MODULI SPACES OF ABELIAN DIFFERENTIALS: THE PRINCIPAL BOUNDARY, COUNTING PROBLEMS AND THE SIEGEL–VEECH CONSTANTS

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Abstract. A holomorphic 1-form on a compact Riemann surface S naturally defines a flat metric on S with cone-type singularities. We present the following surprising phenomenon: having found a geodesic segment (saddle connection) joining a pair of conical points one can find with a nonzero probability another saddle connection on S having the same direction and the same length as the initial one. The similar phenomenon is valid for the families of parallel closed geodesics.

We give a complete description of all possible configurations of parallel saddle connections (and of families of parallel closed geodesics) which might be found on a generic flat surface S. We count the number of saddle connections of length less than L on a generic flat surface S; we also count the number of admissible configurations of pairs (triples,...) of saddle connections; we count the analogous numbers of configurations of families of closed geodesics. By the previous result of [EMa] these numbers have quadratic asymptotics $c \cdot (\pi L^2)$. Here we explicitly compute the constant c for a configuration of every type. The constant c is found from a Siegel–Veech formula.

To perform this computation we elaborate the detailed description of the principal part of the boundary of the moduli space of holomorphic 1-forms and we find the numerical value of the normalized volume of the tubular neighborhood of the boundary. We use this for evaluation of integrals over the moduli space.

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Introduction

1. Flat Surfaces and Geodesics on Flat Surfaces

In this paper we shall consider flat metrics with isolated conical singularities on a closed orientable surface of genus g. An important class of such flat metrics corresponds to the *translation surfaces*: those surfaces for which the flat metric on the surface has trivial linear holonomy. These surfaces have been studied by several authors in various guises; in the context of Abelian differentials (and more generally quadratic differentials) on compact Riemann surface in [S], in the context of rational billiards in [KaZe], [KMS], [Ve3], and [GJ]. Moduli spaces of these structures have been studied in [Ve4], [MS], [Ko], [KonZo], and [EMa]. A survey is given in [MT].

Now the trivial holonomy means, in particular, that the parallel transport of a vector along a small loop going around a conical point brings a vector back to itself. This implies that all cone angles of such surface are integer multiples of 2π.

Chose a direction in the tangent space to some base point $x_0 \in S$ of the flat surface and transport this direction using the parallel transport to all nonsingular points of the surface. Since the monodromy is trivial this parallel transport does not depend on the path. Thus any direction is globally defined on the flat surface punctured at the singularities. Throughout this paper by a *flat surface* we mean a flat surface having trivial holonomy representation in $\mathrm{SO}(2, \mathbb{R})$ endowed with a choice of direction. Following a tradition we often call the conical points the *saddles*.

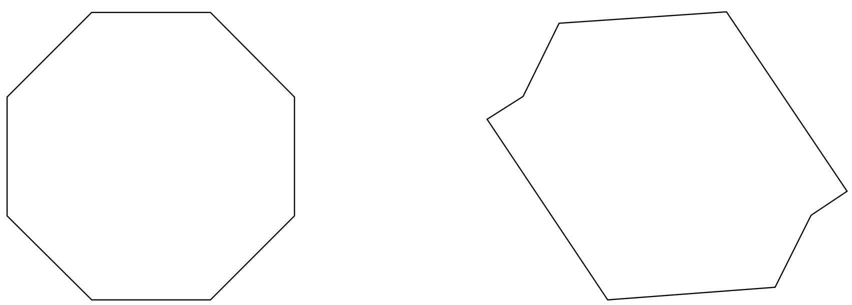

Figure 1. Identifying the opposite sides of a regular octagon by parallel translations we obtain a flat surface of genus $g = 2$. Deforming the regular octagon in such way that the sides are organized into pairs of parallel sides of equal length we get a *family* of flat surfaces of genus $g = 2$ each having a single conical singularity with the cone angle 6π.

A flat torus endowed with a choice of direction gives an example of a flat surface without any saddles. One can glue a flat torus from a parallelogram by identifying opposite sides. A generalization can be found by considering a polygon in the plane having the property that its sides are distributed into pairs that are parallel and of equal length. Identifying the pairs by translations one gets a closed oriented flat surface. For example, identifying the opposite sides of a regular octagon one gets a surface of genus 2, see Figure 1. All vertices of the octagon are identified to a single saddle which has an angle of 6π. We may assume that we have cartesian coordinates in the polygon; by convention we chose the vertical direction as the distinguished one.

In these coordinates away from the saddles, geodesics on the surface S are straight lines. The behavior of geodesics on flat surfaces is in many aspects similar to the behavior of geodesics on a flat torus. In particular, a geodesic cannot change the direction, and thus cannot have self intersections. Generically a geodesic is dense in the surface which is analogous to an irrational winding line on the torus.

However, a flat surface different from the torus must have saddles; some geodesics hit the saddles. Some geodesics hit the saddles going both in forward and in backward directions. Such a geodesic segment joining a pair of saddles and having no saddles in its interior is called a *saddle connection*.

A geodesic leaving a regular point x may return back to x without meeting any singularities. Since it cannot change its direction, we necessarily get a *closed regular geodesic* in this case. Note that as in the case of the torus, any geodesic leaving a nearby point in a parallel direction will be also closed and will have the same length as the initial one. Thus closed regular geodesics appear in families of parallel geodesics of the same length. However, unlike the case of the torus, such parallel closed geodesics do not typically fill all the surface, but only a cylindrical subset. Each boundary component of such a cylinder contains a saddle. For example, choosing the vertical direction on the flat surface obtained from a regular octagon (see Figure 1) we get two families of regular closed geodesics, and each of the two families is filling a cylinder.

Note that the shortest representative in a homotopy class of curves joining two saddles exists. Typically, even for the class of a simple closed curve, one gets a broken line containing many geodesic segments. The segments of these broken lines pass through the saddles changing directions there. We want to stress that throughout this paper we consider only those saddle connections which contain a single geodesic segment and thus no saddles in its interior. However, we do not exclude the situation when the endpoints of a saddle connection coincide. We call such saddle connections the *closed geodesics*.

Now fix a flat surface S. Let $\hat{S}$ be the universal cover of $S - \Sigma$, where Σ is the collection of saddles. There is an isometric map of $\hat{S}$ to $\mathbb{R}^2$ called the *developing map* which maps a lift $\hat{\gamma}$ of an oriented curve γ on S to a curve in $\mathbb{R}^2$. We denote by $hol(\gamma)$ the difference of the endpoints of the image. The vector $hol(\gamma) \in \mathbb{R}^2$ is called the *holonomy* vector of γ.

Remark 1.1. By definition a flat surface has trivial holonomy representation in the linear group

$$\pi_1(S) \to H_1(S, \mathbb{Z}) \to \mathrm{SO}(2, \mathbb{R}).$$

However, the holonomy representation in the *affine* group is nontrivial: its image belongs to the group of translations. From now on we reserve the notion of *holonomy* for this second representation.

Now fix a pair of saddles on a flat surface S, and some length L. Consider all those saddle connections joining the fixed saddles whose length is shorter then L. We are interested in the asymptotics for the number of saddle connections when the bound L tends to infinity. In the model case of the torus it is sufficient to pass to the universal covering plane to see that the number of geodesic segments joining a generic pair of points grows quadratically: same as the volume of a ball, that is as πL^2.

It is proved in [EMa] that the growth rate of the number of saddle connections for a generic flat surface also has quadratic asymptotics $c\cdot(\pi L^2)$, and, moreover, almost all flat surfaces in some natural families of flat surfaces share the same constant c in the asymptotics. One of the principal goals of this paper is to compute these constants for all natural families (which are explicitly defined below). We also compute the constants in the quadratic asymptotics for the number of families of closed regular geodesics.

It is relatively easy to show that for a generic flat surface one can never find a pair of saddle connections having the same direction but different lengths. However, somewhat surprisingly, the configurations of pairs of parallel saddle connections of the *same* length and same direction can be found on almost every flat surface, and, what may seem even more surprising, their number also has quadratic asymptotics (with another constant c).

One can ask now the similar questions about the triples of saddle connections, or about some more specific configurations of saddle connections, or similar questions concerning closed geodesics. We give a complete description of those configurations which can be found on a generic surface; we show that each of them has quadratic asymptotics (this was essentially already proved in [EMa]), and in every case we compute the corresponding constant in the quadratic asymptotics.

Remark 1.2. A rational billiard table gives rise to such a flat surface. See for example [KaZe], [KMS], or [MT]. However, rational billiards are not generic in these natural families and so the constants described in this paper do not apply to the billiard problem, with an exception for some particular classes off billiards, see [EZo].

1.1. **Flat Surfaces and Abelian Differentials.** If we cut a flat surface S successively along an appropriate collection of saddle connections we can decompose it into polygons contained in $\mathbb{R}^2 = \mathbb{C}$, see, for example, Figure 1. We may then view S as a union of polygons glued along parallel sides by means of parallel translations. Note that we have endowed every polygon with a complex coordinate. Since our gluing rules are just translations, the transition functions in these complex coordinates have the form

$$z \to z + c.$$

Thus any flat surface with the conical singularities removed is endowed with a natural complex structure. Moreover, consider a holomorphic 1-form $\omega = dz$ on every polygon. Since $dz = d(z + c)$ we obtain a globally defined holomorphic 1-form on the surface with removed singularities. It is an easy exercise to show that the complex structure and the holomorphic differential can be extended to the singularities; the holomorphic 1-form (or, what is the same, the *Abelian differential*) ω has a zero at every conical point.

Conversely, given a pair (M, ω) where M is a Riemann surface, ω is a holomorphic Abelian differential on M, and a point $P \in M$ such that $\omega(P) \neq 0$, there exists a local coordinate z near P such that $\omega = dz$. Such a local coordinate is unique up to change of coordinates $z \to z + c$. Thus $|dz|^2$ is a flat metric on M; this metric develops conical singularities at the zeroes of ω. At a zero of order k the total angle is $2\pi(k + 1)$. Furthermore, such a coordinate chart near every nonsingular point determines a structure $S = (M, \omega)$ of a "translation surface" (which we agree to call just a "flat surface") on M, namely an atlas of coordinate charts which cover the surface away from the singularities, such that the transition functions are translations

$$z \to z + c.$$

Specifying a direction means in this language that we distinguish between the flat surface defined by ω and $e^{i\theta}\omega$.

1.2. **Holonomy and the sets $V_{cyl}(S)$ and $V_{sc}(S)$.** Note that in terms of the local coordinates (x, y) defining the flat structure, $\omega = dz = dx + idy$. This has an important consequence: For any oriented curve β on S, the *holonomy* of β (as defined above in terms of the developing map) coincides with the period $\int_\beta \omega$ of ω over β, (and we have identified $\mathbb{R}^2$ and $\mathbb{C}$). In particular, if β and γ are homologous, then $hol(\gamma) = hol(\beta)$.

Let $\mathrm{Cyl}(S)$ denote the set of cylinders of regular closed geodesics on S. For a cylinder $C \in \mathrm{Cyl}(S)$, let $hol(C) \in \mathbb{R}^2$ denote the holonomy of a regular closed geodesic γ in C. The vector $hol(C)$ is independent of the choice of γ in C since all regular closed geodesics belonging to a given cylinder are homologous. We now define $V_{cyl}(S) \subset \mathbb{R}^2$ as

$$V_{cyl}(S) = hol(\mathrm{Cyl}(S)).$$

For example if S_0 is the standard flat torus (i.e. the unit square with opposite sides identified), then $V_{cyl}(S_0) = \{(a, b) \in \mathbb{Z}^2 \ : \ \gcd(a, b) = 1\}$.

In general $V_{cyl}(S)$ is a discrete subset of $\mathbb{R}^2$. The number of cylinders on S of length at most L (counted without multiplicity) is exactly the cardinality of the intersection of $V_{cyl}(S)$ with the ball of radius L centered at the origin.

We note that the map $hol : \mathrm{Cyl}(S) \to V_{cyl}(S)$ need not be one to one (see Figure 2 for an example where there are two cylinders with the same holonomy). We define the *multiplicity* of a point $v \in V_{cyl}(S)$ to be the number of distinct cylinders C such that $hol(C) = v$. Somewhat surprisingly, higher multiplicity is very common even for a generic surface. For example we will show that if the genus is at least 3, the number of multiplicity 2 vectors in $V_{cyl}(S)$ of length at most L also grows quadratically in L.

We can repeat the same construction for the saddle connections. Let $\mathrm{SC}(S)$ denote the set of saddle connections on S, and let $V_{sc}(S) = hol(SC(S))$. Again, the map $hol : SC(S) \to V_{sc}(S)$ is far from one to one. For example, in Figure 2, there are four distinct saddle connections which map to the vector $(1, 0)$. We will handle the multiplicites more carefully in §3.1.

2. Moduli Spaces of Abelian Differentials

2.1. **Stratification.** Let α be a partition of $2g - 2$ (i.e. a representation of $2g - 2$ as a sum of positive integers). Let $\mathcal{H}(\alpha)$ denote the moduli space of pairs (M, ω) where M is a closed Riemann surface of genus g, and ω is a holomorphic Abelian differential on M such that the orders of its zeroes is given by α. The zeroes on each surface are assumed to be *named*. We say that two such structures are *equivalent* if there is an isomorphism from one to the other which takes zeroes to zeroes, preserving the naming. The set of equivalence classes is denoted $\mathcal{H}(\alpha)$ and is called a *stratum*. This term is justified by the fact that the space of all Abelian differentials on Riemann surfaces of genus g is stratified by the spaces $\mathcal{H}(\alpha)$, as α

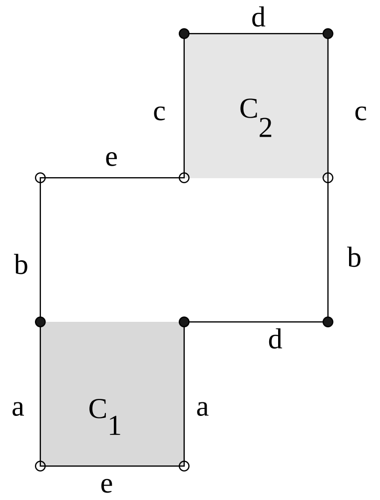

FIGURE 2. Consider four unit squares with sides glued as indicated. The result is a closed surface S of genus 2. The five points indicated by the filled circle symbol are identified, as are the five points indicated by the unfilled circle symbol. Hence S has two conical singularities, each of total angle 4π. Each of the shaded squares corresponds to a cylinder C_i of horizontal regular closed geodesics of length 1. Hence, $hol(C_1) = hol(C_2) = (1, 0)$. Note, however, that the corresponding closed regular geodesics are not homologous. This means that the relation above would not survive under a generic deformation of this flat surface. Figure 9 gives an example when such relation is stable under any small deformation of the flat surface.

varies over the partitions of $2g - 2$. The stratum corresponding to the partition $(1, \ldots, 1)$ is called the *principal stratum*: it corresponds to holomorphic differentials with simple zeroes. For example, the surface in Figure 2 belongs to $\mathcal{H}(1, 1)$. The stratum might be nonconnected, although the components have been completely classified (see [KonZo]). The classification involves the notion of parity of spin structure and issues of hyperellipticity. These components will be described later in the introduction.

2.2. **Local Coordinates and Volume Element.** The stratum $\mathcal{H}(\alpha)$ can be topologized and given a natural "Lebesgue" measure as follows. For a flat surface $S_0 \in \mathcal{H}(\alpha)$ with conical points $P_1, \ldots, P_n$, choose a basis of cycles for the relative homology $H_1(S_0, \{P_1, \ldots, P_n\}; \mathbb{Z})$. This basis may be chosen so that each element is represented by a saddle connection. Equivalently, the saddle connections cut S into a union of polygons. For any S near S_0 the vectors associated to these saddle connections serve as local coordinates, see, for example, Figure 1. By construction these vectors are the relative *periods* of the Abelian differential ω (i.e. the integrals of ω along the paths joining points P_i and P_j, where $i = j$ is also allowed). So, we, actually, use a domain in the space of relative cohomology $H^1(S_0, \{P_1, \ldots, P_n\}; \mathbb{C})$

as a local coordinate chart. Ignoring the complex structure we get a domain in $\mathbb{R}^n = H^1(S_0, \{P_1, \dots, P_n\}; \mathbb{C})$. Note that we have a natural cubic lattice in our coordinates given by the "integer" cohomology

$$H^1(S_0, \{P_1, \dots, P_n\}; \mathbb{Z} \oplus i\mathbb{Z}) \subset H^1(S_0, \{P_1, \dots, P_n\}; \mathbb{C}) = \mathbb{R}^n$$

We define a measure $d\nu(S)$ on $\mathcal{H}(\alpha)$ as Lebesgue measure defined by these coordinates, normalized so that the volume of a unit cube in $\mathbb{R}^n$ is 1. One easily checks that the measure is well-defined independent of choice of basis. Let $\mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha)$ be the hypersurface in $\mathcal{H}(\alpha)$ of unit area flat surfaces.

Choose a symplectic homology basis of closed curves A_i, B_i, $i = 1, \dots, g$ on the surface $S = (M, \omega)$. The area of the flat surface S defined by the Abelian differential ω is given by the Riemann bilinear relation,

$$\int_S |\omega|^2 dxdy = \frac{i}{2} \int_S \omega \wedge \bar{\omega} = \frac{i}{2} \sum_i \left(\int_{A_i} \omega \int_{B_i} \bar{\omega} - \int_{A_i} \bar{\omega} \int_{B_i} \omega \right)$$

Thus the unit area surfaces represent a "hyperboloid" in terms of the coordinates chosen above.

Throughout this paper we shall mostly work with the "hyperboloid" $\mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha)$. The volume element (and the measure) in the embodying space $\mathcal{H}(\alpha)$ induce naturally the volume element (and the measure) on the hypersurface $\mathcal{H}_1(\alpha)$ in the following way. There is a natural $\mathbb{C}^*$-action on $\mathcal{H}(\alpha)$: having $\lambda \in \mathbb{C} - \{0\}$ we associate to the flat surface $S = (M, \omega)$ the flat surface $\lambda \cdot S = (M, \lambda \cdot \omega)$. In particular, we can represent any $S \in \mathcal{H}(\alpha)$ as $S = rS'$, where $r \in \mathbb{R} - \{0\}$, and where S' belongs already to the "hyperboloid": $S' \in \mathcal{H}_1(\alpha)$. Geometrically this means that vectors associated to saddle connections on S' are multiplied by r to give vectors on S. It means also that $\text{area}(S) = r^2 \cdot \text{area}(S') = r^2$, since $\text{area}(S') = 1$. We define the *volume element* $d\,\text{vol}(S')$ on the "hyperboloid" $\mathcal{H}_1(\alpha)$ by disintegration of the volume element $d\nu(S)$ on $\mathcal{H}(\alpha)$:

$$d\nu(S) = r^{n-1}\, dr d\,\text{vol}(S')\,,$$

where $n = \dim_{\mathbb{R}} \mathcal{H}(\alpha)$. Using this volume element we define the total *volume of the stratum* $\mathcal{H}_1(\alpha)$:

$$\text{Vol}(\mathcal{H}_1(\alpha)) := \int_{\mathcal{H}_1(\alpha)} d\,\text{vol}(S')\,. \tag{1}$$

Since the measure on $\mathcal{H}_1(\alpha)$ is induced by the volume element, the total volume of the stratum coincides with the total measure.

For a subset $E \subset \mathcal{H}_1(\alpha)$ we let $C(E) \subset \mathcal{H}(\alpha)$ denote the "cone" based on E:

$$C(E) := \{S = rS' \mid S' \in E,\ 0 < r \le 1\}\,. \tag{2}$$

Our definition of the volume element on $\mathcal{H}_1(\alpha)$ is consistent with the following normalization:

$$\text{Vol}(\mathcal{H}_1(\alpha)) = n \cdot \nu(C(\mathcal{H}_1(\alpha))\,, \tag{3}$$

where $n = \dim_{\mathbb{R}} \mathcal{H}(\alpha)$, and $\nu(C(\mathcal{H}_1(\alpha))$ is the total volume (total measure) of the "cone" $C(\mathcal{H}_1(\alpha)) \subset \mathcal{H}(\alpha)$ measure by means of the volume element $d\nu(S)$ on $\mathcal{H}(\alpha)$ defined above.

It is a result of [M], [MS] and [Ve2] that these volumes are finite. The values have been calculated in [EOk], see also [Zo] for the values of the volumes of some low-dimensional strata. These will be needed for the calculations of the asymptotic constants. Tables of their values for various low genus examples are presented in section 6.4.

3. Evaluation of the Constants

3.1. **Configurations of Saddle Connections.** It is convenient to formalize the problem of counting saddle connections and the problem of counting closed geodesics in the following way.

For the first problem consider the set of vectors in $\mathbb{R}^2$ associated to a saddle connection that starts at a zero of order m_1 and ends at a distinct zero of order m_2. For some vectors there is a single saddle connection whose holonomy is that vector, and we say the vector has multiplicity one. There, however, may be a set of saddle connections connecting the two zeroes that map to the same vector under the holonomy map. In that case we say that the vector has higher multiplicity. It follows from the definition of the volume element that if a generic surface has saddle connections with higher multiplicity, then those saddle connections are homologous. Thus from now on we can assume the saddle connections are homologous in the case of higher multiplicity.

Now suppose that we have precisely p homologous saddle connections $\gamma_1, \dots, \gamma_p$ joining z_1 of order m_1 to z_2 of order m_2. All the γ_i, $1 \le i \le p$ have the same holonomy (where the orientation of each γ_i is from z_1 to z_2). By convention the cyclic order of the γ_i at z_1 is clockwise in the orientation defined by the flat structure. Let the angle between γ_i and γ_{i+1} at z_1 be $2\pi(a'_i + 1)$; let the angle between γ_i and γ_{i+1} at z_2 be $2\pi(a''_i + 1)$. We call this the *configuration*

$$\mathcal{C} = (m_1, m_2, a'_i, a''_i)$$

of the p homologous saddle connections. If there is just a single saddle connection ($p = 1$) then $\mathcal{C} = (m_1, m_2)$.

Remark 3.1. Since the flat structure is defined by a 1 form ω, and the holonomy of a curve β is given by $\int_\beta \omega$, two homologous curves always have the same holonomy. However as mentioned previously the geodesic representative of a homotopy class may consist of a broken line of saddle connections. Thus the higher multiplicity phenomenon occurs when homologous, nonhomotopic geodesics are each represented by a single saddle connection.

Now given a surface S and a configuration $\mathcal{C}$, let $V_{\mathcal{C}}(S)$ denote the vectors v in $\mathbb{R}^2$ such that precisely the p saddle connections in the configuration map to v under the holonomy map. We want to compute the asymptotics as $T \to \infty$ of

$$|V_{\mathcal{C}}(S) \cap B(T)|,$$

where $B(T)$ is the ball of radius T.

Now consider the second problem of this paper. We have a surface S and a saddle connection γ_1 joining a zero to itself. For this problem we shall count the primitive elements. There may be other saddle connections $\gamma_2, \dots, \gamma_m$ which are homologous to γ_1. Some of them may start and end at the same zero, the others may start and end at the other zeroes. Each γ_i returns to its zero at some angle θ_i, which must be an odd multiple of π. The return angles of π and $-\pi$ are special. In these cases, the saddle connection bounds a cylinder. The other saddle connection bounding the cylinder may or may not be based at the same zero. For some vectors there is a single saddle connection whose holonomy is that vector, or pair if they bound a cylinder, and in either case, we say the vector has multiplicity one.

Suppose the curves γ_i bound q cylinders. The complement of the curves and cylinders are $p = m - q$ surfaces. We say the multiplicity is p. The complement of the curves and cylinders are $p = m - q$ surfaces $S_1, \dots, S_p$. We can associate a set $J = \{i_1, \dots, i_q\} \subset \{1, \dots, p\}$, where $0 \le q \le p$ with the property that if $i_j \in J$ then there is a cylinder between γ_{i_j} and γ_{i_j+1}. By convention, when $J = \emptyset$ we let $q = 0$, that is we have no cylinders at all.

Now suppose a surface S_i is bounded by two homologous curves that leave and return to the same zero. The two curves bound angles $2(a_i' + 1)\pi$ and $2(a_i'' + 1)\pi$ at the zero. The two boundary circles are joined together at the zero to form a "figure eight". If the two homologous curves that bound S_i return to different zeroes, then they bound angles b_i' and b_i'' at those zeroes. The boundary circles are disjoint in that case.

Altogether this gives a *configuration* $\mathcal{C}$

$$\mathcal{C} = (J, a_i', a_i'', b_k', b_k'')$$

that are associated to this collection of saddle connections.

As in the first problem for a surface S and configuration $\mathcal{C}$ let $V_{\mathcal{C}}$ denote the vectors in $\mathbb{R}^2$ such that the saddle connections in $\mathcal{C}$ have holonomy equal to that vector. Again we shall compute the asymptotics as $T \to \infty$ of

$$|V_{\mathcal{C}}(S) \cap B(T)|,$$

where $B(T)$ is the ball of radius T.

For either of the problems, given the configuration $\mathcal{C}$, for each component of each $\mathcal{H}_1(\alpha)$, there is a constant $c = c(\alpha, \mathcal{C})$ (depending on the connected component of $\mathcal{H}(\alpha)$) such that by [EMa] for almost all S in that component of $\mathcal{H}_1(\alpha)$,

$$|V_{\mathcal{C}}(S) \cap B(T)| \sim \pi c T^2.$$

Thus the growth rate is quadratic for a generic surface. What is perhaps surprising is that this formula says that the growth rate is quadratic for a generic surface even for multiple homologous saddle connections. By contrast, the definition of the measure ν implies that a generic surface does not have *any* pairs of saddle connections that determine vectors in the same direction but with *different* lengths.

Remark 3.2. The issue of higher multiplicity and specifying the angles is not explicitly addressed in [EMa]. However, the set $V_{\mathcal{C}}(S)$ defined above, satisfies all of the axioms of that paper, so the asymptotic formula from [EMa] is applicable to the problems formulated above.

One of the main goals of this paper is to evaluate the constant c. We shall have numerous formulae throughout the paper which will refer to the quadratic asymptotic constant computed above. In all cases they will hold for almost every surface in the component.

We also note that these constants have been verified *numerically*. A formula based on the formula of Kontsevich (see [Ko]) expresses the sum of the Lyapunov exponents of the Teichmüller geodesic flow on each connected component of every stratum $\mathcal{H}(\alpha)$ in terms of some specific linear combination of the constants $c(\alpha, \mathcal{C})$ for the corresponding component of $\mathcal{H}(\alpha)$. These Lyapunov exponents have been computed numerically and the constants achieved in this paper are consistent with these exponents and with Kontsevich's formula.

3.2. **Siegel–Veech Formula and Computation of the Constant.** In [EMa] it was proved that for each configuration $\mathcal{C}$, the constant on each component of $\mathcal{H}_1(\alpha)$ satisfies the following Siegel–Veech formula:

$$\frac{1}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}\int_{\mathcal{H}_1(\alpha)} \hat{f}(S)\, d\,\mathrm{vol}(S) = c\int_{\mathbb{R}^2} f \tag{4}$$

where f is a continuous function of compact support in $\mathbb{R}^2$ and

$$\hat{f}(S) = \sum_{\vec{v}\in V_{\mathcal{C}}(S)} f(\vec{v}).$$

The terms in the sum are counted with multiplicity; that is, we also count vectors associated to saddle connections that are not in the configuration if they are in the support of f. Thus a major part of this paper can be thought of as finding the constant in the Siegel–Veech formula. The strategy for evaluating c consists of letting f be the characteristic function of $B(\epsilon)$ and computing the leading term of order ϵ^2 in the asymptotics of the left-hand-side of (4) as $\epsilon \to 0$. The support of $\hat{f}$ is the space, denoted $\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})$. This is the set of flat surfaces such that the saddle connections with the given configuration have length at most ϵ. This space is not compact, since as we let $\epsilon\to 0$, one leaves compact sets. The space $\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})$ is also difficult to analyze since there may be saddle connections shorter than ϵ other than those in the configuration $\mathcal{C}$.

Accordingly, let $\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$ denote the set of flat surfaces in $\mathcal{H}_1^{\epsilon}(\alpha)$ such that all saddle connections or closed geodesics that are not homologous to the ones in the configuration $\mathcal{C}$ have length greater than 2ϵ. (The explanation for the superscript "*th*" is that we think of these surfaces as lying in the part of the stratum which except for the curves in the configuration is "thick".) The point of this definition is that $\hat{f}$ is identically 1 on this space. The space will also prove to be easier to analyze since we shall be able to associate a boundary to it.

We shall find the asymptotics of $\int_{\mathcal{H}_1^{\epsilon,\mathcal{C}}(\alpha)} \hat{f}(S)\, d\,\mathrm{vol}(S)$ by breaking the domain of integration $\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})$ into $\mathcal{H}_1^{\epsilon,th}(\alpha)$ and its complement, the *thin* part $\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C}) - \mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$. We now claim that

$$\int_{\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})-\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})} \hat{f}(S)\ d\,\mathrm{vol} = o(\epsilon^2). \tag{5}$$

To prove (5), we decompose $\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C}) - \mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$ into a union of sets where a saddle connection or closed geodesic has length between $(2\epsilon)^{n+1}$ and $(2\epsilon)^n$, where n is a positive integer. Since by definition, on each surface in this set there is a saddle with length between $(2\epsilon)^{n+1}$ and $(2\epsilon)^n$ and a nonhomologous saddle with length at most ϵ, the volume of that set is $O(2^n\epsilon^{2n+2})$. We have the following estimate proved in [EMa] which is needed to bound the integrand. Let $l(S)$ denote the length of the shortest saddle connection or closed geodesic on S. Then there are constants c' and $0<\delta<1$ so that for $S\in\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})$

$$\hat{f}(S)\le c'(l(S)^{-1-\delta}),$$

These two estimates imply that the integral of $\hat{f}$ on the set where there is a saddle of length between $(2\epsilon)^{n+1}$ and $(2\epsilon)^n$ is

$$O(\epsilon^2 2^n \epsilon^{n(1-\delta)})$$

Since $\delta < 1$, summing over n, we find that (5) holds.

