathrm{Nbhd}(F'_{j,\sigma}(2a, b-a), a)$. Hence, by (21),

$$hd(\mathcal{P}_{i,\sigma}(s_1,r_1), \mathcal{P}_{j,\sigma}(s_1,r_1)) = O(r_1) = O(\eta r_0) \quad \text{for all } i,j \in \sigma \tag{22}$$

Write $\mathcal{P}_\sigma = \mathcal{P}_{i,\sigma}(s_1,r_1)$ where $i = \min_{j\in\sigma} j$. Hence, combining (22) and (17),

$$hd\left(\bigcup_{j=1}^{M} F'_j, \bigcup_{\sigma\subset\{1,\ldots,M\}} \mathcal{P}_\sigma\right) = O(\eta r_0) \tag{23}$$

Hence (after changing $\tilde{\phi}$ by $O(\eta r_0)$) we may view $\tilde{\phi}$ as an (κ, C') quasi-isometry $\tilde{\phi} : B \to \bigcup \mathcal{P}_\sigma$, where $C' = O(\eta r_0)$ and B is a ball containing $\tilde{\phi^{-1}}(B_R)$. We would like to apply Lemma A.3; we now verify that for any $r \gg \eta r_0$ the estimate (46) is satisfied for (the convex pieces of) the $\mathcal{P}_\sigma$, with C' in place of C. First suppose σ and τ are not disjoint; choose $j \in \sigma \cap \tau$. By (22), the Hausdorff distance from $\mathcal{P}_\sigma$ to $\mathcal{P}_{j,\sigma}(s_1,r_1)$ is $O(\eta r_0)$, and the Hausdorff distance from $\mathcal{P}_\tau$ to $\mathcal{P}_{j,\tau}(s_1,r_1)$ is also $O(\eta r_0)$. But $\mathcal{P}_{j,\sigma}(s_1,r_1)$ and $\mathcal{P}_{j,\tau}(s_1,r_1)$ are disjoint polyhedra lying in the flat F'_j. Then, the required estimate clearly holds for any $r \gg C'$. If σ and τ are disjoint, then pick $j \in \tau \setminus \sigma$. Since

$$\mathrm{Int}(\mathcal{P}_{i,\sigma}(s_1,r_1), r) \subset \mathcal{P}_{i,\sigma}(s_1, r_1 + r) \subset F'_{i,\sigma}(\lambda_1^{-1}s_1, \lambda_2^{-1}(r_1+r))$$

for any $x \in \mathrm{Int}(\mathcal{P}_{i,\sigma}, r)$, and since $j \notin \sigma$, we have $d(x, F'_j) \ge \lambda_2^{-1}(r_1 + r) \ge \lambda_2^{-1} r$. Hence, for any $y \in B(x, 4\kappa\eta_1\eta_2 C')$, $d(y, F'_j) \ge \lambda_2^{-1} r - 4\kappa\eta_1\eta_2 C' \succ r$ if $r \gg C'$. But, by (19), for any $z \in \mathcal{P}_\tau$, $d(z, F_j) = O(\eta r_0) = O(C')$. Hence, $y \notin \mathcal{P}_\tau$. Thus (46) holds. Now the lemma follows from Lemma A.3. ∎

Lemma 3.7 (Closeness sets of polyhedra). *Suppose $F_1, F_2 \in \mathcal{F}$ are flats, $x \in \Omega_{F_1} \cap \Omega_{F_2}$ and $F_2 \in \mathrm{Trans}_{F_1}(x)$. Choose any R such that $R \gg \rho$, and write B_R for $B(\tilde{\phi}(x), R)$. Let $\mathcal{P}_1, \mathcal{P}_2 \subset X$ be polyhedra and suppose there exist subsets $U_i \in F_i$ such that $hd(\tilde{\phi}(U_i), \mathcal{P}_i \cap B_R) = O(\epsilon R)$. Then, for any r with $\epsilon R \ll r \ll R$:*

(a) *The injectivity radius of $\mathcal{P}_1 \cap \mathrm{Nbhd}(\mathcal{P}_2, r) \cap B_R$ is $O(r)$.*
(b) *The exists a hyperplane $L \subset \mathcal{P}_1$ such that*

$$\mathcal{P}_1 \cap \mathrm{Nbhd}(\mathcal{P}_2, r) \cap B_R \subset \mathrm{Nbhd}(L, O(r)) \cap B_R.$$

Proof. Let $d_0 = d(F_1', F_2')$ where the F_j' are flats containing $\mathcal{P}_j$. Let $d(r) = r$ when $0 < r < d_0$, and $d(r) = \lambda_0(1 + d_0) + r - d_0$ when $r \geq d_0$. Then $r \leq d(r) \leq \lambda_0(r+1)$ and the range of $d(r)$ does not contain the interval $[d_0, \lambda_0(1+d_0))$ where Lemma B.1 cannot be applied. If $\epsilon R \ll r \ll R$ then $d(r) = O(r)$ and by Lemma B.1 part (iii) there exists a polyhedron $\mathcal{Q} \subset \mathcal{P}_1$ such that for $hd(\mathcal{Q}, \mathcal{P}_1 \cap \mathrm{Nbhd}(\mathcal{P}_2, d(r))) = O(r)$.

By assumption, $\tilde{\phi}$ has a coarse inverse $\phi^* : (\mathcal{P}_1 \cup \mathcal{P}_2) \cap B_R \to U_1 \cup U_2$, which is a (κ, C') quasi-isometric embedding with $C' = O(\epsilon R)$. By Lemma A.4,

$$\begin{aligned}(24)\quad \phi^*(\mathcal{P}_1 \cap \mathrm{Nbhd}(\mathcal{P}_2, d(r) \cap B_R) &\subset \mathrm{Nbhd}(U_1, O(r)) \cap \mathrm{Nbhd}(U_2, O(r)) \subset \\ &\subset \mathrm{Nbhd}(F_1, O(r)) \cap \mathrm{Nbhd}(F_2, O(r))\end{aligned}$$

By Lemma 3.1, the injectivity radius of $\mathrm{Nbhd}(F_1, O(r)) \cap \mathrm{Nbhd}(F_2, O(r))$ is $O(r)$. Hence, since ϕ^* is a quasi-isometry, the injectivity radius of $\mathcal{P}_1 \cap \mathrm{Nbhd}(\mathcal{P}_2, d(r)) \cap B_R$ and hence of $\mathcal{Q}$ is $O(r)$. Now the assertion (b) follows from Lemma B.9 and the fact that $d(r) \geq r$. ☐

Proof of Lemma 3.5: Choose R_0 sufficiently large so that for any $y \in F$ outside B_{R_0}, there exists a point $x \in \Omega_F^*$ with $d(x,y) = O(\epsilon d(y, x_0))$, and suppose $R > 4\kappa R_0$.

If L does not intersect $B = B(x_0, 4\kappa R)$ then $\tilde{\phi}(L)$ does not intersect $B(\phi(x_0), 2R)$ and there is nothing to prove. If L does intersect then B we can replace L by a parallel hyperplane L_1 such that $hd(L \cap B, L_1 \cap B) = O(\epsilon R)$, and L_1 contains a point $x \in \Omega_F^*$. Choose $F_0 \in \mathrm{Trans}_F(x)$ such that $F \cap F_0 = L_1$.

By Lemma 3.6, there exist finite unions of polyhedra, $\mathcal{P}_1, \dots, \mathcal{P}_M$ and $\mathcal{Q}_1, \dots, \mathcal{Q}_M$ such that $hd(\tilde{\phi}(F) \cap A_R, \bigcup \mathcal{P}_i \cap A_R) = O(\epsilon R)$ and $hd(\tilde{\phi}(F_0) \cap A_R, \bigcup \mathcal{Q}_j \cap A_R) = O(\epsilon R)$. By Lemma A.4,

$$\begin{aligned}(25)\quad \tilde{\phi}(L_1) \cap A_R \subset \mathrm{Nbhd}(\bigcup_{i=1}^M \mathcal{P}_i, O(\epsilon R)) \cap \mathrm{Nbhd}(&\bigcup_{j=1}^M \mathcal{Q}_j, O(\epsilon R)) \cap A_R \subset \\ &\subset \bigcup_{i=1}^M \bigcup_{j=1}^M \mathrm{Nbhd}(\mathcal{P}_i, O(\epsilon R)) \cap \mathrm{Nbhd}(\mathcal{Q}_j, O(\epsilon R)) \cap A_R\end{aligned}$$

But by Lemma 3.7 (b) with $U_1 = \tilde{\phi}^{-1}(\mathcal{P}_i \cap B_{4R})$, $U_2 = \tilde{\phi}^{-1}(\mathcal{Q}_j \cap B_{4R})$, for every i, j there exists a hyperplane $L_{ij}' \subset \mathcal{P}_i$ such that

$$\mathrm{Nbhd}(\mathcal{P}_i, O(\epsilon R)) \cap \mathrm{Nbhd}(\mathcal{Q}_j, O(\epsilon R)) \cap A_R \subset \mathrm{Nbhd}(L_{ij}', O(\epsilon R)) \cap A_R$$

This implies that the lemma holds for L_1. Hence it also holds for L. ☐

The same argument as in the proof of Lemma 3.5 also shows the following variant of Lemma 3.5:

Lemma 3.8. *Suppose F is a flat such that $\tilde{\phi}(F)$ is close to a single flat F' (in the sense of Step 2 (c)). Suppose $x \in \Omega_F^*$, and L is a hyperplane in F passing through x. Suppose $R \gg \rho$, where ρ is as* *in Step 2. Let $A_R = A(R, 2R; \phi(x)) \subset X$, Then there exists $M' > 0$ (depending only on κ, X) such that there exist at most M' hyperplanes $L'_1 \cdots L'_{M'} \subset F'$ with*

$$\tilde{\phi}(L) \cap A_R \subset \mathrm{Nbhd}\left(\bigcup_{j=1}^{M'} L'_j, O(\epsilon R)\right) \cap A_R$$

3.2. **A certain class of quasi-isometries $\mathbb{R}^n \to \mathbb{R}^n$.**

Lemma 3.9 (Complexes). *Let $S_1, \ldots, S_r$ be r fixed hyperplanes passing through the origin in $\mathbb{R}^n$; their intersections form a cellular complex Y, whose maximal cells have dimension $n-1$. Suppose f is a $1-1$ cellular map from Y to Y (i.e f maps cells to cells, and preserves adjacency relations). Then either f sends each hyperplane to a hyperplane, or there exist distinct hyperplanes L and M such that $f(L)$ separates $\mathbb{R}^n$ into two connected components H_1 and H_2, and $f(M)$ lies entirely in the closure of H_1.*

Proof. We may assume that f is a continuous map $Y \to Y$ which respects the simplicial complex structure. By the Jordan curve theorem, $f(L)$ separates $\mathbb{R}^n$ into two connected components H_1 and H_2. Suppose $f(L)$ is not a hyperplane. Then there exist two adjacent cells C_1 and C_2 in L such that $f(C_1)$ and $f(C_2)$ do not lie in the same hyperplane. Since C_1 and C_2 are adjacent, there exists a subspace C of codimension 2 containing $C_1 \cap C_2$. Pick $x_0 \in C_1 \cap C_2$ away from the origin, and let H_0 be a 2 dimensional plane passing through $f(x_0)$ and orthogonal to $f(C)$. Denote by $D_0 = D_0(\epsilon R)$ a disk in H_0 centered at x_0 and of radius ϵR where R is the distance from x_0 to the origin and $\epsilon > 0$ is small.

Then for any hyperplane $M \supset C$, $f(M) \cap D_0$ is "broken line", i.e. a one-dimensional piecewise straight curve, with the (possible) turn at $f(x_0)$. For $M = L$, there is indeed a "turn" at $f(x_0)$, i.e $f(L) \cap D_0$ consists of two half-lines intersecting at $f(x_0)$. Note that $f(L) \cap D_0$ divides D_0 into two open connected components $D_1 \subset H_1$ and $D_2 \subset H_2$, and the number of half-lines in D_1 and D_2 differs by at least 2. For definiteness suppose the number of half-lines in D_1 is larger than in D_2. Then for at least one hyperplane M containing C, $f(M) \cap D_0$ is contained entirely in $\bar{D}_1$. We now claim that $f(M) \subset \bar{H}_1$. Let M_+ and M_- be the two open half-hyperplanes into which L divides M, so that $M = M_+ \cup M_- \cup (M \cap L)$. Note that $f^{-1}(D_1)$ intersects both M_+ and M_- nontrivially. Hence, any point $x \in M \setminus (M \cap L)$ can be connected to a point in $f^{-1}(D_1)$ by a path $\gamma \subset M$ which does not cross $M \cap L$. But then $f(\gamma)$ is a path connecting $f(x)$ to a point in $f(M) \cap D_0 \subset \bar{H}_1$. Since f is one-to-one, this path does not cross $f(L)$, hence $f(x) \in \bar{H}_1$. □

Lemma 3.10 (No Turns). *Suppose q is a (continuous) (κ, C) quasi-isometry from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$. Suppose B' is a simply connected open subset of Euclidean space (with the Euclidean metric), contained in the image of q, and let $B = q^{-1}(B')$. Let $S_1, \ldots, S_r$ be r fixed hyperplanes passing through the origin. We call any hyperplane parallel to one of the S_i a "marked" hyperplane. Suppose there exist constants m and $C' > C$ such that for every marked hyperplane L in B there exist m marked hyperplanes $L'_1, \ldots, L'_m$ in B' such that $q(L) \subset \mathrm{Nbhd}(L'_1 \cup \ldots L'_m, C')$. Then there exists $\beta = \beta'(\kappa, X)C'$ such that for every L intersecting B, and every subset $B'_1 \subset B$ satisfying $d(B'_1, \partial B) \gg \beta$, $q(L) \cap B'_1$ is within β of a single marked hyperplane L' in B'. Conversely, for every marked hyperplane L' intersecting B'_1 there exists a hyperplane L in B such that $q^{-1}(L' \cap B'_1)$ is within β of L.*

Proof. Since we will consider only marked hyperplanes, we will call them simply hyperplanes. Let r denote the number of hyperplanes passing through a point, and let $s = 2^m r$. Pick constants η'_j and λ'_{jk}, $0 \le j \le s$, $0 \le k \le s$, such that $1 = \eta'_0 \ll \cdots \ll \eta'_{s-1} \ll \eta'_s = \beta'$, and $\eta'_j = \lambda'_{j0} \ll \lambda'_{j1} \cdots \ll$

$\lambda'_{js} = \eta'_{j+1}$. Write $\eta_j = \eta'_j C'$, and $\lambda_{jk} = \lambda'_{jk} C'$. Suppose there is a hyperplane L for which the conclusion of the lemma is false with the choice of $\beta = \eta_s$.

For each j suppose $L_1, \dots, L_b$ are hyperplanes such that $q(L) \subset \mathrm{Nbhd}(\bigcup_{k=1}^b L_k, \eta_j)$. The union of the L_k has the structure of a cellular complex Y_j in $\mathbb{R}^n$ of dimension $n-1$; we shall refer to cells of dimension $n-1$ as simply "cells". We may compose q with nearest point projection onto Y_j to get a (κ, η_j) quasi-isometry $q_j : L \to Y_j$. Using the standard "connect-the-dots" construction, we may assume that q_j is continuous.

By Lemma A.3 (applied to q composed with the nearest point projection onto Y_j) there exists a parameter ρ_j, $\eta_j \ll \rho_j \ll \eta_{j+1}$, such that

$$hd\left(q(L) \cap B'_1, \bigcup_{Q \in \mathcal{I}} Q \cap B'_1\right) \leq \rho_j \tag{26}$$

where $\mathcal{I}$ is a certain subset of the cells of Y_j. We define $d(Y_j)$ to be the minimum possible cardinality of the subset $\mathcal{I}$, such that (26) holds (with fixed ρ_j). Then clearly $d(Y_j)$ is less then the total number of cells, which is bounded by 2^b. Let d_j be the minimum of $d(Y_j)$ over all possible Y_j. Note that $d_{j+1} \leq d_j$. Since $d_1 \leq 2^m$ and $s > 2^m$, we claim that there exists j such that $d_j = d_{j+1}$.

Now fix such a j, and let $d = d_j = d_{j+1}$. Let $Q'_1, \dots, Q'_d$ denote the sells in $\mathcal{I}$. We claim that there exists a point p on $q(L)$ such that within η_j of p there exists at least two cells say Q'_1 and Q'_2 which do not lie on the same hyperplane (indeed if this was false, $q(L)$ would be within η_j of one hyperplane). The cells Q'_1 and Q'_2 have diameter at least $\eta_{j+1} - \eta_j$ (otherwise $d_{j+1} < d_j$).

Let $L_1, \dots L_r$ be the hyperplanes passing through $q^{-1}(p)$. By assumption $q(L_j) \cap B'$ is within the C' neighborhood of at most m hyperplanes $S_{j1}, \dots, S_{jm}$. Denote by L_{jk} hyperplanes parallel to S_{jk} so that all these hyperplanes pass though p.

If the hyperplane S_{ji} intersects $B(p, \lambda_{jk})$ then the C'-neighborhood of S_{ji} will be within $2\lambda_{jk}$-neighborhood of L_{ji} since $\lambda_{jk} \gg C'$. Consider the ball $B(p, \lambda_{j1})$. If not all hyperplanes S_{ji} intersecting $B(p, \lambda_{j1})$ intersect $B(p, \lambda_{j0})$ then we consider $B(p, \lambda_{j2})$. Again if not all hyperplanes intersecting $B(p, \lambda_{j2})$ intersect $B(p, \lambda_{j1})$ then consider $B(p, \lambda_{j3})$ and etc. Doing this for each $1 \leq l \leq r$ and since $s > mr$ we get that there exists an integer k such that for each l, $1 \leq l \leq r$, $q(L_l) \cap B(p, \lambda_{j,k+1})$ is within a distance $2\lambda_{jk}$ of a union of hyperplanes passing through p.

We now compose q with nearest-point projection to get a map (which we call $\hat{q}$) from $Y = \bigcup_{i=1}^r L_i \cap B(q^{-1}(p), \lambda_{j,k+1})$ to $Y' = \bigcup_{i=1}^r L'_i \cap B(p, \lambda_{j,k+1})$, where the L'_i are the hyperplanes passing through p. Note that $\hat{q}$ is a $(\kappa, 2\lambda_{jk})$ quasi-isometry.

Denote by Z the $n-2$-skeleton of Y, i.e the union of all intersections $L_i \cap L_m$, $1 \leq i, m \leq r$, and by Z' the union of all intersections $L'_i \cap L'_m$, $1 \leq i, m \leq r$. Let B'_p denote the ball $B(p, \frac{1}{2}\lambda_{j,k+1})$ and $B_p = B(q^{-1}(p), \frac{1}{2}\lambda_{j,k+1})$. By (26), for each hyperplane L_i there exists a union of cells $\mathcal{D}_i \subset Y'$ such that $hd(q(L_i) \cap B'_p, \mathcal{D}_i) \leq \lambda^{(4)}$ where $\lambda_{jk} \ll \lambda^{(4)} \ll \lambda_{j,k+1}$. Note that since $\hat{q}$ is a $(\kappa, 2\lambda_{jk})$ quasi-isometric embedding, and the diameters of the cells in Y' are $\succ \lambda_{j,k+1} \gg \lambda^{(4)}$, for $i \neq m$ the cells in $\mathcal{D}_i$ and $\mathcal{D}_m$ are distinct. Thus, $\mathcal{D}_i \cap \mathcal{D}_m \subset Z'$. Applying Lemma A.4 part (ii) with $U = L_i \cap \hat{q}^{-1}(B'_p)$, $V = L_m \cap \hat{q}^{-1}(B'_p)$, $U' = L_i \cap B'_p$, $V' = L_m \cap B'_p$, and letting i, m vary, we get that exists a constant $\lambda^{(3)}$, with $\lambda^{(4)} \ll \lambda^{(3)} \ll \lambda_{j,k+1}$, such that

$$\hat{q}(Z) \cap B'_p \subset \mathrm{Nbhd}(Z' \cap B'_p, \lambda^{(3)}) \tag{27}$$

Let Q be any cell of Y. By a volume preservation argument, there exists a cell Q' of Y' such that $\hat{q}(Q)$ intersects $\mathrm{Int}(Q', \lambda^{(2)})$, where $\lambda^{(3)} \ll \lambda^{(2)} \ll \lambda_{j,k+1}$. In view of (27) and the fact that $\hat{q}$ is a $(\kappa, 2\lambda_{jk})$ quasi-isometric embedding, $\hat{q}^{-1}(\mathrm{Int}(Q', \lambda^{(2)}) \cap B'_p) \subset \mathrm{Int}(Q, \lambda^{(3)})$. Thus, we may apply Corollary A.2 with $U = Q$ and $V = Q'$ to show that $\mathrm{Int}(Q', \lambda^{(2)}) \cap B'_p \subset \mathrm{Nbhd}(q(Q), \lambda^{(1)}) \cap B'_p$, where $\lambda^{(2)} \ll \lambda^{(1)} \ll \lambda_{j,k+1}$. Since $\hat{q}$ is a quasi-isometric embedding, the images of distinct cells must consist of distinct cells. Hence, since the number of cells in Y' is equal to the number of cells in Y, for any cell $Q_i \in Y$ there exists a unique cell $Q_i' \in Y'$ such that $hd\left(\hat{q}(Q_i) \cap B_p', Q_i' \cap B_p'\right) \leq \lambda^{(1)}$. The adjacency relations are preserved since by Lemma A.4 part (ii), for any cells Q_l, Q_m of Y, $hd(q(Q_l \cap Q_m) \cap B_p', Q_l' \cap Q_m' \cap B_p') \leq \lambda^{(0)}$, where $\lambda^{(1)} \ll \lambda^{(0)} \ll \lambda_{j,k+1}$. Hence, by a volume preservation argument, $\dim(Q_l \cap Q_m) = \dim(Q_l' \cap Q_m')$.

Thus $\hat{q}$ induces a one-to-one map of complexes $f : Y \to Y'$. Since we know that $\hat{q}(L)$ is not contained in a single hyperplane, by Lemma 3.9, there exists a hyperplane $M \in Y$ such that $f(M)$ does not cross $f(L)$. Pick $x_1, x_2 \in M$ on opposite sides of L such that $\lambda^{(0)} \ll d(x_i, \hat{q}^{-1}(p)) \ll \lambda_{j,k+1}$, and $d(x_i, L) \gg \lambda^{(0)}$, $i = 1, 2$. Note that $q(x)$ and $\hat{q}(x)$ differ by $O(\lambda_{jk}) \ll \lambda^{(0)}$ on Y, and q is onto B_p' by Lemma A.1. Then $d(q(x_i), f(L)) \gg \lambda^{(0)}$. Since $f(L)$ is a finite union of disjoint cells, and the points $\hat{q}(x_1) \in \hat{q}(M)$ and $\hat{q}(x_2) \in \hat{q}(M)$ are on the same side of $f(L)$, there exists a path $\gamma \subset B_p'$ connecting $q(x_1)$ and $q(x_2)$ with $d(\gamma, f(L)) \gg \lambda^{(0)}$.

Then $\gamma_0 = q^{-1}(\gamma) \subset q^{-1}(B_p')$ is a path connecting x_1 and x_2 with $d(\gamma_0, L) \gg \lambda^{(0)}$. This is a contradiction since x_1 and x_2 are on opposite sides on L.

To prove the converse let L' be a hyperplane in B_1', and pick a point $x' \in L'$. Let $x = \phi^{-1}(x')$ and consider all planes $L_1, ..., L_r$ passing through x. As it was proven above for each $q(L_j)$, $1 \leq j \leq r$, there exists L_j' such that $q(L_j) \cap B_1' \subset \mathrm{Nbhd}(L_j', \beta) \cap B_1'$. Since $x' \in q(L_j)$ we can by taking a parallel hyperplane and changing β to 2β assume that L_j' passes through x'. Since q is $(\kappa, 2\beta)$ quasi-isometry we have that distinct L_j correspond distinct L_j'. Therefore there is L such that $q(L) \cap B_1' \subset \mathrm{Nbhd}(L', 2\beta) \cap B_1'$. Modifying q on L by composing it with the nearest point projection on L' and using Lemma A.1, we get $L' \cap B_1' \subset \mathrm{Nbhd}(q(L), \beta')$. Therefore, since q is a quasi-isometry, $q^{-1}(L' \cap B_1') \subset \mathrm{Nbhd}(L, \beta_1)$. □

3.3. Hyperplanes and Weyl chambers.

Lemma 3.11 ($\tilde{\phi}(F)$ is close to cone over limit set). *Suppose $F \in \mathcal{F}$ is a flat, and let $\tilde{\phi}$ be the graded quasi-isometric embedding of Step 2. Then,*

$$\tilde{\phi}(F) \sim_{\lambda\epsilon} \mathcal{V}$$

where λ is a constant depending only on κ and X and $\mathcal{V}$ is the union of Weyl chambers passing through the origin:

$$\mathcal{V} = \bigcup_{k \in \mathcal{L}(\phi)} \pi(kA_+) \subset X$$

where the limit set $\mathcal{L}(\phi)$ is defined in [EF, Definition 5.1].

Note. Unlike Lemma 3.6 we do not claim a bound of the type $\tilde{\phi}(F) \sim_{\lambda\epsilon, \phi(x), \lambda\rho} \mathcal{V}$.

Proof. Argue as in the proof of [EF, Lemma 8.3]. □

Lemma 3.12 (Walls into Hyperplanes). *Suppose $F \subset \mathcal{F}^*$ is a flat, and let $\mathcal{V}$ be as in Lemma 3.11. Let L' be a codimension 1 wall such that outside a sufficiently large ball, L' is contained completely in a single chamber $\mathfrak{C}' \subset \mathcal{V}$. Then there exists a hyperplane $L \subset F$ such that, outside a sufficiently large ball, $L' \subset \tilde{\phi}(L)[\lambda\epsilon]$ for some constant λ depending only on κ and X.*

Conversely, if L is a wall in F such that outside a sufficiently large ball $\tilde{\phi}(L)$ is contained completely in $\mathfrak{C}'[\lambda\epsilon]$ where $\mathfrak{C}'$ is a chamber $\mathfrak{C}'$ in $\mathcal{V}$, then there exists a hyperplane L' such that outside a sufficiently large ball, $\tilde{\phi}(L) \cap \mathfrak{C}' \subset L'[\lambda\epsilon]$.

Proof. Let $A'(R_j, 4R_j)$ be an annular domain in the chamber $\mathfrak{C}' = \pi(kA_+)$ centered at e, where $R_j = 2^j R_1$ with $R_1 = 2\kappa R$. Denote by B'_j the part of $\mathfrak{C}' \cap A'(R_j, 4R_j)$ such that

$$d(B'_j, \partial\mathfrak{C}') \gg \beta'(\kappa, X)C'$$

where $\beta'(\kappa, X)$ and $C' = O(\epsilon R_j)$ are the same as in Lemma 3.5 and Lemma 3.10 and let $B_j = \tilde{\phi}^{-1}(B'_j)$. Let L' be a codimension 1 wall contained in $\mathfrak{C}'$. Denote by L'_j a parallel translate of L' such that $L'_j \cap B'_j$ is not empty. Note that we can choose L'_j so that $d(L'_j \cap A'(R_j, 4R_j), \partial\mathfrak{C}') \gg \beta'\epsilon R_j$, and $d(L', L'_j) \prec \epsilon R_j$. Combining Lemma 3.5 with the "converse" statement of Lemma 3.10 (with $B_1 = B_j$ and $B'_1 = B'_j$ defined as above) we get a hyperplane $L_j \subset F$ such that

$$\tilde{\phi}(L_j \cap B_j) \subset L'_j[\lambda\epsilon] \cap B'_j \subset L'[\lambda'\epsilon] \cap B'_j$$

where λ' is independent of ϵ and R_j. Then by the local packing lemma (Lemma A.1),

$$\tilde{\phi}(L_j \cap B_j)[\lambda''\epsilon] \supset L'_j \cap B'_j \tag{28}$$

Let U'_j be a ball in $L'_j \cap A'(2R_j, 3R_j)$ of radius $\succ R_j$. Since L'_j and L'_{j+1} are parallel there exists a ball $U'_{j+1} \in L'_{j+1}$ such that $hd(U'_j, U'_{j+1}) = O(\epsilon R_j)$. By (28) there exist subsets $U_j \in B_j \cap L_j$ and $U_{j+1} \in B_j \cap L_{j+1}$ so that $hd(\tilde{\phi}(U_j), U'_j) = O(\epsilon R_j)$ and $hd(\tilde{\phi}(U_{j+1}), U'_{j+1}) = O(\epsilon R_j)$, Thus, $U_j \subset L_j$, $U_{j+1} \subset L_{j+1}$ and $hd(U_j, U_{j+1}) = O(\epsilon R_j)$, Also, since $\tilde{\phi}$ is a quasi-isometric embedding, U_j contains a ball of radius $\succ R_j$. This implies that L_j and L_{j+1} are parallel, and $d(L_j, L_{j+1}) \le \lambda\epsilon R_j$. Hence

$$d(L_0, L_j) \le \sum_{k=1}^{j} d(L_{k-1}, L_k) \le \sum_{k=1}^{j} \lambda\epsilon 2^k R_1 \le 2\lambda\epsilon R_j$$

and

$$\tilde{\phi}(L_0) \cap B'_j \subset \tilde{\phi}(L_j)[\lambda_1\epsilon] \cap B'_j \subset L'[\lambda_2\epsilon].$$

Analogously, let L be a wall in F such that $\tilde{\phi}(L) \subset \mathfrak{C}'[\lambda\epsilon]$. Let $A_j = A(R_j, 4R_j)$ denote an annular domain in F centered at $\pi(x)$. Let $B_j \subset A_j$ denote the part of A_j such that $\tilde{\phi}(B_j) \subset \mathfrak{C}'$ and $d(\tilde{\phi}(B_j), \partial\mathfrak{C}') \gg \beta'(\kappa, X)C$. Let $B'_j = \tilde{\phi}(B_j)$.

Denote by L_j a hyperplane parallel to L such that $d(L, L_j) \prec \epsilon R_j$ and $L_j \cap B_j$ is not empty. By Lemma 3.5 and Lemma 3.10 there exists a hyperplane L'_j such that $\tilde{\phi}(L_j \cap B_j) \subset L'_j[\lambda\epsilon] \cap B'_j$. Since $L_j \cap B_j \cap B_{j+1}$ is not empty and contains balls of radius $\succ R_j$ we get as above that L'_j and L'_{j+1} are parallel hyperplanes and $d(L'_j, L'_{j+1}) = O(\epsilon R_j)$. Therefore $d(L'_0, L'_j) \prec \epsilon R_j$ and $\tilde{\phi}(L \cap A_j) \subset \tilde{\phi}(L_j \cap B_j)[\lambda\epsilon] \subset L'_j[\lambda'\epsilon] \cap B'_j \subset L'_0[\lambda''\epsilon] \cap B'_j \subset L'_0[\lambda''\epsilon]$. Since j is arbitrary we have $\tilde{\phi}(L) \cap \mathfrak{C}' \subset L'_0[\lambda''\epsilon]$. ☐

Using Lemma 3.8 instead of Lemma 3.5 in the above proof we get the following lemma:

Lemma 3.13 (Hyperplanes to hyperplanes)**.** *Let $F \subset \mathcal{F}^*$ be a flat and suppose $\tilde{\phi}(F)$ is close (in the sense of Step 2 (c)) to a single flat F'. Suppose $x \in \Omega^*_F$, and let $L \supset F$ be a hyperplane passing through x. Then, there exists a constant λ depending only on κ and X and a hyperplane $L' \subset F'$ such that*

$$\tilde{\phi}(L) \sim_{\lambda\epsilon, \phi(x), \lambda\rho} L'$$

Lemma 3.14 (Chambers to Chambers)**.** *Suppose $\mathcal{F} \subset \mathcal{F}^*$, and let $\mathcal{V}$ be as in Lemma 3.11. Let $\mathfrak{C}$ be a chamber in F. Then there exists a chamber $\mathfrak{C}'$ in $\mathcal{V}$ such that $\tilde{\phi}(\mathfrak{C}) \sim_{\lambda\epsilon} \mathfrak{C}'$. The chamber $\mathfrak{C}'$ is unique up to equivalence.*

Proof. Let $\mathfrak{C}'$ be an arbitrary chamber in $\mathcal{V}$. Taking a parallel translate of $\mathfrak{C}'$ in $\mathcal{V}$ (and denoting it again by $\mathfrak{C}'$) we can assume that $\tilde{\phi}$ packs $\mathfrak{C}'$, i.e $\tilde{\phi}(F) \supset \mathfrak{C}' \setminus \partial\mathfrak{C}'[\lambda\epsilon]$ (see [EF, Lemma 8.3] or Corollary A.2) By Lemma 3.12, for each wall L_i' in $\partial\mathfrak{C}'$ there exists a hyperplane L_i in F such that outside a sufficiently large ball, $L_i' \subset \tilde{\phi}(L_i)[\lambda\epsilon]$. By replacing L_i by a parallel translate, we may assume that all L_i pass through the point $x \in F$, and still, outside a sufficiently large ball, $L_i' \subset \tilde{\phi}(L_i)[\lambda\epsilon]$. Let $\mathcal{P}_j$ be the polyhedra bounded by the L_i. By Corollary A.2 we may assume that (outside a sufficiently large ball), $\tilde{\phi}^{-1}(\mathfrak{C}')$ packs one of these polyhedra, which we denote $\mathcal{P}$, i.e $\tilde{\phi}^{-1}(\mathfrak{C}') \supset \mathcal{P} \setminus \partial\mathcal{P}[\lambda\epsilon]$. Since the diameter of $\mathfrak{C}' \cap A(R, 2R)$ is $\succ R$, and $hd(\tilde{\phi}^{-1}(\mathfrak{C}') \cap A(R, 2R), \mathcal{P} \cap A(R, 2R)) = O(\epsilon R)$ (recall $\epsilon \ll 1$), $\mathcal{P}$ must contain a Weyl chamber $\mathfrak{C}$. Hence $\tilde{\phi}(\mathfrak{C}) \subset \tilde{\phi}(\mathcal{P}) \subset \mathfrak{C}'[\lambda\epsilon]$.

Now consider a wall L_i bounding $\mathfrak{C}$. Since $\tilde{\phi}(L_i) \subset \mathfrak{C}'[\lambda\epsilon]$, arguing as in the proof of Lemma 3.12, we may conclude that $\tilde{\phi}(L_i)$ is contained in $L_i'[\lambda\epsilon]$ for some hyperplane L_i' belonging to a flat $F' \supset \mathfrak{C}'$. We may replace the L_i' by parallel translates all passing through the same point; still, outside a sufficiently large ball, $\tilde{\phi}(L_i) \subset L_i'[\lambda\epsilon]$. By the local packing lemma, $\tilde{\phi}$ restricted to $\mathfrak{C}[\lambda\epsilon]$ packs one of the polyhedra bounded by the L_i', say $\mathcal{P}'$. Since $\tilde{\phi}(\mathfrak{C}) \subset \mathfrak{C}'[\lambda\epsilon]$, $L_i' \subset \mathfrak{C}'[2\lambda\epsilon]$, hence $\mathcal{P}' \subset \mathfrak{C}'[\lambda'\epsilon]$. Also since the injectivity radius of $\mathcal{P}' \cap A(R, 2R) \succ R$, $\mathcal{P}'$ contains a Weyl chamber $\mathfrak{C}_1' \subset \mathfrak{C}'[\lambda\epsilon]$. By Lemma B.7 this can only happen if $\mathfrak{C}_1'$ is equivalent to $\mathfrak{C}'$. Thus, outside of a large ball, $\mathfrak{C}' \subset \tilde{\phi}(\mathfrak{C})[\lambda\epsilon]$.

Since $\tilde{\phi}^{-1}(\mathfrak{C}') \supset \mathcal{P} \setminus \partial\mathcal{P}[\lambda\epsilon]$ we have $\tilde{\phi}^{-1}(\mathfrak{C}')[\lambda\epsilon] \supset \mathcal{P} \supset \mathfrak{C}$. The uniqueness of $\mathfrak{C}$ follows from the last assertion of Lemma B.7.

Therefore we have a map from chambers in $\mathcal{V}$ to equivalence classes of chambers in F. This map is one-to-one and onto: the first property follows immediately from the fact that $\tilde{\phi}^{-1}$ is a graded quasi-isometry, and the second from the fact that $F \subset \tilde{\phi}^{-1}(\mathcal{V})[\lambda\epsilon]$ outside of a large ball. □

Proof of Proposition 3.3. By Lemma 3.14 one has a well defined map ϕ_0 from the equivalence classes of chambers in F (i.e $F(\infty) \subset \hat{X}$) to equivalence classes of chambers in $\mathcal{V}$, i.e. $\mathcal{L}(\phi) \subset \hat{X}$.