Then, since $\hat{f}$ is zero outside $\mathcal{H}_1^\epsilon(\alpha, \mathcal{C})$ and $\hat{f} = 1$ on $\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C})$, we have

$$\int_{\mathcal{H}_1(\alpha)} \hat{f}(S)\; d\,\mathrm{vol} = \int_{\mathcal{H}_1^\epsilon(\alpha,\mathcal{C})} \hat{f}(S)\; d\,\mathrm{vol} = \mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})) + o(\epsilon^2). \tag{6}$$

As we shall see, $\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C}))$ is always of the order ϵ^2, which is the order of the leading term that we are trying to find. Thus in all cases in this paper, after substituting (6) into (4) we see that

$$\pi c = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \frac{\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}.$$

Thus the goal of this paper will be to calculate the asymptotics of the numerator. Now as one lets $\epsilon \to 0$, the flat surfaces in $\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C})$ degenerate to simpler surfaces. In the case of a single saddle connection joining a pair of distinct zeroes the zeroes coalesce to a higher order zero giving a surface in a stratum of higher codimension. If there are multiple saddle connections joining distinct zeroes, or as in the second problem there are closed geodesics, the result are flat surfaces of lower genus which may be highly degenerate. In any of the cases we shall say that the resulting surfaces belong to the *principal boundary* of the original stratum. Each configuration will determine a component of this principal boundary. A major part of this paper then will be to describe th is principal boundary, an object of independent interest.

Conversely, given surfaces in the principal part of the boundary, we will describe a set of constructions that allow us to recover surfaces in $\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C})$. The first is a splitting apart of a higher order zero into simpler zeroes that is the opposite of coalescing. The second is a slit and gluing construction which is the opposite of the degeneration of multiple saddle connections. To construct surfaces with homologous closed geodesics, we shall also need a pair of constructions, called the parallelogram and figure eight constructions.

In order to completely describe the way the configurations of saddles can arise from the principal boundary, we shall need to describe in combinatorial terms the number of ways these constructions can be performed. Because we are interested in components of strata we shall also need in certain situations to pay attention to parity of spin structures and hyperellipticity. Once this is done, the correspondence between configurations and boundary surfaces will allow us to compute $\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C}))$ in terms of the volumes of components of strata of these simpler surfaces and the combinatorial information about how many ways there are to build surfaces form the simpler surfaces. There will also be a normalizing factor which is the volume of the component of the stratum of the given $\mathcal{H}_1(\alpha)$.

One can think of every stratum as an analogue of a complex polyhedron; the boundary of the polyhedron has "faces" of complex codimension 1, "edges" of complex codimension 2, etc. We may think of our calculations as follows. We could define a *tubular ϵ-neighborhood* of the boundary as the subset of those flat surfaces which contain a geodesic element of length at most ϵ. We could similarly define a *tubular ϵ-neighborhood of the "edges"* of the boundary as the subset of those flat surfaces which contain at least two nonhomologous geodesic elements of length at most ϵ. Thus our calculation shows that for a moderately growing function near the boundary, we can ignore the contribution in a tubular neighborhood of the edges.

Remark 3.3. One can consider these same problems in the context of *quadratic differentials*. To avoid overloading the paper we prefer to treat them in a subsequent one.

4. Readers Guide

In the first half of the paper we solve these problems for saddle connections joining distinct zeroes. We describe the multiplicity one case first, because it is the easiest and illustrates the computations. This is done in Sections 6.1–6.3. We then describe the principal boundary in the higher multiplicity case, describing the *slit* and *gluing* constructions and again give the constants. This is done in Sections 7.1–7.2 and 7.5. Sections 8.1–8.3 as well as the corresponding preliminaries from Section 5.1 are devoted to describing the constants in the cases of strata with several components and could be skipped by the reader interested only in the general thrust of the paper.

In the second half of the paper we consider the case of closed geodesics. We describe the principal boundary using the *parallelogram* and *figure eight* constructions. The reader interested only in the structure of the principal boundary may consult Sections 9–10. The computation of the combinatorial factor is performed in Section 10.5 and the computation of the constant c is given in Section 11.1. The rest of Section 11 is devoted to numerous examples and Section 12 is devoted to the special case of strata with several connected components.

We are illustrating all constructions by numerous examples. All possible strata and all possible configurations in genera $g = 2$ and $g = 3$ and all related computations are presented as the examples in the main body of the text. The complete list of admissible configurations in genus $g = 4$, and the values of the corresponding constants are presented in the Appendix. We also present in the Appendix the values of the constants for the two distinguished strata up to genus $g = 30$.

5. Connected Components of the Strata

5.1. Parity of a Spin Structure. As noted earlier, the strata $\mathcal{H}_1(\alpha)$ are not necessarily connected. To classify the components one needs the notion of parity of spin structure. Consider a smooth simple curve β on the flat surface S which does not contain any zeroes. We define the index $ind(\beta) \in \mathbb{N}$ of the vector field tangent to β to be the degree of the corresponding Gauss map: the total change of the angle between the vector tangent to the curve and the vector tangent to the horizontal foliation is $2\pi \cdot ind(\beta)$.

Now take a symplectic homology basis $\{A_i, B_i\}, i = 1, \dots, g$, such that the intersection matrix has the canonical form: $A_i \circ A_j = B_i \circ B_j = 0$, $A_i \circ B_j = \delta_{ij}$, $1 \le i, j \le g$. Though such basis is not unique, traditionally it is called a *canonical basis*. Consider a collection of smooth closed curves representing the chosen basis. Denote them by the same symbols A_i, B_i.

The *parity of the spin structure* $\phi(S)$ of the flat structure is defined to be

$$\phi(S) := \sum_{i=1}^{g} (ind(A_i) + 1)(ind(B_i) + 1) \quad (mod\ 2)$$

Now the index of a small loop around a single zero of order k is $k+1$ so the parity is only defined if the zeroes have *even* order, which we now assume. It follows from the results of D.Johnson [J] that the parity of the spin structure $\phi(S)$ does not depend neither on the choice of representatives, nor on the choice of the canonical homology bases.

Moreover, it follows from the results of M.Atiyah [At] that $\phi(S)$ is invariant under continuous deformations of the flat structure of the translation surface, (see [KonZo] for details).

Since we do not need the notion of a spin structure itself, we do not discuss it in this paper; the reader may consider the notion "parity of the spin structure" as a single term. Note that it is applicable only to those flat structures ω which have set of zeros α containing even numbers only.

5.2. **Connected Components of the Strata.** The components are classified as follows (see [KonZo]).

Consider the general case, when $g > 3$, and α is different from two exceptional cases: $\alpha \neq (2g-2)$, $\alpha \neq (g-1, g-1)$. Under these assumptions the stratum $\mathcal{H}_1(\alpha)$ is connected whenever collection α contains an odd number. When $g > 3$, $\alpha \neq (2g-2)$, $\alpha \neq (g-1, g-1)$, and α has only even elements, there are exactly two connected components corresponding to even and odd spin structures. We shall denote them by $\mathcal{H}_1^{even}(\alpha)$ and $\mathcal{H}_1^{odd}(\alpha)$ correspondingly.

The strata $\mathcal{H}_1(2g-2)$ and $\mathcal{H}_1(g-1, g-1)$ have in addition a special component — the hyperelliptic one. It consists of flat structures on hyperelliptic surfaces. In both cases the hyperelliptic involution τ sends the flat structure ω to $-\omega$; for $\mathcal{H}_1(g-1, g-1)$ we also require that τ interchanges two zeroes of ω.

For $g > 3$ the stratum $\mathcal{H}_1(2g-2)$ has three connected components: the hyperelliptic one — $\mathcal{H}_1^{hyp}(2g-2)$, and two nonhyperelliptic components — $\mathcal{H}_1^{even}(2g-2)$ and $\mathcal{H}_1^{odd}(2g-2)$ corresponding to even and odd spin structures. For even $g > 3$ the stratum $\mathcal{H}_1(g-1, g-1)$ has two connected components: the hyperelliptic one, and the nonhyperelliptic one $\mathcal{H}_1^{nonhyp}(g-1, g-1)$. For odd $g > 3$ the stratum $\mathcal{H}_1(g-1, g-1)$ has three connected components: the hyperelliptic one, denoted by $\mathcal{H}_1^{hyp}(g-1, g-1)$, and two nonhyperelliptic ones — $\mathcal{H}_1^{even}(g-1, g-1)$ and $\mathcal{H}_1^{odd}(g-1, g-1)$ corresponding to even and odd spin structures.

Genera 2 and 3 are special. In genus 2, all flat structures are hyperelliptic, so each of the $\mathcal{H}_1(2)$ and $\mathcal{H}_1(1,1)$ have one component which is hyperelliptic. In genus 3, only hyperelliptic flat structures may have even parity of the spin structure. Thus $\mathcal{H}_1(4)$ and $\mathcal{H}_1(2,2)$ each have two connected components, the hyperelliptic one and the one with odd spin structure. The other strata in genus 3 are connected.

The flat surfaces from hyperelliptic strata have the following parity of the spin-structure (see [KonZo]). The parity of the spin structure determined by a flat structure from the hyperelliptic component $S \in \mathcal{H}_1^{hyp}(2g-2)$ equals

$$\phi(S) = \left[\frac{g+1}{2}\right] \pmod 2 \tag{7}$$

where the square brackets denote the integer part of a number. The parity of the spin structure determined by a flat structure ω from the hyperelliptic component

$\omega \in \mathcal{H}_1^{hyp}(g-1, g-1)$, for odd genera g equals

$$\phi(S) = \binom{\frac{g+1}{2}} \ (mod\ 2) \quad \text{for odd } g \tag{8}$$

5.3. **Strata of Surfaces with Marked Points.** In this paper we shall often consider the strata $\mathcal{H}_1(\alpha)$ of surfaces $S = (M, \omega)$ where we not only fix the zeroes of the Abelian differential ω, but we also mark one or more regular points on the surface. Say, $\mathcal{H}_1(3, 1, 0)$ will denote the surfaces of genus $g = 3$ endowed with an Abelian differential with zeroes of orders 3 and 1 having one additional marked point ("zero of order 0 of the Abelian differential").

Throughout this paper we assume that all zeroes and all marked points are always numbered. Thus a stratum with one or more marked points has the natural structure of a fiber bundle over the corresponding stratum without marked points with a direct product (minus diagonals) of several copies of the translation surface as a fiber, where the number of copies equals the number of marked points. For example, the *universal curve* $\mathcal{H}_1(3, 1, 0)$ fibers over $\mathcal{H}_1(3, 1)$ with a fiber S.

In particular, there is the following formula for the dimension of the strata with marked points. Let g be the genus and let $|\alpha|$ denote the number of entries in α. Suppose that $0 \notin \alpha$. Then

$$\dim_{\mathbb{C}} \mathcal{H}(\alpha, \underbrace{0, \dots, 0}_{n}) = \begin{cases} \dim_{\mathbb{C}} \mathcal{H}(\alpha) + n = 2g - 1 + |\alpha| + n & \text{when } \alpha \neq \emptyset \\ \dim_{\mathbb{C}} \mathcal{H}(\emptyset) + n - 1 = n + 1 & \text{when } \alpha = \emptyset \end{cases} \tag{9}$$

(The case $\alpha = \emptyset$ is exceptional, since by the homogeneity of the torus we can translate one of the marked points to any prescribed point.)

The natural measure on the stratum with marked points disintegrates in the product measure, where the measure along the fiber is just the Lebesgue measure on S (correspondingly product of several copies of S) induced by the flat metric on S, and the measure on the base is the natural measure on the corresponding stratum taken already without marked points.

Considering the volumes of the strata we confine ourselves to the subspaces $\mathcal{H}_1(\alpha)$ of the strata for which the area of every surface S (measured in the flat metric) is equal to one. The observation above implies that *volumes of the strata with marked points coincide with the volumes of the corresponding strata without marked points*:

$$\mathrm{Vol}(\mathcal{H}_1(\alpha)) = \mathrm{Vol}(\mathcal{H}_1(\alpha, 0)) = \mathrm{Vol}(\mathcal{H}_1(\alpha, 0, 0)) = \dots \tag{10}$$

Part 1. Saddle Connections Joining Distinct Zeroes

In this part we describe the possible configurations of saddle connections joining distinct zeroes and compute the constants in the corresponding asymptotics.

To separate the basic construction and numerous details we start with the easiest case. In the first section of this part we assume that the saddle connection joining a pair of zeroes P_i, P_j has *multiplicity one*, i.e., there are no other saddle connections in the same direction joining the same pair of zeroes. In this section we also assume that the translation surface under consideration belongs to a connected stratum.

In the second section of this part we consider the problem in full generality for the translation surfaces from connected strata. In particular we give the explicit general formula for the surfaces from the principal stratum.

In the last section of this part we treat the surfaces from the strata which are not connected.

6. Saddle Connections of Multiplicity One. Connected Strata

Throughout this section we assume that the flat surface S under consideration belongs to a *connected* stratum $\mathcal{H}_1(\alpha)$.

6.1. **Collapsing a Pair of Zeroes, Principal Boundary.** Suppose we have a surface $S \in \mathcal{H}_1(\alpha)$, a saddle connection of length at most ϵ joining distinct zeroes w_1, w_2 of orders m_1 and m_2 and no saddle connection homologous to it. Let $z_1, \dots, z_l$ be the other zeroes. Choose a basis of cycles in the relative homology group $H_1(S, \{w_1, w_2, z_1, \dots, z_l\}; \mathbb{C})$. This basis can be represented by a collection of curves β_i on the surface S. One of these curves, β_0 is the saddle joining w_1 and w_2. We suppose that all other curves except (at most one) miss w_1, w_2; the remaining curve (if it exists), say, β_1 is not closed; it has w_1 as an endpoint. (If w_1, w_2 are the only zeroes of S, then all curves other than the saddle are closed and this β_1 does not exist. One can construct a flat surface S' by collapsing the saddle connection of length ϵ to a zero w of order $m = m_1 + m_2$. If β_1 exists then we can replace β_1 with a curve joined to w. One can do this deformation while keeping the holonomy fixed along the entire basis other than β_0. All named zeroes on S not affected by the collapse are given the same names on S'. We may think of this as a *Whitehead* move on the foliation in direction γ. The resulting surface S' belongs to a stratum $\mathcal{H}_1(\alpha')$. By construction the partition α' is obtained from the partition α by replacing the pair of entries m_1, m_2 by the entry m. The stratum $\mathcal{H}_1(\alpha')$ is in the closure of $\mathcal{H}_1(\alpha)$ inside the moduli space of all flat structures on the surfaces of genus g. We say $\mathcal{H}_1(\alpha')$ is the *principal* boundary of $\mathcal{H}_1(\alpha)$ corresponding to this configuration.

6.2. **Breaking up a Zero.** Conversely, suppose we are given a flat surface S' defined by ω', a zero w of order $m \geq 2$, a pair of positive integers m_1, m_2, such that $m = m_1 + m_2$, and a vector $\gamma \in \mathbb{R}^2$ of length ϵ. Further suppose that S' does not have any saddle connection or closed geodesic of length smaller than 2ϵ. Let $w, z_1, \dots, z_l$ be the set of all zeroes of the flat structure on S'. Choose a basis of cycles in the relative homology group $H_1(S', \{w, z_1, \dots, z_l\}; \mathbb{C})$. Let this basis be represented by a collection of curves β_i on the surface S'. We suppose that all curves except (at most one) miss the point w; the remaining curve (if it exists), say, β_1 is not closed; it has w as an endpoint. (If w is the only zero on S', then all the basic curves are closed, and β_1 does not exist.)

We take a disc of radius ϵ about w that misses all other zeroes. We may break up the zero w of order m on S' into two zeroes z', z'' of orders m_1 and m_2 correspondingly with a vector γ joining them constructing a flat surface S. Again we can describe this breakup as a Whitehead move on the foliation in direction γ.

If $m_1 \neq m_2$, the number of ways of effecting the breakup is $2m+2$. However we have $\overrightarrow{z'z''} = \gamma$ for only half of the resulting surfaces, i.e. for $m+1$ ones; for another $m+1$ surfaces we have $\overrightarrow{z''z'} = \gamma$. If $m_1 = m_2 = m/2$ the number of the resulting surfaces is again $m+1$. for every such surface there are two ways to assign "names"

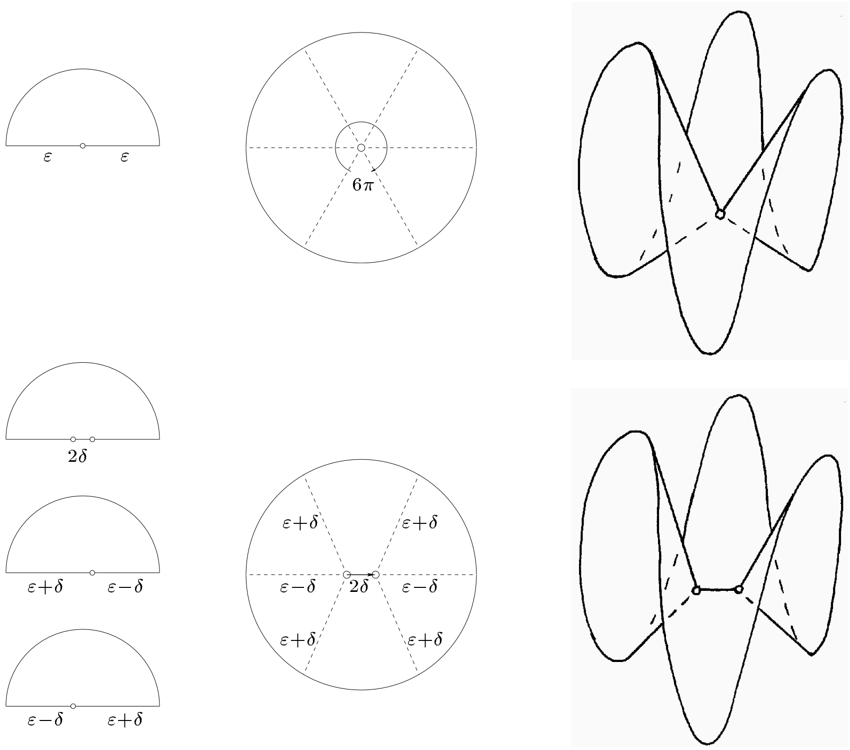

FIGURE 3. Breaking up a zero of degree 2 into two simple zeroes. Note that the surgery is local: we do not change the flat metric outside of the neighborhood of the zero.

z', z'' to the newborn zeroes of order $m/2$. This doubles the number of resulting surfaces with "named" zeroes. However we again have only $m+1$ surfaces with $\overrightarrow{z'z''} = \gamma$; for another $m+1$ ones we have $\overrightarrow{z''z'} = \gamma$. We again choose only those $m+1$ surfaces for which we have $\overrightarrow{z'z''} = \gamma$.

Again we let the curve which had the endpoint at w (if it was present) keep the corresponding endpoint at z' during the deformation. We may do this in a manner that preserves the holonomy along the homology basis of S'.

We denote the assignment by

$$(S', \gamma, m) \to (S, m_1, m_2)$$

We denote by $\mathcal{H}_1(\alpha)$ the stratum which contains the resulting flat surface S. By construction the partition α is obtained from the partition α' by replacing the entry m by two entries m_1 and m_2.

Recall that all the zeroes on the surface S' are "named". For all the zeroes on the new surface S different from the zero which was just broken we keep the same "names" on S as their initial "names" on S'.

Now choose a simply connected subset of $\mathcal{H}_1(\alpha')$ of full volume and remove the set of flat surfaces with a saddle connection or closed geodesic of length at most 2ϵ.

Call the resulting set $\mathcal{F}' \subset \mathcal{H}_1(\alpha')$. We have

$$\mathrm{Vol}(\mathcal{H}_1(\alpha') - \mathcal{F}') = O(\epsilon^2)$$

Choose a homology basis that is valid for all $S' \in \mathcal{F}'$. Thus distinct S_1', S_2' have different holonomy on some basis element. As in Section 3.1 let $\mathcal{H}_1^{\epsilon,th}(\alpha) \subset \mathcal{H}_1(\alpha)$ be the set of flat surfaces with a saddle connection of length at most ϵ joining the named zeroes and no other saddle connections of length smaller than 2ϵ. We have

Lemma 6.1. *For γ a vector in $\mathbb{R}^2$ of length at most ϵ, except for a set of $S' \in \mathcal{F}'$ of volume 0, there are precisely $m+1$ surfaces in $\mathcal{H}_1^{\epsilon,th}(\alpha)$. that are the result of the assignment $(S', \gamma, m) \to (S, m_1, m_2)$. Moreover every surface in $\mathcal{H}_1^{\epsilon,th}(\alpha)$ is the result of such an assignment.*

Proof. We have addressed every issue except the statement that there are $m+1$ surfaces that are the result of the assignment. First notice that if $S_1' \neq S_2'$, the holonomy of the two resulting S_1, S_2 built out of S_1', S_2' differ on the homology basis. Thus if they are isomorphic, the isomorphism does not respect the relative homology basis. This implies that S_1', S_2' must have a saddle connection of length at most ϵ. However we have removed surfaces in $\mathcal{H}_1(\alpha')$ with saddle connections or closed curves of length at most 2ϵ. The remaining obstacle is that for fixed S', two or more of the $m+1$ surfaces S built from S' are isomorphic by an isomorphism that fixes the short saddle connection. But this isomorphism is induced by an isomorphism of S' that fixes the zero, and the set of flat surfaces that have automorphisms has measure 0. ☐

Remark 6.2. Note that the cohomology class $[\omega] \in H^1(S, \{z', z_1, \dots, z_l\}; \mathbb{C})$ together with $\overrightarrow{z'z''} = \gamma$ determines an element of the relative cohomology group $[\omega] \in H^1(S, \{z', z'', z_1, \dots, z_l\}; \mathbb{C})$. The lemma above claims, actually, that the mapping $\mathcal{H}_1(\alpha) \to H^1(S, \{z', z'', z_1, \dots, z_l\}; \mathbb{C})$, defined by $S \to [\omega]$ is a ramified covering of order $m+1$ with a ramification locus at $\mathcal{H}_1(\alpha')$.

6.3. **Computing the Asymptotic Constants.** In this section we derive formulae for the constant c in quadratic asymptotics of the number of saddle connections joining a zero of order m_1 to a zero of order m_2 in multiplicity one. Recall from the introduction that this means that we have to find the asymptotics of $\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha))$. Recall also that we are assuming that the zeroes are named. Since we may have numerous zeroes of orders m_1 and m_2 we, actually, have two different counting problems:

Problem 1. Count the constant in the quadratic asymptotics for the number of saddle connections joining a *fixed* zero z_1 of order m_1 to a *fixed* zero z_2 of order m_2. (When $m_1 = m_2$ we require that $z_1 \neq z_2$.)

Problem 2. Count the constant in the quadratic asymptotics for the number of saddle connections joining *some* zero of order m_1 to *some other* zero of order m_2. (This is, in fact, equivalent to counting the saddle connections joining unnumbered zeroes.)

Let us start by assuming that $\mathcal{H}_1(\alpha)$ is *connected.* The case of strata that are not connected will be postponed until the end of the section on higher multiplicity.

Again let $n = \dim_{\mathbb{R}} \mathcal{H}(\alpha)$, $n' = \dim_{\mathbb{R}} \mathcal{H}(\alpha')$. Then $n' = n - 2$. Recall that $d\nu'(S')$ is the natural measure on $\mathcal{H}(\alpha')$. For $S' \in C(\mathcal{F}')$ we set $S' = rS''$ where

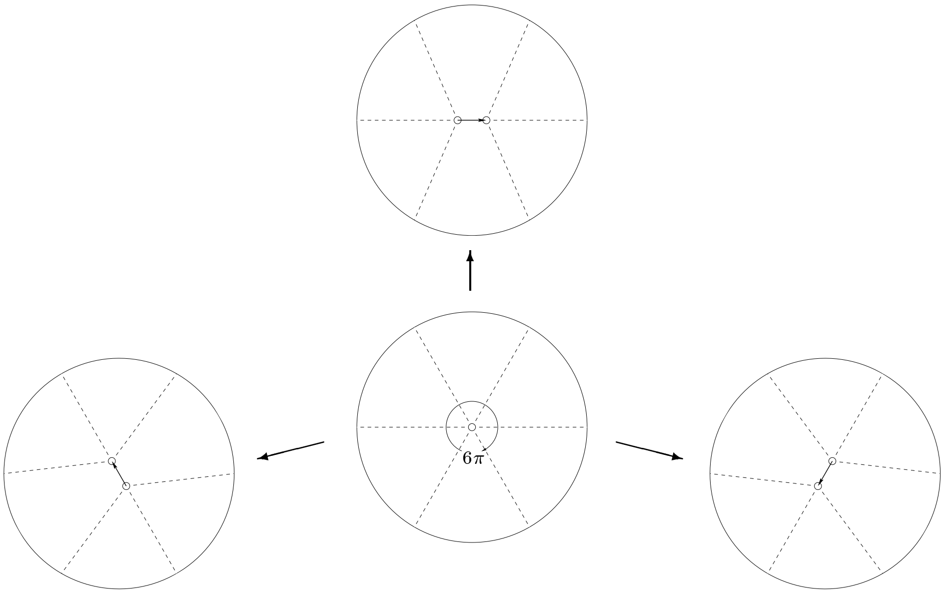

FIGURE 4. The cone angle corresponding to a zero of order $m = 2$ is equal to $(m+1)\cdot 2\pi = 6\pi$. Thus we have $(m+1) = 3$ different ways of breaking up a zero of order 2 in a direction $\vec{\gamma}$. In this way (generically) we get $m = 3$ different flat surfaces

$\mathrm{area}(S'') = 1$ so $\mathrm{area}(S') = r^2$. Then by Lemma 6.1,

$$d\nu(S) = (m_1 + m_2 + 1) d\nu'(S')\, d\gamma = (m_1 + m_2 + 1) r^{n'-1}\, dr\, d\,\mathrm{vol}'(S'')\, d\gamma$$

Again by Lemma 6.1,

$$\begin{aligned}\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))) &= (m_1 + m_2 + 1)\,\mathrm{Vol}(\mathcal{F}') \int_0^1 r^{n'-1} \int_{B(\epsilon r)} d\gamma\, dr \\ &= \pi\epsilon^2 (m_1 + m_2 + 1)\,\mathrm{Vol}(\mathcal{F}') \int_0^1 r^{n'+1}\, dr. \\ &= \frac{\pi\epsilon^2}{n'+2}(m_1 + m_2 + 1)\,\mathrm{Vol}(\mathcal{F}').\end{aligned}$$

Since $n = n' + 2$ we have

$$\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})) = n\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha))) = (m_1 + m_2 + 1)\pi\epsilon^2\,\mathrm{Vol}(\mathcal{F}')$$

Thus, the constant from the Problem 1 has the following form:

$$\begin{aligned}c = \lim_{\epsilon\to 0} \frac{\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))}{\pi\epsilon^2\,\mathrm{Vol}(\mathcal{H}_1(\alpha))} = \lim_{\epsilon\to 0} \frac{(m_1 + m_2 + 1)\,\mathrm{Vol}(\mathcal{F}'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))} = \\ = \frac{(m_1 + m_2 + 1)\,\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}.\end{aligned}$$

Note that having solved the Problem 1 we get a solution of Problem 2 by an elementary combinatorial calculation. Let $o(m_i)$ be the number of zeroes of order

m_i in the stratum $\mathcal{H}(\alpha)$. If $m_1 \neq m_2$, then there are $o(m_1)$ ways of choosing the zero of order m_1 and $o(m_2)$ ways of choosing the zero of order m_2. Thus we get an additional factor $o(m_1) \cdot o(m_2)$ in comparison with Problem 1.

Formula 6.1. *The constant in any connected stratum $\mathcal{H}_1(\alpha)$ for the number of saddle connections of multiplicity one joining two zeroes of orders $m_1 \neq m_2$ is equal to*

$$c = o(m_1)o(m_2) \cdot (m_1 + m_2 + 1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))} \quad \text{when } m_1 \neq m_2 \tag{11}$$

If $m_1 = m_2$, then there are $o(m_1)(o(m_1) - 1)/2$ ways of choosing an unordered pair of zeroes of order m_1. Having chosen an unordered pair, we choose, which of two zeroes would called be z_1, and which one would be z_2 in arbitrary way: the number of saddle connections is obviously symmetric with respect to interchange of the points z_1, z_2. Thus, when $m_1 = m_2$ the answer to Problem 2 is given as the answer for Problem 1 multiplied by an additional factor $o(m_1)(o(m_1) - 1)/2$.

Formula 6.2. *The constant for the number of saddle connections of multiplicity one joining two distinct zeroes of the same orders m_1 is equal to*

$$c = \frac{o(m_1)(o(m_1) - 1)}{2} \cdot (2m_1 + 1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))} \qquad \text{when } m_1 = m_2 \tag{12}$$

Note that we may view the entire combinatorial calculation in terms of the naming of the zeroes. We can assume that on S' we have broken up the zero of $m = m_1 + m_2$ at the first "named" zero of order m. The remaining zeroes are then given the same names on S,. Other zeroes of orders m_1, m_2 already have names on S. There are $o(m_1)o(m_2)$ ways of adding additional names to zeroes of orders m_1, m_2 if $m_1 \neq m_2$ and $o(m_1)(o(m_1) - 1)$ if $m_1 = m_2$.

Note that in the consideration above the stratum $\mathcal{H}_1(\alpha')$ might be nonconnected. In this case $\mathrm{Vol}(\mathcal{H}_1(\alpha'))$ is the sum of the volumes of connected components. For example if we start with the stratum $\mathcal{H}_1(5,3)$, then $\mathcal{H}_1(\alpha')$ has three connected components $\mathcal{H}_1^{hyp}(8), \mathcal{H}_1^{even}(8)$ and $\mathcal{H}_1^{odd}(8)$, and the sum of the volumes of these must be in the numerator.

6.4. **Examples: Constants for Connected Strata in Genus 3.** As an illustration of the formulae above we now give the explicit values for these constants in a number of cases.

In Table 1 we present the normalized volumes $\frac{1}{\pi^{2g}}\mathrm{Vol}(\mathcal{H}_1(\alpha))$ of the strata with *numbered* zeroes, see [EOk], see also some values in [Zo]. These values will be used for computations in all the examples.

Both strata in genus $g = 2$ are hyperelliptic; they will be treated later on in section 8. In genus $g = 3$ there are three connected strata: $\mathcal{H}_1(3,1), \mathcal{H}_1(2,1,1)$, and $\mathcal{H}_1(1,1,1,1)$.

Example 6.3. **Stratum $\boldsymbol{\mathcal{H}_1(3,1)}$.** After collapsing zeroes we obtain a flat surface $S' \in \mathcal{H}_1(4)$, where both components $\mathcal{H}_1^{hyp}(4)$ and $\mathcal{H}_1^{odd}(4)$ occur. We have $m_1 =$

TABLE 1. Normalized volumes $\frac{1}{\pi^{2g}}\mathrm{Vol}(\mathcal{H}_1(\alpha))$ of the strata in small genera

Genera $g = 1, 2$

$\mathcal{H}_1(\emptyset)$	$\mathcal{H}_1(2)$	$\mathcal{H}_1(1,1)$
$\frac{1}{3}$	$\frac{1}{120}$	$\frac{1}{135}$

Genus $g = 3$

$\mathcal{H}_1^{hyp}(4)$	$\mathcal{H}_1^{odd}(4)$	$\mathcal{H}_1(3,1)$	$\mathcal{H}_1^{hyp}(2,2)$	$\mathcal{H}_1^{odd}(2,2)$	$\mathcal{H}_1(2,1,1)$	$\mathcal{H}_1(1,1,1,1)$
$\frac{1}{6720}$	$\frac{1}{2430}$	$\frac{16}{42525}$	$\frac{1}{9450}$	$\frac{1}{4320}$	$\frac{1}{3780}$	$\frac{1}{4860}$

Genus $g = 4$

$\mathcal{H}_1^{hyp}(6)$	$\mathcal{H}_1^{odd}(6)$	$\mathcal{H}_1^{even}(6)$	$\mathcal{H}_1(5,1)$
$\frac{1}{580608}$	$\frac{1}{37800}$	$\frac{32}{1913625}$	$\frac{1}{36750}$
$\mathcal{H}_1^{odd}(4,2)$	$\mathcal{H}_1^{even}(4,2)$	$\mathcal{H}_1(4,1,1)$	$\mathcal{H}_1^{nonhyp}(3,3)$
$\frac{1}{79380}$	$\frac{1}{107520}$	$\frac{11}{653184}$	$\frac{1}{51030}$
$\mathcal{H}_1^{hyp}(3,3)$	$\mathcal{H}_1(3,2,1)$	$\mathcal{H}_1(3,1,1,1)$	$\mathcal{H}_1^{odd}(2,2,2)$
$\frac{1}{992250}$	$\frac{1}{70875}$	$\frac{62}{5740875}$	$\frac{31}{4354560}$
$\mathcal{H}_1^{even}(2,2,2)$	$\mathcal{H}_1(2,2,1,1)$	$\mathcal{H}_1(2,1,1,1,1)$	$\mathcal{H}_1(1,\dots,1)$
$\frac{37}{6804000}$	$\frac{131}{13608000}$	$\frac{1}{136080}$	$\frac{377}{67359600}$

$3, m_2 = 1$, thus the combinatorial factor equals $o(m_1)o(m_2)(m_1 + m_2 + 1) = 5$. We get

$$c = 5 \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(4))}{\mathrm{Vol}(\mathcal{H}_1(3,1))} = 5 \cdot \frac{(\mathrm{Vol}(\mathcal{H}_1^{hyp}(4)) + \mathrm{Vol}(\mathcal{H}_1^{odd}(4))}{\mathrm{Vol}(\mathcal{H}_1(3,1))} = \frac{7625}{1024} \approx 7.45$$

Example 6.4. **Stratum $\boldsymbol{\mathcal{H}_1(2,1,1)}$**. There are two cases here in multiplicity 1. The first case is a saddle connection joining the zero of degree 2 with any of the two zeroes of degree 1. Collapsing the zeroes we obtain a surface $S' \in \mathcal{H}_1(3,1)$. For $m_1 = 2$, $m_2 = 1$, $o(m_1) = 1$, $o(m_2) = 2$. Thus the combinatorial constant equals 8. We get

$$c = 8\frac{\mathrm{Vol}(\mathcal{H}_1(3,1))}{\mathrm{Vol}(\mathcal{H}_1(2,1,1))} = \frac{512}{45} \approx 11.4$$

The second case consists of collapsing the pair of simple zeroes to a zero of order 2. We obtain a surface $S' \in \mathcal{H}_1(2,2)$. Any surface in any component of $\mathcal{H}_1(2,2)$ can be found by collapsing a pair of simple zeroes in this manner. Here $m_1 = m_2 = 1$ and $o(m_1) = 2$ so the combinatorial constant is 3. We get

$$c = 3 \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2,2))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = 3 \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2,2)) + \operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{153}{40} \approx 3.83$$

6.5. **Constants for the Principal Stratum $\mathcal{H}_1(1,\dots,1)$.** The surface of genus g has $2g-2$ simple zeroes; $o(1) = 2g-2$. The corresponding surface $S' \in \mathcal{H}_1(2,1,\dots,1)$ has $2g-4$ simple zeroes. The factor $o(m_1)\,(o(m_1)-1)\,/2$ gives $(g-1)(2g-3)$, and the factor $(2m_1+1)$ gives 3. Applying formula (12)) we get

Formula 6.3. *The constant for the number of saddle connections of multiplicity one joining two distinct zeroes is equal to*

$$c = 3(g-1)(2g-3) \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2,\overbrace{1,\dots,1}^{2g-4}))}{\operatorname{Vol}(\mathcal{H}_1(\underbrace{1,\dots,1}_{2g-2}))}$$

In the table below we present some values of the constant in this case.

TABLE 2. Principal stratum $\mathcal{H}_1(1,\dots,1)$; values of the constants for saddle connections of multiplicity one joining distinct zeroes

	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$
$c=$	$\frac{27}{8}$	$\frac{162}{7}$	$\frac{22275}{377}$	$\frac{2594700}{23357}$	$\frac{2954056635}{16493303}$	$\frac{13229971542}{50280671}$	$\frac{14740938123723}{40593663941}$
$c\approx$	3.375	23.14	59.08	111.1	179.1	263.1	363.1

7. Multiple Homologous Saddle Connections. Connected Strata

7.1. **Principal Boundary.** Consider a surface $S \in \mathcal{H}_1(\alpha)$ with a fixed pair of zeroes z_1 and z_2 of orders m_1 and m_2 correspondingly. Suppose that we have a configuration $\mathcal{C}$ (see Section 3) of precisely p homologous saddle connections $\gamma_1,\dots,\gamma_p$ joining z_1 to z_2. A pair γ_i and γ_{i+1} bounds a surface S_i. The surfaces S_i and S_{i+1} share the saddle connection γ_{i+1}. By convention the cyclic order of γ_i at z_1 is clockwise in the orientation defined by the flat structure. The angle between γ_i and γ_{i+1} at z_1 be $2\pi(a_i'+1)$; the angle between γ_i and γ_{i+1} at z_2 be $2\pi(a_i''+1)$.

Cut the surface S along all γ_i, $i = 1,\dots,p$. Gluing together the two sides γ_i and γ_{i+1} of the boundary of S_i we get a flat surface with two distinguished zeroes z_i', z_i'' of orders a_i' and a_i'' correspondingly joined by a saddle connection, $\overrightarrow{z_i'z_i''} = \gamma$. When one of a_i', a_i'' is equal to zero the corresponding point is just a marked point. If $a_i' = a_i'' = 0$, then both points are marked points.

By construction any resulting flat surface S_i has a saddle connection γ_i joining the pair of zeroes and does not have any other saddle connections homologous to it. This implies that contracting the saddle connection γ_i to a point we get a flat surface S_i' with a distinguished zero (marked point) w_i of order $a_i = a_i' + a_i''$.

Let $S_i' \in \mathcal{H}(\alpha_i')$. We say that $\mathcal{H}(\alpha_i')$ is the principal boundary of this configuration.

7.2. **Slit Construction.** We would like to reverse the above degeneration and build surfaces with multiple homologous saddle connections out of surfaces in $\mathcal{H}(\alpha_i')$. We need the following construction. Let S' be a surface with a zero of order $a \geq 0$. Suppose there are no saddle connections of length shorter than 2ϵ. Let $a', a'' \geq 0$ such that $a' + a'' = a$. Let γ be a vector of length ϵ. If $a', a'' \neq 0$ we may break up the zero of order a into zeroes of orders a', a'' with a saddle connection determining γ joining them as before. If $a' = 0$ take a point z' of the form $w - \gamma$ (this means along a geodesic from w in direction $-\gamma$ and distance $|\gamma|$) and join it to $z'' = w$ on S'. If $a'' = 0$ we take a segment from $z' = w$ to $z'' = w + \gamma$ on S'.

In either case we may slit the resulting surface along the saddle connection joining the two zeroes z', z''. In this way we build a surface with one boundary component consisting of two arcs, denoted γ' and γ'', joining the endpoints of the slit. The angles between γ' and γ'' at the points z' and z'' are $2\pi(a'+1)$ and $2\pi(a''+1)$ correspondingly.

Note that the flat structure on S' fixes the choice of the orientation. By convention we give the "names" γ' and γ'' to the arcs in such a way that turning around z' in a clockwise direction from γ'' to γ' we do not leave the surface.

Conversely, consider a flat surface with a single boundary component consisting of two arcs joining a pair of vertices. The convention on the choice of the "names" γ' and γ'' means that as soon as we know which of two vertices is z', and which is z'' we can determine which of two arcs is γ', and which is γ''.

We describe now how to build surfaces with multiple homologous saddle connections using the slit construction.

7.3. **Building Surfaces With Multiple Saddle Connections.** Suppose $S' = S_1', \sqcup \dots \sqcup S_p'$ is a disconnected flat surface and on each S_i' there is a zero or a regular point which we think of as a *marked point* w_i of order a_i. Given pairs a_i', a_i'' with $a_i = a_i' + a_i''$, and a vector γ, we perform the slit construction on each surface. We obtain surfaces with one boundary component each of which consists of two arcs γ_i' and γ_i''. We glue γ_i' to γ_{i+1}'', calling this curve γ_i, (and γ_p' to γ_1''). This gives a closed surface S, a pair of zeroes z_1 and z_2 of orders m_1 and m_2, where $\sum a_i' = m_1 + 1 + p$ and $\sum a_i'' = m_2 + 1 - p$, and a set of homologous curves γ_i, $i = 1, \dots, p$ joining z_1 to z_2. The angle between γ_{i-1} and γ_i at z_i is $2\pi(a_i' + 1)$ and the angle between them at z_2 is $2\pi(a_i'' + 1)$. We denote this assignment by

$$(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$$

Our convention on the choice of γ_i', γ_i'' (see the section above) implies that the cyclic order $\dots \to \gamma_{i-1} \to \gamma_i \to \gamma_{i+1} \to \dots$ is clockwise at z_1 with respect to the orientation defined by the flat structure.

Fix a basis for the relative homology on each surface S_i'. A relative homology basis for S is given by:

- A relative homology basis of curves β_i for S'.
- A curve connecting the two zeroes. The integral of ω along this curve is γ.

We introduce $\bar{\alpha}_i'$ as follows:

$$\alpha_i' = \begin{cases} \bar{\alpha}_i' \sqcup \{a_i\} & \text{when } a_i \neq 0 \\ \bar{\alpha}_i' & \text{when } a_i = 0 \end{cases}$$

Having a collection of integers α we denote by $|\alpha|$ their sum, say, $|\alpha| = 2g - 2$.

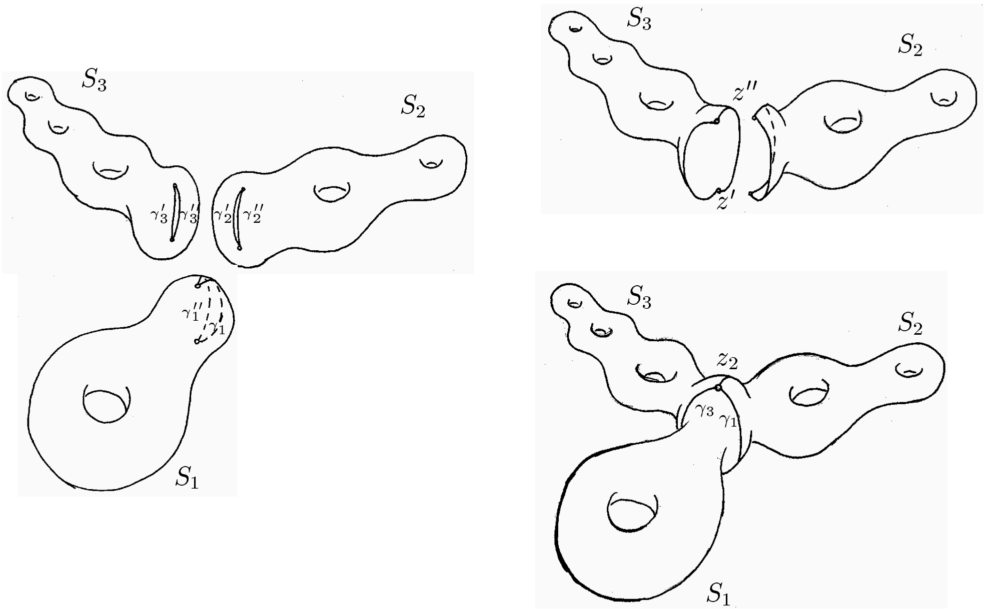

FIGURE 5. Multiple homologous saddle connections.

The construction above implies the following conditions on partitions α_i' and the numbers a_i', a_i''.

Lemma 7.1. *An assignment $(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$ satisfies the following necessary conditions on the collection (α_i', a_i', a_i''):*

$$a_1' + \cdots + a_p' = m_1 + 1 - p$$
$$a_1'' + \cdots + a_p'' = m_2 + 1 - p$$
$$\bar{\alpha}_1' \sqcup \cdots \sqcup \bar{\alpha}_p' \sqcup \{m_1\} \sqcup \{m_2\} = \alpha$$
$$|\bar{\alpha}_i'| = a_i' + a_i'' \ (mod\ 2) \text{ for } i = 1, \ldots, p$$

Moreover, when the stratum $\mathcal{H}_1(\alpha)$ is connected, these conditions are sufficient: every surface $S \in \mathcal{H}_1(\alpha)$ with a configuration of p homologous saddle connections joining the pair of zeroes can be obtained by an assignment $(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$ with appropriate $(S', \gamma, a_i', a_i'')$.

We shall introduce notation to describe the degeneration pattern for the surfaces S_i.

Notation. The integer represented as a sum of two integers corresponds to a_i represented as $a_i' + a_i''$. Here the order of appearances of the summands $a_i' + a_i''$ is significant. The cyclic order of the surfaces is represented by $S_1 \succ S_2 \succ S_3 \succ \ldots \succ S_p \succ S_1$.

Example 7.2. Let $S \in \mathcal{H}_1(4, 3, 2, 1)$; $m_1 = 3, m_2 = 4$; $p = 3$. There are 15 possible pictures for the three homologous saddle connections joining the zero of order 3 to the zero of order 4 depending on the returning angles (Table 3) Thus for example, the first line in table 3 refers to 3 surfaces in cyclic order. The first is a torus (no zeroes) where we have broken up a marked point into $(0 + 0)$; the second surface

Degeneration pattern	α'_1	α'_2	α'_3
$(0+0) \succ (1+1) \succ (0+1,2,1) \succ$	$(\emptyset)$	(2)	$(2,1,1)$
$(1+1) \succ (0+0) \succ (0+1,2,1) \succ$	(2)	$(\emptyset)$	$(2,1,1)$
$(0+0) \succ (0+0) \succ (1+2,2,1) \succ$	$(\emptyset)$	$(\emptyset)$	$(3,2,1)$
$(0+2) \succ (0+0) \succ (1+0,2,1) \succ$	(2)	$(\emptyset)$	$(2,1,1)$
$(0+0) \succ (0+2) \succ (1+0,2,1) \succ$	$(\emptyset)$	(2)	$(2,1,1)$
$(1+1) \succ (0+1,1) \succ (0+0,2) \succ$	(2)	$(1,1)$	(2)
$(0+0) \succ (1+0,1) \succ (0+2,2) \succ$	$(\emptyset)$	$(1,1)$	$(2,2)$
$(0+2) \succ (1+0,1) \succ (0+0,2) \succ$	(2)	$(1,1)$	(2)
$(0+0) \succ (1+2,1) \succ (0+0,2) \succ$	$(\emptyset)$	$(1,3)$	(2)
$(0+0) \succ (0+1,1) \succ (1+1,2) \succ$	$(\emptyset)$	$(1,1)$	$(2,2)$
$(1+1) \succ (0+0,2) \succ (0+1,1) \succ$	(2)	(2)	$(1,1)$
$(0+0) \succ (1+1,2) \succ (0+1,1) \succ$	$(\emptyset)$	$(2,2)$	$(1,1)$
$(0+0) \succ (0+0,2) \succ (1+2,1) \succ$	$(\emptyset)$	(2)	$(1,3)$
$(0+2) \succ (0+0,2) \succ (1+0,1) \succ$	(2)	(2)	$(1,1)$
$(0+0) \succ (0+2,2) \succ (1+0,1) \succ$	$(\emptyset)$	$(2,2)$	$(1,1)$

TABLE 3. Possible degenerations for $S \in \mathcal{H}_1(4,3,2,1)$ with $m_1 = 3, m_2 = 4$; $p = 3$

has a single zero of order 2 which we have broken into two simple zeroes indicated by $(1+1)$, and the third surface has zeroes of $2,1,1$, where we have broken one of the simple zeroes into a simple zero and a zero of order 0.

7.4. **Stratum Interchange and $\gamma \to -\gamma$ Symmetry.** In this section we discuss the possible symmetries of the assignment

$$(S', \gamma, a'_i, a''_i) \to (S, m_1, m_2, a'_i, a''_i)$$

Let us first specify our problem.

Problem 1. We assume that the zeroes $z_1, \dots, z_l$ of the surface S are numbered. We fix the zeroes $z_1, z_2 \in S$, $z_1 \neq z_2$, of orders m_1 and m_2 correspondingly.

We start with the case when we fix also the following data. We assume that the zeroes $z_3, \dots, z_{l_1}$ belong to S_1, the zeroes $z_{l_1+1}, \dots, z_{l_1+l_2}$ belong to S_2, ..., the zeroes $z_{l_1+\dots+l_{p-1}+1}, \dots, z_{l_1+\dots+l_p}$ belong to S_p. Here $l_i + 1 = card(\alpha'_i)$, when $a_i \neq 0$, and $l_i = card(\alpha'_i)$, when $a_i = 0$. We have $l = l_1 + \dots + l_p = card(\alpha)$.

In this setting the only thing that is not determined is the case when S has exactly two zeroes: z_1 and z_2. Then every S'_i either has a single zero of order a_i or it is a torus with a marked (regular) point. Note that we fix only the cyclic order of the collection (a'_i, a''_i), but not the numbering of each of these pair of numbers. The natural action of the cyclic group of order p on the collection of ordered pairs (a'_i, a''_i) organized in a cyclic order, may have nontrivial stabilizer, which we denote by Γ. We get symmetry of order $|\Gamma|$. This type of symmetry we call the *stratum interchange.*

Problem 2. Now consider the problem with fewer constrains. We count the number of occurrences of the following phenomenon: *some* zero of S of order m_1 is joined to *some* zero of order m_2 by precisely p homologous saddle connections

$\gamma_1, \dots, \gamma_p$ as above. We fix the type α'_i of the flat surface S'_i obtained as described above, but we make no assumptions on the distribution of the zeroes $z_1, \dots, z_l$ by the surfaces S'_i. As before we fix only the cyclic order of appearances of the surfaces S_i.

Remark 7.3. Problem 2 may be considered as generalization of Problem 1 to the case when the zeroes of S and of S'_i are *not* numbered.

Remark 7.4. In both settings the $\mathcal{H}_1(\alpha'_i)$ might be nonconnected. We could also specify the connected component of the $\mathcal{H}_1(\alpha'_i)$ when this occurs. We prefer the setting where this data is not specified.

Consider the natural action of the cyclic group of order p on the collection

$$(\alpha'_1, a'_1, a''_1) \succ \cdots \succ (\alpha'_p, a'_p, a''_p) \succ$$

organized in a cyclic order. If it has a nontrivial stabilizer we denote it by the same symbol Γ. We get a symmetry of order $|\Gamma|$ which we also call the *stratum interchange.*

Example 7.5. Let $\alpha = (27, 15, 1, 1)$, $m_1 = 27$, $m_2 = 15$. Consider the degeneration of S into 6 surfaces of the following types $\alpha'_1 \succ \cdots \succ \alpha'_6 \succ$

$$(2) \succ (7,1) \succ (12) \succ (2) \succ (7,1) \succ (12) \succ$$

The possible symmetry in this case is a symmetry of order 2, where subgroup Γ is a shift by 3.

The degeneration

$$(0+2) \succ (4+3,1) \succ (7+5) \succ (0+2) \succ (4+3,1) \succ (7+5) \succ$$

possesses this symmetry, while the degeneration

$$(0+2) \succ (3+4,1) \succ (7+5) \succ (2+0) \succ (4+3,1) \succ (7+5) \succ$$

does not.

When $m_1 = m_2$ we may have an additional symmetry in Problem 2. Let P, Q be two zeroes of ω of order $m_1 = m_2$ joined by precisely p homologous saddle connections. Let the homology class of the saddle connection be represented by a vector γ. Assigning the "names" $z_1 := P$ and $z_2 := Q$ to P and Q we get a decomposition $(S'_i, \gamma, \alpha'_i, a'_i, a''_i)$.

Consider now the very same surface S with the same configuration of p homologous saddle connections joining the same two zeroes P and Q. We may declare now that the correspondence of zeroes is inverse with respect to the previous assignment: $z_1 := Q$ and $z_2 := P$. Since our saddle connections are oriented (from z_1 to z_2) the homology class of the same saddle connection (from P to Q) is represented now by the vector $-\gamma$. Since the cyclic order of "pieces" is determined by the cyclic order at the point z_1 the new identification reverses the cyclic order in the collection (α'_i), as well as the order in the pairs (a'_i, a''_i).

We say, that we have a $\gamma \to -\gamma$ *symmetry* if and only if the assignment $z_1 := Q$ and $z_2 := P$ gives a decomposition with the same (up to a cyclic reenumeration) collection (α'_i, a'_i, a''_i) as before.

It is easy to see the following combinatorial criterion of the $\gamma \to -\gamma$ symmetry. Interchange simultaneously all $a'_i \leftrightarrow a''_i$, and change the cyclic order of the resulting collection (α'_i, a''_i, a'_i) to the opposite one. We possess a $\gamma \to -\gamma$ symmetry if and only if the result of this operation gives us the initial collection (α_i', a_i', a_i'') up to a cyclic reenumeration. We denote $\gamma \to -\gamma$ symmetry by Γ_-, where $|\Gamma_-| = 2$. We let $|\Gamma_-| = 1$ when there is no $\gamma \to -\gamma$ symmetry.

The condition $m_1 = m_2$ is an obvious necessary condition for $\gamma \to -\gamma$ symmetry. It will be always assumed in the discussions of $\gamma \to -\gamma$ symmetry.

Under this condition every degeneration of multiplicity 1 possesses a $\gamma \to -\gamma$ symmetry. There is a $\gamma \to -\gamma$ symmetry in multiplicity 2 if and only if at least one the following conditions is valid: either

— $a_1' = a_1''$ (and then $a_2' = a_2''$ automatically); or

— $\alpha_1' = \alpha_2'$ and $a_1' = a_2''$ (and then $a_1'' = a_2'$ automatically).

Example 7.6. Let $\alpha = (11, 11)$. Consider degeneration of S into 4 surfaces of the following types $\alpha_1' \succ \alpha_2' \succ \alpha_3' \succ \alpha_4'$:

$$(2) \succ (6) \succ (2) \succ (6) \succ$$

The degeneration

$$(0+2) \succ (3+3) \succ (2+0) \succ (3+3) \succ$$

possesses the $\gamma \to -\gamma$ symmetry, but does not have the stratum interchange, so $|\Gamma_-| = 2$, $|\Gamma| = 1$.

The degeneration

$$(0+2) \succ (4+2) \succ (0+2) \succ (4+2) \succ$$

does not have a $\gamma \to -\gamma$ symmetry, but allows the stratum interchange, so $|\Gamma_-| = 1$, $|\Gamma| = 2$.

The degeneration

$$(1+1) \succ (3+3) \succ (1+1) \succ (3+3) \succ$$

has both symmetries: $|\Gamma_-| = 2$, $|\Gamma| = 2$.

Remark 7.7. Describing the assignments $(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$ one can consider *all* possible assignments, and then take into account corresponding symmetries, or one can deal with the classes, eliminating the symmetry whenever it is possible.

We have chosen the second way. For example the assignment

$$(2+0) \succ (0+0) \succ (0+1, 2, 1)$$

is not presented in the Table 3 since it is symmetric to the assignment

$$(0+0) \succ (0+2) \succ (1+0, 2, 1)$$

(which is in the list) by composition of $\gamma \to -\gamma$ symmetry with the stratum interchange.

As in the single saddle connection situation for small ϵ, we let $\mathcal{F}'$ be a simply connected domain in $\mathcal{H}_1(\alpha')$ such that for every surface in $\mathcal{F}'$, each saddle connection and closed geodesic has length at least 2ϵ. Again $\mathrm{Vol}(\mathcal{F}')$ differs from $\mathrm{Vol}(\mathcal{H}_1(\alpha'))$ by $O(\epsilon^2)$. Let $C(\mathcal{F}')$ the cone over $\mathcal{F}'$. Let $\mathcal{H}_1^{\epsilon,th}(\alpha, \mathcal{C})$ be the subset of $\mathcal{H}_1(\alpha)$ consisting of those $(S, m_1, m_2, a_i', a_i'')$ with the prescribed set of homologous curves joining two zeroes of length at most ϵ and no other saddle connections or closed geodesic has length smaller than 2ϵ.

Lemma 7.8. *For almost all $S' \in \mathcal{F}'$ and γ a vector in $\mathbb{R}^2$ of length at most ϵ, there are exactly $\prod_{i=1}^{p}(a_i+1)$ flat surfaces S that are the result of the assignment $(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$, where $a_i = a_i' + a_i''$. There are precisely $2|\Gamma|$ assignments that result in the same flat surface if there is a $\gamma \to -\gamma$ symmetry, $|\Gamma|$ to 1 otherwise. The assignment is onto $C(\mathcal{H}_1^{\epsilon,th}(\alpha))$.*

Proof. We may choose a homology basis valid for all of $\mathcal{F}'$. For distinct S' and S'' the holonomy must differ for some basis curve. The surfaces that correspond to S' and S'' by the assignment can be isomorphic only if the isomorphism does not preserve the homology basis. If the curves in the configuration are preserved, then the possibilities are that we have the $\gamma \to -\gamma$ symmetry, or we have the stratum symmetry. If the set of configurations curves are not preserved then S' contains a curve of length at most ϵ in length. However the last possibility has been ruled out by the choice of $\mathcal{F}'$. The only remaining possibility is that for fixed S' two or more of the resulting S are isomorphic by an isomorphism that preserves the set of homologous curves. But such an isomorphism must come from an automorphism of S' and the set of such S' has measure 0. For each pair a_i', a_i'' there are $a_i + 1$ ways to perform a slit construction at a zero of order $a_i = a_i' + a_i''$, providing $a_i + 1$ surfaces. The Lemma now follows from Lemma 7.1.

□

7.5. **The Constant in Higher Multiplicity, Connected Strata.** We start with the study of the constant c in the setting where we fix the partition of numbered zeroes of S by components S_i' (see Problem 1 in section 7.4). We assume first that $\mathcal{H}_1(\alpha)$ is *connected.*

By Lemma 7.8 the data

$$(S', \gamma, a_i', a_i'')$$

defines

$$M = \frac{1}{|\Gamma_-| \cdot |\Gamma|} \cdot \prod_{i=1}^{p}(a_i+1)$$

surfaces S in the setting of Problem 1 of section 6.2.

We write $S_i = r_i S_i'$, where $\operatorname{area} S_i' = 1$. Then $\operatorname{area} S_i = r_i^2$.