From the proof of Lemma 3.14 it is clear that neighboring chambers in F map to neighboring chambers (here neighboring means sharing a codimension 1 wall). This implies that if $\mathfrak{C}_1, \mathfrak{C}_2$ are chambers in F, and k_1, k_2 the corresponding points in $F(\infty)$, then then $d_c(\phi_0(k_1), \phi_0(k_2)) \leq d_c(k_1, k_2)$. (Recall d_c is the combinatorial distance in the Tits building $\mathcal{T}(X)$, i.e the length of the minimal gallery connecting the two chambers). Indeed there exists a minimal gallery γ connecting k_1 and k_2 and lying in F; then $\phi_0(\gamma)$ is a gallery connecting $\phi_0(k_1)$ and $\phi_0(k_2)$ and $\ell(\phi_0(\gamma)) \leq \ell(\gamma)$.

From Lemma 3.11, and the proof of [EF, Theorem 1.1] (see [EF, §7]) there exist two limit points $a', b' \in \mathcal{L}(\phi)$ such that $a' \notin b'\mathcal{S}$, i.e. $d_c(a', b')$ is maximal. Let F' denote the unique flat such that $a', b' \subset F'(\infty)$ (see [EF, §7.1]). Let a, b be points in $F(\infty)$ such that $\phi_0(a) = a'$ and $\phi_0(b) = b'$. Then $d_c(a, b) \geq d_c(a', b')$, and since the latter is maximal, $d_c(a, b)$ is maximal. Let γ denote any minimal gallery connecting a and b; then $\phi_0(\gamma)$ is a minimal gallery connecting a' and b'; since $F'(\infty)$ is an apartment and apartments are convex (see [B, Proposition, §IV.4, p. 88]), $\phi(\gamma) \subset F'(\infty)$. Since any chamber is contained in such a minimal gallery (see [B, Lemma, §IV.5, p. 92]), this implies that $\phi_0(F(\infty)) \subset F'(\infty)$, i.e $\mathcal{L}(\phi) \subset F'(\infty)$. Now part (i) follows easily from Theorem 1.5 and [EF, Corollary 7.4].

Part (ii) follows immediately from combining part (i) with Lemma 3.14. □

Proof of Proposition 3.4. Part (a) is immediate from Proposition 3.3 and Lemma 3.13. To get part (b) one may argue as follows: Let $Y \subset F$ denote the union of the hyperplanes passing though x, let $x' \in F'$ denote the nearest point in F' to $\phi(x)$, and let $Y' \subset F'$ denote the union of the hyperplanes passing through x'. By part (a), $\tilde{\phi}(Y) \sim_{\lambda\epsilon, x', \lambda\rho} Y'$. Note that $F \setminus Y$ is the union of the interiors of the chambers passing through x. For any chamber $\mathfrak{C} \subset F$ based at x we may pick a point $u \in \mathfrak{C}$ such that $d(u,x) \gg \rho$ and $d(u,Y) \succ d(u,x)$. Then $d(\tilde{\phi}(u), Y') \succ d(\tilde{\phi}(u), x')$. Now part (b) follows from the graded version of Corollary A.2. ◻

4. The boundary map

Definition 4.1 (The boundary map $\phi_0 : \mathcal{U} \to \hat{X}$)**.** Let $\mathcal{U} \subset \hat{X}$ denote the set of equivalence classes of Weyl chambers of flats in $\mathcal{F}^*$. Let $\phi_0 : \mathcal{U} \to \hat{X}$ be the map defined on these Weyl chambers by part (ii) of Step 3.

We note that ϕ_0 is well defined. Indeed, let $\mathfrak{C}_1 \subset F_1$ and $\mathfrak{C}_2 \subset F_2$ be equivalent Weyl chambers, i.e. $hd(\mathfrak{C}_1, \mathfrak{C}_2) < c < \infty$, and the flats F_1 and F_2 belong to $\mathcal{F}^*$. Pick $x_0 \in F_1$. By Step 2 part (c) there exists a $(\kappa, \lambda\epsilon, C')$ graded q.i. embedding $\tilde{\phi}$ based at x_0 from $F_1 \cup F_2$ to X, and λ depends only on κ and X. For any $x_1 \in \mathfrak{C}_1$ there exists $x_2 \in \mathfrak{C}_2$ such that $d(x_1, x_2) \leq c$. Then since $\tilde{\phi}$ is a graded q.i. embedding, $d(\tilde{\phi}(x_1), \tilde{\phi}(x_2)) \prec d(x_1,x_2) + \epsilon d(x_1,x_0) + \epsilon d(x_2,x_0) \prec \epsilon d(\tilde{\phi}(x_1), \tilde{\phi}(x_0))$, where we assumed that $d(x_1, x_0) \gg \max(c/\epsilon, C')$. Since $x_1 \in \mathfrak{C}_1$ is arbitrary, this shows that $\tilde{\phi}(\mathfrak{C}_1) \subset \tilde{\phi}(\mathfrak{C}_2)[\lambda\epsilon]$, outside of a large ball. Analogously one has $\tilde{\phi}(\mathfrak{C}_2) \subset \tilde{\phi}(\mathfrak{C}_1)[\lambda\epsilon]$. Therefore, $\tilde{\phi}(\mathfrak{C}_1) \sim_{\lambda\epsilon} \tilde{\phi}(\mathfrak{C}_2)$. Hence if we denote $\phi_0(\mathfrak{C}_i)$ by $\mathfrak{C}'_i$ then $\mathfrak{C}'_1 \sim_{\lambda\epsilon} \mathfrak{C}'_2$. Then $\mathfrak{C}'_1$ is equivalent to $\mathfrak{C}'_2$ by the last assertion of Lemma B.7. This shows that ϕ_0 is well defined.

Recall that the cells of maximal dimension in the Tits building $\mathcal{T}(X)$ are equivalence classes of Weyl chambers, i.e points in K/M. For $\sigma \subset \Delta$ and $k \in K$, kK_σ can be viewed as a cell in $\mathcal{T}(X)$ of codimension $|\sigma|$ (here for a subset σ of Δ, $|\sigma|$ is the cardinality of σ).

Definition 4.2 (Tits Map defined on a subset)**.** Let E be a subset of $\hat{X}$, and let $\psi : E \to \hat{X}$ be a map. We say that ψ is a *Tits map defined on* E if for all $k \in E$ and any $\sigma \subset \Delta$ there exists $\tau \subset \Delta$ with $|\tau| = |\sigma|$ such that $\psi(kK_\sigma \cap E) \subset \psi(k)K_\tau$.

In other words, on its domain of definition, ψ sends adjacent simplices in $\mathcal{T}(X)$ into adjacent simplices, and preserves the dimension of the intersection. Note that we do not require ψ to be continuous.

Step 4 (ϕ_0 preserves partial ordering in $\mathcal{T}(X)$)**.** The map $\phi_0 : \mathcal{U} \to \hat{X}$ defined in Definition 4.1 is a Tits map defined almost everywhere.

The proof uses the following:

Lemma 4.3 (Walls to walls)**.** *Suppose $F \in \mathcal{F}^*$ is a flat and let F' be as in Step 3. Let $L \subset F$ be a σ-wall. Then there exists a τ-wall L' in F' such that $|\sigma| = |\tau|$, and $\tilde{\phi}(L) \sim_{\lambda\epsilon} L'$. The constant λ depends only on κ and X.*

Proof of Lemma 4.3. The wall L may be viewed as the intersection of a collection of Weyl chambers $\mathfrak{C}_1, \dots, \mathfrak{C}_r \subset F$. By Step 3 there exist Weyl chambers $\mathfrak{C}'_i \subset F'$ such that $\tilde{\phi}(\mathfrak{C}_i) \sim_{\lambda\epsilon} \mathfrak{C}'_i$. We may assume all the $\mathfrak{C}'_i$ are based at the same point $x' \in F'$ (otherwise replace them by equivalent Weyl chambers). Let $L' = \cap_{i=1}^r \mathfrak{C}'_i$. From the geometry of the root system (i.e all angles bounded from below), it is clear that (the graded version of) Lemma A.4 part (ii) can be applied repeatedly. Thus, $\tilde{\phi}(L) \sim_{\lambda\epsilon} L'$. The assertion that $|\sigma| = |\tau|$ follows easily by volume considerations, or the packing lemma (cf. Lemma A.1). ◻

Proof of Step 4. Suppose $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{U}$, and let $k_i \in K$ be such that $\pi(k_iA^+)$ is equivalent to $\mathfrak{C}_i$ for $i = 1, 2$. By assumption $k_2 \in k_1K_\sigma$. This implies that there exist σ-walls $L_i \subset \partial\mathfrak{C}_i$ such that $hd(L_1, L_2) < \infty$. Let $\mathfrak{C}'_i = \phi_0(\mathfrak{C}_i)$.

By Lemma 4.3 there exists $\tau \subset \Delta$ with $|\tau| = |\sigma|$ and a τ-wall L'_1 such that $\tilde{\phi}(L_1) \sim_{\lambda\epsilon} L'_1$. Since L_1 and L_2 are equivalent, this also implies that $\tilde{\phi}(L_2) \sim_{\lambda\epsilon} L'_1$. Hence, outside a sufficiently large ball, $L'_1 \subset \mathfrak{C}'_1[\lambda'\epsilon] \cap \mathfrak{C}'_2[\lambda'\epsilon]$.

Let $k'_1, k'_2, k' \in K$ be such that $hd(\pi(k'_i A^+), \mathfrak{C}'_i) < \infty$ and $hd(\pi(k' A^+_\tau), L'_1) < \infty$. Then, outside a sufficiently large ball, $\pi(k' A^+_\tau) \subset \pi(k'_1 A^+)[\lambda''\epsilon] \cap \pi(k'_2 A^+)[\lambda''\epsilon]$. Then by Lemma B.7, $k'_i \subset k' K_\tau$. □

5. Uniform continuity of the boundary map

Step 5 (ϕ_0 is bi-Hölder)**.** There exists a set $\mathcal{U}_2 \subset \mathcal{U}_1 \subset K/M$ of full (K-invariant) measure such that for $x, y \in \mathcal{U}_2$, the map ϕ_0 satisfies the Hölder type inequalities

$$c^{-1} d_K(x,y)^\lambda \le d_K(\phi_0(x), \phi_0(y)) \le c\, d_K(x,y)^{1/\lambda} \qquad x, y \in \mathcal{U}_2$$

Here d_K is the left K-invariant metric on $\hat{X} \cong K/M$, and λ and c do not depend on x, y.

Remark. The identification of $\hat{X}$ with K/M and thus d_K depends on the choice of origin. However, since the different possible metrics are conjugate by diffeomorphisms and thus equivalent, the notion of bi-Hölder is independent of choices. For chambers $\mathfrak{C}_1, \mathfrak{C}_2 \subset X$, we let $d_K(\mathfrak{C}_1, \mathfrak{C}_2) = d_K(\mathfrak{D}_1, \mathfrak{D}_2)$, where $\mathfrak{D}_i$ is equivalent to $\mathfrak{C}_i$ and based at e.

The motivation for Step 5 is the following well known:

Lemma 5.1. *Let Y be a rank 1 symmetric space, and let $q : Y \to Y$ be a quasi-isometry. Then the induced boundary map $q_0 : \hat{Y} \to \hat{Y}$ is bi-Holder. (Here $\hat{Y}$ is the boundary at infinity, i.e. the equivalence classes of geodesic rays).*

Proof of Lemma 5.1. Without loss of generality, $q(e) = e$. Let $\gamma_1 = \pi(k_1 A^+)$ and $\gamma_2 = \pi(k_2 A^+)$ be geodesic rays starting at e, where $k_1, k_2 \in K$. Write $\theta = d_K(k_1, k_2)$; then θ is the angle between γ_1 and γ_2. We parameterize the γ_i by arc length. Pick a sufficiently large $Q > 0$, and let $t = \sup\{s \;:\; s > 0 \text{ and } d(\gamma_1(s), \gamma_2(s)) < Q\}$. By Lemma B.5, $c_1 |\log\theta| < t < c_2 |\log\theta|$ where c_1, c_2 are independent of θ.

By Mostow's lemma, there exist geodesic rays γ'_i and a constant N depending only on Y and the q.i. constants of q such that $hd(q(\gamma_i), \gamma'_i) < N$. Without loss of generality, the γ'_i also start at e, and $Q \gg N$. Let $t' = \sup\{s \;:\; s > 0 \text{ and } d(\gamma'_1(s), \gamma'_2(s)) < 2\kappa Q\}$, and $t'' = \sup\{s \;:\; s > 0 \text{ and } d(\gamma_1(s), \gamma_2(s)) < Q/(2\kappa)\}$. Since q is a quasi-isometry, and $Q \gg N$, $t \prec t'' \le t' \prec t$. Hence by Lemma B.5, $c'_1 |\log\theta'| \le |\log\theta| \le c'_2 |\log\theta'|$ where c'_1 and c'_2 are independent of θ and θ' is the angle between γ'_1 and γ'_2. After exponentiating, we get the desired estimate. □

The proof of Lemma 5.1 carries over easily to the higher rank case – cf. Lemma 5.4. However, the presence of holes may create difficulties, since the entire argument in the proof of Lemma 5.1 may take place in a hole, e.g. when trying to prove continuity across the endpoint of a horosphere. In the higher rank case, this difficulty can be overcome as follows:

Suppose we consider two Weyl chambers $\mathfrak{C}_1$ and $\mathfrak{C}_2$ based at e which share a codimension 1 wall L_0. Let F_1 be a flat containing $\mathfrak{C}_1$ and F_2 a flat containing $\mathfrak{C}_2$; then F_1 and F_2 intersect in a hyperplane L which contains L_0. Note that F_1 and F_2 are contained in a totally geodesic submanifold Z of X which is a metric product $Z = L \times Y$, where Y is a rank 1 symmetric space. In fact, $F_1 \cup F_2 = L \times (\gamma_1 \cup \gamma_2)$, where γ_1 and γ_2 are two geodesics in Y passing through the same point. The angle $d_K(\mathfrak{C}_1, \mathfrak{C}_2)$ is the same as the angle between the geodesics γ_1 and γ_2 and can be measured above any point of L.

The strategy of the proof of Step 5 is the following: suppose the common base point of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is in or near a hole (in F_1 or F_2). By the Moore ergodicity theorem, applied to the subgroups A_σ of A, we may assume that L is uniformly distributed; hence there is a point p of L_0 which is away from the holes in F_1 and F_2. We then apply the argument in the proof of Lemma 5.1 to $p \times (\gamma_1 \cup \gamma_2) \subset p \times Y \cong Y$.

For this argument, we need results on images of hyperplanes similar to Step 3 part (i) (Proposition 3.4 (a) is not sufficient). This, together with some related issues is done in §5.1. The program sketched in the previous paragraph is carried out in §5.3. The assumption that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ share a codimension 1 wall is removed in §5.4.

5.1. Mapping hyperplanes.

Lemma 5.2 (Hyperplanes to hyperplanes, Uniform Version). *There exists a constant $N = N(\kappa, \epsilon, C)$, a family of hyperplanes $\mathcal{L}$, and for each $L \in \mathcal{L}$ nonempty subsets $\Omega_L \subset L$, such that the following holds:*

(a) *$\mathcal{L}$ is a Γ-invariant i.e. if $\gamma \in \Gamma$ and $L \in \mathcal{L}$ then $\gamma L \in \mathcal{L}$.*
(b) *$\mathcal{L}$ has full measure, i.e. there exists a set $E \subset \Gamma\backslash G$ with $\mu(E) = 1$ such that for any $y \in p^{-1}(E)$ and any $\alpha \in \Sigma$, $\pi(yA_\alpha) \in \mathcal{L}$.*
(c) *For every $L \in \mathcal{L}$, there exist a hyperplane L' such that $\phi(\Omega_L)$ lies within a distance N of L'.*
(d) *For every $L \in \mathcal{L}$, and every $x \in L$,*

$$\liminf_{r\to\infty} \frac{|B(x,r) \cap \Omega_L|}{|B(x,r) \cap L|} \geq 1 - |\Sigma|\epsilon \tag{29}$$

(e) *For every $y \in p^{-1}(E)$, $\pi(yA) \in \mathcal{F}^*$ and for any $\alpha \in \Sigma$*

$$\liminf_{r\to\infty} \frac{|\Omega_F \cap B(y,r) \cap L_\alpha|}{|B(y,r) \cap L_\alpha|} \geq 1 - |\Sigma|\epsilon$$

Proof. Let Ω^* and E^* be as in the proof of Lemma 3.2. Let χ_{Ω^*} be the characteristic function of Ω^*. By the ergodic theorem Theorem 1.4, there exists a set $E \subset E^* \subset \Gamma\backslash G$ with $\mu(E) = 1$ such that for all $x \in E$, and all $\alpha \in \Sigma$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \chi_{\Omega^*}(x \exp v)\, d\nu_\alpha(v) = \mu(\Omega^*) \geq 1 - |\Sigma|\epsilon \tag{30}$$

where ν_α is Lebesgue measure on the Lie algebra of $A_\alpha \cong \mathbb{R}^{n-1}$, and here B_r is the ball of radius r is this Lie algebra. Let $\mathcal{L} = \{\pi(yA_\alpha) \;:\; y \in p^{-1}(E)\}$. Then (a) and (b) hold by construction. For $L = \pi(zA_\alpha) \in \mathcal{L}$, let $\Omega_{z,L} = \{\pi(z \exp v) \;:\; p(z \exp v) \in \Omega^* \text{ and } v \in A_\alpha\}$, and $\Omega_L = \bigcup_{z \text{ with } L = \pi(zA_\alpha)} \Omega_{z,L}$. Then, (d) holds (already for $\Omega_{z,L}$) by (30). Also (e) holds since $\Omega_{z,L} \subset \Omega_F$ and (30).

It remains to prove (c). In fact we will show that (c) holds already for $\Omega_{z,L}$; Ω_L is introduced only to simplify notation.

Suppose $x \in \Omega_{z,L}$. Let $F_1 = \pi(zA)$. Then $x \in \Omega_{F_1}$, and there exists $F_2 \in \mathrm{Trans}_{F_1}(x)$ such that $F_1 \cap F_2 = L$ and $x \in \Omega_{F_2}$. Let F_1', F_2' be as in Step 3.