For the computations in this section and the section on saddles joining a zero to itself, we shall use the following notation. By (9) we get

$$d_i = \begin{cases} n_i = \dim_{\mathbb{R}} \mathcal{H}_1(\alpha_i') & \text{when } a_i \neq 0 \text{ or when } \alpha_i' = \emptyset \\ \dim_{\mathbb{R}} \mathcal{H}_1(\alpha_i') + 2 & \text{when } \alpha_i' \neq \emptyset \text{ and } a_i = 0 \end{cases} \tag{13}$$

In case there are no marked points, we have

$$d\nu(S) = M \prod_{i=1}^{p} d\nu_i'(S_i)\, d\gamma = M \prod_{i=1}^{p} \left(r_i^{n_i-1}\, dr_i\right) \prod_{i=1}^{p} d\operatorname{vol}'(S_i')$$

In case z_i is a marked point (i.e. $a_i = 0$), the formula for $d\nu(S)$ is multiplied by dz_i since dz_i gives the area element of the varying point z_i. In that case the last form for the measure has an extra factor of r_i^2. Set

$$W = \prod_{i=1}^{p} \operatorname{Vol}(\mathcal{H}_1(\alpha_i)).$$

Then, in all cases up to an error term that is $O(\epsilon^2)$,

$$\begin{aligned}\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})) &= MW\int_{D(1)}\prod_{j=1}^{p} r_j^{d_j-1}\int_{B(\epsilon\sqrt{\sum r_j^2})} d\gamma \prod_{j=1}^{p} dr_j \\ &= MW\int_{D(1)}\prod_{j=1}^{p} r_j^{d_j-1}\pi\epsilon^2(\sum r_j^2)\prod_{j=1}^{p} dr_j \\ &= MW\pi\epsilon^2\int_{D(1)}\sum_{j=1}^{p} r_j^{d_j+1}\prod_{i\neq j} r_i^{d_i-1}\prod_{j=1}^{p} dr_j\end{aligned}$$

We now make the change of variable $x_i = r_i^2$ to evaluate each term in the sum. For each j, let $b_j = d_j/2$ and $b_i = d_i/2-1$, so that $r_j^{d_j+1}\,dr_j = (1/2)x_j^{b_j}\,dx_j$ and $r_i^{d_i-1}\,dr_i = (1/2)x_i^{b_i}\,dx_i$. Then each term in the sum becomes

$$\frac{1}{2^p}\int_{\sum_i x_i\le 1} x_1^{b_1}\dots x_p^{b_p}\,dx_1\dots dx_p$$

We have seen that this integral yields

$$\frac{1}{2^p}\frac{b_1!\dots b_p!}{(b_1+\dots+b_p+p)!}$$

Since $b_1+\dots+b_p+p = \sum(d_i/2)+1 = n/2$, summing the terms gives

$$\begin{aligned}\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})) = \frac{\pi\epsilon^2 WM}{2^{p-1}}\frac{(\sum(\frac{d_i}{2})(\frac{d_1}{2}-1)!\dots(\frac{d_p}{2}-1)!}{(\frac{n}{2}-1)(\frac{n}{2}-2)!} = \\ = \frac{\pi\epsilon^2 WM}{2^{p-1}}\frac{(\frac{d_1}{2}-1)!\dots(\frac{d_p}{2}-1)!}{(\frac{n}{2}-2)!}\end{aligned}$$

Hence,

$$c = \lim_{\epsilon\to 0}\frac{\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))}{\pi\epsilon^2\,\mathrm{Vol}(\mathcal{H}_1(\alpha))} = \frac{M}{2^{p-1}(\frac{n}{2}-2)!\,\mathrm{Vol}(\mathcal{H}_1(\alpha))}\prod_{i=1}^{p}(\tfrac{d_i}{2}-1)!\,\mathrm{Vol}(\mathcal{H}_1(\alpha_i))$$

Thus, finally we get the following expression for the constant c in the setting where we fix the partition of named zeroes of S by components S_i' (see Problem 1 in section 7.4):

$$c = \frac{1}{|\Gamma|}\cdot\prod_{j=1}^{p}(a_j+1)\cdot\frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!}\cdot\frac{\prod_{i=1}^{p}\mathrm{Vol}(\mathcal{H}_1(\alpha_i))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}$$

Let us study now the constant c in the setting where we do not specify which zeroes S are to be joined. (see Problem 2 in section 7.4). This counting problem can be reduced to the previous one by a purely combinatorial computation. Consider $m\neq m_1$, $m\neq m_2$, $m\neq a_i$, $i=1,\dots,p$. Then these zeroes are all “inherited” by the surfaces S_i. Since all the zeroes $z_1,z_2,\dots,z_l$ are named, the number of ways to distribute $o(m)$ zeroes of order m into groups of $o_1(m),\dots,o_p(m)$ zeroes equals

$$\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}$$

If $m\neq m_1$ and $m\neq m_2$ but $m=a_j$ for some j then one of the $o_j(m)$ zeroes of order m which lives on S_j is newborn; that is, arising from the degeneration, while the other $o_j(m)-1$ zeroes of order m come from the corresponding zeroes of order m on S. Thus the corresponding factor in the denominator becomes $(o_j(m)-1)!$. Multiplying numerator and denominator by $o_j(m)$ we get the following factor:

$$\frac{o(m)!}{\prod_{j=1}^{p} o_j(m)!}\cdot\prod_{j\ \mid\ a_j=m} o_j(m)$$

(Equivalently, we may say that we have performed the slit construction at the first "named" zero of order a_j on S_j. The rest of the zeroes of that order inherit their names on S, but then we count the number of ways of merging the names from different S_i onto S.) Consider now the case when $m=m_1$ or $m=m_2$. When $m_1\neq m_2$ and $m=m_1$ (resp. $m=m_2$) we have the arrangements of the other $o(m_1)-1$ (resp. $o(m_2)-1$) zeroes of order m. When $m_1=m_2$ we have an arrangement of the other $o(m_1)-2$ zeroes of order m. This changes the numerator in the formula above by the corresponding factorial. However, in these cases we have to count the number of ways to chose a zero of degree m_1 and a zero of degree m_2 on the initial surface S. This number is equal to $o(m_1)\cdot o(m_2)$ if $m_1\neq m_2$ and $o(m_1)(o(m_1)-1)$ if $m_1=m_2$. Thus for zeroes of these degrees we have this extra factor $o(m_1)\cdot o(m_2)$ which combined with previous considerations gives $o(m)!$ in this case as well. Taking into account the possible symmetries (see section 7.4), we finally get the following expression for the constant c in the setting where the zeroes of S are not numbered (see Problem 2 in section 7.4).

Formula 7.1.

$$c=\frac{1}{|\Gamma_-|}\cdot\frac{1}{|\Gamma|}\cdot\prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{j=1}^{p} o_j(m)!}\right)\cdot\prod_{\substack{j=1\\ a_j\neq 0}}^{p} o_j(a_j)\cdot \tag{14}$$
$$\cdot\prod_{j=1}^{p}(a_j+1)\cdot\frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!}\cdot\frac{\prod_{i=1}^{p}\mathrm{Vol}(\mathcal{H}_1(\alpha_i))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}$$

We give some examples of this formula in the case of connected strata $\mathcal{H}_1(\alpha)$.

Example 7.9. **Stratum $\boldsymbol{\mathcal{H}_1(3,1)}$**; The multiplicity is $p=2$. After degeneration we get a surface S'_1 of genus 2 with a single zero of order $a_1=2$, where now $a'_1=2, a''_1=0$, and we get a torus S'_2, with $a_2=a'_2=a''_2=0$.

The saddle connections γ_1 and γ_2 partition the cone angle 8π at z_1 in two sectors with angles 2π and 6π; they partition the cone angle 4π at z_2 in two sectors with angles 2π and 2π. There is no stratum interchange, so $|\Gamma|=1$. We get

$$c=3\cdot\frac{1}{2}\cdot\frac{3!\cdot 1!}{5!}\frac{\mathrm{Vol}(\mathcal{H}_1(2))\cdot\mathrm{Vol}(\mathcal{H}_1(\emptyset))}{\mathrm{Vol}(\mathcal{H}_1(3,1))}=\frac{567}{1024}\approx 0.554$$

Example 7.10. **Stratum $\boldsymbol{\mathcal{H}_1(2,1,1)}$**; genus $g=3$, multiplicity $p=2$. First suppose that we have a pair of homologous saddle connections joining the zero of degree $m_1=2$ with one of two zeroes of degree $m_2=1$. After degeneration we get two surfaces $S'_1\in\mathcal{H}_1(a_1,1)$, and $S'_2,\in\mathcal{H}_1(a_2)$, where $a_1+a_2=m_1+m_2+2-2p=1$. Thus $a_1=1$, $a_2=0$. There is no stratum interchange, so $|\Gamma|=1$. Altogether we get

$$c=4\cdot\frac{1}{2}\cdot\frac{4!\cdot 1!}{6!}\cdot\frac{\mathrm{Vol}(\mathcal{H}_1(1,1))\,\mathrm{Vol}(\mathcal{H}_1(\emptyset))}{\mathrm{Vol}(\mathcal{H}_1(2,1,1))}=\frac{28}{45}\approx 0.622$$

Suppose now that we have a pair of homologous saddle connections joining two simple zeroes. After degeneration we get two surfaces $S_1', \in \mathcal{H}(a_1, 2)$, and $S_2', \in \mathcal{H}(a_2)$, where $a_1 + a_2 = m_1 + m_1 + 2 - 2p = 0$. Thus S' is genus 2 with a marked point and with a zero of degree 2, while S_2' is a torus with a marked point. Here we have $a_i' = a_i'' = 0$, $i = 1, 2$.

There is no stratum interchange, so $|\Gamma| = 1$. We now get

$$c = \frac{1}{2} \cdot 1 \cdot \frac{4! \cdot 1!}{6!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2)) \operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{40} = 0.175$$

7.6. **Principal stratum.** The cone angle at any simple zero equals 4π. Thus for the principal stratum we may have at most two homologous saddle connections joining a pair of distinct zeroes. Hence in the higher multiplicity case, the surface $S, \in \mathcal{H}_1(1, \dots, 1)$ of genus g degenerates to a pair of surfaces S_1', S_2' of genera $g_1 + g_2 = g$. Each of S_1', S_2' has only simple zeroes and one additional marked point. We always have the $\gamma \to -\gamma$ symmetry, so $|\Gamma_-| = 2$. When $g_1 = g_2$ we have a symmetry $|\Gamma| = 2$ due to the stratum interchange; there is no other symmetry otherwise.

Applying formula (14) and (13) we finally get the following answer.

Formula 7.2. *The constant for the number of saddle connections of multiplicity two joining two distinct zeroes in the principal stratum is*

$$c = \frac{1}{2|\Gamma|} \cdot \frac{(2g-2)!}{(2g_1-2)!(2g_2-2)!} \cdot 1 \cdot 1 \cdot \frac{1}{2^{2-1}} \cdot \frac{(4g_1-3)!(4g_2-3)!}{(4g-5)!} \cdot$$

$$\cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\overbrace{1,\dots,1}^{2g_1-2})) \operatorname{Vol}(\mathcal{H}_1(\overbrace{1,\dots,1}^{2g_2-2}))}{\operatorname{Vol}(\mathcal{H}_1(\underbrace{1,\dots,1}_{2g-2}))} =$$

$$= \frac{1}{4|\Gamma|} \cdot \frac{(2g-2)!\,(4g_1-3)!\,(4g_2-3)!}{(2g_1-2)!\,(2g_2-2)!\,(4g-5)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\overbrace{1,\dots,1}^{2g_1-2})) \operatorname{Vol}(\mathcal{H}_1(\overbrace{1,\dots,1}^{2g_2-2}))}{\operatorname{Vol}(\mathcal{H}_1(\underbrace{1,\dots,1}_{2g-2}))}$$

$$\text{where} \quad g_1 + g_2 = g,\ g_1, g_2 \ge 1, \qquad |\Gamma| = \begin{cases} 1 & \text{when } g_1 \neq g_2 \\ 2 & \text{when } g_1 = g_2 \end{cases}$$

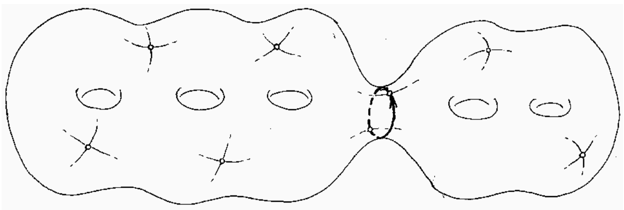

FIGURE 6. Saddle connection of multiplicity 2 on a surfaces from the principal stratum $\mathcal{H}_1(1, \dots, 1)$.

Table 4 presents the numerical values for these constants. More values are presented in Table 8 in the Appendix. (We present only approximate values of constants in Table 8 just to save the space.)

TABLE 4. Principal stratum; values of the constants for saddle connections of multiplicity two joining a pair of distinct zeroes

	$g_2=1$	$g_2=2$	$g_2=3$	$g_2=4$	$g_2=5$
$g_1=1$	$\frac{5}{8}$	$\frac{6}{7}$	$\frac{315}{377}$	$\frac{19604}{23357}$	$\frac{13897415}{16493303}$
$g_1=2$	$\frac{6}{7}$	$\frac{30}{377}$	$\frac{1680}{23357}$	$\frac{686140}{16493303}$	$\frac{9529656}{351964697}$
$g_1=3$	$\frac{315}{377}$	$\frac{1680}{23357}$	$\frac{154350}{16493303}$	$\frac{352872}{50280671}$	$\frac{389127620}{121780991823}$
$g_1=4$	$\frac{19604}{23357}$	$\frac{686140}{16493303}$	$\frac{352872}{50280671}$	$\frac{336277214}{365342975469}$	$\frac{435911877856}{704782787198207}$
$g_1=5$	$\frac{13897415}{16493303}$	$\frac{9529656}{351964697}$	$\frac{389127620}{121780991823}$	$\frac{435911877856}{704782787198207}$	$\frac{19865637635886}{249020069093788675}$

8. Strata that are not Connected

The analysis of admissible assignments, structure of local surgeries, and most of the elements of our constructions are basically the same for the strata having several connected components. However, we need to establish the correspondence between connected components and parts of the boundary of the stratum they are adjacent to. In other words we need to describe the admissible constructions, which lead to the flat surfaces in the prescribe connected component.

8.1. **Parity of the Spin Structure in the Slit Construction. Admissible Assignments for Surfaces from Hyperelliptic Components.** We start with outlining the relation between the parities of spin structures of the resulting surface and its components in the slit construction.

Lemma 8.1. *Suppose S constructed above has only even zeroes so has a spin structure. Then each component surface S'_i has only even zeroes and so also has a spin structure. The parity of the spin structure $\phi(S)$ is equal to the sum of the parities of the spin structures of the components:*

$$\phi(S) = \sum_{i=1}^{p} \phi(S'_i) \ (mod\ 2) \tag{15}$$

Proof. For each S'_i, all the zeroes except the distinguished one come from the corresponding zeroes on S unchanged; in particular they have the same degrees as the initial zeroes of S. Thus for each S'_i all the zeroes but at most one, have even degrees. Note that the total sum of the degrees of the zeroes of S'_i is even: it equals $2g_i+2$, where g_i is the genus of the surface S'_i. Hence the remaining zero has even degree as well. (By usual convention we consider a marked point as a "zero of degree 0".)

For each surface S'_i consider a collection of $2g_i$ smooth simple closed curves on S'_i representing the canonical basis of cycles. Breaking up the zero z_i we make a local surgery of the flat structure, i.e., we do not change the flat structure outside of a small domain $U_i' \subset S_i'$ containing the zero z_i. Thus for each i we can deform if necessary the curves from the corresponding collection in such a way that they stay outside of a neighborhood of the zero z_i. Thus we may assume that for each $i = 1, \dots, p$ none of the chosen curves on S_i' intersect U_i'. This means that all the curves survive under the surgery, moreover, the union of collections for all i gives us a canonical basis on S. By construction the index of any resulting curve in the flat structure ω is the same as the index of the corresponding curve in the flat structure ω_i'. Hence we get the desired relation (15). The lemma is proved. □

Lemma 8.2. *Let S belong to one of the components $\mathcal{H}_1^{odd}(\alpha)$ or $\mathcal{H}_1^{even}(\alpha)$. Suppose that $\alpha \neq (g-1, g-1)$. Then the surface S can be obtained by an assignment $(S', \gamma, a_i', a_i'') \to (S, m_1, m_2, a_i', a_i'')$, where $m_1, m_2 \in \alpha$, if and only if the collection (α_i', a_i', a_i'') satisfies the conditions of Lemma 7.1, all α_i' are even, and the parities of the spin structures of S_i' satisfy the condition of Lemma 8.1.*

Note that the surfaces S_i' from the *hyperelliptic* component $\mathcal{H}_1^{hyp}(\alpha)$ and from the *hyperelliptic* component $\mathcal{H}_1^{hyp}(g-1, g-1)$, where g is odd, also have parity of the spin structure (see formulae (7), (8)). In general these surfaces are also involved in assignments from Lemma 8.2 producing *nonhyperelliptic* components $\mathcal{H}_1^{even}(\alpha)$ and $\mathcal{H}_1^{odd}(\alpha)$. However, in the exceptional case of $\alpha = (g-1, g-1)$ with odd g some assignments lead to surfaces S from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(g-1, g-1)$. These assignments are classified in the lemma below.

Lemma 8.3. *Let a flat surface S from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(g-1, g-1)$ be obtained by an assignment*

$$(S', \gamma, a_i', a_i'') \to (S, g-1, g-1, a_i', a_i'').$$

Then for almost every S the assignment has one of the following two types:

- *— The multiplicity is 1; the flat surface S' belongs to the hyperelliptic component $\mathcal{H}_1^{hyp}(\alpha)$, and $a' = a'' = g-1$.*
- *— The multiplicity is 2; the flat surfaces S_i', $i = 1, 2$ belong to the hyperelliptic components $\mathcal{H}^{hyp}(2g_i - 2)$, where $g_1 + g_2 = g$, and $a_i' = a_i'' = g_i - 1$, $i = 1, 2$.*

The assignment is $|\Gamma|$ to 1 and is onto the corresponding set of hyperelliptic surfaces.

Proof. The set of hyperelliptic structures such that the involution interchanges some components has positive codimension, so we may disregard these and assume the involution fixes each component. This implies that all flat surfaces S_i' belong to components $\mathcal{H}^{hyp}(2g_i - 2)$. Since the hyperelliptic involution sends a vector v to $-v$, but does not change the orientation of the surface, if the multiplicity is greater than one, it cannot map any of the γ_i' or γ_i'' to themselves. It must therefore interchange γ_i' and γ_i'' for each i. This implies that $i = 2$. □

Corollary 8.4. *Almost any flat surface S from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(g-1, g-1)$ has no saddle connections of multiplicity greater then two.*

8.2. **Constants for the Components of the Stratum $\mathcal{H}_1(g-1, g-1)$.** We start with the stratum $\mathcal{H}_1(g-1, g-1)$ which is exceptional: it has the hyperelliptic connected component.

8.2.1. *Hyperelliptic Connected Component* $\mathcal{H}_1^{hyp}(g-1,g-1)$. The admissible assignments for the surfaces from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(g-1,g-1)$ are described by Lemma 8.3. The possible multiplicities are one and two. In multiplicity one modifying (12) in accordance with Lemma 8.3 we get the following formula.

Formula 8.1. *The constant for the number of saddle connections of multiplicity one joining distinct zeroes in* $\mathcal{H}_1^{hyp}(g-1,g-1)$ *is given by*

$$c = (2g-1)\cdot\frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g-2))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(g-1,g-1))}$$

In other words, collapsing a saddle connection of multiplicity one on a surface $S \in \mathcal{H}_1^{hyp}(g-1,g-1)$ we get a surface S' from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(\alpha)$. Breaking up the zero of a surface $S' \in \mathcal{H}_1^{hyp}(\alpha)$ we necessarily get a surface S from the hyperelliptic component $\mathcal{H}_1^{hyp}(g-1,g-1)$.

Table 5. Hyperelliptic component $\mathcal{H}_1^{hyp}(g-1,g-1)$; values of the constants for saddle connections of multiplicity one joining distinct zeroes

	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$	$g=9$
$c=$	$\frac{27}{8}$	$\frac{225}{32}$	$\frac{6125}{512}$	$\frac{297675}{16384}$	$\frac{3361743}{131072}$	$\frac{9018009}{262144}$	$\frac{372683025}{8388608}$	$\frac{59836330125}{1073741824}$
$c\approx$	3.375	7.03125	11.9629	18.1686	25.6481	34.401	44.4273	55.7269

Example 8.5. **Component $\mathcal{H}_1^{hyp}(1,1)$.** In genus $g=2$ the component $\mathcal{H}_1^{hyp}(1,1)$ coincides with the whole principal stratum $\mathcal{H}_1(1,1)$. The formula above gives the same constant as formula 6.3 for the principal stratum in genus $g=2$:

$$c = 3\cdot\frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(1,1))} = 3\cdot\frac{\operatorname{Vol}(\mathcal{H}_1(2))}{\operatorname{Vol}(\mathcal{H}_1(1,1))} = \frac{27}{8} = 3.375$$

Example 8.6. **Component $\mathcal{H}_1^{hyp}(2,2)$.** In genus 3 we have the connected component $\mathcal{H}_1^{hyp}(2,2)$. After collapsing zeroes we obtain a hyperelliptic surface $S' \in \mathcal{H}_1^{hyp}(4)$. We get

$$c = 5\cdot\frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(4))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2,2))} = \frac{225}{32} \approx 7.03$$

We present more values in table 5.

Consider now multiple saddle connections for a flat surface S from the connected component $\mathcal{H}_1^{hyp}(g-1,g-1)$. By Lemma 8.3 we only have $p=2$ in the higher multiplicity, and $S_i' \in \mathcal{H}^{hyp}(2g_i-2)$, $i=1,2$, where $g_1+g_2=g$. Here we always have the $\gamma\to-\gamma$ symmetry, so $|\Gamma_-|=2$. When $g_1=g_2$ we also have the stratum interchange symmetry, so $|\Gamma|=2$, and $|\Gamma|=1$ otherwise. Thus, after appropriate modification of formula (14) we get the following constant

Formula 8.2. *The constant in multiplicity* 2 *for saddle connections joining distinct zeroes in* $\mathcal{H}_1^{hyp}(g-1,g-1)$ *is given by*

$$c = \frac{(2g_1-1)(2g_2-1)}{2|\Gamma|} \cdot \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(g-1,g-1))}$$

Example 8.7. **Component $\boldsymbol{\mathcal{H}_1^{hyp}(1,1)}$.** Saddle connections of multiplicity two joining distinct zeroes on a surface from $\mathcal{H}_1^{hyp}(1,1) = \mathcal{H}_1(1,1)$ were actually, already considered in the section treating the principal stratum. Here after decomposition we get a pair of tori S_1', S_2' each with a marked point. We get the following value for the corresponding constant:

$$c = \frac{1\cdot 1}{2\cdot 2} \cdot \frac{1!\cdot 1!}{3!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))\operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1(1,1))} = \frac{5}{8} = 0.625$$

TABLE 6. Hyperelliptic component $\mathcal{H}_1^{hyp}(g-1,g-1)$; values of the constants for saddle connections of multiplicity two joining distinct zeroes

	$g_2=1$	$g_2=2$	$g_2=3$	$g_2=4$	$g_2=5$	$g_2=6$
$g_1=1$	$\frac{5}{8}$	$\frac{63}{32}$	$\frac{375}{128}$	$\frac{67375}{16384}$	$\frac{361179}{65536}$	$\frac{1867635}{262144}$
$g_1=2$	$\frac{63}{32}$	$\frac{567}{512}$	$\frac{22275}{8192}$	$\frac{441441}{131072}$	$\frac{1083537}{262144}$	$\frac{10496871}{2097152}$
$g_1=3$	$\frac{375}{128}$	$\frac{22275}{8192}$	$\frac{96525}{65536}$	$\frac{875875}{262144}$	$\frac{4027725}{1048576}$	$\frac{1187146125}{268435456}$
$g_1=4$	$\frac{67375}{16384}$	$\frac{441441}{131072}$	$\frac{875875}{262144}$	$\frac{14889875}{8388608}$	$\frac{2083228875}{536870912}$	$\frac{18432519875}{4294967296}$
$g_1=5$	$\frac{361179}{65536}$	$\frac{1083537}{262144}$	$\frac{4027725}{1048576}$	$\frac{2083228875}{536870912}$	$\frac{8749561275}{4294967296}$	$\frac{74923166241}{17179869184}$
$g_1=6$	$\frac{1867635}{262144}$	$\frac{10496871}{2097152}$	$\frac{1187146125}{268435456}$	$\frac{18432519875}{4294967296}$	$\frac{74923166241}{17179869184}$	$\frac{624359718675}{274877906944}$

Example 8.8. **Component $\boldsymbol{\mathcal{H}_1^{hyp}(2,2)}$.** Consider saddle connections of multiplicity two joining distinct zeroes on a surface from $\mathcal{H}_1^{hyp}(2,2)$. After decomposition we get a surface $S_1' \in \mathcal{H}_1(2)$ of genus 2 with a single zero of order 2 and a torus $S_2' \in \mathcal{H}(\emptyset)$ with a marked point. The cone angle between γ_1 and γ_2 is the same at z_1 and z_2; it equals 2π. The cone angle between γ_2 and γ_1 is also the same at z_1 and z_2, but now it equals 4π. The constant is given by

$$c = \frac{3\cdot 1}{2\cdot 1} \cdot \frac{3!\cdot 1!}{5!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2))\operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2,2))} = \frac{63}{32} \approx 1.97$$

We present more values in table 6 and in table 9 in the Appendix.

8.2.2. *Connected Component $\mathcal{H}_1^{nonhyp}(g-1,g-1)$; Even Genus.* For $g \ge 4$, g even, the stratum $\mathcal{H}_1(g-1,g-1)$ has exactly two connected components: $\mathcal{H}_1^{hyp}(g-1,g-1)$ and $\mathcal{H}_1^{nonhyp}(g-1,g-1)$. Thus, dealing with the connected component $\mathcal{H}_1^{nonhyp}(g-1,g-1)$ we have to exclude all the assignments producing surfaces S in the stratum $\mathcal{H}_1(g-1,g-1)$ those that actually belong to the component $\mathcal{H}_1^{hyp}(g-1,g-1)$. The latter problem has already been solved. Thus we get the following answers.

Formula 8.3. *For any even genus $g \geq 4$ any flat surfaces S from the connected component $\mathcal{H}_1^{nonhyp}(g-1, g-1)$ having a saddle connection of multiplicity one joining distinct zeroes may be obtained from the corresponding surface S' from one of the components $\mathcal{H}_1^{even}(2g-2)$, $\mathcal{H}_1^{odd}(2g-2)$.*

The corresponding constant is equal to

$$c = (2g-1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{even}(2g-2)) + \mathrm{Vol}(\mathcal{H}_1^{odd}(2g-2))}{\mathrm{Vol}(\mathcal{H}_1^{nonhyp}(g-1, g-1))} =$$

$$= (2g-1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(2g-2)) - \mathrm{Vol}(\mathcal{H}_1^{hyp}(2g-2))}{\mathrm{Vol}(\mathcal{H}_1(g-1, g-1)) - \mathrm{Vol}(\mathcal{H}_1^{hyp}(g-1, g-1))}$$

We proceed with multiplicity two. Here we get the formula analogous to formula 8.2 for the component $\mathcal{H}_1^{hyp}(g-1, g-1)$.

Formula 8.4. *The constant for multiplicity 2 joining two zeroes in the stratum $\mathcal{H}^{nonhyp}(g-1, g-1)$ is given by*

$$c = \frac{(2g_1-1)(2g_2-1)}{2|\Gamma|} \cdot \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} \cdot$$

$$\cdot \frac{\mathrm{Vol}(\mathcal{H}_1(2g_1-2))\,\mathrm{Vol}(\mathcal{H}_1(2g_2-2)) - \mathrm{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\,\mathrm{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))}{\mathrm{Vol}(\mathcal{H}_1^{nonhyp}(g-1, g-1))},$$

For multiplicity $p \geq 3$ it is easy to account for the impact of the hyperelliptic component (see Corollary 8.4). We get the following particular case of formula 14 in the case of $\mathcal{H}^{nonhyp}(g-1, g-1)$

Formula 8.5.

$$c = \frac{1}{2|\Gamma|} \cdot \prod_{i=1}^{p}(2g_i-1) \cdot \frac{\prod_{i=1}^{p}(2g_i-1)!}{(2g-1)!} \cdot \frac{\prod_{i=1}^{p}\mathrm{Vol}(\mathcal{H}_1(2g_i-2))}{\mathrm{Vol}(\mathcal{H}_1^{nonhyp}(g-1, g-1))}$$

8.2.3. *Components $\mathcal{H}^{even}(g-1, g-1)$ and $\mathcal{H}^{odd}(g-1, g-1)$.* We consider the multiplicity one case first. Assume that S belongs to one of the nonhyperelliptic components $\mathcal{H}^{even}(g-1, g-1)$ or $\mathcal{H}^{odd}(g-1, g-1)$. Contracting the saddle connection joining the two zeroes, we merge the two zeroes into one and we get a surface S' in $\mathcal{H}^{even}(2g-2)$ or $\mathcal{H}^{odd}(2g-2)$ correspondingly. Conversely breaking up a single zero of a surface S' from a nonhyperelliptic connected component $\mathcal{H}^{even}(2g-2)$ or $\mathcal{H}^{odd}(2g-2)$ into two zeroes of degrees $g-1$ we get a surface from the component $\mathcal{H}^{even}(g-1, g-1)$ or $\mathcal{H}^{odd}(g-1, g-1)$ correspondingly. Thus we have to modify formula (12) in the following way.