Since $\Omega_{z,L} \subset L \cap \Omega_{F_1} \cap \Omega_{F_2}$, by Step 3 (i), $\phi(\Omega_{z,L}) \subset \mathrm{Nbhd}(F_1', N) \cap \mathrm{Nbhd}(F_2', N)$. Choose $r \gg N$, and let $\mathcal{P}'$ be as in Lemma B.1 part (i) re. the flats F_1' and F_2'. Then,

$$\phi(\Omega_{z,L}) \subset \mathrm{Nbhd}(\mathcal{P}', O(r)) \tag{31}$$

Let $\tilde{\phi} : F_1 \cup F_2 \to X$ be as in Step 2 part (c). Note that since (d) holds for $\Omega_{z,L}$, there exists λ depending only on X such that $\Omega_{z,L}[\lambda\epsilon] = L$. Then (31) implies that (outside a sufficiently large ball),

$$\tilde{\phi}(L) \subset \mathcal{P}'[\lambda'\epsilon]. \tag{32}$$

Conversely, choose any $x \in \mathcal{P}'$ with $\epsilon d(x,z) \gg r$, and let $\tilde{\phi}^* : F_1' \cup F_2' \to F_1 \cup F_2$ be the coarse inverse of $\tilde{\phi}$. Then, since $\tilde{\phi}^*$ is a graded quasi-isometry, $\tilde{\phi}^*(x)(F_i') = F_i$, and $x \in \mathrm{Nbhd}(F_1', O(r)) \cap$

Nbhd$(F_2', O(r))$, we have $\tilde{\phi}^*(x) \in F_1[\lambda_1\epsilon] \cap F_2[\lambda_1\epsilon]$. Hence by Lemma 3.1, $\tilde{\phi}^*(x) \in L[\lambda_1'\epsilon]$. Hence, combining with (32), we get $\tilde{\phi}(L) \sim_{\lambda'\epsilon} \mathcal{P}'$.

By Proposition 3.4, there exists a hyperplane L' such that $\tilde{\phi}(L) \sim_{\lambda\epsilon} L'$. Hence $\mathcal{P}' \sim_{\lambda''\epsilon} L'$. This implies that $\mathcal{P}$ is a strip between two hyperplanes parallel to L'.

In view of (31), to show (c) we need only show that the width of the strip is $O(r)$. This can be checked as follows: Choose R so that $r \ll R \ll r/\epsilon$. Then, by Lemma 3.7 part (a), the injectivity radius of $\mathcal{P}' \cap B_R$ is $O(r)$. Since $\mathcal{P}'$ is a strip, this shows that the width of the strip is $O(r)$. □

5.2. **Choice of origin.** Choose $x_0 \in \Gamma\backslash G$. Let $\mathcal{U}'_{x_0} \subset K/M$ denote the set of $k \in K$ such that $\pi(x_0 k) \in E$ where E is as in Lemma 5.2. Note that for almost all choices of x_0,

$$m_K(\mathcal{U}'_{x_0}) = 1, \tag{33}$$

(where m_K is the K-invariant measure on K/M). We choose x_0 so that (33) holds, and denote $\mathcal{U}'_{x_0}$ by $\mathcal{U}'$. For simplicity of notation we set $x_0 = e$ (this can always be accomplished by replacing Γ with a suitable conjugate).

Let $\mathcal{U}_1$ denote the set of points k in K/M such that $k \in \mathcal{U}'$ and for all $\sigma \subset \Delta$, $m_K(\mathcal{U}' \cap kK_\sigma) = m_K(kK_\sigma)$. By Fubini's theorem, $m_K(\mathcal{U}_1) = 1$.

We normalize ϕ so that $\phi(e) = e$.

5.3. **Angles between flats.** The following lemma is the main estimate used in the proof of Step 5.

Lemma 5.3 (Angle between flats is bi-Hölder). *Suppose $\alpha \in \Delta$, $k \in \mathcal{U}_1$, and $x_1, x_2 \in kK_\alpha \cap \mathcal{U}'$. Then,*

$$C^{-1} d_K(x_1, x_2)^{1/\lambda} \le d_K(\phi_0(x_1), \phi_0(x_2)) \le C d_K(x_1, x_2)^\lambda$$

where $C = C(k)$ and λ is independent of x_1, x_2 and k.

The proof uses the following:

Lemma 5.4. *Suppose $F_1, F_2 \in \mathcal{F}^*$ are flats, and $\mathfrak{C}_1 \subset F_1$ and $\mathfrak{C}_2 \subset F_2$ are Weyl chambers both based at the point $p \in F_1 \cap F_2$. Suppose $\mathfrak{C}_1$ and $\mathfrak{C}_2$ share a codimension 1 wall, and $p \in \Omega^*_{F_1} \cap \Omega^*_{F_2}$. Suppose $p' \in B(\phi(p), O(N))$. Let $g_0 \in G$ be such that $g_0 \cdot p = e$, and let $g_0' \in G$ be such that $g_0' \cdot p' = e$. Then,*

$$C^{-1} d_K(g_0\mathfrak{C}_1, g_0\mathfrak{C}_2)^{1/\lambda} \le d_K(g_0' \cdot \phi_0(\mathfrak{C}_1), g_0' \cdot \phi_0(\mathfrak{C}_2)) \le C d_K(g_0\mathfrak{C}_1, g_0\mathfrak{C}_2)^\lambda \tag{34}$$

where C and λ are independent of $\mathfrak{C}_1, \mathfrak{C}_2, F_1, F_2, g_0, g_0'$ and p.

Remarks. The assumption that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ share a codimension 1 wall is not necessary. Also, if $p = e$, then we can let $g_0 = 1$, $g_0' = 1$, and the estimate of Step 5 follows immediately from (34). However, in general, we may not assume that for almost all $k \in \mathcal{U}_1$, $e \in \Omega^*_{\pi(kA)}$, so Lemma 5.4 cannot be applied directly.

Proof of Lemma 5.4. By Proposition 3.4 part (b), there exist Weyl chambers $\mathfrak{C}_1''$, $\mathfrak{C}_2''$ such that $\tilde{\phi}(\mathfrak{C}_i) \sim_{\lambda\epsilon,\phi(x),\rho} \mathfrak{C}_i''$. By Step 3, since $p \in \Omega_{F_i}$, $d(p', F_i') = O(N)$. By Lemma B.8 for $i = 1, 2$ we may replace $\mathfrak{C}_i''$ by an equivalent chamber $\mathfrak{C}_i'$ such that $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ are both based at p', and still $\tilde{\phi}(\mathfrak{C}_i) \sim_{\lambda\epsilon,p',\rho'} \mathfrak{C}_i'$ for any $\rho' \gg \max(\rho, N)$. For $i = 1, 2$, let $\mathfrak{D}_i = g_0\mathfrak{C}_i$, $\mathfrak{D}_i' = g_0'\mathfrak{C}_i'$. Then all four of these Weyl chambers are based at e, hence there exist k_i, $k_i' \in K$ such that $\mathfrak{D}_i = \pi(k_i A^+)$, $\mathfrak{D}_i' = \pi(k_i' A^+)$, $i = 1, 2$. By Step 4, $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ (hence also $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$) share a codimension 1 wall. Let $\theta = d_K(k_1, k_2)$, $\theta' = d_K(k_1', k_2')$. Hence (34) is equivalent to $C^{-1}\theta^{1/\lambda} \le \theta' \le C\theta^\lambda$.

Let $\tilde{\psi} = g_0' \circ \tilde{\phi} \circ g_0^{-1}$. Then, $\tilde{\psi} : \mathfrak{D}_1 \cup \mathfrak{D}_2 \to \mathrm{Nbhd}(\mathfrak{D}_1', N) \cup \mathrm{Nbhd}(\mathfrak{D}_2', N)$ is a graded quasi-isometric embedding, and $\tilde{\psi}(\mathfrak{D}_i) \sim_{\lambda\epsilon, e, \rho'} \mathfrak{D}_i'$, $i = 1, 2$. It follows from Proposition 3.4 (cf. the proof of Lemma 4.3) that

$$\tilde{\psi}(\partial \mathfrak{D}_i) \sim_{\lambda\epsilon, e, \rho'} \partial \mathfrak{D}_i'. \tag{35}$$

Let $Q \gg 1$ and $0 < \nu_3 < 1$ be as in Lemma B.6. In the following paragraph, the implicit constants do not depend on ν_3 and Q. Without loss of generality, θ is sufficiently small so that $\nu_3 |\log \theta| \gg \max(Q/\epsilon, N/\epsilon, \rho'/\epsilon)$. By Lemma B.6, there exist points $z_i \in \mathfrak{D}_i$ such that $d(z_1, z_2) \le Q$, $d(z_i, e) = O(|\log \theta|)$ and $d(z_i, \partial \mathfrak{D}_i) \ge \nu_3 |\log \theta|$. Then, since $\tilde{\psi}$ is a graded q.i. embedding and by (35), $d(\tilde{\psi}(z_i), \partial \mathfrak{D}_i') \succ \nu_3 |\log \theta|$. Also,

$$d(\tilde{\psi}(z_1), \tilde{\psi}(z_2)) = O(d(z_1, z_2)) + O(\epsilon d(z_1, e)) + O(\epsilon d(z_2, e)) \ll \nu_3 |\log \theta|$$

Let z_i' denote the nearest point projection of $\tilde{\psi}(z_i)$ onto $\mathfrak{D}_i'$. Then, there exists $r \succ \nu_3 |\log \theta|$ (with the implied constant depending only on κ, X) such that $d(z_1', z_2') \le \lambda_2 r$, $d(z_i', \partial \mathfrak{D}_i') \ge \nu' r$, where λ_2, ν' are as in Lemma B.4. Hence, by Lemma B.4, $\theta' \le c e^{-\nu r}$. Hence, $\theta' \le c\theta^{\lambda}$, where λ depends only on κ, X (and ν_3, but ν_3 and Q depend only on X).

The opposite inequality follows from applying the same argument to the map $\tilde{\psi}^* : \mathfrak{D}_1' \cup \mathfrak{D}_2' \to \mathfrak{D}_1 \cup \mathfrak{D}_2$ which is the coarse inverse of $\tilde{\psi}$ composed with nearest point projection. □

Proof of Lemma 5.3. We write $x_1 = kk_1$, $x_2 = kk_2$, where $k_i \in K_\alpha$. Let $L = \pi(kA_\alpha)$. Since $k \in \mathcal{U}_1$, $L \in \mathcal{L}$, it follows from Lemma 5.2 that there exists a hyperplane L' such that $\phi(\Omega_L) \subset \mathrm{Nbhd}(L', N)$. Pick a point x_0 in Ω_L. Then, in particular, $\phi(x_0) \in \mathrm{Nbhd}(L', N)$. Let $x_0' \in G$ be such that $L' = \pi(x_0' A_\beta)$ for some $\beta \in \Sigma$, and $\pi(x_0') \in L$ is the closest point to $\phi(x_0)$. Let $g_1' = x_0'^{-1}$, so that $g_1' \pi(x_0') = e$. Note that g_1', x_0' and x_0 are independent of x_1 and x_2.

Since $kk_i \in \mathcal{U}'$, by Lemma 5.2 parts (d) and (e) $\liminf_{r \to \infty} |\Omega_{F_1} \cap \Omega_{F_2} \cap \Omega_L \cap B(x_0, r)| \ge (1 - 3|\Sigma|\epsilon) |B(x_0, r) \cap L|$. Hence there exists a point $p \in L \cap \Omega_{F_1} \cap \Omega_{F_2} \cap \Omega_L$.

Since $p \in L$, $p = \pi(ka)$ for some $a \in A_\alpha$. For $i = 1, 2$ let $\mathfrak{C}_i$ denote the Weyl chambers $\pi(kak_iA^+)$. Then the $\mathfrak{C}_i$ are based at p. Since $k_i \in K_\alpha$, $a \in A_\alpha$ and K_α and A_α commute, $\mathfrak{C}_i$ is equivalent to $\pi(kk_iA^+)$. Hence $d_K(\mathfrak{C}_1, \mathfrak{C}_2) = d_K(k_1, k_2)$, and

$$\phi_0(x_i) = \phi_0(\pi(x_iA^+)) = \phi_0(\mathfrak{C}_i). \tag{36}$$

Let $g_0 = (ka)^{-1}$, so that $g_0 p = e$. Then $g_0 \mathfrak{C}_i = \pi(k_iA^+)$, hence

$$d_K(g_0\mathfrak{C}_1, g_0\mathfrak{C}_2) = d_K(k_1, k_2) = d_K(x_1, x_2). \tag{37}$$

Now let p' be the closest point in L' to $\phi(p)$. We have $d(p', \phi(p)) \le N$. Also, since $p' \in L'$, there exists $a' \in A_\beta$ such that $p' = \pi(x_0'a')$. Let $g_0' = (x_0'a')^{-1}$ so that $g_0'p' = e$.

For $i = 1, 2$, let $\mathfrak{C}_i'$ denote the Weyl chamber in the equivalence class $\phi_0(\mathfrak{C}_i)$ based at p'. Let L_+ denote the common codimension 1 wall of $\mathfrak{C}_1$ and $\mathfrak{C}_2$. By Step 4, $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ share a common codimension 1 wall H_+, and by Lemma 4.3, $\tilde{\phi}(L_+) \sim_{\lambda\epsilon} H_+$. But $\tilde{\phi}(L_+) \subset \tilde{\phi}(L) \sim_{\lambda\epsilon} L'$, hence outside a large ball, $H_+ \subset L'[\lambda\epsilon]$. Since H_+ is based at p' and $p' \in L'$, this implies that $H_+ \subset L'$. Hence $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ share a codimension 1 wall contained in L'.

Hence $\mathfrak{C}_i' = \pi(x_0'a'k_i'A^+)$, where $k_i' \in K_\beta$. Let $\mathfrak{D}_i' = \pi(x_0'k_i'A^+)$, so that $\mathfrak{D}_i'$ is based at $\pi(x_0')$. Since $a' \in A_\beta$, $k_i' \in K_\beta$ and A_β and K_β commute, $\mathfrak{C}_i'$ is equivalent to $\mathfrak{D}_i'$, so we may write $\phi_0(\mathfrak{C}_i) = \mathfrak{D}_i'$. Also note that $g_0'\mathfrak{C}_i' = \pi(k_i'A^+) = g_1'\mathfrak{D}_i'$, so we may write

$$g_0' \cdot \phi_0(\mathfrak{C}_i) = g_0'\mathfrak{C}_i' = g_1'\mathfrak{D}_i' = g_1' \cdot \phi_0(\mathfrak{C}_i) \tag{38}$$

By Lemma 5.4,

$$C^{-1} d_K(g_0\mathfrak{C}_1, g_0\mathfrak{C}_2)^{1/\lambda} \le d_K(g_0' \cdot \phi_0(\mathfrak{C}_1), g_0' \cdot \phi_0(\mathfrak{C}_2)) \le C d_K(g_0\mathfrak{C}_1, g_0\mathfrak{C}_2)^{\lambda} \tag{39}$$

Combining (36), (37), (38) and (39), we get

$$C^{-1} d_K(x_1, x_2)^{1/\lambda} \leq d_K(g_1' \cdot \phi_0(x_1), g_1' \cdot \phi_0(x_2)) \leq C d_K(x_1, x_2)^{\lambda} \tag{40}$$

Since g_1' acts as a diffeomorphism of the boundary, and is independent of x_1, x_2, this implies

$$c_2'' d_K(x_1, x_2)^{1/\lambda} \leq d_K(\phi_0(x_1), \phi_0(x_2)) \leq c_1'' d_K(x_1, x_2)^{\lambda}$$

where c_1'', c_2'' depend on g_1', hence L. ☐

5.4. **Differentiability and $\mathcal{T}(X)$.** In this section we complete the proof of Step 5. In view of Lemma 5.3 and Step 4 it is enough to prove the following:

Proposition 5.5. *Let $\mathcal{U}_1 \subset \hat{X}$ be a subset with $m_K(\mathcal{U}_1) = 1$, and let $\phi_0 : \mathcal{U}_1 \to \hat{X}$ be a Tits map defined on $\mathcal{U}_1$. Suppose ϕ_0 satisfies the conclusion of Lemma 5.3. Then, there exists a subset $\mathcal{U}_2 \subset \mathcal{U}_1$ with $m_K(\mathcal{U}_2) = 1$ such that the restriction of ϕ_0 to $\mathcal{U}_2$ is bi-Holder.*

The proof of Proposition 5.5 does not involve either quasi-isometries or ergodic theory; instead it deals with some extra structure of the building $\mathcal{T}(X)$. Lemma 5.3 does not immediately imply the statement of Step 5 because of the restriction to Weyl chambers with a common codimension 1 wall L, and because the constant C depends on L. The idea of the proof is to use Step 4 to show that finitely many applications of Lemma 5.3 imply the statement for ϕ_0.

Lemma 5.6 (Image of uniformly distributed set under ϕ_0 is dense). *Let E be a subset of full measure in K/M. Then, $\phi_0(E \cap \mathcal{U}_1)$ is dense in K/M.*

Proof. Without loss of generality, we may assume that $E \subset \mathcal{U}_1$. Let $E_1 = E \cap \{k \in K/M : m_{K_\alpha}(kK_\alpha \cap E) = 1,\ \forall \alpha \in \Delta\}$, where m_{K_α} is normalized Haar measure on K_α/M, and we denote the push forward measure on kK_α/M also by m_{K_α}. Then, by Fubini's theorem, $m_K(E_1) = 1$.

Suppose $k \in E_1$. Then we claim that $\phi_0(E)$ is dense in $\bigcup_{\alpha\in\Delta} k' K_\alpha$, where $k' = \phi_0(k)$. Indeed by Step 4 for each $\alpha \in \Delta$ there exists $\beta \in \Delta$ such that $\phi_0(kK_\alpha \cap E) \subset k' K_\beta$, and by Lemma 5.3 the associated map $\psi_\alpha = \ell_{k'}^{-1} \circ \phi_0 \circ \ell_k : K_\alpha/M \to K_\beta/M$ (where ℓ_k denotes left translation by k) is bi-Hölder on its domain of definition (i.e. $k^{-1}\mathcal{U}_1 \cap K_\alpha/M$). Hence ψ_α extends to an injective continuous map $\overline{\psi_\alpha} : K_\alpha/M \to K_\beta/M$. Choose α so that K_α/M has maximal dimension: then by invariance of domain (see e.g. [Mu, Theorem 36.5]) $\overline{\psi_\alpha}$ is a homeomorphism, and $\dim(K_\beta/M) = \dim(K_\alpha/M)$. Since $\overline{\psi_\alpha}$ is a homeomorphism, the image of the full measure (hence dense) set $E \cap kK_\alpha/M$ is dense. Proceeding in this way from maximal to minimal dimension of K_α/M, we see that for any $\alpha \in \Delta$, $\phi_0(E \cap kK_\alpha/M)$ is dense in $\phi_0(k)K_\beta/M$. This implies the claim.

Similarly, for $j \geq 2$, let $E_j = E_{j-1} \cap \{k \in K/M : m_{K_\alpha}(kK_\alpha \cap E_{j-1}) = 1\}$. Then for any j, $m_K(E_j) = 1$, and for any $k \in E_j$, $\phi_0(E_{j-1} \cap k \cup_{\alpha\in\Delta} K_\alpha/M)$ is dense in $\bigcup_{\alpha\in\Delta} \phi_0(k) K_\alpha$

To finish the proof of the lemma, note that the maximal combinatorial distance in the Tits building is $\ell < \infty$. Hence for every two elements $x, y \in K/M$ there exists a gallery of length ℓ connecting x and y, i.e. a sequence of elements $x = x_0, x_1, \ldots, x_\ell = y$, with $x_{i+1} \in x_i K_{\alpha_i}$, with $\alpha_i \in \Delta$. We fix $x \in \phi_0(E_\ell)$, and let y denote an arbitrary point in K/M. By the claim, there exists $x_1' \in \phi_0(E_{\ell-1})$ arbitrarily close to x_1. By the same argument, there exists $x_2' \in \phi_0(E_{\ell-2})$ arbitrarily close to x_2, etc. Thus there exists a point $x_\ell' \in \phi_0(E)$ arbitrarily close to y. This proves the lemma. ☐

Lemma 5.7 (The function f_α). *Suppose $t \in K_\alpha$, $s \in \mathcal{S}$, and $y = ts \notin \mathcal{S}$.*

(a) *tM is uniquely determined by y; we denote it by $f_\alpha(y)$. Also for $m \in M$, $f_\alpha(ym) = f_\alpha(y)$. There is a well defined function $f_\alpha : \mathcal{S}^c/M \to K_\alpha/M$.*
(b) *There exists a Zariski closed proper subset S_α of K/M such that the restriction of f_α to $\mathcal{S}_\alpha^c$ is real analytic.*
(c) *$f_\alpha(my) = mf_\alpha(y)$ for $m \in M$ and $y \in \mathcal{S}^c$.*

Proof. Since $s \in \mathcal{S}$, $s \in K \cap Bw'B$ for some $w' \in W$, where $w' = w_{\alpha_1} \dots w_{\alpha_r}$ and $r < \ell$, the maximal combinatorial distance (see the "algebraic preliminaries"). Then $s = k_{\alpha_1} \dots k_{\alpha_r}$ where $k_{\alpha_i} \in K \cap Bw_{\alpha_i}B = K_{\alpha_i}$. Consider the path (of length $r+1$) $x_0, \dots, x_{r+1}$ where $x_0 = eM$, $x_1 = tM$, $x_i = tk_1 \dots k_{i-1}M$, $1 \leq i \leq r+1$, so that $x_{r+1} = yM$. Thus $r+1 \geq d_c(yM, eM)$. But since $y \notin \mathcal{S}$, $d_c(y,e) = \ell$, hence $r = \ell - 1$. Hence $(x_0, \dots, x_{r+1})$ is a minimal gallery connecting the chambers eM and yM.

Since apartments in the Tits building are convex (see [B, Proposition, §IV.2, p. 88]), any minimal gallery connecting two chambers always lies in an apartment containing the two chambers. The apartment $\mathcal{A}$ containing eM and yM is unique since $y \notin \mathcal{S}$. Thus the chamber $x_1 = tM$ is uniquely determined as the chamber in $\mathcal{A}$ which is adjacent to the chamber eM and obtained from it by reflection in the wall associated to w_α. Hence $tM \in K_\alpha/M$ is determined uniquely by $yM \in \mathcal{S}^c/M$. This proves (a).

Since $r = \ell - 1$, s lies in the interior of a maximal non-open Bruhat cell. Let $K'_\alpha \subset K_\alpha$ be a submanifold containing t transversal to tM, so that near tM, K_α/M is locally diffeomorphic to K'_α. Then the product map $\Psi_\alpha : K'_\alpha \times \mathcal{S}/M \to K/M$ is real analytic; by part (a) it is one-to-one near (t,s), and $f_\alpha(\Psi_\alpha(t', s'M)) = t'M$. Since Ψ_α is one-to-one near (t,s), we can have $D\Psi_\alpha = 0$ only on a (possibly empty) proper Zariski closed subset V_α; then by the implicit function theorem, f_α is differentiable away from the proper Zariski closed subset $\Psi_\alpha(V_\alpha)$. It is easy to check that $\Psi_\alpha(V_\alpha)$ does not in fact depend on the choice of K'_α. This proves (b).

To show (c) note that if $m \in M$, and $t = f_\alpha(y)$, i.e. $t \in K_\alpha/M$ and $(eM, tM, tk_1M, \dots, tk_1 \dots k_{\ell-1}M = yM)$ is a minimal gallery connecting eM and yM, then $(meM, mtM, mtk_1M, \dots, mtk_1 \dots k_{\ell-1}M = myM)$ is a minimal gallery connecting $meM = eM$ to myM. Hence $f_\alpha(my) = mf_\alpha(y)$ for $m \in M$. $\square$

For $k \in K$, $\alpha \in \Delta$, we write $f_{k,\alpha}(y) = f_\alpha(k^{-1}y)$.

Lemma 5.8 (Coordinates on K/M).
- (i) *For any $x, y \in K/M$ with $x \neq y$, there exists $k \in K$ and $\alpha \in \Delta$ such that $f_{k,\alpha}(x) \neq f_{k,\alpha}(y)$.*
- (ii) *For any $x \in K/M$ there exist $\alpha_1, \dots, \alpha_m \in \Delta$ and an open full measure subset V of $\prod_{i=1}^m K$, invariant under right-multiplication by $\prod_{i=1}^m M$, such that for any $(k_1, \dots, k_m) \in V$ there exists an open neighborhood U of x such that the map $\Phi : U \to K_{\alpha_1}/M \times \dots K_{\alpha_m}/M$ given by $\Phi(y) = (f_{k_1,\alpha_1}(y), \dots, f_{k_m,\alpha_m}(y))$ is a diffeomorphism onto its image.*

Proof. To prove (i), we may assume without loss of generality that $x = e$. Suppose $f_\alpha(k^{-1}) = f_\alpha(k^{-1}y)$ for all α and all k such that both sides are defined. Now suppose $s \in \mathcal{S}_{\ell-1}$ where $\mathcal{S}_{\ell-1}$ is the subset of $\mathcal{S}$ with $d_c(s,e) = \ell - 1$. Then there exists $\alpha \in \Delta$ such that s forms a minimal gallery together with almost all elements $t \in K_\alpha$; then $f_\alpha(ts)$ is defined. Writing $k^{-1} = ts$, we get $t = f_\alpha(k^{-1}) = f_\alpha(k^{-1}y) = f_\alpha(tsy)$, i.e $sy \in \mathcal{S}$. Note that $f_\alpha(tsy)$ is defined for almost all t if s in a dense subset $\mathcal{S}'_{\ell-1}$ of $\mathcal{S}_{\ell-1}$. Hence we have $sy \in \mathcal{S}$ for all $s \in \mathcal{S}'_{\ell-1}$. However, $\mathcal{S}$ is closed and $\mathcal{S}'_{\ell-1}$ is dense in $\mathcal{S}$. Hence, $\mathcal{S}y \subset \mathcal{S}$. But this is clearly impossible unless $y = e$. Indeed, let $\mathcal{A}$ be an apartment containing e and y^{-1}, and let w denote the chamber in $\mathcal{A}$ opposite e. Then there exists a minimal gallery connecting e and w and passing though y^{-1} (see [B, Lemma, §IV.5, p. 92]). Thus $w = y^{-1}s$ for some $s \in \mathcal{S}$, hence $w^{-1} = s^{-1}y$. Since $\mathcal{S}^{-1} = \mathcal{S}$, $s^{-1} \in \mathcal{S}$. However $w^{-1} \notin \mathcal{S}$ hence we have a contradiction. This proves (i). Note that (i) implies that for any $x \neq y$ there exists $\alpha \in \Delta$ such that the set $\{k \in K \ : \ f_{k,\alpha}(x) \neq f_{k,\alpha}(y)\}$ is open and dense K.

Suppose (ii) does not hold. Then there exists a point $y \in K/M$ and a vector $X \in T_y(K/M)$ such that for all all $\alpha \in \Delta$ and all k with $k^{-1}y \in \mathcal{S}^c_\alpha$, $Df_{k,\alpha}(X) = 0$. Since for any $k_0 \in K$, $f_{k_0k,\alpha}(k_0y) = f_{k,\alpha}(y)$, the vector $k_0X \in T_{k_0y}(K/M)$ satisfies the same condition, i.e $Df_{k,\alpha}(k_0X) = 0$ for all $\alpha \in \Delta$ and all $k \in K$ with $k^{-1}(k_0y) \in \mathcal{S}^c_\alpha$. We thus define a vector field on K/M by choosing the vector k_0X in $T_{k_0y}(K/M)$. Let $\gamma : [0,1] \to K/M$ be a flow line for this vector field; then

$f_{k,\alpha}(\gamma(0)) = f_{k,\alpha}(\gamma(1))$ for all $\alpha \in \Delta$ and $k \in K$ such that both sides are defined. This contradicts (i). $\square$

Proof of Proposition 5.5. For $k \in \mathcal{U}_1$, $\phi_0(kK_\alpha) \subset \phi_0(k)K_\beta$, for some $\beta \in \Delta$.

Suppose $x \in \mathcal{U}_1 \cap \mathcal{S}^c$. Let $t = f_{k,\alpha}(x)$, i.e $k^{-1}x = ts$, $s \in \mathcal{S}$. Write $s = k_1 \dots k_{\ell-1}$ where $k_i \in K_{\alpha_i}$ for some $\alpha_i \in \Delta$. Then the gallery

$$(eM, tM, tk_1M, tk_1k_2M, \dots, tk_1 \dots k_{\ell-1}M)$$

is a minimal gallery connecting e and $k^{-1}x$. Hence

$$(kM, ktM, ktk_1M, ktk_1k_2M, \dots, ktk_1 \dots k_{\ell-1}M)$$

is a minimal gallery connecting k and x.

Suppose this gallery lies completely in $\mathcal{U}_1$ (this will happen for $x \in H_{k,\alpha}$ where $H_{k,\alpha} \subset K/M$ has full measure). Then

$$(\phi_0(k)M, \phi_0(kt)M, \phi_0(ktk_1)M, \phi_0(ktk_1k_2)M, \dots, \phi_0(ktk_1 \dots k_{\ell-1})M)$$

is also a gallery connecting $\phi_0(k)$ and $\phi_0(x)$. If $\phi_0(x) \notin \phi_0(k)\mathcal{S}$, then this gallery is minimal, and (since $\phi_0(kt) \subset \phi_0(k)K_\beta$), $f_{\phi_0(k),\beta}(\phi_0(x)) = \phi_0(k)^{-1}\phi_0(kt)$. Hence

$$f_{\phi_0(k),\beta}(\phi_0(x)) = \phi_0(k)^{-1}\phi_0(kf_{k,\alpha}(x)), \qquad \text{for } x \in H_{k,\alpha} \tag{41}$$

Choose $x \in \mathcal{U}_1$. By Lemma 5.8 for $1 \leq i \leq m$ there exists subsets $E_i \subset K$ of full measure such that if for $1 \leq i \leq m$, $k_i \in E_i$ then condition (ii) of Lemma 5.8 holds for $(k_1, \dots, k_m)$ and a neighborhood U_x of x. Now let V be as in Lemma 5.8 part (ii) but for the point $\phi_0(x)$ instead of x. By Lemma 5.6 the set $\prod_{i=1}^m \phi_0(E_i)$ is dense in $\prod_{i=1}^m K/M$. Since V is open there exist $k_i \in E_i$ such that $(\phi_0(k_1), \dots, \phi_0(k_m)) \in V$. Thus we may assume that the condition of part (ii) of Lemma 5.8 holds for x, the k_i, and for some neighborhood of U_x of x, and also for $\phi_0(x)$, the $\phi_0(k_i)$, and some neighborhood $U_{\phi_0(x)}$ of $\phi_0(x)$. Then for $y \in U_x \cap \phi_0^{-1}(U_{\phi_0(x)}) \cap_{i=1}^m H_{k_i,\alpha_i}$, using (41)

$$\phi_0(y) = \Phi^{-1}(\phi_0(k_1)^{-1}\phi_0(k_1f_{k_1,\alpha_1}(y)), \dots, \phi_0(k_m)^{-1}\phi_0(k_mf_{k_m,\alpha_m}(y))) \tag{42}$$

where Φ is the diffeomorphism from Lemma 5.8 at the point $\phi_0(x)$. By Lemma 5.3, the restriction of ϕ_0 to $k_iK_{\alpha_i} \cap \mathcal{U}_1$ is bi-Hölder. Then by (42), ϕ_0, restricted to $U_x \cap \phi_0^{-1}(U_{\phi_0(x)}) \cap_{i=1}^m H_{k_i,\alpha_i}$ is bi-Hölder, hence uniformly continuous. Then, by compactness, there exists a full measure subset $\mathcal{U}_2 \subset \mathcal{U}_1$ such that ϕ_0 restricted to $\mathcal{U}_2$ is uniformly continuous. Hence $\phi_0|_{\mathcal{U}_2}$ extends to a continuous map $\overline{\phi_0}$ defined on all of K/M. It is clear that $\overline{\phi_0}$ is locally injective; it remains to show that $\overline{\phi_0}$ is injective.

For every $x, y \in K$, there exists k, α so that (i) of Lemma 5.8 holds; for fixed k and α the set of (x, y) which satisfies this condition is open. Hence, using the compactness of $K \times K$ there exists a finite collection of $k_i \in \mathcal{U}_1$ such that if $f_{k_i,\alpha_i}(x) = f_{k_i,\alpha_i}(y)$ for all i for which these functions are defined, then $x = y$.

Suppose $\overline{\phi_0}$ is not injective. Then, since $\overline{\phi_0}$ is locally injective, there exist disjoint neighborhoods U and V such that $\overline{\phi_0}(U) = \overline{\phi_0}(V)$. Then, after passing to smaller neighborhoods if needed, there exists i such that $f_{k_i,\alpha_i}(U)$ and $f_{k_i,\alpha_i}(V)$ are disjoint. Since ϕ_0 restricted to $k_iK_{\alpha_i} \cap \mathcal{U}_1$ is bi-Hölder, $\overline{\phi_0}$ restricted to $k_iK_{\alpha_i}$ is injective. Hence $\overline{\phi_0}(k_if_{k_i,\alpha_i}(U))$ and $\overline{\phi_0}(k_if_{k_i,\alpha_i}(V))$ are disjoint.

The fact that $\overline{\phi_0}$ is locally onto implies that there exist $x \in U$, $y \in V$ such that $z = \overline{\phi_0}(x) = \overline{\phi_0}(y)$. By perturbing k_i we may make sure that $f_{\phi_0(k_i),\beta}(z)$ makes sense for all $\beta \in \Delta$, and still $\overline{\phi_0}(k_if_{k_i,\alpha_i}(x)) \neq \overline{\phi_0}(k_if_{k_i,\alpha_i}(y))$. But by (41),

$$f_{\overline{\phi_0}(k_i),\beta_i}(z) = \overline{\phi_0}(k_i)^{-1}\overline{\phi_0}(k_if_{k_i,\alpha_i}(x)) = \overline{\phi_0}(k_i)^{-1}\overline{\phi_0}(k_if_{k_i,\alpha_i}(y))$$

This contradicts the fact that $\overline{\phi_0}(k_if_{k_i,\alpha_i}(x)) \neq \overline{\phi_0}(k_if_{k_i,\alpha_i}(y))$. $\square$

6. The boundary map and the Tits building

In view of Step 5, there exists a continuous bijection $\overline{\phi_0} : \hat{X} \to \hat{X}$ such that ϕ_0 agrees with $\overline{\phi_0}$ almost everywhere.

Step 6 (ϕ_0 is induced by an isometry)**.** If G has no rank one factors, then there exists an element $g \in \mathrm{Isom}(X)$ such that for almost all $x \in \hat{X}$, $\phi_0(x) = g \cdot x$, where $\cdot$ denotes action on the boundary. If G has rank one factors but $\mathrm{rank}(G) \geq 2$, then ϕ_0 is a factor preserving map (possibly composed with a permutation of isomorphic factors).

Note. If we identify $\hat{X}$ with K/M, we may push forward the measure m_K to obtain a measure ν on $\hat{X}$. Even though ν depends on the identification (i.e. on the choice of basepoint), the measure class of ν is independent of all choices. By "almost all $x \in \hat{X}$" we mean "ν-almost-everywhere".

Proposition 6.1 ($\bar{\phi_0}$ induces a Tits isomorphism $\mathcal{T}(X) \to \mathcal{T}(X)$)**.** *There exists an order preserving bijection $\phi_G : \mathcal{T}(X) \to \mathcal{T}(X)$ such that the restriction of ϕ_G to $\hat{X}$ agrees with $\overline{\phi_0}$ (and hence with ϕ_0 almost everywhere).*

Proof of Proposition 6.1. The only properties of $\overline{\phi_0}$ used in the proof are that it is a uniformly continuous injective map which is Tits almost everywhere.

In view of our realization of $\mathcal{T}(X)$, it is enough to show that for any $\sigma \subset \Delta$, there exists $\tau \subset \Delta$ with $|\tau| = |\sigma|$ such that for any $k \in K$,

$$\overline{\phi_0}(kK_\sigma) = \overline{\phi_0}(k)K_\tau \tag{43}$$

Indeed, if (43) holds, we can define $\phi_G(kK_\sigma) = \overline{\phi_0}(k)K_\tau$; since ϕ_0 is uniformly continuous and injective, ϕ_G is an order preserving injection. The fact that ϕ_G is a bijection follows from (43) and invariance of domain (see e.g. [Mu, Theorem 36.5]).