Formula 8.6. *The constant is equal to*

$$c = (2g-1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{even}(2g-2))}{\mathrm{Vol}(\mathcal{H}_1^{even}(g-1, g-1))}$$

$$c = (2g-1) \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{odd}(2g-2))}{\mathrm{Vol}(\mathcal{H}_1^{odd}(g-1, g-1))}$$

Example 8.9. **Component $\mathcal{H}^{odd}(2,2)$.** After collapsing zeroes we obtain a surface $S' \in \mathcal{H}^{odd}(4)$. We get

$$c = 5 \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{odd}(4))}{\mathrm{Vol}(\mathcal{H}_1^{odd}(2,2))} = \frac{80}{9} \approx 8.89$$

Remark 8.10. Note that the similar nonhyperelliptic connected components with *even* parity of the spins structure do not exist in genus $g = 3$.

Now we consider higher multiplicity. We again have to consider multiplicity two separately because some assignments in multiplicity two produce surfaces from the hyperelliptic connected component $\mathcal{H}^{hyp}(g-1,g-1)$, see Lemma 8.1. We again get the formula analogous to formula 8.2 for the component $\mathcal{H}^{hyp}(g-1,g-1)$ and to formula 8.4 for the component $\mathcal{H}^{nonhyp}(g-1,g-1)$.

It will be convenient to introduce the following notation. We introduce a function $\delta(\alpha,\phi)$ which has values 0 or 1. It is equal to 1 when all of the following three conditions are satisfied: all the zeroes of α (if any) have even degrees; $\mathcal{H}(\alpha)$ contains the hyperelliptic component $\mathcal{H}^{hyp}(\alpha)$; the parity of the spin structure of $\mathcal{H}^{hyp}(\alpha)$ coincides with ϕ. Otherwise $\delta(\alpha,\phi) = 0$.

Formula (7) for the parity of the spin structure of $\mathcal{H}^{hyp}(\alpha')$ shows that

$$\delta(\alpha,\phi) = \begin{cases} 1 & \text{when } \alpha = (2g-2) \text{ and } \left[\dfrac{g+1}{2}\right] = \phi \pmod 2 \\ 1 & \text{when } \alpha = (g-1,g-1),\ g \text{ is odd,} \\ & \text{and } \left(\dfrac{g+1}{2}\right) = \phi \pmod 2 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

We shall also use the following convention: the volume of a nonexistent component is equal to zero. In this notation we get the following six "dummy" volumes for the strata in small genera:

$$\begin{array}{lll} \mathrm{Vol}(\mathcal{H}_1^{odd}(\emptyset)) = 0 & \mathrm{Vol}(\mathcal{H}_1^{odd}(2)) = 0 & \mathrm{Vol}(\mathcal{H}_1^{even}(4)) = 0 \\ \mathrm{Vol}(\mathcal{H}_1^{even}(\emptyset)) = 0 & \mathrm{Vol}(\mathcal{H}_1^{even}(2)) = 0 & \mathrm{Vol}(\mathcal{H}_1^{even}(2,2)) = 0 \end{array} \tag{17}$$

and also we have

$$\mathcal{H}^{hyp}(\alpha) = 0 \quad \text{if } \alpha \neq (2g-2), (g-1,g-1)$$

Formula 8.7. *A surface S from one of the nonhyperelliptic connected components $\mathcal{H}^{even}(g-1,g-1)$ or $\mathcal{H}^{odd}(g-1,g-1)$ may have a saddle connection of multiplicity two joining the two distinct zeroes if and only if it can be obtained from surfaces S'_1 and S'_2 of genus g_1, g_2 where $g_1 + g_2 = g$ and where the following additional requirements are satisfied. At least one of the surfaces S'_1, S'_2 does not belong to the hyperelliptic component $\mathcal{H}^{hyp}(2g_i - 2)$, $i = 1, 2$; the parities of the spin structures of the surfaces satisfy relation* (15)*: $\phi(S'_1) + \phi(S'_2) = \phi(S)$. For $S \in \mathcal{H}^{odd}(g-1,g-1)$ the spin structures of S'_1, S'_2 should have opposite parities. This means in particular that there is no longer a stratum interchange symmetry. For $g_1 \neq g_2$*

$$c=\frac{(2g_1-1)(2g_2-1)}{2}\cdot\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}\cdot\frac{1}{\operatorname{Vol}(\mathcal{H}_1^{odd}(g-1,g-1))}\cdot$$
$$\Bigg(\Big(\big(\operatorname{Vol}(\mathcal{H}_1^{even}(2g_1-2))+\delta((2g_1-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\big)\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(2g_2-2))+\delta((2g_2-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\big)\Big)+$$
$$+\Big(\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(2g_1-2))+\delta((2g_1-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\big)\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{even}(2g_2-2))+\delta((2g_2-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\big)\Big)-$$
$$-\delta((2g_1-2),even)\delta((2g_2-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))-$$
$$-\delta((2g_1-2),odd)\delta((2g_2-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\Bigg)$$

For $g_1=g_2=g/2$, where g is even, we get

$$c=\frac{(2g_1-1)^2}{2}\cdot\frac{\big((2g_1-1)!\big)^2}{(2g-1)!}\cdot\frac{1}{\operatorname{Vol}(\mathcal{H}_1^{odd}(g-1,g-1))}\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{even}(2g_1-2))+\delta((2g_1-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\big)\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(2g_2-2))+\delta((2g_2-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\big)$$

Example 8.11. **Component $\boldsymbol{\mathcal{H}^{odd}(2,2)}$**. From Formula 8.7 we know that the degeneration of multiplicity 2 of a surface from $\mathcal{H}(2,2)$ has the form $\big(\bar{0}+\bar{0}\big)\succ\big(\bar{1}+\bar{1}\big)\succ$. Since the strata $\mathcal{H}(\emptyset)$ and $\mathcal{H}(2)$ are (hyper)elliptic the resulting surface in $\mathcal{H}(2,2)$ is also hyperelliptic, see Lemma 8.1. Hence degeneration of multiplicity 2 is not realizable for flat surfaces from $\mathcal{H}^{odd}(2,2)$.

The expression for the constant for the surfaces from $\mathcal{H}^{even}(g-1,g-1)$ is analogous to the one above with the only exception that now we get the stratum interchange symmetry $|\Gamma|=2$ for $g_1=g_2$:

$$c=\frac{(2g_1-1)(2g_2-1)}{2|\Gamma|}\cdot\frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!}\cdot\frac{1}{\operatorname{Vol}(\mathcal{H}_1^{even}(g-1,g-1))}\cdot$$
$$\Bigg(\Big(\big(\operatorname{Vol}(\mathcal{H}_1^{even}(2g_1-2))+\delta((2g_1-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\big)\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{even}(2g_2-2))+\delta((2g_2-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\big)\Big)+$$
$$+\Big(\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(2g_1-2))+\delta((2g_1-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\big)\cdot$$
$$\cdot\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(2g_2-2))+\delta((2g_2-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\big)\Big)-$$
$$-\delta((2g_1-2),even)\delta((2g_2-2),even)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))-$$
$$-\delta((2g_1-2),odd)\delta((2g_2-2),odd)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_2-2))\Bigg)$$

Finally, for from multiplicity at least three it is easy to account for the impact of the hyperelliptic component (see Corollary 8.4). We get the following particular case of formula 14:

Formula 8.8. *Almost every surface in $\mathcal{H}^{even}(g-1,g-1)$ or in $\mathcal{H}^{odd}(g-1,g-1)$ having saddle connections of multiplicity $p\geq 3$ joining two zeroes can be obtained by assignment*

$$\big(\overline{(g_1-1)})+\overline{(g_1-1)}\big)\succ\ldots\big(\overline{(g_p-1)}+\overline{(g_p-1)}\big)\succ$$

where $g_i\geq 1$, $i=1,\dots,p$, and $g_1+\cdots+g_p=g-1$. The corresponding constant equals

$$c=\frac{1}{|\Gamma|}\cdot\prod_{i=1}^{p}(2g_i-1)\cdot\frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^{p}(2g_i-1)!}{(2g-1)!}\cdot\frac{1}{\operatorname{Vol}(\mathcal{H}_1^{\phi}(g-1,g-1))}\cdot$$
$$\cdot\sum_{\substack{\phi_1,\dots,\phi_p\in\{even,odd\}\\ \phi_1+\cdots+\phi_p=\phi\ (mod\,2)}}\prod_{i=1}^{p}\Big(\operatorname{Vol}(\mathcal{H}_1^{\phi_i}(2g_i-2))+\delta((2g_i-2),\phi_i)\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_i-2))\Big)$$

Note that here we always have the $\gamma\to-\gamma$ symmetry, $|\Gamma_-|=2$, which is cancelled by $o(m)!=o(g-1)!$ (see (14)). Note that in this particular case we prefer to use summation over *all* combinations of parities ϕ_i producing the proper parity of the sum. Thus the stratum interchange symmetry Γ depends only on the collection of g_i, and on their cyclic order.

Example 8.12. **Component $\mathcal{H}^{odd}(2,2)$.** Consider a saddle connection joining the two zeroes of a surface $S\in\mathcal{H}^{odd}(2,2)$, and suppose that the saddle connection has multiplicity 3. Each of S_i', for $i=1,2,3$ is a torus with a marked point. The angle between any consecutive saddle connections at z_1 and z_2 equals 2π. There is a symmetry of order 3 coming from stratum interchange, so $|\Gamma|=3$. Taking into consideration $1/4$ from $1/2^{p-1}$ we get

$$c=\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1!\cdot1!\cdot1!}{5!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^3}{\operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))}=\frac{1}{9}\approx 0.111$$

Remark 8.13. Let $g\geq 4$ be odd. We see that the maximal multiplicity of a saddle connection joining the distinct zeroes of a surface from the stratum $\mathcal{H}(g-1,g-1)$ is different for different connected components: it equals 2 for almost all surfaces from $\mathcal{H}^{hyp}(g-1,g-1)$, it equals $g-1$ for almost all surfaces in $\mathcal{H}^{even}(g-1,g-1)$; it equals g for almost all surfaces in $\mathcal{H}^{odd}(g-1,g-1)$.

8.3. **Other Strata.** The description for all other strata $\mathcal{H}(\alpha)$, where all the entries of α are even, is similar to the one just presented.

Formula 8.9. *The constant for the nonhyperelliptic connected component $\mathcal{H}^{\phi}(\alpha)$, $\phi\in\{even,odd\}$, $\alpha\neq(g-1,g-1)$, is given by*

$$c = \frac{1}{|\Gamma_-|} \cdot \frac{1}{|\Gamma|} \cdot \prod_{m\in\alpha} \left(\frac{o(m)!}{\prod_{j=1}^{p} o_j(m)!} \right) \cdot \prod_{\substack{j=1\\ a_j\neq 0}}^{p} o_j(a_j) \cdot \prod_{j=1}^{p} (a_j+1) \cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!} \cdot$$

$$\cdot \frac{1}{\operatorname{Vol}(\mathcal{H}_1^{\phi}(\alpha))} \cdot \sum_{\substack{\phi'_1,\dots,\phi'_p\in\{even,\,odd\}\\ \phi'_1+\dots+\phi'_p=\phi\,(mod\,2)}} \prod_{i=1}^{p} \Big(\operatorname{Vol}(\mathcal{H}_1^{\phi_i}(\alpha'_i)) + \delta(\alpha'_i,\phi'_i) \operatorname{Vol}(\mathcal{H}_1^{hyp}(\alpha'_i)) \Big)$$

Example 8.14. **Stratum $\mathcal{H}^{even}(4,2)$**. In multiplicity one, after collapsing zeroes we obtain a surface $S' \in \mathcal{H}(6)$, where either of the two components $\mathcal{H}^{hyp}(6)$ and $\mathcal{H}^{even}(6)$ having even parity of the spin structure are possible. We get

$$c = 7 \cdot \frac{\big(\operatorname{Vol}(\mathcal{H}_1^{hyp}(6)) + \operatorname{Vol}(\mathcal{H}_1^{even}(6))\big)}{\operatorname{Vol}(\mathcal{H}_1^{even}(4,2))} = \frac{253001}{18225} \approx 13.88$$

In multiplicity two we get two surfaces $S'_1 \in \mathcal{H}(\alpha'_1)$, $S'_2 \in \mathcal{H}(\alpha'_2)$, where $\alpha'_1+\alpha'_2 = m_1+m_2-p = 4$.

There are two possibilities. In the first case $\alpha'_1 = \alpha'_2$. Note that any $S' \in \mathcal{H}(2)$ has odd parity of the spin structure, see (7). Thus the parity of the resulting flat structure S is even, see Lemma 8.1.

The saddle connections γ_1 and γ_2 partition the cone angle 10π at z_1 in two sectors with angles 8π and 2π; they partition the cone angle 6π at z_2 in two sectors with the angles 4π and 2π. Here $a'_1 = 2, a''_1 = 0$ while $a'_2 = a''_2 = 1$ so that while $\mathcal{H}(\alpha'_1) = \mathcal{H}(\alpha'_2) = \mathcal{H}(2)$, there is no stratum interchange since $a'_1 \neq a'_2$. Thus $|\Gamma| = 1$. We get

$$c = 9 \cdot \frac{1}{2} \cdot \frac{3!\cdot 3!}{7!} \cdot \frac{\big(\operatorname{Vol}(\mathcal{H}_1(2))\big)^2}{\operatorname{Vol}(\mathcal{H}_1^{even}(4,2))} = \frac{6}{25} = 0.24$$

In the second case after degeneration we get $S'_1 \in \mathcal{H}_1(\emptyset)$, $S'_2 \in \mathcal{H}_1(4)$ Since $\mathcal{H}(\emptyset)$ has odd parity of the spin structure, $S'_2 \in \mathcal{H}^{odd}(4)$ in order to result in a surface $S \in \mathcal{H}^{even}(4,2)$ (see Lemma 8.1).

The saddle connections γ_1 and γ_2 partition the cone angle 10π at z_1 in two sectors with the angles 2π and 8π; they partition the cone angle 6π at z_2 in two sectors with the angles 2π and 4π. We get

$$c = 5 \cdot \frac{1}{2} \cdot \frac{1!\cdot 5!}{7!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset)) \operatorname{Vol}(\mathcal{H}_1^{odd}(4))}{\operatorname{Vol}(\mathcal{H}_1^{even}(4,2))} = \frac{640}{729} \approx 0.8779$$

Multiplicity 3 is not realizable in the component $\mathcal{H}^{even}(4)$ (see the next subsection).

Example 8.15. **Stratum $\mathcal{H}^{odd}(4,2)$**. This component is similar in many aspects to the previous one, so we skip those details of calculations which are common for both components.

The first case is $\mathcal{H}^{odd}(4,2)$ with multiplicity $p = 1$. After collapsing zeroes we obtain a surface $S' \in \mathcal{H}(6)$, the only component that results is $\mathcal{H}^{odd}(6)$. We get

$$c = 7 \cdot \frac{\big(\operatorname{Vol}(\mathcal{H}_1^{odd}(6))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(4,2))} = \frac{147}{10} = 14.7$$

Consider multiplicity 2. The case of degeneration to two surfaces $S_1', S_2' \in \mathcal{H}(2)$ does not take place here, since the surface $S \in \mathcal{H}(4,2)$ obtained from two surfaces from $\mathcal{H}(2)$ has even parity of spin structure.

Thus, the only possibility is $S_1' \in \mathcal{H}_1(\emptyset)$; $S_2' \in \mathcal{H}_1(4)$, $a_2' = 3$, $a_2'' = 1$. Since $\mathcal{H}(\emptyset)$ has odd parity of the spin structure, S_2' must be in components of $\mathcal{H}(4)$ having even parity of spin structure in order to result a surface $S \in \mathcal{H}^{odd}(4,2)$ (see Lemma 8.1). There is only one such component: $\mathcal{H}^{hyp}(4)$.

Thus we get

$$c = 5 \cdot \frac{1}{2} \cdot \frac{1! \cdot 5!}{7!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset)) \operatorname{Vol}(\mathcal{H}_1^{hyp}(4))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(4,2))} = \frac{15}{64} \approx 0.2344$$

(see analogous calculation for the component $\mathcal{H}^{even}(4,2)$.

Consider now multiplicity 3. It is not hard to check that $S_1' \in \mathcal{H}(2)$, $S_2', S_3' \in \mathcal{H}(\emptyset)$. Note that both $\mathcal{H}(\emptyset)$ and $\mathcal{H}(2)$ have odd parity of the spin structure. Thus the resulting surface $S \in \mathcal{H}(4,2)$ always has odd parity of the spin structure, see Lemma 8.1.

The saddle connections $\gamma_1, \gamma_2, \gamma_3$ partition the cone angle 10π at z_1 in three sectors with the angles 6π, 2π, and 2π; they partition the cone angle 6π at z_2 in three sectors with the angles 2π. We find

$$c = 3 \cdot \frac{1}{4} \cdot \frac{3! \cdot 1! \cdot 1!}{7!} \cdot \frac{\operatorname{Vol}(\mathcal{H}(2)) \big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^2}{\operatorname{Vol}(\mathcal{H}_1^{odd}(4,2))} = \frac{21}{320} \approx 0.06563$$

Part 2. Saddle Connections Joining a Zero to Itself

9. Principal Boundary

In this part we consider the second problem of this paper. As described in Section 3, we have a surface S and a saddle connection γ_1 joining a zero to itself. There may be other saddle connections γ_i which are homologous to γ_1. Suppose the curves γ_i bound q cylinders. The complement of the curves and cylinders are $p = m - q$ surfaces $S_1, \dots, S_p$. The sum of the genera of the S_i is $g-1$. The surfaces S_i are attached in some cyclic order. Thus for any i the "next surface" S_{i+1} and the previous surface S_{i-1} make sense mod p.

We associate a set $J = \{i_1, \dots, i_q\} \subset \{1, \dots, p\}$, where $0 \le q \le p$ with the property that if $i = i_j \in J$ then there is a cylinder between γ_{i_j} and γ_{i_j+1}. Now suppose a surface S_i is bounded by two homologous curves that leave and return to the same zero. The two curves bound angles $2(a_i' + 1)\pi$ and $2(a_i'' + 1)\pi$ at the zero. The two boundary circles are joined together at the zero to form a "figure eight". If the two homologous curves that bound S_i return to different zeroes, then they bound angles b_i' and b_i'' at those zeroes. The boundary circles are disjoint in that case.

Suppose S_i is a surface whose boundary components are joined at a point to form a figure eight; that is we may think of that figure eight as a pair of circles with an identification of a point on one with a point on another. We may separate the boundary circles at the common point, by removing the identification. This gives

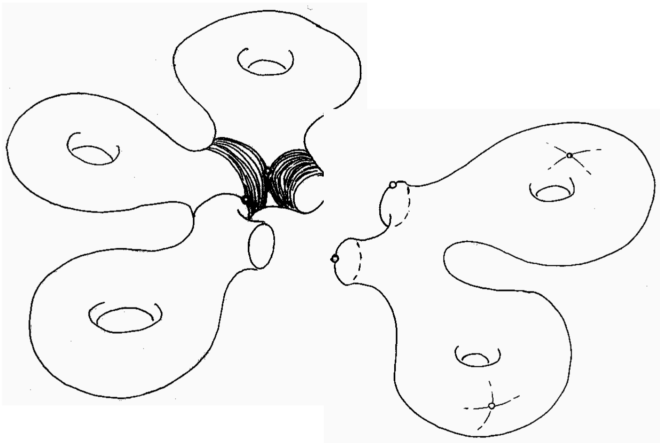

FIGURE 7. Homologous closed geodesics of high multiplicity. Topological picture.

a surface with a single circular boundary component with two singular points on it with angles $2\pi(a_i'+1)$ and $2\pi(a_i''+1)$. The boundary component is made up of a pair of saddle connections of equal length joining the two zeroes. Now we identify the two saddles, giving a closed surface with a saddle connection joining the two zeroes. We then collapse the saddle into a zero of order $a_i = a_i' + a_i''$.

Now let S_i be a surface whose boundary circles are disjoint. There are $2b_i'+1$ and $2b_i''+1$ sectors at the cone points that are not adjacent to the boundary circles which are cyclically ordered. For one circle the first of these is outgoing and for the other incoming. We shall want to "open up" the circles. The construction is not quite canonical and depends on the notion of transversal, which we now discuss.

Definition 9.1. Given a pair of points z_0, z_1 on a surface and a vector γ, a *transversal* joining z_0, z_1 is a smooth path $\rho(t)$, $0 \le t \le 1$, $\rho(0) = z_0$, $\rho(1) = z_1$, such that for all $0 < t < 1$, $\rho(t)$ is a regular point and such that the scalar product of the tangent vector to ρ with γ is strictly positive along the path, $\langle \dot\rho, \gamma\rangle > 0$ for any t.

Notice that given a zero of order k and direction γ there are $k+1$ outgoing sectors at the zero which means that we can connect the zero to a point in the sector by a transversal in the sector. Similarly, there are $k+1$ incoming sectors which means that we can connect a point in the sector to the zero by a transversal in the sector. We need to show that transversals exist joining two zeroes and an outgoing sector at one with an incoming sector at the other. This follows from the Calabi Theorem [C], or from the results in [HM]. However, for the sake of completeness we give a proof here.

Proposition 9.2. *Given a flat surface S a direction γ, a pair of zeroes with an outgoing sector for the first zero and an incoming sector for the second, then there is a transversal joining the two zeroes in those sectors.*

Proof. Suppose ω defines the flat surface. We may define a holomorphic 1-form $\tilde\omega = e^{i\phi}\omega$ such that $Im(\tilde\omega)$ vanishes along the directions γ and $Re(\tilde\omega)$ is positive along the direction γ. Let z_0 and z_1 any two regular points. Let $U = U(z_0)$ consist of the set of points such that there is a transversal ρ joining z_0 to z and such that

$Im(\tilde{\omega}) > 0$ along ρ. Then U is an open set. We show that it is the entire surface punctured at the zeroes, so that it must contain z_1.

It is easy to see that if the boundary ∂U contains a regular point $w \in S$, it contains the trajectory passing in direction γ through w. Let $\gamma^\perp$ be the direction orthogonal to γ, and such that $Im(\tilde{\omega})$ is positive on $\gamma^\perp$. Let $\partial U = \partial U^+ \sqcup \partial U^-$, where $\gamma^\perp$ is the *inner* normal to ∂U^-, where $\gamma^\perp$ is the *inner* normal to ∂U^-, and *outer* normal to ∂U^+. It is easy to see that ∂U^+ contains only the isolated points — the zeroes of ω. Thus the 1-cycle represented by ∂U^- is homologous to zero. Hence, since $Re(\tilde{\omega})$ is a closed 1-form we have $\int_{\partial U^-} Re(\tilde{\omega}) = 0$. On the other hand, if ∂U^- contains some curves, then they all have direction γ and $Re(\tilde{\omega})$ is positive in direction γ and we have a contradiction. Thus ∂U^- contains only zeroes of ω.

Now given a pair of zeroes z and w, choose z_0 in the outgoing sector at z and w_0 in the incoming sector at w. Join z_0 to w_0 by a transversal. Connect up z to z_0 by a transversal and w_0 to w by a transversal. The juxtaposition of these paths is the required transversal joining z to w. □

Choose τ_i to be a transversal joining the cone points in these two sectors. Before proceeding further we also prove the following lemma which says that we may take τ_i to have bounded length.

Lemma 9.3. *There exists M'' such that for all $S \in C(\mathcal{H}_1^\epsilon(\alpha))$ and any two zeroes z_0, z_1 in the same component S_i of the complement of the γ_j, there is a transversal τ_i of length at most M'' joining z_0 to z_1 in that component. Moreover there is a constant c such that $|\tau_i \cdot \beta| \leq c$ for any element β of the homology basis*

Proof. Suppose the result is false. Then there exists a sequence S_n of flat structures, a sequence of directions γ_n, and zeroes z_0, z_1 lying in the same components of the complement of the curves and cylinders on S_n such that the shortest transversals joining z_0 to z_1 in the component of the complement has length $M_n \to \infty$. We can normalize so that S_n has unit area. By passing to subsequences we can assume that there is a surface S_∞, a set of zeroes and distinguished points of S_∞ such that S_n converges to S_∞ and the convergence is uniform from the complement of annular neighborhoods V_n of C_i and the β_i to the complement of a neighborhood U of the marked points and zeroes of S_∞. We can assume z_0, z_1 correspond to zeroes or marked points w_0 and w_1 of S_∞ and the directions γ_n converge to a direction γ. By Proposition 9.2 we join w_0 to w_1 by a transverse path ρ. Let w_0', w_1' be the points where ρ leaves and then returns to U. Call this subsegment ρ'. By uniform convergence there is a transversal ρ_n on S_n for large n close to ρ', leaving and returning to V_n at points z_0' and z_1'. We now can connect up z_0' and z_1' to z_0 and z_1 inside V_n to find a bounded length transversal.

Since the lengths of the transversals τ_i are bounded by M'', the algebraic intersections $\beta \cdot \tau_i$ are bounded by some constant c: if not, there would be sequences of flat structures S with transversals of length at most M such that the algebraic intersection goes to infinity. Passing to subsequences we can assume the flat structures converge and the transversal converge, and we have a contradiction. □

We continue with the discussion of the principal boundary. For S_i whose boundary components consists of a pair of circles, cut S_i along a transversal τ_i of length at most M''. This gives a surface minus a parallelogram with two sides parallel to τ_i and two sides parallel to the boundary circles. Fill in the parallelogram to gives a closed surface S'_i. We note that the resulting S'_i depends on choice of τ_i. Altogether, using the figure eight and parallelogram degenerations the original surface S (of genus g) degenerates into $p = m - q$ disjoint surfaces $S'_1, \dots, S'_p$. The sum of the genera of the S'_i is $g - 1$. The surfaces S'_i are attached in the same cyclic order as the S_i. The resulting surfaces belong to the *principal boundary* corresponding to the configuration.

Finally we note that the holonomy for S and S'_i of a basis of curves β of the S'_i satisfies

$$v_S(\beta) = v_{S'_i}(\beta) + (\sum_i \beta \cdot \tau_i)\gamma$$

10. Constructing surfaces with homologous saddle connections

Our objective now is to reverse the degeneration and build homologous saddle connections with associated configuration $(J, a'_i, a''_i, b'_k, b''_k)$ out of simpler surfaces. We shall start with a collection of closed surfaces and via a "figure eight" and "parallelogram construction" build a closed surface with a curve or set of homologous curves joining a zero to itself.

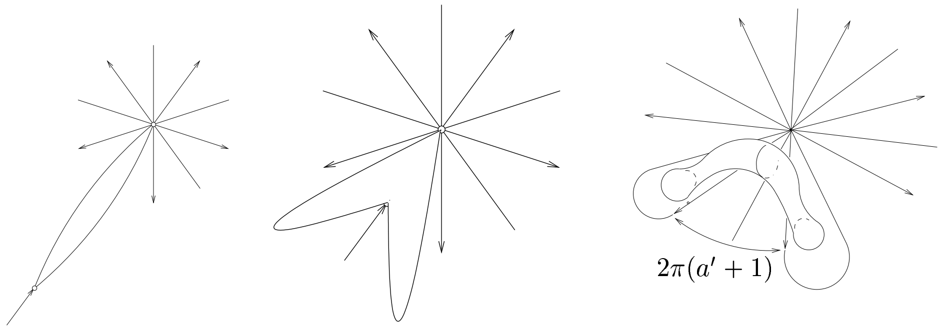

FIGURE 8. Here we applied the figure eight construction with $a' = 0$ and $a'' = 4$ to a zero of order $a = a' + a'' = 4$, and then we glued in a cylinder.

10.1. **Figure Eight Construction.** Given a surface S'_i, a zero (or a marked point) z_i of order a_i, and a pair of numbers $a'_i + a''_i = a_i$ we may break up the zero and perform the slit construction. We may then identify the two endpoints of the slit. This gives a surface with two circular boundary components γ'_i, γ''_i, which are joined at a point. The two circular boundary components separate the total angle at the point into angles $2\pi(a'_i + 1)$ and $2\pi(a''_i + 1)$. By convention on the choice of γ'_i, γ''_i (see section 7.2) turning around z_i in a clockwise direction from γ''_i to γ'_i inside the surface S'_i we turn by the angle $2\pi(a'_i + 1)$.

Recall that we may break the zero of order a_i up in $a_i + 1$ ways. Thus the figure eight construction, actually, gives us $a_i + 1$ surfaces.

In the second construction, given a pair of distinct points and a vector, we shall "open" the surface at the two points using the vector to build a surface with two boundary components. Again the construction is not quite canonical and depends on choice of transversal.

10.2. **Parallelogram Construction.** Suppose z and w are zeroes of orders $b', b'' \geq 0$ on a surface S', γ is a vector of length ϵ. Given separatrices leaving z and w in direction γ the notion of z' as the point $z+\gamma$ and w' as the point $w+\gamma$ make sense for small γ. At the point z there are two sectors adjacent to the segment from z to z'. Choose the outgoing sector. Similarly choose the incoming sector at w adjacent to the segment from w to w'. Let ρ be a transversal joining z to w transversal to γ in those sectors. For $|\gamma|$ small enough parallel translations of ρ by the vectors $s\gamma$, $0 < s \leq 1$ are well defined, and the translated curves are adjacent to the segments from z to z' and w to w' Denote by ρ' the parallel translation of ρ by γ. Now, cut S' along the segments $\overline{zz'}$, $\overline{ww'}$, and along the curves ρ, ρ'. Remove the interior of a resulting "parallelogram" with vertices z, z', w, w'. Now identify the two sides of the "parallelogram" parallel to ρ. This identifies z with z' and w with w' and creates a surface with two circular boundary pieces which we denote $\overline{zz'}$ and $\overline{ww'}$. We note that the resulting flat structure depends on the choice of the separatrices leaving each zero, the choice of transversal as well as the other given data.