We now prove (43). In particular, we need to show that τ in (43) (and Step 4) depends only on σ and not on $k \in K/K_\sigma$. Let π_σ denote the projection $K/M \to K/K_\sigma$. Then π_σ is clearly continuous. Let

$$E \subset \mathcal{U}_1 = \{k \in \mathcal{U}_1 \ : \ m_K(kK_\theta \cap \mathcal{U}_1) = m_K(kK_\theta) \text{ for all } \theta \subset \Delta\},$$

Then $m_K(E) = 1$, and thus in particular E is dense. For each τ with $|\tau| = |\sigma|$, let

$$F_\tau = \{z \in K/K_\sigma \ : \ \overline{\phi_0}(\pi_\sigma^{-1}(z)) \subset k'K_\tau \text{ for some } k' = k'(z)\} \tag{44}$$

Suppose for now that σ is such that $\dim K/K_\sigma$ is maximal, i.e $\dim K/K_\theta \leq \dim K/K_\sigma$ for all $\theta \subset \Delta$ with $|\theta| = |\sigma|$. Since $\overline{\phi_0}$ is continuous and bijective, invariance of domain implies that for every τ with F_τ non-empty, equality holds in (44), i.e $\overline{\phi_0}(\pi_\sigma^{-1}(z)) = k'K_\tau$. This also implies that the distinct F_τ are disjoint.

By Step 4, for each $k \in \mathcal{U}_1$ there exists $\tau \subset \Delta$ with $|\tau| = |\sigma|$ and $k' \in K$ such that $\phi_0(kK_\sigma \cap \mathcal{U}') \subset k'K_\tau$ Thus $E \subset \bigcup_{|\tau|=|\sigma|} \pi_\sigma^{-1}(F_\tau) \cap \mathcal{U}_1$, and if $x \in E$ and $\pi_\sigma(x) \in F_\tau$ then $\phi_0(xK_\sigma \cap \mathcal{U}') \subset k'K_\tau$ for some $k' \in K$.

We now claim that

$$\overline{\pi_\sigma(E) \cap F_\tau} \subset F_\tau. \tag{45}$$

Suppose $\pi_\sigma(x) \in \bar{F_\tau}$. Let $y \in K$ be any point such that $\pi_\sigma(x) = \pi_\sigma(y)$. Let x_j and y_j be two sequences in $E \cap \pi_\sigma^{-1}(F_\tau)$ converging to x and y respectively. Since $x_j \in E$ and $\lim \pi_\sigma(x_j) = \lim \pi_\sigma(y_j)$, there exists $y'_j \in E \cap x_jK_\sigma$ such that $d(y'_j, y_j) \to 0$. Since $\pi_\sigma(x_j) = \pi_\sigma(y'_j) \in F_\tau$, $\pi_\tau(\overline{\phi_0}(y'_j)) = \pi_\tau(\overline{\phi_0}(x_j))$. Since $\overline{\phi_0}$ and π_τ are both continuous, $\pi_\tau(\overline{\phi_0}(x)) = \lim \pi_\tau(\overline{\phi_0}(x_j)) = \lim \pi_\tau(\overline{\phi_0}(y'_j)) = \lim \pi_\tau(\overline{\phi_0}(y_j)) = \pi_\tau(\overline{\phi_0}(y))$. Thus, since $y \in \pi_\sigma^{-1}(\pi_\sigma(x))$ is arbitrary, $\pi_\sigma(x) \in F_\tau$. This proves (45).

We now claim that each (non-empty) F_τ is open. Indeed, suppose $x \in F_\tau$. Let B_j be a sequence of balls shrinking to x. If all sufficiently small B_j contained points in $F_{\tau'} \cap \pi_\sigma(E)$, where $\tau' \neq \tau$, then by (45) we would have $x \in F_{\tau'}$, which would contradicts the disjointness of the distinct F_τ. Hence for some n, $B_n \cap \pi_\sigma(E) \subset F_\tau$. Since $\pi_\sigma(E)$ is dense, this implies $B_n \subset F_\tau$, hence F_τ is open. Now by connectedness, one of the F_τ is all of K/K_σ, and the rest are empty. Hence (43) holds for σ.

Note that if (43) holds, then σ and τ determine each other. Thus to prove (43) for all σ (without the assumption that K/K_σ has maximal dimension), one can proceed by reverse induction on the dimension of K/K_σ. $\square$

Proof of Step 6. It is enough to prove that for all x in $K/M \cong \hat{X}$, $\overline{\phi_0}(x) = g \cdot x$ for some $g \in \text{Isom}(X)$. By Proposition 6.1, $\overline{\phi_0}$ induces a bijection ϕ_G of the Tits building associated to X into itself; the map on the cells of maximal dimension (i.e chambers) is $\overline{\phi_0}$, hence a homeomorphism. Thus, ϕ_G is an order preserving bijection of the Tits building onto itself which is also a homeomorphism of the Furstenberg boundary into itself. By the theorem of Tits ([Ti], see also [Mo, Corollary 16.2 and §17]) any such map is induced by an isometry as long as G has no rank one factors, and is factor preserving (up to a permutation) even if G has rank one factors. $\square$

7. The quasi-isometry ϕ is close to an isometry

In §7, §8 and §9 we assume that G has no rank one factors; we will return to the case where G has rank one factors in §10.

Step 7 (ϕ is close to an isometry)**.** There exists an isometry g and a number $C > 0$ such that for all $\gamma \in \Gamma$, $d(\phi(\gamma), \pi(g\gamma)) < C$.

We first note a corollary of Step 6:

Corollary 7.1. *Let $g \in \text{Isom}(X)$ be as in Step 6. Then there exists a subset $E_0 \subset G$ with the following properties:*

(a) *E_0 has full measure, i.e for any compact $U \subset G$, $m(E_0 \cap U) = m(U)$ where $m(\cdot)$ denotes Haar measure on G.*
(b) *Let $\mathcal{F}_0 \subset \mathcal{F}^*$ denote the family of flats $\{\pi(xA) \;:\; x \in E_0\}$. Then, for any $F \in \mathcal{F}_0$ and any $x \in \Omega_F$, we have $d(\phi(x), gF) \leq N$.*

Proof of Corollary 7.1. Let

$$E_0 = \{x \in p^{-1}(E^*) \text{ such that for every chamber } \mathfrak{C} \subset \pi(xA),\ \phi_0(\mathfrak{C}) = g \cdot \mathfrak{C}.\}$$

where E^* is as in the proof of Lemma 3.2. Then, part (a) holds since the condition $\phi_0(\mathfrak{C}) = g \cdot \mathfrak{C}$ holds a.e. by Step 6. The condition (b) holds by Step 3, since we may identify F' as gF. $\square$

We also note the following structural lemma, analogous to Lemma 3.1:

Lemma 7.2. *For every $\delta > 0$ there exists a subset $Q \subset K/M \times K/M$ with $m_K(Q) > 1 - \delta$ such that for $(k_1, k_2) \in Q$ and $k \in K$ we have $(kk_1, kk_2) \in Q$ and for any $h \in G$ the following hold:*

(a) *The flats $F_1 = \pi(hk_1A)$ and $F_2 = \pi(hk_2A)$ satisfy $F_1 \cap F_2 = \{\pi(h)\}$.*
(b) *The flats F_1 and F_2 are "almost orthogonal", i.e. there exists constants λ_1, λ_2 depending only on X and δ, such that if $r > \lambda_2$ then*

$$\text{Nbhd}(F_1, r) \cap \text{Nbhd}(F_2, r) \subset B(\pi(h), \lambda_1 r)$$

Proof. Without loss of generality, we may assume $h = 1$. Let $R = \bigcup_{w \in W} \bigcup_{\sigma \subset \Delta} K_\sigma w$. We choose Q of the form $\{k \in K \ : \ d_K(k, R) \geq \nu\}$, where $\nu = \nu(\delta, X)$. Suppose $x_1 \in F_1$, $x_2 \in F_2$. Then we may write $x_i = \pi(k_i' a_i)$ where $a_i \in A^+$ and $k_i' \in k_i W$. Note that for every $\sigma \subset \Delta$, $d_K(k_1', k_2' K_\sigma) > \theta$ where θ depends only on ν and X.

For $i = 1, 2$ there exists $\alpha_i \in \Delta$ such that $\alpha_i(x_i) \succ d(x_i, e)$. Let $\sigma = \{\alpha_1\} \cup \{\alpha_2\}$. Then, by Lemma B.3, if $s = \min_{i=1,2} d(x_i, e)$ is sufficiently large, then $d(x_1, x_2) \succ s$. This implies (b). $\square$

Proof of Step 7. Let $\mathcal{D} \subset G$ denote a fundamental domain for $\Gamma \backslash G$. For $u \in \mathcal{D}$ let $S_u = \{k \in K \ : \ p(uk) \in \Omega\ \}$, where Ω is as in Step 2 (e). By Step 2 (e) and Fubini's theorem there exists a compact subset U of $\mathcal{D}$ of positive measure such that for any $u \in U$, $m_K(S_u) \geq 1 - 3|\Sigma|\epsilon$. Then, in view of Lemma 7.2, for any $u \in U$, $m_K(Q \cap (S_u \times S_u)) > 0$.

Now $\gamma \in \Gamma$ be arbitrary. Let E_0 be as in Corollary 7.1. In view of Corollary 7.1 (a), we may choose $u \in U$ and $(k_1, k_2) \in Q \cap (S_u \times S_u)$ so that $\gamma u k_i \in E_0$ for $i = 1, 2$. Let $h = \gamma u$ and for $i = 1, 2$ let $F_i = \pi(h k_i A)$. Note that since U is compact, $d(\pi(\gamma), \pi(h)) < c$ where c depends only on U (and thus only on X).

Since $p(hk_i) \in \Omega$, $\pi(h) \in \Omega_{F_1} \cap \Omega_{F_2}$. Hence, by Corollary 7.1 (b),

$$\phi(\pi(h)) \subset \mathrm{Nbhd}(gF_1, N) \cap \mathrm{Nbhd}(gF_2, N)$$

Since g acts by isometries, and F_1, F_2 are almost orthogonal, the flats gF_1 and gF_2 are also almost orthogonal, and intersect at $\pi(gh) = g\pi(h) \in X$. Hence, by Lemma 7.2,

$$\phi(\pi(h)) \subset \mathrm{Nbhd}(gF_1, N) \cap \mathrm{Nbhd}(gF_2, N) \subset \mathrm{Nbhd}(gF_1 \cap gF_2, N') = \mathrm{Nbhd}(\pi(gh), N')$$

where N' depends only on N and X. Since $d(\pi(\gamma), \pi(h)) < c$, ϕ is a quasi-isometry and g is an isometry, for any $\gamma \in \Gamma$, $d(\phi(\gamma), \pi(g\gamma))$ is bounded independently of γ. $\square$

Remark: We actually proved that $d(\phi(\gamma), \pi(g\gamma))$ is bounded by a constant depending only on κ, C, X, and Γ.

8. The quasi-isometry ϕ is close to a commensuration

Step 8 (ϕ is close to a commensuration)**.** The element g of Step 7 belongs to $Comm(\Gamma)$.

Lemma 8.1. *Suppose $g \in Isom(X)$ is such that the orbit $p(g\Gamma) \subset \Gamma \backslash Isom(X)$ is bounded. Then $p(g\Gamma)$ is finite, and $g \in Comm(\Gamma)$.*

Proof. This lemma, and its equivalence to the statement of Step 8 are due to Nimish Shah, who proved a much more general statement in [Sh]. The proof uses ideas from the theory of unipotent flows. Recall that a finite index subgroup of Γ is generated by unipotent elements. By [Sh], $\overline{p(g)\Gamma} = p(F)$ for a closed subgroup F of $Isom(X)$ containing Γ. By Borel density, F^0 is normal in G, and $p(F^0)$ is closed. Hence $F^0 = G$ or $F^0 = \{1\}$. Since $p(F^0)$ is compact, $F^0 \neq G$. Hence F^0 is the identity, and thus $p(g\Gamma)$ is finite.

$\square$

Proof of Step 8. This follows immediately from Lemma 8.1. Indeed $\phi : \Gamma \to \Gamma$ is a quasi-isometry, and by Step 7 there exists $g \in Isom(X)$ and $c < \infty$ such that for all $\gamma \in \Gamma$, $d(\pi(g\gamma), \phi(\gamma)) < c$. Since $\phi(\gamma) \in \Gamma$, we get $d(p(g\gamma), p(\Gamma)) < c$. Hence the orbit $p(g\Gamma)$ is bounded in $\Gamma \backslash Isom(X)$. By Lemma 8.1 this implies $g \in Comm(\Gamma)$. $\square$

This completes the proof of Theorem 0.1.

9. Proof of Corollary 0.2

Proof of Corollary 0.2. This is by now standard; the relevant argument appeared in [CC], [S1], [FS], and [S2]. We give only a short outline for completeness. Suppose Λ is finitely generated, and we have a (κ, c) quasi-isometry $q : \Lambda \to \Gamma$. For each $\lambda \in \Lambda$, left multiplication by λ is an isometry $\ell_\lambda : \Lambda \to \Lambda$; so $q \circ \ell_\lambda \circ q^{-1}$ is a quasi-isometry from Γ to Γ. Thus we have a map $\Psi : \Lambda \to QI(\Gamma)$ sending λ to $q \circ \ell_\lambda \circ q^{-1}$, which one can verify to be a homomorphism. By Theorem 0.1 there is an isomorphism i of $QI(\Gamma)$ with $Comm(\Gamma)$. Let $\Phi = i \circ \Psi$. Thus $\Phi(\Lambda)$ is a subgroup of $Comm(\Gamma)$.

The fact that Λ is finitely generated and the definition of Φ imply that for any $r \gg 1$, the set $\{\lambda \in \Lambda \ : \ d(\Phi(\lambda)1, 1) < r\}$ is finite. Hence the kernel of Φ is finite, and also $\Phi(\Lambda)$ is discrete.

To show $\Phi(\Lambda)$ is a lattice, one may argue as follows. By the construction, uniformly for $\gamma \in \Gamma$, $d(\gamma, \Psi(q^{-1}(\gamma))1) < \infty$.

Note that for any $\lambda \in \Lambda$, $\Psi(\lambda)$ is a (κ, C) quasi-isometry $\Gamma \to \Gamma$, with constants κ and C which are independent of λ. Hence, since our bound in Theorem 0.1 depended only on Γ and the quasi-isometry constants, $d(\Psi(\lambda), \Phi(\lambda)) < \infty$ uniformly for $\lambda \in \Lambda$. Hence, uniformly for $\gamma \in \Gamma$, $d(\gamma, \Phi(q^{-1}(\gamma))1) < \infty$, i.e. every point $\gamma \in \Gamma$ is within a finite distance from a point in $\Phi(\Lambda)$. Hence $\Phi(\Lambda)$ has finite co-volume and is thus a lattice. $\square$

10. Rank One Factors

In this section we prove the following:

Proposition 10.1 (ϕ extends to all of G and is factor preserving)**.** *Let G be a semisiple Lie group with finite center and rank at least 2; suppose $G = G_1 \times G_2 \times \ldots \times G_n$ where each G_i is simple. Let Γ be an irreducible nonuniform lattice in G, and let ϕ be a quasi-isometry of Γ to itself, in the word metric. Then for each i there exists a quasi-isometry $\psi_i : G_i \to G_i$ (defined on all of G_i) such that ϕ is a bounded distance from the product map $\psi_1 \times \psi_2 \times \ldots \times \psi_n$ (possibly composed with an isometry permuting isomorphic factors). If G_i is any factor with* $\operatorname{rank} G_i \geq 2$ *then ψ_i can be taken to be an isometry.*

This is an intermideate result which is intended to be combined with results from [S1], [S2] and [FS] to give a generalization of Theorem 0.1 and Corollary 0.2 to the case where $\operatorname{rank} G \geq 2$ but G has rank one factors; see [Fa] for an outline of the argument. The proof of Proposition 10.1 is a variation on the argument in §7. We begin with the following:

Lemma 10.2. *Let G be as in Proposition 10.1. Fix i, $1 \leq i \leq n$, and let π_i denote the projection $G \to G_i/K_i$, where K_i is the maximal compact subgroup of G_i. Let $\mathcal{C}$ be a compact subset of $\Gamma \backslash G$. Suppose $x \in p^{-1}(\mathcal{C})$, $y \in p^{-1}(\mathcal{C})$, and $\pi_i(x) = \pi_i(y)$. Then there exist flats F_1, F_2, H_1, H_2 and points $x_0, y_0 \in G$ such that the following hold:*

(a) *$d(\pi(x), \pi(x_0)) < c_0$ and $d(\pi(y), \pi(y_0)) < c_0$ where c_0 is independent of x and y.*
(b) *$\{\pi(x_0)\} = F_1 \cap F_2$ and $\{\pi(y_0)\} = H_1 \cap H_2$.*
(c) *F_1 and F_2 are almost orthogonal (see Lemma 7.2); so are H_1 and H_2.*
(d) *$\pi(x_0) \in \Omega_{F_1} \cap \Omega_{F_2}$ and $\pi(y_0) \in \Omega_{H_1} \cap \Omega_{H_2}$, where sets of the type Ω_F are as in Step 3.*
(e) *For every chamber $\mathfrak{C}$ contained in F_1 or F_2 or H_1 or H_2, $\phi_0(\mathfrak{C}) = \overline{\phi_0}(\mathfrak{C})$.*
(f) *$\pi_i(F_1) = \pi_i(H_1)$ and $\pi_i(F_2) = \pi_i(H_2)$.*

Proof of Lemma 10.2. Let E_0 be as in the proof of Corollary 7.1, i.e.

$$E_0 = \{x \in p^{-1}(E^*) \text{ such that for every chamber } \mathfrak{C} \subset \pi(xA),\ \phi_0(\mathfrak{C}) = \overline{\phi_0}(\mathfrak{C})\}$$

where E^* is as in the proof of Lemma 3.2. Then E_0 has full measure. Let $E_0' = \{x \in G$ such that $xk \in E_0$ for almost all $k \in K\}$. Then E_0' also has full measure.

Let $\mathcal{D}$ be a fundamental domain for $\Gamma\backslash G$. Let $U \subset \mathcal{D}$ be as in the proof of Step 7. Since $x, y \in p^{-1}(\mathcal{C})$, $\mathcal{C}$ and U are compact, and E_0' has full measure, there exists $z \in G$ with $d(\pi(z), e) < c_0$, $xz \in p^{-1}(U) \cap E_0'$ and $yz \in p^{-1}(U) \cap E_0'$. Let $x_0 = xz$ and let $y_0 = yz$.