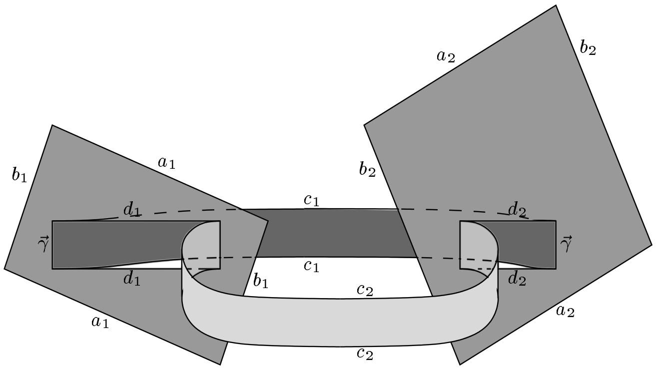

FIGURE 9. In this example we apply the *parallelogram construction* simultaneously to two tori (represented by large parallelograms with identified opposite sides) and then glue in a pair of cylinders (represented by the narrow stripes with identified opposite sides).

Example 10.1. In the example presented at Figure 9 we take two flat tori. On the picture they are represented by the large parallelograms with identified opposite sides a_1 to a_1, b_1 to b_1, and a_2 to a_2, b_2 to b_2. In fact, zeroes z_1, w_1, and z_2, w_2 are fake here: all these points are just the marked points of the tori. In this example we chose vector $\vec{\gamma}$ to be vertical.

We cut a rectangular domain out of the interior of each of the two tori in such way that the vertical side of each of the two removed rectangles equals to $\vec{\gamma}$. Then we identify the horizontal sides d_1 to d_1 and d_2 to d_2. We get a pair of tori, each provided with a pair of holes.

Finally, we join the hole $\overline{w_1 w_1'}$ on the first torus to the hole $\overline{z_2 z_2'}$ on the second torus by a cylinder (on the picture the cylinder is represented by a narrow stripe with identified opposite sides c_1 to c_1); we also join the hole $\overline{w_2 w_2'}$ on the second torus to the hole $\overline{z_1 z_1'}$ on the first torus by a cylinder (on the picture this second cylinder is represented by a narrow stripe with identified opposite sides c_2 to c_2).

We get a flat surface of genus $g = 3$ with four simple zeroes. By construction the surface contains two cylinders filled with closed regular geodesics homologous to $\vec{\gamma}$ (these cylinders are represented by narrow stripes with horizontal sides c_1 and c_2). In other words the saddle connection represented by γ has multiplicity 2. Note that the two cylinders would survive under any sufficiently small deformation of the surface, which means that this configuration is generic.

10.3. **Admissible Constructions.** Consider the following data:

- For $1 \le i \le p$: surfaces $S'_i \in \mathcal{H}(\alpha_i)$ with area $S'_i < 1$.
- A vector $\gamma \in \mathbb{R}^2$, with $|\gamma| < \epsilon$.
- For $1 \le i \le p$: a pair of points $z_i, w_i \in S'_i$ which are zeroes of orders b'_i, b''_i or marked points. The points may coincide. If they are distinct, we are also given a transversal ρ_i joining z_i and w_i. The ρ_i may be chosen so that for some fixed M and fixed homology basis of curves β,

$$|\beta \cdot \rho_i| \le M \tag{18}$$

  where $\beta \cdot \rho_i$ is the algebraic intersection number.
- If $w_i = z_i$ and is a zero of order a_i, then a pair of numbers a'_i, a''_i with $a_i = a'_i + a''_i$.
- A subset $J = \{i_1, \dots, i_q\} \subset \{1, \dots, p\}$, where $0 \le q \le p$. By convention, when $J = \emptyset$ we let $q = 0$.
- For $1 \le j \le q$: a number $h_{i_j} > 0$ such that $\sum h_{i_j}|\gamma| + \operatorname{area} S_{i_j} < 1$.
- For $1 \le j \le q$: a number t_{i_j} with $0 \le t_{i_j} < |\gamma|$.

Given the above data we construct a flat surface S with homologous saddles of multiplicity m with associated data $(J, a'_i, a''_i, b'_k, b''_k)$. Namely, for each i, if $z_i = w_i$ we use the figure eight construction and if $z_i \ne w_i$ we use the parallelogram construction giving two boundary curves $\gamma'_i = \overline{z_i z_{i+1}}$ and $\gamma''_i = \overline{w_i w'_i}$. For each i either glue γ'_i directly to γ''_{i+1} or, if $i = i_j \in \{i_1, \dots, i_q\} = J$ glue in cylinders C_{i_j} of "width" $|\gamma|$, "height" h_{i_j} and "twist" t_{i_j} to connect $\overline{w_{i_j} w'_{i_j}}$ to $\overline{z_{i_j+1} z'_{i_j+1}}$ ($i+1$ considered mod p).

The resulting surface S also depends on the data h_{i_j}, t_{i_j} and ρ_i. Denote by S' the union of the pieces S'_i, h the vector of h_{i_j}, t the vector of t_{i_j} and ρ the collection of ρ_i. Then we have an *admissible construction*

$$(S', J, a'_i, a''_i, b'_i, b''_i, \rho, \gamma, h, t) \to S$$

A relative homology basis for S is given by:

- A relative homology basis of curves β for each component S'_i of S'.
- The curve ρ_i on each component S'_i.
- A "horizontal" curve on each cylinder C or closed curve. The integral of ω along this curve is γ.
- A "vertical" curve on each cylinder. The integral of ω along this curve is $h + ir$.

The holonomy vectors $v_S(\beta)$ on S of the set of curves β that live on S'_i is given by

$$v_S(\beta) = v_{S'}(\beta) + (\beta \cdot \rho_i)\gamma \tag{19}$$

Lemma 10.2. *Suppose that we*
— either use at least one parallelogram construction; or

— *if we use only figure eight constructions, we use at least one cylinder.*
Then the resulting surface S is a nondegenerate flat surface.

Given the admissible construction $(S', J, a_i', a_i'', b_k', b_k'', \rho, \gamma, h, t) \to S$ let us study in more detail how the resulting flat surface S is made up of subsurfaces. The resulting surface S has two groups of zeroes: the ones inherited from the components S_i' without any changes, and the newborn ones. Every newborn zero has at least one curve γ_i passing through it; the curves γ_i do not pass through the zeroes of the first group. Actually, it is easy to see from our construction, that the collection of curves γ_{i_j} passing through a zero of S has the form $\gamma_i, \gamma_{i+1}, \dots, \gamma_{i+s}$, with consecutive indices (where as usual $p+1$ is identified with 1, so p is followed by 1). It is easy to give a description of the nature of the corresponding set $\{i, i+1, \dots, i+s\}$.

Any newborn zero is formed by one of the following constructions:

• **Type I.** There is a chain of consecutive surfaces $S_{j_r+1} \to S_{j_r+2} \to \cdots \to S_{j_r+s}$ glued directly. To each surface of the chain we apply the figure eight construction. The first surface of the chain S_{j_r+1} is glued to the preceding surface S_{j_r} by a cylinder and similarly for the last surface of the chain S_{j_r+s} and its successor. In our notation we have $j_r + s = j_{r+1}$, where both $j_r, j_{r+1} \in J$, and $s \geq 0$.

In this case the newborn zero has order

$$\sum_{k=1}^{s} (a_{j_r+k} + 2), \tag{20}$$

where $a_{j_r+k} = a'_{j_r+k} + a''_{j_r+k}$. By convention $s = 0$ stands for the case when multiplicity p is equal to 1, $J = \{1\}$, and we have a single surface S_1 to which we apply figure eight construction and glue in a cylinder.

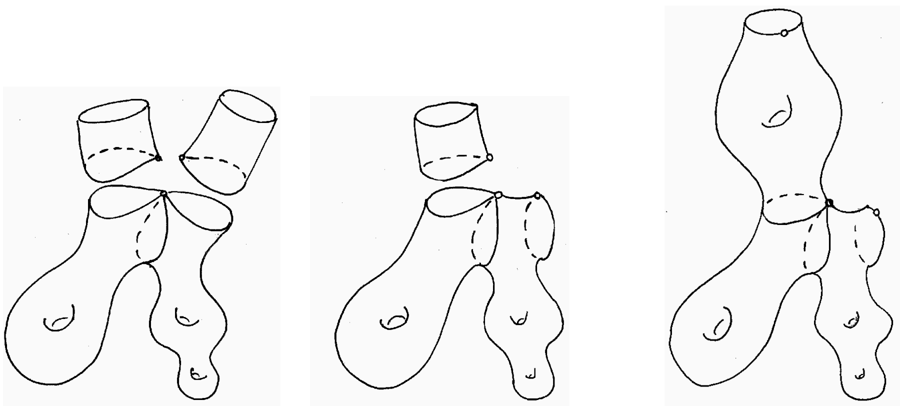

FIGURE 10. Newborn zeroes of three different types.

• **Type II.** Now we have the following chain of surfaces $S_{j_r+1}, S_{j_r+2}, \dots, S_{j_r+s+1}$ glued directly. We apply the figure eight construction to all surfaces except the last, to which we apply the parallelogram construction. The first surface S_{j_r+1} of the chain is glued to S_{j_r} by a cylinder. The value $s = 0$ is allowed: in this case we have a single surface S_{j_r+1} to which we apply a parallelogram construction, and then attach a cylinder to the zero z_{j_r+1} of order $b'_{j_r+1} + 1$. In this case the newborn zero has order

$$\sum_{k=1}^{s}(a_{j_r+k}+2) \;+\; (b''_{j_r+s+1}+1) \tag{21}$$

We may have the symmetric picture, when the surface S_j to which we apply the parallelogram construction is at the beginning of the chain, then we glue directly the surfaces $S_{j+1},\dots,S_{j+s}$ to which we apply figure eight construction, and to the last surface S_{j+s} (where $j+s\in J$ equals some j_r) we glue a cylinder. We have a symmetric formula for the order of the zero:

$$(b'_j+1)+\sum_{k=1}^{s}(a_{j+k}+2)$$

• **Type III.** Finally, we may have surfaces $S_k,\dots,S_{k+s+1}$ to which we apply the parallelogram construction boarding both ends of the chain; we apply the figure eight construction to all interior surfaces $S_{k+1}\dots S_{k+s}$; we glue all surfaces directly. The value $s=0$ is admissible: by convention it means that we have only two surfaces S_k and S_k+1 in the chain to which we apply parallelogram constructions and then glue directly γ''_k to γ'_{k+1}. In this case we get the newborn zero of the following order

$$(b'_k+1)+\sum_{i=1}^{s}(a_{k+i}+2)\;+(b''_{k+s+1}+1) \tag{22}$$

We now adopt the following notation to describe how the surfaces are put together.

Notation The symbol $\to$ corresponds to direct gluing of the surfaces; the symbol $\Rightarrow$ corresponds to a cylinder joining two consecutive surfaces. If a number is written as a sum of two numbers $a'_i+a''_i$ with a bar over the sum, we apply the figure eight construction to a zero of order $a_i=a'_i+a''_i$ breaking it up into zeroes of a'_i and a''_i; if there are two multiplicity numbers, $\bar b'_i,\bar b''_i$, we apply the parallelogram construction. By convention b'_i is the left one, b''_i is the right one. Say, in $(4,\bar 3,2,\bar 1)$ we have $b'_1=3, b''_1=1$, and in $(\overline{0+0})$ we have $a_2=a'_2=a''_2=0$.

Example 10.3. Consider the following collection of flat surfaces organized in a cyclic order

$$\begin{array}{cccccccccccccccccccc}
\to & (4,\bar 3,2,\bar 1) & \to & (\overline{0+0}) & \to & (\overline{4+2},2) & \to & (\bar 9,8,\bar 7) & \to & (\bar 5,\bar 3) & \Rightarrow & (\overline{2+2},2) & \to & (5,\overline{1+2}) & \Rightarrow & (\overline{1+0},1) & \to & (2,\overline{0+0}) & \to \\
\to & S_1 & \to & S_2 & \to & S_3 & \to & S_4 & \to & S_5 & \Rightarrow & S_6 & \to & S_7 & \Rightarrow & S_8 & \to & S_9 & \to
\end{array}$$

In the example above we glue S_5 to S_6 with a cylinder, and S_7 to S_8 with a cylinder; all the other gluings are direct. Thus in this example we have $p=9$, and $J=\{5,7\}$. The chain $S_1\to S_2\to S_3\to S_4$ gives a newborn zero of type III. It has order $(1+1)+\big((0+2)+(6+2)\big)+(9+1)=22$; there are 3 separatrix loops $\gamma_1,\gamma_2,\gamma_3$ passing through this newborn zero. The loops γ_1 and γ_2 bound the surface S_2 and bound angles 2π with each other. The loops γ_2,γ_3 bound S_3 with angles 10π and 6π. The next chain $S_4\to S_5$ gives a newborn zero of type III; it has order $(7+1)+(5+1)=14$; there is only one separatrix loop γ_4 passing through this zero. The next chain is $S_5\Rightarrow$. It is of type II; it gives a newborn zero of order $3+1=4$ with a separatrix loop γ_5 passing through it. The next chain $\Rightarrow S_6\to S_7\Rightarrow$ gives a newborn zero of order $(4+2)+(3+2)=11$ of type I, and three separatrix loops $\gamma_6,\gamma_7,\gamma_8$ passing through it. The surface S_6 is bounded by γ_6 and γ_7 which make angles 8π and 4π at the zero of order 11, while S_7 is bounded by γ_7 and γ_8

which make angles 6π and 4π at the zero. Finally, the last chain $\Rightarrow S_8 \to S_9 \to S_1$ produces a newborn zero of type II. Its order is $(1+2)+(0+2)+(3+1)=9$; there are three separatrix loops $\gamma_9, \gamma_{10}, \gamma_{11}$ passing through it. The loops γ_9 and γ_{10} bound the surface S_8 making angles 4π and 2π with each other while γ_{10} and γ_{11} bound S_9 making angles 2π and 2π with each other.

The resulting surface inherits the zeroes of orders $4, 2, 2, 8, 2, 5, 1, 2$ which are coming unchanged from the components S_i (the integers which are not barred.) It also has the newborn zeroes of orders $22, 14, 4, 11, 9$. Thus the resulting surface S belongs to the stratum $\mathcal{H}(22, 14, 11, 9, 8, 5, 4, 4, 2, 2, 2, 2, 1)$.

10.4. **Stratum Interchange and $\gamma \to -\gamma$ Symmetry.** In this section we discuss the possible symmetries of the admissible construction

$$(S, a_i`, a_i'', b_k', b_k'', \rho, \gamma, h, t) \to S$$

In many aspects these symmetries are analogous to the symmetries discussed in section 7.4. We again consider two settings: the first one with more restrictions, the second one with fewer restrictions.

Problem 1. We assume that all zeroes $z_1, \dots, z_l$ of the surface S are numbered. We fix the zeroes $z_1, \dots, z_r \in S$, $z_i \neq z_j$, for $i \neq j$, of orders $m_1, \dots, m_r$ correspondingly. We assume that the flat surface S possesses exactly s loops of saddle connections $\gamma_1, \dots, \gamma_s$ homologous to some fixed loop γ. We assume that every loop γ_i starts and ends at one of the zeroes $z_1, \dots, z_r$.

Cutting along the loops γ_i, $i = 1, \dots, r$ we perform the surgery as described above decomposing S into the collection S_i'. The collection is organized in a cyclic order as described above.

We fix the types α_i' of the surfaces S_i', the distinguished zeroes (marked points) $w_i, z_i \in S_i'$, and the numbers (a_i', a_i''), (b_k', b_k'') where both types of pairs are ordered. We fix, whether S_j' is joined to S_j' by a cylinder, or directly.

We start with the case when we fix also the following data. We assume that $z_1, \dots, z_r$ are the newborn zeroes numbered with respect to this cyclic order. We assume that the zeroes $z_{r+1}, \dots, z_{l_1}$ lie in S_1, the zeroes $z_{l_1+1}, \dots, z_{l_1+l_2}$ lie in S_2, ..., the zeroes $z_{l_1+\dots+l_{p-1}+1}, \dots, z_{l_1+\dots+l_p}$ lie in S_p. Here $l = l_1 + \dots + l_p = card(\alpha)$.

Problem 2. Now consider the problem with fewer constraints. We count the number of occurrences of configurations of loops of saddle connections described in the previous problem with the only difference that now we assume all the zeroes to be "anonymous". In other words, we keep all geometric information on configuration (number r of zeroes involved; number s of loops of homologous saddle connections; types α_i' of the surfaces S_i', and the cyclic order of their appearances; numbers (a_i', a_i''), (b_j', b_j''), where both types of pairs are ordered; the way (by means of a cylinder or directly) S_i' is joined to S_{i+1}').

However, if we have two zeroes P_i, P_j of the same degree on the surface S, and, neither of them is "newborn", in the setting of Problem 2 we do not distinguish cases, when P_i lies in S_q' and P_j gets to S_t', and the opposite case.

Remark 10.4. Problem 2 may be considered as generalization of Problem 1 to the case when the zeroes of S and of S_i' are *not* numbered.

Remark 10.5. In both settings the $\mathcal{H}(\alpha_i')$ may be disconnected. We could also specify the connected component of the $\mathcal{H}(\alpha_i')$ when this occurs. We prefer the setting where this data is not specified.

Consider the natural action of the cyclic group of order p on the elements of the assignment organized in a cyclic order. If it has a nontrivial stabilizer we denote it by the same symbol Γ as before. We get a symmetry of order $|\Gamma|$ which we again call the *stratum interchange*. Note that this symmetry might be different in the settings of Problem 1 and Problem 2.

Example 10.6. The assignment

$$(\bar{1}, \bar{1}, 2) \to (\bar{1}, \bar{1}) \Rightarrow (\bar{1}, \bar{1}, 2) \to (\bar{1}, \bar{1}) \Rightarrow$$

does not possess the stratum interchange symmetry in the setting of Problem 1 since the unchanged zeroes of order 2 are named, and so they identify the components of the type $(2, 1, 1)$. However, in the setting of Problem 2 this configuration has the stratum interchange symmetry of order 2.

The assignment

$$(\bar{1}, \bar{1}) \to (\bar{1}, \bar{1}) \Rightarrow (\bar{1}, \bar{1}) \to (\bar{1}, \bar{1}) \Rightarrow$$

possesses the stratum interchange symmetry of order 2 in both settings.

Recall that for saddle connections joining distinct zeroes we also might also have the $\gamma \to -\gamma$ symmetry. It could appear only when $m_1 = m_2$, for in this case we have no invariant way of choosing the orientation of the saddle connection. When we have only saddle connections joining a zero to itself, there is no geometric way of choosing the orientation of saddle connection. The two different choices of the orientation give two decompositions of S. Thus we get the action of the group $\mathbb{Z}/2\mathbb{Z}$ on the collection of assignments. We call this action the $\gamma \to -\gamma$ *symmetry*. This action is easily described in combinatorial terms: one has to change the cyclic order in the assignment to the opposite one, as well as the order in each pair (a'_i, a''_i) and (b'_j, b''_j). Some assignments may stay invariant under this operation. In this case we say that they have $\gamma \to -\gamma$ symmetry. This property also depends whether we consider the assignments in the setting of Problem 1 or of Problem 2.

Example 10.7. None of the assignments in Example 10.6 have the $\gamma \to -\gamma$ symmetry.

An assignment of multiplicity 1 has $\gamma \to -\gamma$ symmetry if and only if $a' = a''$ when we apply the figure eight construction and if and only if $b' = b''$ when we the apply parallelogram construction.

The assignment

$$(\bar{2}, \bar{2}, 2) \Rightarrow (\bar{2}, \bar{2}, 2) \Rightarrow$$

has the $\gamma \to -\gamma$ symmetry in both settings, while the assignment

$$(\bar{2}, \bar{2}, 2) \Rightarrow (\bar{2}, \bar{2}, 2) \Rightarrow (\bar{2}, \bar{2}, 2) \Rightarrow$$

has the $\gamma \to -\gamma$ symmetry only in the setting of Problem 2.

Remark 10.8. In describing the construction $(S', J, a'_i, a''_i, b'_i, b''_i, \rho, \gamma, h, t) \to S$ one can consider *all* possible assignments, and then take into account corresponding symmetries, or one can deal with the classes, eliminating the symmetry whenever it is possible.

We have chosen the second way. For example, the assignment $(\bar{1}, \bar{0}, 1) \to$ (see section 11.2) is symmetric to the assignment $(\bar{0}, \bar{1}, 1) \to$ by the $\gamma \to -\gamma$ symmetry.

Our main objective is to show now that the degeneration of a surface to a point on the principal boundary and the admissible construction of a surface from a point on the principal boundary are essentially inverse operations. The complicating issue is that both depend on choices of transversals and so they need not be exact inverses. However going from one transversal to another amounts to a change of basis and so can be described.

In particular the admissible construction of S depends on a choice of transversals ρ_i for each flat surface S'_i. If we fix a pair of zeroes of a given S'_i and a transversal joining them, then there may be a large set of other flat structures in $\mathcal{H}(\alpha'_i)$ for which there is no transversal in the same homotopy class. Thus on different parts of $\mathcal{H}(\alpha'_i)$ we may be forced to choose different transversals. Thus we need to study how the construction depends on the choice of transversal.

Now fix a small κ. Let $\mathcal{F}'$ denote a compact simply connected subset of $\mathcal{H}_1(\alpha')$ such that for $S' \in \mathcal{F}'$ every saddle connection has length at least κ. Its complement has volume $O(\kappa^2)$. Let $\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$ denote the subset of $\mathcal{H}_1(\alpha,\mathcal{C})$ consisting of those flat surfaces such that the saddle connections in the configuration have length at most ϵ and every other saddle has length at least κ.

Choose a relative homology basis for all $S' \in \mathcal{F}'$. The coordinates provided by the holonomy along the homology basis β, allows one to define a metric d on $\mathcal{F}'$. Let $M' = M + M''$ where M is given by (18) and M'' is given by Lemma 9.3. For $\epsilon < \kappa/2$ find a finite collection of balls $B(x_j, r_j)$ centered at x_j of radius r_j such that

- the smaller balls $B(x_j, r_j - M'\epsilon)$ cover $\mathcal{F}'$
- for each $B(x_j, r_j)$, for each component S'_i of S' for which $z_i \neq w_i$ there is a fixed class of transversals $\rho_{i,j}$ joining z_i to w_i.

We may use the cover by balls $B(x_j, r_j - M'\epsilon)$ to define a partition of $\mathcal{F}'$ by a finite collection of sets $U_j \subset B(x_j, r_j - M'\epsilon)$. Now we can give a well defined assignment

$$\sigma : \mathcal{F}' \to \mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$$

Namely each S' belongs to a unique $S' \in U_j$. Then use the transversal $\rho_{i,j}$ to assign a collection of surfaces $S = \sigma(S')$.

As in the previous section, we have the possibility of a construction for $-\gamma$ coinciding with a construction with γ and also the possibility of stratum interchange. However even without considering these symmetries, it is possible because of the change of transversals from one set U_j to another, that the construction is not $1-1$. We prove

Proposition 10.9. *There is $E' \subset \mathcal{F}'$ such that*

- $\mathrm{Vol}(\mathcal{F}' - E') = O(\epsilon^2)$,
- *On a subset of E' of full volume, σ is $1-1$ except for the $\gamma \to -\gamma$ and stratum interchange symmetries.*
- $\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}) - \sigma(E')) = O(\epsilon^4)$

Proof. Choose E' such that

- every point in E' lies in some ball $B(x_j, r_j - M'\epsilon)$
- every point in E'' is at least $M'\epsilon$ from the boundary of any ball $B(x_k, r_k)$.

The second condition means that if $S' \in E' \cap U_j$ then the $M'\epsilon$ neighborhood of S' is contained in U_j. It also implies

$$\mathrm{Vol}(\mathcal{F}' - E') = O(\epsilon^2)$$

We prove σ is $1-1$ on almost all of E' except for possible stratum interchange and $\gamma \to -\gamma$ symmetries. If $S_1' \neq S_2'$ belong to the same U_j then the collection of surfaces $\sigma(S_1')$ and $\sigma(S_2')$ are constructed with the same transversals and yet differ on holonomy for a set of basis curves. The surfaces can be isomorphic only if the isomorphism does not preserve the basis. If the set of configuration curves are preserved then we have the stratum interchange or $\gamma \to -\gamma$ symmetry. If they are not preserved, then S' contains a curve of length at most ϵ and this possibility has been ruled out by the definition of $\mathcal{F}'$. The remaining possibility is if for a fixed S' two or more of the remaining surfaces in $\sigma(S')$ are isomorphic by an isomorphism that preserves the set of curves in the configuration. But such an isomorphism must arise from an automorphism of S' and such S' have volume 0.

If $S_1', S_2' \in E'$ belong to different U_j, then their distance apart in the metric d is at least $M'\epsilon$, and then $\sigma(S_1')$ and $\sigma(S_2')$ again differ on the homology basis and we have the same conclusion as before.

We now show the third statement. Define $E'' \subset E'$ as in the definition of E' except that we require points to lie in balls $B(x_j, r_j - 2M'\epsilon)$ and be at least $2M'\epsilon$ from the boundary of any ball. Given the transversals τ_i from Lemma 9.3, let E be the subset of $\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C})$ consisting of those S whose degeneration yields surfaces S' in E''. We have

$$\mathrm{Vol}(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}) - E) = O(\epsilon^4)$$

For some j, the resulting surface $S' \in U_j \cap B(x_j, r_j - 2M'\epsilon)$. However since it may not be the case that $\tau_i = \rho_{i,j}$, it may not be true that $\sigma(S') = S$. The third statement in the Proposition follows from the claim that there is some $S'' \in E'$ such that $S = \sigma(S'')$. To prove the claim, notice that since $B(x_j, r_j)$ is a ball and since $|\beta \cdot \tau_i| \leq M''$ and $|\beta \cdot \rho_{i,j}| \leq M$, the point in $\mathbb{R}^{n'}$ defined by the collection of vectors

$$v_{S'}(\beta) + (\sum_i (\beta \cdot \tau_i) - (\beta \cdot \rho_{i,j}))\gamma$$

as β runs over the homology basis, corresponds to the holonomy for a unique $S'' \in B(x_j, r_j - M'\epsilon)$. By definition of E'', we have $S'' \in E'$. But now $\sigma(S'')$ has the same holonomy as S along the basis and so is isomorphic to it. This proves the claim. $\square$

from S''' to S'' changes distances at most

10.5. **Combinatorial Factor, Connected Strata.** We need to compute the combinatorial factor M, which computes for fixed data $(S', J, a_i', a_i'', b_k', b_k'', \gamma, h, t)$ the number of $S = \sigma(S')$ that can be built. We start with the setting of Problem 1 in section 10.4. We first suppose that the stratum $\mathcal{H}(\alpha)$ is connected.

Consider a surface S_i' such that $z_i = w_i$ is a zero of order $a_i = a_i' + a_i''$. We apply the figure eight construction by breaking $z_i = w_i$ into zeroes of orders a_i' and a_i''. There are $a_i + 1$ ways of doing this, see section 10.1. The corresponding boundary components γ_i', γ_i'' are octagon. The component γ_i' is attached to S_{i+1} (directly or by means of a cylinder depending on whether or not $i \in J$); the component γ_i'' is attached to S_{i-1} (directly or by means of a cylinder depending on whether or not $i-1 \in J$).

If $z_k \neq w_k$ we perform the parallelogram construction. The orders of the zeroes are b_k'' and b_k' correspondingly, so there are $b_k'' + 1$ ways of choosing the vector γ

at z_k and b'_k+1 ways of choosing the vector γ at w_i so this gives $(b'_k+1)(b''_k+1)$ choices.

We may have the stratum interchange symmetry or (and) $\gamma \to -\gamma$ symmetry, see section 10.4. Thus the resulting combinatorial factor in the setting of Problem 1 in section 10.4 equals

$$c = \frac{1}{|\Gamma_-|} \cdot \frac{1}{|\Gamma|} \cdot \prod_{\substack{1\le i\le p\\ z_i=w_i}} (a_i+1) \cdot \prod_{\substack{1\le k\le p\\ z_k\neq w_k}} (b'_k+1)(b''_k+1)$$

We now compute the combinatorial factor in the setting of Problem 2 in section 10.4. For each integer m, let $o(m)$ denote its multiplicity. For each i, $i = 1,\dots,p$, let $o_i(m)$ denote the multiplicity of m in α'_i.

Consider an integer m different from the order of any newborn zero on S (see formulae 20, 21, 22), and different from any a_i or b'_k, b''_k. Since all the zeroes $z_1, z_2, \dots, z_l$ of S are numbered, the number of ways to arrange $o(m)$ zeroes of order m into groups of $o_1(m),\dots,o_p(m)$ zeroes equals

$$\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}$$

where we let $0! = 1$ by convention.

Let now $s \ge 1$ be the number of the newborn zeroes in S of order m, and suppose that m is different from any a_i, b'_k, b''_k. Now we have to chose s of $o(m)$ numbered zeroes to be the newborn ones, and then we have to arrange the remaining $o(m)-s$ ordered zeroes into groups of $o_1(m),\dots,o_p(m)$ zeroes. For such m we get the factor

$$o(m)(o(m)-1)\cdot\dots\cdot(o(m)-s+1) \ \cdot\ \frac{(o(m)-s)!}{\prod_{i=1}^{p} o_i(m)!} = \frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}$$

which coincides with the previous one. Hence, we do not need to distinguish this case from the previous one.

If m is different from any b'_k, b''_k, but is equal to some a_i, then there are only $o_i(m)-1$ zeroes of order m on S'_i inherited from S. Thus the corresponding factor in the denominator in the formula above equals $(o_i(m)-1)!$. Multiplying numerator and denominator by $o_i(m)$ we get the following factor for such m:

$$\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!} \cdot \prod_{\substack{1\le i\le p\\ a_i=m\\ z_i=w_i}} o_i(m)$$

We have to make similar correction for those m which coincide with some b'_k or b''_k. In the case when m equals one of the $b'_k \neq b''_k$ the correction is completely analogous to the previous one. In the case when for some k we have $m = b'_k = b''_k$, we get only $o(m)-2$ zeroes of order m on S'_k, inherited from S. Thus the corresponding factor in the denominator in the formula above equals $(o_k(m)-2)!$. Multiplying numerator and denominator by $o_k(m)(o_k(m)-1)$ and collecting all the correction terms we finally get the following combinatorial factor for the setting of Problem 2 in section 10.4.

$$M=\frac{1}{|\Gamma_-|}\cdot\frac{1}{|\Gamma|}\cdot\prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{i=1}^{p}o_i(m)!}\right)\cdot$$

$$\cdot\prod_{\substack{1\le i\le p\\ z_i=w_i\\ a_i\neq 0}}o_i(a_i)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k\\ b'_k\neq b''_k\\ b'_k\neq 0}}o_k(b'_k)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k\\ b'_k\neq b''_k\\ b''_k\neq 0}}o_k(b''_k)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k\\ b'_k=b''_k\neq 0}}o_k(b'_k)(o_k(b'_k)-1)\cdot$$

$$\cdot\prod_{\substack{1\le i\le p\\ z_i=w_i}}(a_i+1)\cdot\prod_{\substack{1\le k\le p\\ z_k\neq w_k}}(b'_k+1)(b''_k+1)$$

Note that in general the symmetry groups Γ and Γ_- corresponding to possible stratum interchange symmetry and to $\gamma\to-\gamma$ symmetry are different in Problems 1 and 2, see section 10.4.

There will also be additional factors coming from the spin structures and the hyperelliptic strata. We shall discuss these separately.

11. Computing the Constants for Connected Strata

Recall that q is the number of cylinders we shall attach. Let $n_i=\dim_{\mathbb{R}}\mathcal{H}(\alpha_i)$. We also define d_i to equal n_i except in the following situations. If α_i has a single marked point which is not a zero and S_i is not the torus, we have $d_i=n_i+2$. If there are a pair of marked point, and the surface is not the torus $d_i=n_i+4$. If the surface is a torus then $d_i=n_i+2$. Then,

$$\dim_{\mathbb{R}}\mathcal{H}(\alpha)=2q+2+\sum_{i=1}^{p}d_i,$$

where α is the stratum such that $S\in\mathcal{H}(\alpha)$.

11.1. **Computation of the constants.** Recall that $d\nu(S)$ is the measure in $\mathcal{H}(\alpha)$ and $d\nu(S'_i)$ is the measure in $\mathcal{H}(\alpha_i)$. We first assume that $\mathcal{H}(\alpha)$ is connected. If we choose a homology basis β_i for the S'_i, and let $hol_S(\beta_i)$ be the holonomy of β_i on the surface S, then the measure $d\nu$ restricted to the image $\sigma(E')$ under the admissible construction is the product measure of $Md\gamma\ dt_i dh_i$ and $\prod d(hol_S(\beta_i))$. By Equation (19) the latter coincides with $d\nu'$ up to an error that is $O(\epsilon^2)$. Since $|\gamma|\le\epsilon$, up to an error $O(\epsilon^4)$ we can write

$$d\nu(S)=Md\gamma\prod_{i=1}^{p}d\nu'(S'_i)\prod_{i=1}^{q}dh_i dt_i$$

except for the following situations. If z_i is a marked point and w_i is a zero then we have the additional factor $dz(S_i)$ in the term $d\nu_i$. If z_i is a zero and w_i is a marked point we have $dw(S_i)$. If $w_i=z_i$ is a marked point, we have $dz(S_i)$ and if $z_i\neq w_i$ are both marked points we have the factor $dz(S_i)\ dw(S_i)$.

Note that we could also consider strata with one or two marked points. The stratum would be a fiber space over the stratum $\mathcal{H}_1(\alpha)$. The fiber over a point would be the flat surface if there is one marked point or product of flat surfaces if there are two, the flat surface representing that point in $\mathcal{H}_1(\alpha)$. The measure on the fiber space is then locally product measure of the base with the area (or product of area measures) on the fiber.

Note that $(\operatorname{area} S) = \sum_i (\operatorname{area} S_i) + h_i|\gamma|$, and if $S \in C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))$ then $|\gamma| \leq \epsilon\sqrt{(\operatorname{area} S)}$. Hence,

$$|\gamma|^2 \leq \epsilon^2 \sum_i (\operatorname{area} S_i) + h_i|\gamma|,$$

i.e.

$$\sum_i h_i \geq \frac{|\gamma|}{\epsilon^2} - \frac{\sum_i (\operatorname{area} S_i)}{|\gamma|}.$$

The expression on the left is positive when $|\gamma| > \epsilon\sqrt{(\sum_i \operatorname{area} S_i)}$.

Let

$$W = \prod_i \operatorname{Vol}(\mathcal{H}_1(\alpha_i))$$

For $s > 0$, let $D(s) = \{(r_1,\dots,r_p) \ : \ \sum_i r_i^2 < s\}$. Then by Proposition 10.9, and the fact that $\operatorname{Vol}(\prod_i \mathcal{H}_1(\alpha_i) - \mathcal{F}') = O(\kappa^2)$, up to an error term that is $O(\epsilon^2(\epsilon^2+\kappa^2))$ we have

$$\begin{aligned}
&\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha))) = WM\cdot \\
&= \Bigg(\int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\, dr_i\right) \int_{\gamma\in B(\epsilon\sqrt{\sum_i r_i^2})} \int_{0\leq \sum_i h_i \leq \frac{1-\sum_i r_i^2}{|\gamma|}} \prod_{i=1}^q \left(\int_0^{|\gamma|} dt_i\, dh_i\right) d\gamma + \\
&+ \int_{D(1)} \prod_{i=1}^p r_i^{d_i-1}\, dr_i \int_{\gamma\in B(\epsilon)-B(\epsilon\sqrt{\sum_i r_i^2})} \int_{\frac{|\gamma|}{\epsilon^2}-\frac{\sum_i r_i^2}{|\gamma|}\leq\sum_i h_i\leq\frac{1-\sum_i r_i^2}{|\gamma|}} |\gamma|^q dh_i\, d\gamma \Bigg) = \\
&= WM\cdot \Bigg(\int_{D(1)} \prod_{i=1}^p r_i^{d_i-1}\, dr_i \int_{\gamma\in B(\epsilon\sqrt{\sum_i r_i^2})} \frac{1}{q!}(1-\sum_i r_i^2)^q\, d\gamma + \\
&+ \int_{D(1)} \prod_{i=1}^p r_i^{d_i-1}\, dr_i \int_{\gamma\in B(\epsilon)-B(\epsilon\sqrt{\sum_i r_i^2})} \frac{1}{q!}\left[(1-\sum_i r_i^2)^q - \left(\frac{|\gamma|^2}{\epsilon^2}-\sum_i r_i^2\right)^q\right] d\gamma \Bigg)
\end{aligned}$$

Performing the integral over γ, up to this error term, we obtain

$$\begin{aligned}
&\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))) = \\
&\qquad = \frac{\pi\epsilon^2 WM}{q!} \int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\right)\left(1-\sum_i r_i^2\right)^q \left(\sum_i r_i^2\right) \prod_i dr_i \\
&\qquad + \frac{\pi\epsilon^2 WM}{q!} \int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\right)\left(1-\sum_i r_i^2\right)^q \left(1-\sum_i r_i^2\right) \prod_i dr_i \\
&\qquad - \frac{\pi WM}{q!} \int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\right) \int_{\epsilon\sqrt{\sum_i r_i^2}}^{\epsilon} \left(\frac{s^2}{\epsilon^2}-\sum_i r_i^2\right)^q 2s\, ds \prod_i dr_i \\
&\qquad = \frac{\pi\epsilon^2 WM}{q!} \int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\right)\left(1-\sum_i r_i^2\right)^q \prod_i dr_i \\
&\qquad - \frac{\pi\epsilon^2 WM}{(q+1)!} \int_{D(1)} \left(\prod_{i=1}^p r_i^{d_i-1}\right)\left(1-\sum_i r_i^2\right)^{q+1} \prod_i dr_i
\end{aligned}$$

where we have used the change of variable $u = (s^2/\epsilon^2 - \sum r_i^2)$ in the last line. We now make the change of variable $x_i = r_i^2$. Let $b_i = d_i/2 - 1$, so that $r_i^{d_i-1}\,dr_i = (1/2)x_i^{b_i}\,dx_i$. Then,

$$\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))) = \frac{\pi\epsilon^2 WM}{2^p}\int_{\sum_i x_i \le 1} x_1^{b_1}\dots x_p^{b_p}\left[\frac{1}{q!}\left(1-\sum_i x_i\right)^q - \frac{1}{(q+1)!}\left(1-\sum_i x_i\right)^{q+1}\right]dx_1\dots dx_p$$

Evaluating the integral from the inside out via repeated application of the identity

$$\int_0^u x^a(u-x)^b\,dx = \frac{a!\,b!}{(a+b+1)!}u^{a+b+1}$$

yields

$$\begin{aligned}\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))) &= \\ &= \frac{\pi\epsilon^2 WM}{2^q}\left[\frac{b_1!\dots b_q!}{(b_1+\dots+b_p+p+q)!} - \frac{b_1!\dots b_p!}{(b_1+\dots+b_p+p+q+1)!}\right] = \\ &= \frac{\pi\epsilon^2 WM}{2^p}\frac{b_1!\dots b_p!}{(b_1+\dots+b_p+p+q-1)!}\cdot\frac{1}{b_1+\dots+b_p+p+q+1}\end{aligned}$$

Since $b_1+\dots+b_p+p+q+1 = n/2$, up to an error term that is $O(\epsilon^2(\epsilon^2+\kappa^2))$, we have

$$\nu(C(\mathcal{H}_1^{\epsilon,th}(\alpha,\mathcal{C}))) = \frac{\pi\epsilon^2 WM}{2^{p-1}}\frac{b_1!\dots b_p!}{(\frac{n}{2}-2)!}$$

Now first taking the limit as $\epsilon\to 0$ for fixed κ, and then taking the limit as $\kappa\to 0$, we get

$$c = \lim_{\kappa\to 0}\lim_{\epsilon\to 0}\frac{\nu(C(\mathcal{H}_1^{\epsilon}(\alpha,\mathcal{C})))}{\pi\epsilon^2\nu_1(\mathcal{H}_1(\alpha))} = \\ = M\cdot\frac{1}{2^{p-1}}\cdot\frac{\prod_{i=1}^p(\frac{d_i}{2}-1)!}{(\frac{n}{2}-2)!}\cdot\frac{\prod_{i=1}^p\operatorname{Vol}(\mathcal{H}_1(\alpha_i'))}{\operatorname{Vol}(\mathcal{H}_1(\alpha))}$$

We finally get the following expression for the constant c in the setting where the zeroes of S are not numbered (see Problem 2 in section 10.4):

Formula 11.1. *The list of possible configurations of homologous saddle connections joining zeroes to themselves is the same for almost all surfaces in any connected stratum $\mathcal{H}(\alpha)$. The possible configurations are described in section 10.3. The constant in the quadratic asymptotics for the number of saddle connections with associated data $(J, a_i', a_i'', b_k', b_k'')$ is presented by the following formula:*

$$c = \frac{1}{|\Gamma_-|} \cdot \frac{1}{|\Gamma|} \cdot \prod_{m \in \alpha} \left(\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!} \right) \cdot \tag{23}$$
$$\cdot \prod_{\substack{1 \le i \le p \\ z_i = w_i \\ a_i \in \alpha}} o_i(a_i) \cdot \prod_{\substack{1 \le k \le p \\ z_k \neq w_k \\ b'_k \neq b''_k \\ b'_k \in \alpha}} o_k(b'_k) \cdot \prod_{\substack{1 \le k \le p \\ z_k \neq w_k \\ b'_k \neq b''_k \\ b''_k \in \alpha}} o_k(b''_k) \cdot \prod_{\substack{1 \le k \le p \\ z_k \neq w_k \\ b'_k = b''_k \in \alpha}} o_k(b'_k)(o_k(b'_k) - 1) \cdot$$
$$\cdot \prod_{\substack{1 \le i \le p \\ z_i = w_i}} (a_i + 1) \cdot \prod_{\substack{1 \le k \le p \\ z_k \neq w_k}} (b'_k + 1)(b''_k + 1) \cdot$$
$$\cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_{i=1}^{p} (\frac{d_i}{2} - 1)!}{(\frac{d}{2} - 2)!} \cdot \frac{\prod_{i=1}^{p} \operatorname{Vol}(\mathcal{H}_1(\alpha'_i))}{\operatorname{Vol}(\mathcal{H}_1(\alpha))}$$

11.2. **Examples: Constants for Strata in Genus 3. Stratum $\boldsymbol{\mathcal{H}(3,1)}$**

$\rightarrow (\overline{\mathbf{1}}, \overline{\mathbf{0}}, \mathbf{1}) \rightarrow$

The first possibility is multiplicity 1 with no cylinder. After degeneration we get a genus 2 surface with a distinguished simple zero, a marked point and another simple zero. We have $d_1 = 12$. We employ the parallelogram construction to the distinguished simple zero and marked point producing two circles which we glue directly. Since $1 = b'_1 \neq b''_1 = 0$ we do not have the $\gamma \to -\gamma$ symmetry. Thus $M = (b'_1 + 1)(b''_1 + 1) = 2$. We have

$$c = 2 \cdot \frac{(5 + 1 - 1)!}{(7 - 2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(3,1))} = \frac{105}{16} \cdot \frac{1}{\zeta(2)} \approx 3.990$$

$\Rightarrow (\overline{\mathbf{1} + \mathbf{0}}, \mathbf{1}) \Rightarrow$

The next possibility is multiplicity 1 with a cylinder where the other side also returns to the zero of order 3. After degeneration we get a surface of genus 2 with a distinguished simple zero on which we perform the figure eight construction, gluing in a cylinder. Since $1 = a'_1 \neq a''_1 = 0$ we do not have the $\gamma \to -\gamma$ symmetry. Thus $M = (a_1 + 1) = 2$. Now $d_1 = 10$ so

$$c = 2 \cdot \frac{(5 - 1)!}{(7 - 2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(3,1))} = \frac{21}{16} \cdot \frac{1}{\zeta(2)} \approx 0.7979$$

$\Rightarrow (\overline{\mathbf{2}}, \overline{\mathbf{0}}) \Rightarrow$

If the other side returns to the simple zero, then after degeneration, we get a genus 2 surface with a double zero and a marked point on which we perform the parallelogram construction. Again $d_1 = 10$. Since $2 = b'_1 \neq b''_1 = 0$ we do not have the $\gamma \to -\gamma$ symmetry. Thus $M = (b'_1 + 1)(b''_1 + 1) = 3$. We have

$$c = 3 \cdot \frac{(4 + 1 - 1)!}{(7 - 2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2))}{\operatorname{Vol}(\mathcal{H}_1(3,1))} = \frac{567}{256} \cdot \frac{1}{\zeta(2)} \approx 1.346$$

$\Rightarrow (\overline{\mathbf{0} + \mathbf{0}}) \rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow$

Now we consider multiplicity 2 in this stratum. Then we have a cylinder whose boundary components are curves returning to the zero of order 3 and zero of order one. The homologous curve returns to the zero of order 3. The degenerating surfaces are tori, one with a single marked point, the other with 2 marked points. On the torus with one point, we perform the figure eight construction and on the other the parallelogram construction. We glue in one cylinder and one pair of circles is glued directly.

There is no $\gamma \to -\gamma$ symmetry in this case.

In this case we get

$$c = 1 \cdot \frac{1}{2^{2-1}} \cdot \frac{(3-1)! \cdot (2-1)!}{(7-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset)))^2}{\operatorname{Vol}(\mathcal{H}_1(3,1))} = \frac{105}{256} \cdot \frac{1}{\zeta(2)} \approx 0.2493$$

Stratum $\mathcal{H}(2,1,1)$

$\rightarrow (\mathbf{1}, \mathbf{1}, \overline{\mathbf{0}}, \overline{\mathbf{0}}) \rightarrow$

The multiplicity 1 case with no cylinder degenerates to a genus 2 surface with 2 marked points. We apply the parallelogram construction to the marked points, gluing the circles directly to each other. There is the $\gamma \to -\gamma$ symmetry.

We get

$$c = \frac{1}{2} \cdot \frac{(5+2-1)!}{(8-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{3} \cdot \frac{1}{\zeta(2)} \approx 1.418$$

$\Rightarrow (\mathbf{1}, \mathbf{1}, \overline{\mathbf{0}+\mathbf{0}}) \Rightarrow$

For the case of a cylinder with multiplicity 1 where the other side returns to the double zero the degenerating surface has genus 2 with a single marked point. We apply figure The eight construction at the marked point. We have the $\gamma \to -\gamma$ symmetry.

$$c = \frac{1}{2} \cdot \frac{(5+1-1)!}{(8-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{18} \cdot \frac{1}{\zeta(2)} \approx 0.2364$$

$\Rightarrow (\mathbf{1}, \overline{\mathbf{1}}, \overline{\mathbf{0}}) \Rightarrow$

If the other side of the cylinder returns to a simple zero, then after degeneration we get a surface of genus 2 with a marked point and a simple zero (as well as another simple zero on the surface). We apply the parallelogram construction at the marked point and a zero gluing in a cylinder. Now we have $b_1' = 0$, $b_1'' = 1$. Since $b_1' \neq b_2''$ we do not have a $\gamma \to -\gamma$ symmetry.

$$c = 4 \cdot \frac{(5+1-1)!}{(8-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{28}{9} \cdot \frac{1}{\zeta(2)} \approx 1.891$$

$\Rightarrow (\mathbf{2}, \overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow$

If the cylinder joins two simple zeroes, then after degeneration we get a surface of genus 2 with a zero of order 2 and two marked points. We apply the parallelogram construction at the marked points. We have $b_1' = b_1'' = 0$. Here we have $\gamma \to -\gamma$ symmetry.

$$c = 1 \cdot \frac{(4+2-1)!}{(8-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2))}{\operatorname{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{8} \cdot \frac{1}{\zeta(2)} \approx 0.5319$$

$\Rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow$

There are two multiplicity $p = 2$ cases. The first is if there is a cylinder with boundary curves returning to the simple zeroes and a homologous curve returning to the double zero. We get 2 tori each with 2 marked points so $p = 2, d_1 = d_2 = 6$. We perform the parallelogram construction on each and then glue in one cylinder and glue two circles directly. We have the $\gamma \to -\gamma$ symmetry, and we do not have a stratum interchange.

$$c = 1 \cdot \frac{1}{2^{2-1}} \cdot \frac{(3-1)! \cdot (3-1)!}{(8-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\emptyset))^2}{\mathrm{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{36} \cdot \frac{1}{\zeta(2)} \approx 0.1182$$

$\Rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{0} + \mathbf{0}}) \Rightarrow$

The second case is if there are 2 cylinders. In that case, after degeneration, we get a torus with 2 marked points and a torus with one marked point. We apply the figure eight construction to the torus with one marked point and the parallelogram construction to the other. We then glue in cylinders to each pair of circles. Again we have the $\gamma \to -\gamma$ symmetry, and we do not have a stratum interchange.

$$c = 1 \cdot \frac{1}{2^{2-1}} \cdot \frac{(3-1)! \cdot (2-1)!}{(8-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\emptyset))^2}{\mathrm{Vol}(\mathcal{H}_1(2,1,1))} = \frac{7}{72} \cdot \frac{1}{\zeta(2)} \approx 0.05910$$

11.3. **Principal stratum.** Since any figure eight construction produces a "newborn" zero of order at least two, here we cannot have any figure eight constructions. All the surfaces S'_i, belong to the principal strata in lower genera g_i; there are a pair of marked points $z_i \neq w_i$ on each surface, and we always apply the parallelogram construction with $b'_i = b''_i = 0$. All the curves γ_i bound cylinders.

11.3.1. *Saddle Connections of Multiplicity* 1. In multiplicity one there is a single surface $S' \in \mathcal{H}(\alpha')$ of genus $g-1$ and S' has two marked points $z \neq w$. We recover S from S' by applying the parallelogram construction followed by gluing in a cylinder; α' is the partition of $2(g-1)-2$ into ones.

We always have the $\gamma \to -\gamma$ symmetry, so $|\Gamma_-| = 2$ in all cases. We use the setting of Problem 2 of section 10.4). The combinatorial constant is

$$M = \frac{1}{2} \cdot \frac{(2g-2)!}{(2g-4)!} \cdot (0+1)(0+1)$$

Applying (23) we get the following value for the constant c:

$$c = \frac{(2g-2)(2g-3)}{2} \cdot \frac{((4g-3-2)-1)!}{((4g-3)-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\alpha'))}{\mathrm{Vol}(\mathcal{H}_1(\alpha))}$$

Thus, finally,

$$c = \frac{(g-1)(2g-3)}{4g-5} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1(\overbrace{1,\ldots,1}^{2g-4}))}{\mathrm{Vol}(\mathcal{H}_1(\underbrace{1,\ldots,1}_{2g-2}))}$$

In the table below we present the values of the constant for saddle connections of multiplicity one joining a zero to itself, for flat surfaces living in the principal strata in small genera.

TABLE 7. Principal stratum $\mathcal{H}(1,\dots,1)$; values of the constants in the quadratic asymptotics for the number of closed geodesics of multiplicity one.

	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$	$g=7$	$g=8$
$c\cdot\zeta(2)=$	$\frac{5}{2}$	$\frac{36}{7}$	$\frac{3150}{377}$	$\frac{274456}{23357}$	$\frac{250153470}{16493303}$	$\frac{6531347988}{351964697}$	$\frac{8007196856750}{365342975469}$
$c\approx$	1.51982	3.12648	5.07950	7.14344	9.22041	11.2812	13.3239

11.3.2. *Higher Multiplicity in the Principle Stratum.* The surface S breaks up into a union of p surfaces S_i. of genus g_i. The S_i have simple zeroes. Let α_i denote the partition of $2(g_i-2)$ into ones. Each S_i has two marked points z_i and w_i.

We reconstruct S by the parallelogram construction and then by gluing in cylinders C_i to connect the boundary circles.

There is again always a factor of $1/2$ due to the symmetry $\gamma\to-\gamma$ since all points are marked points. The combinatorial factor is

$$M=\frac{(2g-2)!}{\prod_{i=1}^{p}(2g_i-2)!}$$

If all S_i lie in the same stratum then we have the further stratum interchange symmetry

$$S_1\to S_2\to\dots S_p\to$$

and we need to further divide by p in this case.

Thus applying our formula with M as above, we have

$$c=\frac{M}{2^{p-1}(4g-5)!\operatorname{Vol}(\mathcal{H}_1(\alpha))}\prod_{i=1}^{p}(4g_i-2)!\operatorname{Vol}(\mathcal{H}_1(\alpha_i))$$

To complete this section we present some examples of these computations.

Example 11.1. **Stratum $\boldsymbol{\mathcal{H}(1,1,1,1)}$.** In the stratum $\mathcal{H}(1,1,1,1)$ in genus 3, the only higher multiplicity is $p=2$ so S_1', S_2' have genus 1. We have the stratum interchange symmetry factor of $|\Gamma|=2$; as in all these cases we also have the $\gamma\to-\gamma$ symmetry, $|\Gamma_-|=2$. Hence we have the combinatorial factor

$$M=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4!}{0!\cdot 0!}=6$$

Thus

$$c=6\cdot\frac{1}{2^{2-1}}\cdot\frac{2!\cdot 2!}{7!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^2}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1))}=\frac{3}{14}\cdot\frac{1}{\zeta(2)}\approx 0.1303$$

Example 11.2. **Stratum $\boldsymbol{\mathcal{H}(1,1,1,1,1,1)}$.** Now $g=4$, and we have two different nontrivial partitions of $g-1=3$ arranged in a cyclic order: in multiplicity 2, a surface of genus 2 and a surface of genus 1 and in multiplicity 3 yielding three tori. In the first case we do not have the stratum interchange symmetry, $|\Gamma|=1$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{6!}{0!\cdot 2!}$$

Thus

$$c=\frac{6!}{4}\cdot\frac{1}{2^{2-1}}\cdot\frac{2!\cdot 6!}{11!}\cdot\frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1))}=\frac{90}{377}\cdot\frac{1}{\zeta(2)}\approx 0.1451$$

In the second case we have the stratum interchange symmetry, $|\Gamma|=3$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{6!}{0!\cdot 0!\cdot 0!}=5!$$

Thus

$$c=5!\cdot\frac{1}{2^{3-1}}\cdot\frac{2!\cdot 2!\cdot 2!}{11!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^3}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1))}=\frac{5}{754}\cdot\frac{1}{\zeta(2)}\approx 0.004031$$

Example 11.3. **Stratum $\mathcal{H}(1,1,1,1,1,1,1,1)$.** Proceeding to genus 5 we get the following nontrivial partitions of $g-1=4$ arranged in a cyclic order: in multiplicity 2 giving a surface of genus 1 and one of genus 3; in multiplicity 2 yielding two surfaces of genus 2, in multiplicity 3 yielding two tori and a genus 2 surface and in multiplicity 4 yielding 4 tori.

In the first case we do not have any stratum interchange symmetry, $|\Gamma|=1$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{8!}{0!\cdot 4!}$$

Thus

$$c=\frac{8!}{2\cdot 4!}\cdot\frac{1}{2^{2-1}}\cdot\frac{2!\cdot 10!}{15!}\cdot\frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))\operatorname{Vol}(\mathcal{H}_1(1,1,1,1))}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1,1,1))}=\frac{4200}{23357}\cdot\frac{1}{\zeta(2)}\approx 0.1093$$

For the other multiplicity 2 case $\alpha_1'=\alpha_2'=(1,1)$. We have the stratum interchange symmetry, $|\Gamma|=2$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{8!}{2!\cdot 2!}$$

Thus

$$c=\frac{8!}{16}\cdot\frac{1}{2^{2-1}}\cdot\frac{6!\cdot 6!}{15!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(1,1))\big)^2}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1,1,1))}=\frac{720}{23357}\cdot\frac{1}{\zeta(2)}\approx 0.01874$$

For the third case we do not have any stratum interchange symmetry, $|\Gamma|=1$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{8!}{0!\cdot 0!\cdot 2!}$$

Thus

$$c=\frac{8!}{4}\cdot\frac{1}{2^{3-1}}\cdot\frac{2!\cdot 2!\cdot 6!}{15!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^2\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1,1,1))}=\frac{120}{23357}\cdot\frac{1}{\zeta(2)}\approx 0.003123$$

In the last case we have stratum a interchange symmetry, $|\Gamma|=4$. Hence we get the following combinatorial factor

$$M=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{8!}{0!\cdot 0!\cdot 0!\cdot 0!}=7!$$

Thus

$$c=7!\cdot\frac{1}{2^{4-1}}\cdot\frac{2!\cdot 2!\cdot 2!\cdot 2!}{15!}\cdot\frac{\big(\operatorname{Vol}(\mathcal{H}_1(\emptyset))\big)^4}{\operatorname{Vol}(\mathcal{H}_1(1,1,1,1,1,1,1,1))}=\frac{5}{46714}\cdot\frac{1}{\zeta(2)}\approx 0.00006507$$

12. Strata that are not Connected

12.1. **Parity of the Spin Structure of the Compound Surface.** Consider a flat surface S constructed in the previous section. Throughout this section we suppose that all zeroes of ω are of even orders, so the resulting flat surface S has a spin structure. In this section we discuss the conditions on components S'_i , and on the gluing rules between them which allow us to obtain a surface S with even α. We also compute the parity of the spin structure of S. This will allow us to find constants in the case of components with spin structures.

Lemma 12.1. *Let a nondegenerate flat surface S be obtained from a collection S'_i by applying the figure eight and parallelogram constructions. The resulting flat surface S has zeroes of even orders if and only if the following conditions are valid:*

- *All zeroes of those surfaces S'_i to which we apply the figure eight construction have even orders;*
- *If there is at least one surface to which we apply the parallelogram construction, then there is the following additional alternative. Either*
  - *all zeroes of all surfaces are of even order and every newborn zero of S is of type III;*

  or
  - *the orders b'_k, b''_k of all zeroes $z_k \neq w_k$ to which we apply the parallelogram construction are odd while all the other zeroes of any S_i are even.*

Proof. First note that all zeroes inherited by S from S'_i without changes are of even orders. Thus, if we apply to S'_i the figure eight construction, than all the zeroes of S'_i different from z_i must have even orders. Since the total sum of orders equals $2g_i + 2$, it means that a_i — the order of z_i — is even as well.

Suppose now that there is a surface S'_k to which we apply the parallelogram construction with at least one of b'_k, b''_k even. Without loss of generality we may assume that b'_k is even. By the same arguments as above all zeroes of S'_k different from z_k and w_k have even orders. Since the total sum $2g_k + 2$ of all orders is even, the remaining zero has even order b''_k as well.