Let $Q \subset K/M \times K/M$ be as in Lemma 7.2. For $u \in U$ let $S_u \subset K/M$ be as in the proof of Step 7. By an argument identical to the one in §7, $m_K(Q \cap (S_{p(x_0)} \times S_{p(x_0)}) \cap (S_{p(y_0)} \times S_{p(y_0)})) > 0$. Hence there exist $(k_1, k_2) \in K/M \times K/M$ such that $(k_1, k_2) \in Q \cap (S_{p(x_0)} \times S_{p(x_0)}) \cap (S_{p(y_0)} \times S_{p(y_0)})$ and for $j = 1, 2$, $x_0 k_j \in E_0$ and $y_0 k_j \in E_0$. By construction, the flats $F_j = \pi(x_0 k_j A)$ and $H_j = \pi(y_0 k_j A)$ satisfy the conditions of the lemma. ☐

Lemma 10.3. *Let G be as in Proposition 10.1. Fix i, $1 \le i \le n$, and let $\pi_i(\cdot)$ and $\mathcal{C}$ be as in Lemma 10.2. Then if $x \in p^{-1}(\mathcal{C})$, $y \in p^{-1}(\mathcal{C})$, and $\pi_i(x) = \pi_i(y)$, then $d(\pi_i(\phi(x)), \pi_i(\phi(y))) < C$, where C does not depend on x and y.*

Proof of Lemma 10.3. Let x_0, y_0, F_1, F_2, H_1 and H_2 be as in Lemma 10.2. By Step 3, for $j = 1, 2$ there exist flats F_j', H_j' such that $\phi(\Omega_{F_j}) \subset \mathrm{Nbhd}(F_j', N)$ and $\phi(\Omega_{H_j}) \subset \mathrm{Nbhd}(H_j', N)$, Hence $\phi(x_0) \subset \mathrm{Nbhd}(F_1', N) \cap \mathrm{Nbhd}(F_2', N)$ and $\phi(y_0) \subset \mathrm{Nbhd}(H_1', N) \cap \mathrm{Nbhd}(H_2', N)$.

Note that since F_1, F_2 and H_1, H_2 are almost orthogonal, there exist points x', y' and a constant C' independent of x and y such that $\mathrm{Nbhd}(F_1', N) \cap \mathrm{Nbhd}(F_2', N) \subset \mathrm{Nbhd}(x', C')$ and $\mathrm{Nbhd}(H_1', N) \cap \mathrm{Nbhd}(H_2', N) \subset \mathrm{Nbhd}(y', C')$. Thus $d(\phi(x_0), x') < C''$, and $d(\phi(y_0), y') < C''$, where C'' does not depend on x and y.

Since $\pi_i(F_1) = \pi_i(H_1)$ and $\pi_i(F_2) = \pi_i(H_2)$, by Step 6 we have $\pi_i(F_1') = \pi_i(H_1')$ and $\pi_i(F_2') = \pi_i(H_2')$. Hence $d(\pi_i(\phi(x)), \pi_i(\phi(y))) < C$ as required. ☐

Proof of Proposition 10.1. Since Γ is irreducible, $\pi_i(p^{-1}(\mathcal{C})) = G_i/K_i$. If $u \in G_i/K_i$, choose $x \in \pi_i^{-1}(u) \cap p^{-1}(\mathcal{C})$, and let $\psi_i(u) = \pi_i(\phi(x))$. In view of Lemma 10.3, this (up to a fixed additive constant) does not not depend on the choice of x. To see that ψ_i is a quasi-isometry, we may argue as follows: suppose $u_1, u_2 \in G_i/K_i$. It follows, e.g. from the Moore ergodicity theorem applied to the action of the Cartan subgroup of $\prod_{j \ne i} G_j$ on $\Gamma\backslash G$ that there exist points $x_1, x_2 \in p^{-1}(\mathcal{C})$ such that $\pi_i(x_1) = u_1$, $\pi_i(x_2) = u_2$, and $\pi_j(x_1) = \pi_j(x_2)$ for $j \ne i$. Note that $d(x_1, x_2) = d(u_1, u_2)$, and by Lemma 10.3, $|d(\psi_i(u_1), \psi_i(u_2)) - d(\phi(x_1), \phi(x_2))| < C''$ where C'' is fixed. Since ϕ is a quasi-isometry, this implies that ψ_i also is. ☐

A. Quasi-isometries

A.1. **The packing lemma and its corollaries.** The following lemma is a coarse version of invariance of domain. The proof given in [EF, §8.2] is a variation of an argument due to Geoff Mess[1]. See also [FS, Corollary 5.3].

Lemma A.1 (Local Packing)**.** *Let $q : B \to \mathbb{R}^n$ be a continuous map which is a (κ, C)-quasi-isometric embedding from the ball $B = B(x, r_1) \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Then there exist constants $\eta_1 > 1, \eta_2 > 1$ depending only on κ such that for any $r < r_1$,*

$$\mathrm{Nbhd}(q(B), \eta_1 C) \supset B(q(x), r/\eta_2)$$

(i.e. $q(B)$ essentially contains the ball $B(q(x), r/\eta_2)$).

Corollary A.2 (Packing of metric interiors)**.** *Let U, V be subsets of $\mathbb{R}^n$ and let $q : U \to \mathbb{R}^n$ be a (κ, C) quasi-isometric embedding. Suppose there exists $R > 0$ with $\mathrm{Int}(V, R)$ path connected and $q^{-1}(\mathrm{Int}(V, R))$ a non-empty subset of $\mathrm{Int}(U, 4\eta_1\eta_2 C)$. Then $\mathrm{Int}(V, R) \subset \mathrm{Nbhd}(q(U), \eta_1 C)$.*

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[1] Thanks to Shmuel Weinberger for telling us about this argument.

Proof. Let $W = \mathrm{Nbhd}(q(U), \eta_1 C)$. Take any $y \in q(U) \cap \mathrm{Int}(V, R)$. Let $r_1 = 4\eta_1\eta_2 C$. By assumption, $q^{-1}(y) \in \mathrm{Int}(U, r_1)$, hence by Lemma A.1, $W \supset B(y, 3\eta_1 C)$. Since $W \cap \mathrm{Int}(V, R)$ is non-empty and $\mathrm{Int}(V, R)$ is connected, this implies that $W \supset \mathrm{Int}(V, R)$. □

Lemma A.3 (Packing of complexes)**.** *Let $\mathcal{P}_1, \cdots, \mathcal{P}_m$ be isometrically embedded d-dimensional convex Euclidean polyhedra in X. Let B be a ball in X, and suppose $q : \mathbb{R}^d \to X$ is a (κ, C) quasi-isometric embedding such that*

$$q(\mathbb{R}^d) \cap B \subset \bigcup_{i=1}^{m} \mathcal{P}_i$$

Suppose the $\mathcal{P}_i$ "come close only near their boundaries", i.e. there exists a constant $r > (4\kappa\eta_1\eta_2+1)C$ (where η_1, η_2 are as in Lemma A.1) such that for all $i \neq j$,

$$\mathrm{Int}(\mathcal{P}_i, r) \cap \mathrm{Nbhd}(\mathcal{P}_j, (4\kappa\eta_1\eta_2 + 1)C) = \emptyset \tag{46}$$

Suppose there is a constant λ'' such that for all i,

$$hd(\mathrm{Int}(\mathcal{P}_i, r), \mathcal{P}_i) \leq \lambda'' r. \tag{47}$$

Then there exists a subset σ of $\{1, \ldots, m\}$ and a constant λ' (depending only on λ'', κ, m and the number faces of the $\mathcal{P}_i$) such that

$$hd\left(q(\mathbb{R}^d) \cap B', \bigcup_{i \in \sigma} \mathcal{P}_i \cap B'\right) \leq \lambda' r$$

where $B' = \mathrm{Int}(B, \lambda' r)$.

Proof. Let $V_j = \mathcal{P}_j \cap B$, and let $\sigma = \{j \in [1, m] \ : \ q(\mathbb{R}^d) \cap \mathrm{Int}(V_j, r) \neq \emptyset\}$. By convexity V_j is connected for every $j \in \sigma$.

For $j \in \sigma$, let $U_j = q^{-1}(V_j)$. Now suppose $x \in q^{-1}(\mathrm{Int}(V_j, r))$. Then

$$q(B(x, 4\eta_1\eta_2 C)) \subset B(q(x), (4\kappa\eta_1\eta_2 + 1)C) \cap q(\mathbb{R}^d) \cap B \subset V_j$$

by (46). Hence $q^{-1}(\mathrm{Int}(V_j, r)) \subset \mathrm{Int}(U_j, 4\eta_1\eta_2 C)$, and thus by Corollary A.2, $\mathrm{Int}(V_j, r) \subset \mathrm{Nbhd}(q(U_j), \eta_1 C)$. Since $q(U_j) \subset V_j$ and using (47), $hd(q(U_j), V_j) \leq \lambda'' r$. Hence

$$\bigcup_{j \in \sigma} V_j \subset \mathrm{Nbhd}(q(\mathbb{R}^d), \lambda'' r) \tag{48}$$

Suppose $\rho \gg C$, and $x \in q(\mathbb{R}^d) \cap \mathrm{Int}(B, \rho)$. Then, since $q(B(q^{-1}(x), \rho/(2\kappa))) \subset q(\mathbb{R}^d) \cap B(x, \rho)$ and using again Lemma A.1, there exists a constant ν depending only on κ, and d such that

$$\mathrm{Vol}_d(q(\mathbb{R}^d) \cap B(x, \rho)) \geq \nu \rho^d \tag{49}$$

where Vol_d denotes the d-dimensional volume.

Now suppose there exists $x \in q(\mathbb{R}^d)$ such that $x \in B'$, but $d(x, \bigcup_{j \in \sigma} V_j) \geq \rho$. Then, $q(\mathbb{R}^d) \cap B(x, \rho) \subset \bigcup_{j \notin \sigma} V_j$. But then, since $q(\mathbb{R}^d) \cap \mathrm{Int}(V_j, r) = \emptyset$ for $j \notin \sigma$, $\mathrm{Vol}_d(q(\mathbb{R}^d) \cap B(x, \rho)) \leq \lambda'' r \rho^{d-1}$, where λ'' depends only on m, d, and the number of faces of the $\mathcal{P}_i$. When $\rho > (\lambda''/\nu) r$, this contradicts (49). Let $\lambda' = \lambda''/\nu$. Then,

$$q(\mathbb{R}^d) \cap B' \subset \mathrm{Nbhd}(\bigcup_{j \in \sigma} V_j, \lambda' r) \tag{50}$$

The estimates (48) and (50) imply the lemma. □

A.2. **Closeness sets and intersections.**

Lemma A.4 (Quasi-isometries preserve closeness sets)**.** *Suppose X and Y are metric spaces, $q : X \to Y$ is a (κ, C) quasi-isometric embedding, U and V are subsets of X, and U', V' are subsets of Y.*

(i) *Suppose $q(U) \subset U'$, $q(V) \subset \mathrm{Nbhd}(V', C')$, and $r > C' > C$. Then*
$$q(U \cap \mathrm{Nbhd}(V, r)) \subset U' \cap \mathrm{Nbhd}(V', 3\kappa r)$$

(ii) *Suppose $hd(q(U), U') < C'$, $hd(q(V).V') < C'$ and $r > 3\kappa C' > 3\kappa C$. Suppose also that there exists a constant λ such that*

$$U' \cap \mathrm{Nbhd}(V', 3\kappa r) \subset \mathrm{Nbhd}(U' \cap V', \lambda r) \text{ and } \tag{51}$$
$$U \cap \mathrm{Nbhd}(V, 3\kappa r) \subset \mathrm{Nbhd}(U \cap V, \lambda r)$$

Then,
$$hd\,(q(U \cap V), U' \cap V') < \lambda' r$$
where λ' depends only on κ and λ.

Proof. Suppose $u \in U \cap \mathrm{Nbhd}(V, r)$, then there exists $v \in V$ with $d(u, v) \le r$. Since $r > C$, by (1), $d(q(u), q(v)) \le 2\kappa r$. By assumption, there exists $v' \in V'$ such that $d(q(v), v') \le C'$. Then, $d(q(u), v') \le 2\kappa r + C' \le 3\kappa r$, hence $q(u) \in U' \cap \mathrm{Nbhd}(V', 3\kappa r)$.

To get (ii) we first modify q by $O(C')$ so that $q(U) \subset U'$. Then by part (i), $q(U \cap V) \subset q(U \cap \mathrm{Nbhd}(V, r)) \subset U' \cap \mathrm{Nbhd}(V', 3\kappa r)$. Hence, $q(U \cap V) \subset \mathrm{Nbhd}(U' \cap V', \lambda r)$. Applying the same argument to the coarse inverse of q shows that $U' \cap V' \subset \mathrm{Nbhd}(q(U \cap V), \lambda' r)$. □

B. Geometric Estimates

B.1. **Geometry of neighboring flats.** In this subsection we prove the following lemma:

Lemma B.1 (Geometry of neighboring flats)**.** *(cf.* [Mo, Theorem 7.8]*.)*

(i) *There exist constants λ_0, λ_1, λ_2 depending only on X such that the following holds: For each pair of flats $F_1, F_2 \subset X$ there exist constants $c_\alpha \ge 0$ such that if we denote*
$$\mathcal{P}(r) = \{x \in F_1 \;:\; c_\alpha e^{\alpha(x)} \le e^r, \quad \forall \alpha \in \Sigma\} \tag{52}$$
then for $r > \lambda_0(1 + d(F_1, F_2))$, we have $\mathcal{P}(\lambda_1 r) \subset F_1 \cap \mathrm{Nbhd}(F_2, r) \subset \mathcal{P}(\lambda_2 r)$. (Note that if $c_\alpha = 0$, then the inequality (52) is always satisfied). In particular, there exists a convex polyhedron $\mathcal{P} \subset F_1$ such that $hd(F_1 \cap \mathrm{Nbhd}(F_2, r), \mathcal{P}) = O(r)$.

(ii) *Let L_1 be a hyperplane in F_1. Then there exists a hyperplane $L_2 \subset F_2$ such that for all $r > \lambda_0(1 + d(F_1, F_2))$, we have $F_2 \cap \mathrm{Nbhd}(L_1, r) \subset \mathrm{Nbhd}(L_2, \lambda r)$ where λ depends only on X.*

(iii) *Let $\mathcal{P}_1$ be a (possibly unbounded) polyhedron in F_1, and $r > \lambda_0(1 + d(F_1, F_2))$. Then there exists a polyhedron $\mathcal{P}_2 \subset F_2$ such that $hd(F_2 \cap \mathrm{Nbhd}(\mathcal{P}_1, r), \mathcal{P}_2) \le \lambda r$, where λ depends only on X.*

The proof of Lemma B.1 uses the following:

Lemma B.2 (Gaussian Elimination)**.** *Let MAN denote the Langlands decomposition of a Borel subgroup B where M is compact. Let $W = N_K(A)/M$ denote the Weyl group associated to (G, A), let w_0 denote the longest element of W and let $\bar{N}$ denote $w_0 N w_0^{-1}$.*

For every compact set $\mathcal{C} \subset G$, there exist compact sets $U_N \subset N$, $U_A \subset A$, and $U_{\bar{N}} \subset \bar{N}$ such that $\mathcal{C} \subset W U_{\bar{N}} M U_A U_N$. The set U_N can be chosen so that $a^{-1} U_N a \subseteq U_N$ for $a \in \overline{A_+}$.

Proof of Lemma B.2. Note that for $w \in W$, $w^{-1}BwB \subset w_0^{-1}Bw_0B$. Then, for $w \in W$, $BwB = w(w^{-1}Bw)B \subset w(w_0^{-1}Bw_0)B$. Hence, using the Bruhat decomposition,

$$G = \bigcup_{w \in W} w(w_0^{-1}Bw_0)B. \tag{53}$$

Now for g in $(w_0^{-1}Bw_0)B$, we can write $g = \bar{n}man$, where $n \in \bar{N}$, $m \in M$, $a \in A$, $n \in N$, and the functions $g \longrightarrow \bar{n}(g)$ etc. are clearly continuous (e.g. one can see this by embedding in $SL(n, \mathbb{R})$). Thus these functions are uniformly continuous on any compact set in the interior of $(w_0^{-1}Bw_0)B$. This, together with (53) implies the lemma.

We remark that the $SL(n, \mathbb{R})$ case of this lemma is the well known "complete pivoting" algorithm for inverting a matrix. In this case N is the set of upper triangular matrices with 1's along the main diagonal, $\bar{N}$ is the set of lower triangular matrices with 1's along the main diagonal, A is the set of diagonal matrices of determinant 1 and M is the finite group of diagonal matrices with ± 1 along the diagonal (and determinant 1). Thus (after a suitable permutation corresponding to W) one must factor a matrix into a product of an upper triangular and a lower triangular matrix; this is done by Gaussian elimination, except that one always performs row exchanges in order to divide by as large a number as possible. □

Proof of Lemma B.1. Without loss of generality, we may assume $F_2 = \pi(A)$, and that there exists a point $\pi(g)$ on F_1 such that $F_1 = \pi(gA)$, and $d(\pi(g), e) < d(F_1, F_2) + 1$. In the sequel we abuse notation by writing for $g_1, g_2 \in G$, $d(g_1, g_2)$ instead of $d(\pi(g_1), \pi(g_2))$.

We first prove the following inequality: there exists $w \in W$ and a constant λ depending only on X such that for any $a \in A^+$

$$\lambda^{-1} d(wa, ga) - O(d(g,1)) \leq d(F_2, \pi(ga)) \leq \lambda d(wa, ga) + O(d(g,1)) \tag{54}$$

(all the implied constants depend only on X). This is proved as follows. Using the Iwasawa decomposition, we may write $g = ka_1n_1$ where $k \in K$, $a_1 \in A$, $n_1 \in N$. Then

$$d(g, 1) = d(a_1n_1, 1) = d(n_1, a_1^{-1}). \tag{55}$$

But $d(n_1, a_1^{-1}) \geq d(n_1, A) \geq \lambda^{-1} d(n_1, 1)$ where the last inequality uses [EF, Lemma A.3] (or [Mo, Lemma 7.6]). Hence

$$d(n_1, 1) \leq \lambda d(g, 1). \tag{56}$$

Then from (55) and (56),

$$d(a_1, 1) = d(1, a_1^{-1}) \leq d(1, n_1) + d(n_1, a_1^{-1}) \leq (\lambda + 1) d(g, 1) \tag{57}$$

Then using Lemma B.2 we may write $k = w\bar{n}ma_2n_2$ where $w \in W$, $\bar{n} \in \bar{N}$, $m \in M$, $a_2 \in A$, $n_2 \in N$, where $d(a_2, 1) = O(1)$ and $d(n_2, 1) = O(1)$. Then, $g = w\bar{n}ma_3n_3$, where $a_3 = a_2a_1 \in A$ and $n_3 = (a_1^{-1}n_2a_1)n_1 \in N$. By (57) and the triangle inequality, $d(a_3, 1) = O(d(g, 1))$. Also by (57), (56) and the triangle inequality, $d(n_3, 1) = O(d(g, 1))$. Thus,

$$\begin{aligned} d(A, ga) &= d(A, w\bar{n}ma_3n_3a) \\ &= d(A, wa(a^{-1}\bar{n}a)(ma_3a^{-1}n_3a)) \\ &= d(a^{-1}w^{-1}A, (a^{-1}\bar{n}a)(ma_3a^{-1}n_3a)) \\ &= d(A, (a^{-1}\bar{n}a)(ma_3a^{-1}n_3a)) && \text{since } wAw^{-1} = A \text{ and } a \in A \\ &= d(A, a^{-1}\bar{n}a) + O(d(ma_3a^{-1}n_3a, 1)) && \text{by the triangle inequality} \\ &= d(A, a^{-1}\bar{n}a) + O(d(g, 1)) \end{aligned} \tag{58}$$

where in the last line we used $d(n_3,1) = O(d(g,1))$, $d(a_3,1) = O(d(g,1))$ and the fact that conjugation by a is non-expanding on N. Similarly,

$$\begin{aligned} d(wa, ga) &= d(wa, w\bar{n}ma_3n_3a) = d(wa, wa(a^{-1}\bar{n}a)(ma_3a^{-1}n_3a)) \\ &= d(1, a^{-1}\bar{n}a) + O(d(g,1)) \qquad (59)\end{aligned}$$

But by [EF, Lemma A.3] (or [Mo, Lemma 7.6]), for $\bar{n}' \in \bar{N}$, $\lambda^{-1}d(1,\bar{n}') \le d(A,\bar{n}') \le \lambda d(1,\bar{n}')$. This, together with (58) and (59) implies (54).

Recall that W acts transitively on the Weyl chambers of A based at e. Hence any chamber based at g of F_1 is of the form $\mathfrak{C}_w = \pi(gw\overline{A_+}w^{-1})$ for some $w \in W$. Then by (54), for each chamber $\mathfrak{C}_w$ of F_1 based at $\pi(g)$, there exists $w' \in W$ such that

$$d(w'a, gwa) = O(r), \qquad \text{when} \qquad \pi(gwa) \in \mathfrak{C}_w \cap \mathrm{Nbhd}(F_2, r).$$

Thus on $\mathfrak{C}_w \cap \mathrm{Nbhd}(F_2,r)$, the map $\psi_{w'} : F_1 \to F_2$ given by $\psi_{w'}(\pi(gwa)) = \pi(w'a)$ moves each point by at most $O(r)$. Putting these together we get a map $\psi : F_1 \cap \mathrm{Nbhd}(F_2, r) \to F_2$ which is "coarsely the identity", i.e it moves each point by at most $O(r)$. This implies that the w' can be chosen in a consistent way, i.e. there exists $w'' \in W$ such that

$$d(\psi(\pi(ga)), \pi(ga)) = d(w''a, ga) = O(r), \qquad \text{when} \qquad \pi(ga) \in F_1 \cap \mathrm{Nbhd}(F_2, r). \qquad (60)$$

The equation (60) implies (ii), since we may take $L_2 = \psi(L_1)$.

We now prove (i). In view of (60),

$$d(w''a, ga) = O(r) \quad \text{if and only if} \quad d(A, ga) = O(r) \qquad (61)$$

First we show that for each $w \in W$ and each $\alpha \in w\Sigma^+$, there exists a constant $c_{\alpha,w} \ge 0$ such that if $\mathcal{P}_w(r) = \{x \in \mathfrak{C}_w \;:\; c_{\alpha,w}e^{\alpha(x)} \le e^r\}$ for all $\alpha \in w\Sigma^+$ then

$$\mathcal{P}_w(\lambda_1''r) \subset \mathfrak{C}_w \cap \mathrm{Nbhd}(F_2, r) \subset \mathcal{P}_w(\lambda_2''r) \qquad (62)$$

To prove (62) we may assume $w = 1$ (to handle the other Weyl chambers in F_1, we may replace g by gw for a suitable $w \in W$). We use (58), for $a \in \overline{A_+}$. Write $\bar{n} = \exp u$, where $u \in Lie(\bar{N})$. Then we may write $u = \sum_{\alpha\in\Sigma^+} u_\alpha X_{-\alpha}$ where the $X_{-\alpha}$ are the bases for the root spaces, and $u_\alpha \in \mathbb{R}$. (For notational simplicity, if the multiplicity m_α of α is greater then 1, we think of α as repeated m_α times in Σ). Then $a^{-1}\bar{n}a = \exp(\sum_{\alpha\in\Sigma^+} u_\alpha e^{\alpha(a)} X_{-\alpha})$.

By [EF, Lemmas A.2 and A.3] (or by [Mo, Lemma 7.6]), for $u \in Lie(\bar{N})$, if either $\|u\| \gg 1$ or $d(\pi(\exp u), \pi(A)) \gg 1$, $d(\pi(\exp u), \pi(A))$ is comparable to $\log \|u\|$, where $\|\cdot\|$ is the Euclidean norm on $Lie(\bar{N})$. In our case $\|u\| \gg 1$ since $r \gg 1$. Hence

$$\lambda_1 \log(\sum_{\alpha\in\Sigma^+} u_\alpha^2 e^{2\alpha(a)}) \le d(e, \pi(a^{-1}\bar{n}a)) \le \lambda_2 \log(\sum_{\alpha\in\Sigma^+} u_\alpha^2 e^{2\alpha(a)})$$

We write $a = \exp x$ for $x \in Lie(A) = \mathbb{R}^n$. Then by definition (see [EF, §2]), $\alpha(a) = \alpha(x)$, and the function α on $\mathbb{R}^n$ is linear. Then

$$\max_{\alpha\in\Sigma^+} \lambda_1 \log(u_\alpha^2 e^{\alpha(2x)}) \le d(e, \pi(a^{-1}\bar{n}a)) \le \max_{\alpha\in\Sigma^+} \lambda_2 \log(C u_\alpha^2 e^{\alpha(2x)})$$

Exponentiating this, and using (58) we get:

$$\max_{\alpha\in\Sigma^+} u_\alpha^{2\lambda_1} e^{\alpha(2x\lambda_1)-\delta} \le \exp d(F_2, \pi(ga)) \le C^{\lambda_2} \max_{\alpha\in\Sigma^+} u_\alpha^{2\lambda_2} e^{\alpha(2x\lambda_2)+\delta}$$

Since $\mathfrak{C} \cap \mathrm{Nbhd}(F_2, r)$ is the set $\{g \exp x \in \mathfrak{C} \mid d(F_2, \pi(g \exp x)) \le r\}$, this implies (62).

Pick $r_0 \gg 1$ (depending only on X), and let $\mathcal{Q} = \bigcup_{w\in W} \mathcal{P}_w(r_0)$. Then, by the structure of $\mathcal{P}_w(r)$, $hd(F_1 \cap \mathrm{Nbhd}(F_2, r), \mathcal{Q}) = O(r)$. We now show that there exists a convex polyhedron $\mathcal{P}$ such that

$hd(\mathcal{P}, \mathcal{Q}) = O(r)$. This would imply (i). First suppose $G = SL(n+1, \mathbb{R})$. In view of (61) (changing w'' to w) it is enough to show that for some polyhedron $\mathcal{P}$,

$$hd(\{\pi(ga) \in F_1 \ : \ d(\pi(ga), \pi(wa)) \leq r\}, \mathcal{P}) = O(r). \tag{63}$$

But,

$$d(\pi(ga), \pi(wa)) = d(\pi(a^{-1}w^{-1}ga), e)$$

Let g_{ij} denote the entry of $w^{-1}g$ at row i column j. Then, by [EF, Lemma A.2], $\lambda^{-1}\max_{ij} |(a_i/a_j)g_{ij}|^2 \leq d(\pi(a^{-1}w^{-1}ga), e) \leq \lambda \max_{ij} |(a_i/a_j)g_{ij}|^2$ where λ depends only on X. But $a_i/a_j = e^{\alpha(\log a)}$ for some root α of $SL(n+1, \mathbb{R})$. This implies (63) for the $SL(n+1, \mathbb{R})$ case. For the general case, note that we may take a linear representation of G, i.e an embedding G in $SL(m, \mathbb{R})$ for some m. Then the result for $SL(m, \mathbb{R})$ implies (63) but a priori $\mathcal{P}$ may be bounded by generalized hyperplanes of the form $\omega(x) \leq c_\omega$, where the ω's are the weights of the representation. Still, there are finitely many weights, and each ω is given by an expression of the form $\omega = \sum_\alpha c_\alpha \alpha$, where the implied constants depend only on X. This, together with the fact that $hd(\mathcal{P}, \mathcal{Q}) = O(r)$ where $\mathcal{Q}$ is a union of (non-generalized) polyhedra implies that $\mathcal{P}$ may also be taken to be bounded by non-generalized hyperplanes.

To get (iii) we can take $\mathcal{P}_2 = \psi(\mathcal{P}_1) \cap \mathcal{P}(r)$ where $\mathcal{P}(r)$ is as in part (i) with F_1 and F_2 interchanged. □

B.2. **Angles between Weyl Chambers.** Let $\Theta(\cdot)$ denote the function defined in [EF, §2.1].

Lemma B.3 (Hyperbolic geometry of X)**.** *There exist constants ν, λ_2 depending only on X such that the following holds: Suppose $x, y \in X$ satisfy $\alpha(x) \geq r$, $\alpha(y) \geq r$ for all $\alpha \in \sigma$, where $\sigma \subseteq \Delta$ is non-empty. Then if $d_K(\Theta(x), \Theta(y)K_\sigma) \geq e^{-\nu r}$ then $d(x, y) \geq \lambda_2 r$.*

Proof. This is stated and proved in [EF] as Lemma A.4 in the case where $G = \mathrm{SL}(n, \mathbb{R})$. This case implies the general case by an embedding argument, as in the proof of [EF, Lemma 4.1]. □

Lemma B.4 (Angles between Weyl chambers)**.** *There exist constants ν, ν', λ_2 depending only on X such that the following holds: Suppose $k_1, k_2 \in K$, $\mathfrak{C}_i = \pi(k_i A^+)$, and there exist points $z_i \in \mathfrak{C}_i$ such that $d(z_1, z_2) \leq \lambda_2 r$, and $d(z_i, \partial \mathfrak{C}_i) \geq \nu' r$. Then $d_K(k_1, k_2) \leq e^{-\nu r}$.*

Proof. This follows from Lemma B.3 with $\sigma = \Delta$. The condition that $\alpha(z_i) \geq r$ for all $\alpha \in \Delta$ is implied by $d(z_i, \partial \mathfrak{C}_i) \succ r$. □

Lemma B.5 (hyperbolic geometry)**.** *Let $\gamma_1(t)$ and $\gamma_2(t)$ be two geodesic rays, parameterized by arc length, emanating from the point e in a space of Y negative curvature (bounded away from zero). Pick a sufficiently large number Q, and let t denote the largest value of t' such that $d(\gamma_1(t'), \gamma_2(t')) \leq Q$. Then (for sufficiently large t),*

$$c_1 \log(1/\theta) < t < c_2 \log(1/\theta)$$

where θ is the angle between γ_1 and γ_2, and c_1 and c_2 depend only on Q and Y.

Proof. Standard. □

Lemma B.5 implies the following partial converse to Lemma B.4:

Lemma B.6. *Suppose $\alpha \in \Delta$, $k_2 \in k_1 K_\alpha$, and let $\theta = d_K(k_1, k_2)$. Then for any sufficiently large Q (depending only on X) there exist $z_i \in \pi(k_i A^+)$ such that $d(z_1, z_2) \leq Q$, $d(z_i, e) \leq \lambda |\log \theta|$ and $d(z_i, \partial \pi(k_i A^+)) \geq \nu_Q |\log \theta|$. The constant ν_Q depends only on Q and X, and the constant λ depends only on X.*

Remark. This lemma holds even without the assumption that $k_2 \in k_1 K_\alpha$.

Proof. Let $t = t(Q, \theta)$ be as in Lemma B.5, relative to the geodesics $\gamma_i(t) = k_i \exp(tH_\alpha)$, where $H_\alpha \in \mathfrak{a}$ is a unit vector orthogonal to the hyperplane $\alpha = 0$. Then H_α belongs to the Lie algebra of M_α, and thus the γ_i lie in $X_\alpha = M_\alpha/(K \cap M_\alpha)$. Since X_α has rank one, Lemma B.5 applies, and (for $\theta \ll 1$),

$$c''_Q |\log\theta| \leq t \leq c'_Q |\log\theta|$$

where c'_Q and c''_Q depend only on Q and X. Pick a generic positive unit vector v in the Lie algebra of A_α (i.e. the hyperplane $\alpha = 0$), and choose $\nu > 0$ such that $v + \nu H_\alpha \in \mathfrak{a}^+$. Then the points $z_i = \exp(k_i t(v + \nu H_\alpha))$ satisfy the conditions of the lemma. □

B.3. **Equivalence of Weyl chambers.**

Lemma B.7 (Containment of chambers). *Suppose $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are Weyl chambers, L is a σ-wall, and suppose that outside a sufficiently large ball, $L \subset \mathfrak{C}_1[\lambda\epsilon] \cap \mathfrak{C}_2[\lambda\epsilon]$. Let $k_1, k_2 \in K/M$ be such that $\mathfrak{D}_i = \pi(k_i A^+)$ is equivalent to $\mathfrak{C}_i$. Then $k_2 \in k_1 K_\sigma$, and $\mathfrak{D}_1$ and $\mathfrak{D}_2$ share a σ-wall. In particular, if $\mathfrak{C}_2 \subset \mathfrak{C}_1[\lambda\epsilon]$, then $k_1 = k_2$.*

Proof. After replacing the $\mathfrak{C}_i$ by equivalent Weyl chambers, we may assume that $\mathfrak{C}_i = \pi(k_i A^+)$. Also, after replacing L by a wall passing through e and a finite Hausdorff distance away, we may assume $L = \pi(kA^+_\sigma)$ for some $k \in K$. For any $r > 0$ there exists $a_r \in \overline{A^+}$ such that $\alpha(a_r) = r$ for $\alpha \in \sigma$, $\alpha(a_r) = 0$ for $\alpha \notin \sigma$. Then $\pi(ka_r) \in L$. Since $L \subset \pi(k_i A^+)[\lambda\epsilon]$, there exist $a_i \in A^+$ such that $d(\pi(k_i a_i), \pi(ka_r)) = O(\epsilon r)$. By [EF, Lemma 3.1], $d(x, y) \succ |\alpha(x) - \alpha(y)|$ for all $x, y \in X$, $\alpha \in \Delta$. Hence we may assume $\alpha(a_i) \geq r/2$ for all $\alpha \in \sigma$. By Lemma B.3 if $k_i \notin kK_\sigma$, for sufficiently large r, $d(\pi(k_i a_i), \pi(ka_r)) \succ r$. This is a contradiction. □

The following lemma is well known

Lemma B.8 (Equivalent Chambers). *Let $\mathfrak{C}$ be a Weyl chamber in X based at p. Then there exists an equivalent Weyl chamber $\mathfrak{C}'$ based at e such that $hd(\mathfrak{C}, \mathfrak{C}') = O(d(p, e))$.*

Proof. We may assume $\mathfrak{C} = \pi(gA^+)$ for $g \in G$. Write $g = kan$ using the Iwasawa decomposition. Let $\mathfrak{C}' = \pi(kA^+)$. Then, for $a_1 \in A^+$,

$$\begin{aligned} d(\pi(ga_1), \pi(ka_1)) &= d(\pi(ka_1(a_1^{-1} a n a_1)), \pi(ka_1)) = \\ &= d(\pi(a_1^{-1} a n a_1), e) \prec d(\pi(an), e) = d(ge, e). \end{aligned}$$

Thus, $hd(\mathfrak{C}, \mathfrak{C}') = O(d(ge, e))$. □

B.4. **Convex Polyhedra in $\mathbb{R}^n$.**

Lemma B.9 (Convex Polyhedra). *Let P be a (possibly unbounded) convex polyhedron in $\mathbb{R}^n$. For every hyperplane L of ∂P, let d_L denote the maximum distance between a point of P and L. Let d_P be the smallest of d_L. Then the injectivity radius R_P of P satisfies*

$$R_P \geq \frac{1}{n+1} d_P$$

Proof. Let $B(c, R)$ be a ball of maximal radius R inscribed in P. Suppose that there are exactly $n+1$ hyperplanes L_k, $k = 1, ..., n+1$ of ∂P that are tangent to $B(c, R)$ and form a simplex Q. Let d_k, $k = 1, ..., n+1$, be the distance between L_k and the opposite vertex of Q and let d_Q be the smallest of d_k. Since $Q \supset P$ we have $d_Q \geq d_P$. We claim that $R \geq \frac{1}{n+1} d_Q$. Indeed, connecting c with all vertices of Q we get $n+1$ pyramids. The volume of k-th pyramid is $\frac{1}{n} R A_k$ where A_k

is the area of $Q \cap L_k$. Therefore for the volume of Q we get $V_Q = \frac{1}{n} R \sum_{k=1}^{n+1} A_k$. Also we have $V_Q = \frac{1}{n} d_k A_k$, $k = 1, ..., n+1$. Therefore $V_Q = \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{1}{n} d_k A_k \geq \frac{1}{n(n+1)} d_Q \sum_{k=1}^{n+1} A_k$. Hence $R \geq \frac{1}{n+1} d_Q \geq \frac{1}{n+1} d_P$.

In general case if the number of hyperplanes L of ∂P tangent to $B(c, R)$ is $\geq (n+1)$ we can achieve, by slightly perturbing these hyperplanes (and hence changing P), that exactly $n+1$ hyperplane are tangent to $B(c, R)$ and they form a simplex. After getting the inequality $R \geq \frac{1}{n+1} d_P$ for the new polyhedron we roll back all changed hyperplanes and get by taking the limit that the inequality holds for the original polyhedron P too.

If the number of hyperplane in P tangent to $B(c, R)$ is less then $n+1$ then there must be a subspace S in $\mathbb{R}^n$ of some dimension $m \geq 1$ that is parallel to all such hyperplanes. Then in the orthogonal complement to S we repeat the estimates above and get $R \geq \frac{1}{n-m+1} d_P$. □

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