Consider the newborn zero of S induced from w'_k. It is of one of the types II or III. If it were of type II, then by (21) it would have odd order. Thus this newborn zero is of the type III. Let S'_{k+s+1} be the surface, to which we apply parallelogram construction, at the opposite end of the chain from S'_k. By (22) b'_{k+s+1} is even. Repeating our arguments we show that all zeroes of all S'_i are even, and that all newborn zeroes are of the type III.

The remaining part of the alternative now becomes obvious. □

Let us calculate the parity of the spin structure of the resulting surface S in all possible cases described above.

Lemma 12.2. *Suppose that we use only figure eight constructions to glue a nondegenerate surface S. Suppose that all α_i are even. The parity of the spin structure of the resulting surface S is defined by the parities of the spin structures of the components, and by the parities of the numbers a'_i in the following way:*

$$\phi(S) = 1 + \sum_{i=1}^{p} \phi(S'_i) + \sum (a'_i + 1) \quad (mod\ 2)$$

Proof. For each surface S_i consider a collection of $2g_i$ smooth simple closed curves on S_i' representing a canonical basis of cycles. As usual we may assume that the curves stay outside a neighborhood of the zeroes and the marked points of S_i'.

We may construct now the following canonical basis of cycles on the resulting surface S. Let g be the genus of S. First take all the cycles represented by the basic curves on components. They give us $g-1$ pairs of cycles. By construction they form incomplete canonical basis of cycles on S.

Since we are using only figure eight constructions we have to use at least one cylinder to obtain a nondegenerate surface S. As a cycle A_g we may use the cycle γ represented by the waist curve of this cylinder. By construction it is independent from the previous ones, and it does not intersect them. To complete the construction of the basis we have to chose a cycle B_g dual to A_g. Consider the following curve representing B_g:
— On any cylinder it follows the direction transversal to γ.
— On the surface S_i' the curve starts at the boundary component γ_i'', then follows in the clockwise direction an arc in a small neighborhood of z_i, and finally arrives at the boundary component γ_i' in direction transversal to γ.

The canonical basis of cycles of S is constructed. By construction of this basis the index of any basic curve A_i, B_i, $i=1,\dots,g-1$ in the flat structure S does not depend on the way in which we applied the prescribed combination of figure eight constructions. Thus we have

$$\sum_{k=1}^{g-1}\big(ind(A_k)+1\big)\big(ind(B_k)+1\big)=\sum_{i=1}^{p}\sum_{j_i}\big(ind(A_{j_i})+1\big)\big(ind(B_{j_i})+1\big)=\sum_{i=1}^{p}\phi(S_i)$$

Recall that the cycle A_g is represented by the waist curve of a cylinder. The tangent vector to the waist curve of the cylinder is constant in the flat structure ω. Thus $ind(A_g)=0$, which implies that

$$\begin{aligned}\phi(S)=\sum_{k=1}^{g}\big(ind(A_k)+1\big)\big(ind(B_k)+1\big)=&\\ =1\cdot(ind(B_g)+1)+\sum_{k=1}^{g-1}\big(ind(A_k)+1\big)\big(ind(B_k)+1\big)=&\\ =ind(B_g)+1+\sum_{i=1}^{p}\phi(S_i)\ (mod\ 2)&\end{aligned}$$

It remains to compute the index $ind(B_g)$. By the construction of B_g, we may assume that the tangent vector to the corresponding curve does not turn in the flat structure ω while it crosses the cylinders. It makes a turn by the angle $2\pi(a_i'+1)$ while it follows a small arc joining two holes of a "figure eight" at the point $z_i=w_i$. This shows that $ind(B_g)=\sum(a_i'+1)\ (mod\ 2)$. □

Lemma 12.3. *Suppose α is even so S has a spin structure. If there is some S_k' to which we apply the parallelogram construction and at least one of the points z_k, w_k is a zero (marked point) of even order, then all α_i' are even, and the parity of the spin structure of the flat surface S is equal to the sum of the parities of the spin* *structures of the components*

$$\phi(S)=\sum_{i=1}^{p}\phi(S_i)$$

Proof. We use the canonical basis on S similar to the one used in the Lemma above. To construct the first $g-1$ pairs of curves we again use the curves living on the component surfaces S'_i. In addition to previous assumptions we require that when we apply parallelogram construction to some S_i, the basic curves do not approach the transversal ρ_i chosen to join z_i to w_i. We extend the description of the curve representing the cycle B_g to the surfaces to which we apply the parallelogram construction as follows: on such surfaces S'_i the curve starts at the boundary component γ''_i in a direction transversal to γ and arrives at the boundary component γ'_i in direction transversal to γ. We chose this curve in such way that it does not intersect any basic curves on S'_i.

It follows from Lemma 12.1 that all the zeroes are of type III; we do not have any glued in cylinders: all the gluings are direct. As a curve representing the cycle A_g we now use a small smooth deformation $\tilde\gamma\subset S'_k$ of the boundary curve γ'_k, such that $\tilde\gamma$ does not pass through any zeroes of ω'_k. We get $ind(A_g)=ind(\tilde\gamma)=(b'_k+1)$. Since the number b'_k is even, we see that $ind(A_g)+1=0\ (mod\ 2)$. Thus the impact of the last pair of cycles to the sum

$$\phi(S)=\sum_{k=1}^{g}\big(ind(A_k)+1\big)\big(ind(B_k)+1\big)$$

equals to zero, and

$$\phi(S)=\sum_{k=1}^{g-1}\big(ind(A_k)+1\big)\big(ind(B_k)+1\big)=$$

$$=\sum_{i=1}^{p}\sum_{j_i}\big(ind(A_{j_i})+1\big)\big(ind(B_{j_i})+1\big)=\sum_{i=1}^{p}\phi(S_i)$$

Lemma 12.3 is proved. □

Note that there is a discrete freedom left in the figure eight or in the parallelogram construction. When we perform a figure eight construction at a zero of order $a=a'+a''$ with fixed a',a'' there are actually $a+1$ ways to perform a figure eight construction, (see section 10.1). Similarly, when we perform a parallelogram construction at a pair of zeroes of orders b',b'' we, actually obtain $(b'_k+1)\cdot(b''_k+1)$ different surfaces. Thus, even when we fix all the elements of the collection

$$(S',J,a'_j,a''_j,b'_k,b''_k)$$

we usually get numerous surfaces S defined by such assignment though some freedom may be cancelled by the symmetry. The collection α of degrees of zeros of the resulting flat structure S is, of course, invariant, but a spin structure may vary.

Lemma 12.4. *Let N be the total number of surfaces $S\in\mathcal{H}(\alpha)$ obtained by the assignment*

$$(S',J,a'_j,a''_j,b'_k,b''_k,\gamma,\rho,h,t)\to S$$

with fixed $(S',J,a'_j,a''_j,b'_k,b''_k,\gamma,\rho,h,t)$. Suppose α is even so every such S has a spin structure. If there is some S'_k to which we apply the parallelogram construction, *and one of the points z_k, w_k is an odd order zero, then exactly $N/2$ of the resulting surfaces have even spin structure and $N/2$ have odd spin structure.*

Proof. Note that if one of b'_k, b''_k is odd, the other one is odd as well.

Consider a canonical basis of cycles similar to one constructed in Lemma 12.3. The only difference is that now we may have some cylinders glued in. As in Lemma 12.2 we assume that the curve representing the cycle B_g stays transversal to the direction of γ while passing through the cylinders.

By the choice of this canonical basis of cycles the index of any basic curve representing A_i, B_i, $i = 1, \dots, g-1$ in the flat structure S does not depend on the way in which we applied the prescribed combination of figure eight and parallelogram constructions. Thus the number

$$\phi_0 := \sum_{i=1}^{g-1}(ind(A_i)+1)(ind(B_i)+1)$$

is invariant of the discrete freedom in the assignment

$$(S', J, a'_j, a''_j, b'_k, b''_k, \gamma, \rho, h, t) \to S$$

We construct a representative of the cycle A_g as in Lemma 12.3 using a small smooth deformation $\tilde{\gamma} \subset S'_k$ of the curve γ'_k. We get $ind(A_g) = ind(\tilde{\gamma}) = (b'_k + 1)$. Since the number b'_k is odd, we see that $ind(\tilde{\gamma}) + 1 = 1 \ (mod\ 2)$. Thus the parity of the spin structure of S computed in the constructed canonical basis is represented as follows:

$$\phi(S) = \phi_0 + (ind(A_g)+1)(ind(B_g)+1) = \phi_0 + ind(B_g) + 1 \ (mod\ 2) \tag{24}$$

Let $z_i \neq w_i$ be odd zero of ω'_i to which we apply parallelogram construction. Since the conical angle at z_i is $2(b''_i + 1)\pi$, there are an even number $r = b''_i + 1$ of distinct points $P_1, \dots, P_r$ of the form $z_i + \gamma$. We number the points in the natural cyclic order. Fix all the other elements of the construction leaving the only freedom in the choice of the point P_j to perform the slit z_iP_j. We get $r = b''_i + 1$ surfaces S. We claim that $r/2$ of them have even parity of the spin structure, while another $r/2$ have odd parity of the spin structure.

To see this compare the surfaces obtained from the slits along z_iP_j and along z_iP_{j+1}. The two surfaces S share the same collection of curves representing the cycles A_i, B_i, $i = 1, \dots, g-1$, and A_g. The curves representing the cycle B_g differ only near the point z_i: the tangent vector to one of the curves makes an extra turn by the angle 2π. Thus by equation (24) these two flat surfaces have opposite parities of the spin structures. □

12.2. **Decomposition of Surfaces from Hyperelliptic Components.** In this section we consider possible decompositions of flat surfaces S from hyperelliptic components corresponding to possible configurations of homologous saddle connections joining a zero to itself. In the next section we compute the constants in the quadratic asymptotics for the number of configurations of each type.

We start with the stratum $\mathcal{H}_1^{hyp}(\alpha)$.

Lemma 12.5. *Surfaces S in the hyperelliptic component $\mathcal{H}_1^{hyp}(\alpha), g \geq 3$ are obtained by an assignment*

$$(S', J, a'_j, a''_j, b'_k, b''_k, \gamma, \rho, h, t) \to S$$

of one of the following three types:

i) $(\overline{g-2},\ \overline{g-2}) \rightarrow$

The multiplicity is 1*; the flat surface* S' *belongs to the hyperelliptic component* $\mathcal{H}_1^{hyp}(g-2, g-2)$*; we apply the parallelogram construction gluing the boundaries directly; the points* $z+\gamma$*,* $w+\gamma$ *in the parallelogram construction are chosen to be symmetric with respect to the hyperelliptic involution on* S'*. For fixed data there precisely* $g-1$ *surfaces* S *that can be built.*

ii) $(\overline{(g-2)+(g-2)}) \Rightarrow$

The multiplicity is 1*; the flat surface* S' *belongs to the hyperelliptic component* $\mathcal{H}_1^{hyp}(2g-4)$*; we apply the figure eight construction gluing in a cylinder;* $a' = a'' = g-2$*. There are* $2g-3$ *surfaces* S *that can be built.*

iii) $(\overline{(g_1-1)+(g_1-1)}) \rightarrow (\overline{g_2-1},\ \overline{g_2-1}) \rightarrow$

The multiplicity is 2*; the flat surfaces belong to the hyperelliptic components* $S_1' \in \mathcal{H}^{hyp}(2g_1-2)$*,* $S_2' \in \mathcal{H}^{hyp}(g_2-1, g_2-1)$*, where* $g_1 + g_2 = g-1$*. We apply the figure eight construction with* $a_1' = a_1'' = g_1 - 1$ *to the first surface and the parallelogram construction with* $b_2' = b_2'' = g_2 - 1$ *to the second one gluing both pairs of components directly. The points* $z_2+\gamma$*,* $w_2+\gamma \in S_2'$ *are chosen to be symmetric with respect to the hyperelliptic involution on* S_2'*. There are* $g_1(2g_2-1)$ *surfaces that can be built.*

The surfaces can be related by a stratum interchange or by a $\gamma \to -\gamma$ *symmetry.*

The proof is analogous to the proof of Lemma 8.3.

Decompositions of the flat surfaces from the hyperelliptic connected component $\mathcal{H}_1^{hyp}(g-1, g-1)$ are described by the following Lemma.

Lemma 12.6. *Surfaces* S *in the hyperelliptic component* $\mathcal{H}_1^{hyp}(g-1, g-1)$ *are obtained by an assignment*

$$(S', J, a_j', a_j'', b_k', b_k'', \gamma, \rho, h, t) \to S$$

of one of the following two types.

i) $(\overline{g-2}, \overline{g-2}) \Rightarrow$

The multiplicity is 1*;* $S' \in \mathcal{H}_1^{hyp}(g-2, g-2)$*. We apply the parallelogram construction gluing in a cylinder;* $b_1' = b_1'' = g-2$*. The points* $z+\gamma$*,* $w+\gamma$ *in the parallelogram construction are chosen to be symmetric with respect to the hyperelliptic involution on* S'*. For fixed data there are precisely* $g-1$ *surfaces* S *that can be built.*

ii) $(\overline{(g_1-1)},\ \overline{(g_1-1)}) \rightarrow (\overline{g_2-1}, \overline{g_2-1}) \rightarrow$

The multiplicity is 2*;* $S_i' \in \mathcal{H}^{hyp}(g_i-1, g_i-1)$*. where* $g_1+g_2 = g-1$*. We apply two parallelogram constructions gluing the components directly. The points* $z_i+\gamma$*,* $w_i+\gamma$ *in the parallelogram constructions are chosen to be symmetric with respect to the hyperelliptic involution on each* S_i'*,* $i = 1, 2$*. For the fixed data the number of surfaces* S *that can be built is* $g_1 \cdot g_2$ *for* $g_1 \neq g_2$ *and* $g_1 \cdot g_2/2$ *for* $g_1 = g_2$*.*

Surfaces can be related by a stratum interchange or by a $\gamma \to -\gamma$ *symmetry.*

The proof is analogous to the proof of Lemma 8.3.

12.3. **Constants for the Hyperelliptic Connected Components.** The computation of the constants is based essentially on (23). However, since the flat surfaces in the hyperelliptic components have extra symmetry we have to make necessary adjustments.

12.3.1. *Hyperelliptic Component* $\mathcal{H}_1^{hyp}(2g-2)$. The admissible assignments for this component are described in Lemma 12.5. We always have the $\gamma \to -\gamma$ symmetry for all three assignments. We never have any stratum interchange symmetry.

The only modification which we need to make in equation (23) is as follows. Applying the parallelogram constructions in this case we need to chose the points $z_i+\gamma$, $w_i+\gamma$ to be symmetric with respect to hyperelliptic involution on each S_i (see Lemma 12.5). Note that for this component we always have $b'_k = b''_k$ in any assignment containing a parallelogram construction. Thus we have to replace each factor $(b'_k+1)(b''_k+1)$ by the corresponding factor (b'_k+1). We get the following

Formula 12.1. *For almost all flat surfaces in a hyperelliptic connected component* $\mathcal{H}_1^{hyp}(\alpha)$, $g \geq 3$, *there are only three possible types of configurations of saddle connections joining zero to itself. The constants in the quadratic asymptotics for the number of saddle connections of each type are presented by the following formulae:*

i) Assignment $(\overline{g-2},\ \overline{g-2}) \to$

$$c = \frac{g-1}{2} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(g-2,g-2))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g-2))}$$

ii) Assignment $(\overline{(g-2)+(g-2)}) \Rightarrow$

$$c = \frac{1}{2} \cdot \frac{2g-3}{2g-2} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g-4))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g-2))}$$

iii) Assignment $(\overline{(g_1-1)+(g_1-1)}) \to (\overline{g_2-1},\ \overline{g_2-1}) \to$

$$c = \frac{(2g_1-1)g_2}{4} \cdot \frac{(2g_1-1)!\,(2g_2)!}{(2g-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g_1-2))\operatorname{Vol}(\mathcal{H}_1^{hyp}(g_2-1,g_2-1))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2g-2))}$$

where $g_1+g_2 = g-1$.

Example 12.7. **Stratum $\mathcal{H}(2)$**. This stratum is connected; it coincides with its hyperelliptic component. Flat surfaces from this stratum have a single zero of order 2. It is easy to see that in genus $g=2$ only multiplicity one is realizable.

$\to (\bar{\mathbf{0}},\bar{\mathbf{0}}) \to$

In this case is a single saddle connection returning with angle 3π so that there is no cylinder. After collapsing this saddle connection we get a torus with two marked points to which we apply the parallelogram construction, gluing the circles directly. Here we have

$$c = \frac{2-1}{2} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1(2))} = \frac{10}{3} \cdot \frac{1}{\zeta(2)} \approx 2.026$$

$\Rightarrow (\bar{\mathbf{0}}) \Rightarrow$

The other possibility is that the saddle connection returns at angle π. Then there is a whole cylinder filled with homologous closed geodesics. In particular, there is another saddle connection returning at the angle π homologous to the initial one. After collapsing the cylinder we get a torus with a single marked point. We apply the figure eight construction.

$$c = \frac{1}{4} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1(2))} = \frac{5}{3} \cdot \frac{1}{\zeta(2)} \approx 1.013$$

Example 12.8. **Component $\mathcal{H}^{hyp}(4)$.** In genus $g = 3$ all three possible assignments (see Lemma 12.5) are already admissible.

$\to (\overline{\mathbf{1}}, \overline{\mathbf{1}}) \to$
The first represents the multiplicity 1 case of a saddle connection returning at angle 5π.

$$c = \frac{3-1}{2} \cdot \frac{\mathrm{Vol}(\mathcal{H}(1,1))}{\mathrm{Vol}(\mathcal{H}^{hyp}(4))} = \frac{224}{27} \cdot \frac{1}{\zeta(2)} \approx 5.044$$

$\Rightarrow (\overline{\mathbf{1}+\mathbf{1}}) \Rightarrow$
If there is a cylinder, by symmetry, the spacing of the angles between the boundaries of the cylinders is 4π. Then after degeneration, we get a single zero of order 2 on a surface of genus 2. We apply the figure eight construction with $a_1' = a_1'' = 1$, and glue in a cylinder. We get

$$c = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\mathrm{Vol}(\mathcal{H}(2))}{\mathrm{Vol}(\mathcal{H}^{hyp}(4))} = \frac{7}{2} \cdot \frac{1}{\zeta(2)} \approx 2.128$$

$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \to$
In the multiplicity $p = 2$ case we have two homologous curves, both returning to z_0 at angles 3π. After degeneration we get two tori, one with two marked points, the other with one. We perform the figure eight construction on one torus and the parallelogram construction on the other, gluing the boundaries directly to each other.

$$c = \frac{(2\cdot 1-1)\cdot 1}{4} \cdot \frac{(2\cdot 1-1)!\,(2\cdot 1)!}{(6-1)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}(\emptyset))^2}{\mathrm{Vol}(\mathcal{H}^{hyp}(4))} = \frac{70}{27} \cdot \frac{1}{\zeta(2)} \approx 1.576$$

12.3.2. *Hyperelliptic Component* $\mathcal{H}^{hyp}(g-1, g-1)$. The admissible assignments for this component are described by Lemma 12.6. We always have the $\gamma \to -\gamma$ symmetry for both assignments. We have the stratum interchange symmetry in the second assignment if and only if $g_1 = g_2$.

The only modification which we need to make in the equation (23) is as follows. Applying the parallelogram constructions in this case we need to chose the points $z_i+\gamma$, $w_i+\gamma$ to be symmetric with respect to hyperelliptic involution on each (S_i, ω_i) (see Lemma 12.6). This means that we have to replace each factor $(b_k'+1)(b_k''+1)$ by the corresponding factor $(b_k'+1)$. Thus we get the following

Formula 12.2. *For almost all flat surfaces in the hyperelliptic connected component $\mathcal{H}^{hyp}(g-1, g-1)$, $g \geq 3$, there are only two possible types of configurations of saddle connections joining any of two zeroes to itself. The constants in the quadratic asymptotics for the number of saddle connections of each type are presented by the following formulae:*

i) Assignment $(\overline{g-2}, \overline{g-2}) \Rightarrow$

$$c = \frac{(g-1)}{(2g-1)} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{hyp}(g-2, g-2))}{\mathrm{Vol}(\mathcal{H}_1^{hyp}(g-1, g-1))}$$

ii) Assignment $(\overline{(g_1-1)},\ \overline{(g_1-1)}) \to (\overline{g_2-1}, \overline{g_2-1}) \to$

$$c = \frac{g_1 g_2}{2|\Gamma|} \cdot \frac{(2g_1)!\,(2g_2)!}{(2g-1)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}_1^{hyp}(g_1-1, g_1-1))\,\mathrm{Vol}(\mathcal{H}_1^{hyp}(g_2-1, g_2-1))}{\mathrm{Vol}(\mathcal{H}_1^{hyp}(g-1, g-1))}.$$

where $g_1 + g_2 = g - 1$, *and*

$$|\Gamma| = \begin{cases} 1 & \textit{when } g_1 \neq g_2 \\ 2 & \textit{when } g_1 = g_2 \end{cases}$$

Example 12.9. **Stratum $\mathcal{H}(1,1)$**. In genus $g = 2$ the stratum $\mathcal{H}(1,1)$ is the principal stratum. It is connected, and it coincides with the hyperelliptic component $\mathcal{H}(1,1) = \mathcal{H}^{hyp}(1,1)$.

Since $g = g_1 + g_2 + 1$, and $g_i \geq 1$, we see that multiplicity two does not occur in genus $g = 2$. The value

$$c = \frac{1}{3} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(1,1))} = \frac{5}{2}$$

of the constant in multiplicity one was computed in section 11.3.

Example 12.10. **Component $\mathcal{H}^{hyp}(2,2)$**.

$\Rightarrow (\overline{1}, \overline{1}) \Rightarrow$

In the multiplicity 1 case there is a cylinder returning to the other zero. Here we get

$$c = \frac{2}{5} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2,2))} = \frac{14}{3} \cdot \frac{1}{\zeta(2)} \approx 2.837$$

$\rightarrow (\overline{0}, \overline{0}) \rightarrow (\overline{0}, \overline{0}) \rightarrow$

In the multiplicity 2 case we have homologous curves, neither of which bounds a cylinder, one returning to each zero. The degenerating surfaces are tori each with 2 marked points on each. We perform the parallelogram construction, gluing the circles pairwise directly. Since $g_1 = g_2$ we get not only the $\gamma \to -\gamma$ but also the stratum interchange, $|\Gamma| = 2$.

$$c = \frac{1 \cdot 1}{2 \cdot 2} \cdot \frac{(2 \cdot 1)!\,(2 \cdot 1)!}{(2 \cdot 3 - 1)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))^2}{\operatorname{Vol}(\mathcal{H}_1^{hyp}(2,2))} = \frac{35}{24} \cdot \frac{1}{\zeta(2)} \approx 0.8866$$

12.4. **Connected Component** $\mathcal{H}^{nonhyp}(g-1, g-1)$**; Even Genus** g**.** When genus $g \geq 4$ is even, the stratum $\mathcal{H}(g-1, g-1)$ has two connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(g-1, g-1)$, which we considered in the previous section, and the nonhyperelliptic connected component $\mathcal{H}^{nonhyp}(g-1, g-1)$. The description of admissible constructions for the connected component $\mathcal{H}^{nonhyp}(g-1, g-1)$ for even $g \geq 4$ is completely analogous to the case of connected strata. However, in the following two assignments

$$(\overline{g-2}, \overline{g-2}) \Rightarrow$$

and

$$(\overline{(g_1 - 1)},\ \overline{(g_1 - 1)}) \rightarrow (\overline{g_2 - 1},\ \overline{g_2 - 1}) \rightarrow$$

we have interference with the hyperelliptic components. Thus in these two cases we have to modify the general formula 11.1 by subtracting the correction term corresponding to the constructions leading to hyperelliptic flat surfaces, see formula 12.2.

Formula 12.3. *For almost all flat surfaces in a nonhyperelliptic connected component* $\mathcal{H}^{nonhyp}(g-1, g-1)$ *for even* $g \geq 4$*, the constants in the quadratic asymptotics for the number of saddle connections of each of the following two types are presented by the following formulae:*

i) Assignment $(\overline{g-2}, \overline{g-2}) \Rightarrow$

$$c = \frac{(g-1)^2}{(2g-1)} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(g-2,g-2))}{\operatorname{Vol}(\mathcal{H}_1^{nonhyp}(g-1,g-1))} - \frac{(g-1)}{(2g-1)} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(g-2,g-2))}{\operatorname{Vol}(\mathcal{H}_1^{nonhyp}(g-1,g-1))}$$

ii) Assignment $(\overline{(g_1-1)},\ \overline{(g_1-1)}) \to (\overline{g_2-1},\ \overline{g_2-1}) \to$

$$c = \frac{(g_1)^2(g_2)^2}{2|\Gamma|} \cdot \frac{(2g_1)!\,(2g_2)!}{(2g-1)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(g_1-1,g_1-1))\operatorname{Vol}(\mathcal{H}_1(g_2-1,g_2-1))}{\operatorname{Vol}(\mathcal{H}_1^{nonhyp}(g-1,g-1))} -$$

$$- \frac{g_1 g_2}{2|\Gamma|} \cdot \frac{(2g_1)!\,(2g_2)!}{(2g-1)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^{hyp}(g_1-1,g_1-1))\operatorname{Vol}(\mathcal{H}_1^{hyp}(g_2-1,g_2-1))}{\operatorname{Vol}(\mathcal{H}_1^{nonhyp}(g-1,g-1))}$$

where $g_1 + g_2 = g-1$, *and*

$$|\Gamma| = \begin{cases} 1 & \text{when } g_1 \neq g_2 \\ 2 & \text{when } g_1 = g_2 \end{cases}$$

iii) The constants for all other assignments are given by equation (23), *with* $\alpha = (g-1, g-1)$ *where* $\operatorname{Vol}(\mathcal{H}(g-1,g-1))$ *in the denominator of the rightmost fraction should be replaced by* $\operatorname{Vol}(\mathcal{H}_1^{nonhyp}(g-1,g-1))$.

As an example of this formula we present the list of possible generic degenerations of a flat surface $S \in \mathcal{H}^{nonhyp}(3,3)$ in Appendix B.

Remark 12.11. Note, that one can represent the constant corresponding to the two exceptional assignments in the formula above as

$$c = c_{regular} - c_{hyp} \cdot \frac{\operatorname{Vol}(\mathcal{H}^{hyp}(g-1,g-1))}{\operatorname{Vol}(\mathcal{H}^{nonhyp}(g-1,g-1))}.$$

Here c_{hyp} in the correctional term is the constant for the corresponding assignment for the *hyperelliptic* component $\mathcal{H}^{hyp}(g-1,g-1)$, see Formula 12.2.

12.5. **Nonhyperelliptic Components with a Spin Structure.** Let now α be even and let S belong to one of the nonhyperelliptic components $\mathcal{H}_1^{even}(\alpha)$ or $\mathcal{H}^{odd}(\alpha)$. The admissible assignments for such flat surfaces are described by the general Lemmas 12.2, 12.3, 12.4.

When α is one of the two partitions $(2g-2), (g-1,g-1)$ there are special assignments where there might be interference with hyperelliptic components, see Lemmas 12.5, and 12.6.

In the formula below we use function $\delta(\alpha', \phi')$ which is equal to 1 when the stratum $\mathcal{H}(\alpha')$ contains a hyperelliptic component, and when, moreover, this hyperelliptic component has parity ϕ' of the spin structure. This function is equal to zero otherwise (see (16) for the explicit definition). By convention we let the volume of several "missing" components in small genera and the volume of nonexisting hyperelliptic components be equal to zero (see (17) for the complete list).

Formula 12.4. *Let all the integers in* α *be even, and let* α *be different from any of the partitions* $(2g-2), (g-1,g-1)$.

For almost all flat surfaces in the connected component $\mathcal{H}^{\phi}(\alpha)$ *the constants in the quadratic asymptotics for the number of saddle connections joining some zero to itself are presented by the following formulae:*

— *If the assignment contains only figure eight constructions*

$$
\begin{aligned}
c = \frac{1}{|\Gamma_-|\cdot|\Gamma|} \cdot \prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}\right) \cdot \prod_{\substack{1\le i\le p\\ a_i\in\alpha}} o_i(a_i) \cdot \prod_{1\le i\le p} (a_i+1) \\
\cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!} \cdot \frac{1}{\operatorname{Vol}(\mathcal{H}_1^{\phi}(\alpha))} \cdot \\
\cdot \sum_{\substack{\phi'_1,\dots,\phi'_p\in\{even,\, odd\}\\ (\phi'_1+\dots+\phi'_p)+\\ +(a'_1+\dot{+}a'_p)+\\ +p+1\equiv\phi\,(mod\,2)}} \prod_{i=1}^{p}\Big(\operatorname{Vol}(\mathcal{H}_1^{\phi_i}(\alpha'_i)) + \delta(\alpha'_i,\phi'_i)\operatorname{Vol}(\mathcal{H}_1^{hyp}(\alpha'_i))\Big)
\end{aligned}
$$

— *If the assignment contains at least one parallelogram construction with even* b'_i

$$
\begin{aligned}
c = \frac{1}{|\Gamma_-|} \cdot \frac{1}{|\Gamma|} \cdot \prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}\right) \cdot \\
\cdot \prod_{\substack{1\le i\le p\\ z_i=w_i\\ a_i\in\alpha}} o_i(a_i) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k\ne b''_k\\ b'_k\in\alpha}} o_k(b'_k) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k\ne b''_k\\ b''_k\in\alpha}} o_k(b''_k) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k=b''_k\in\alpha}} o_k(b'_k)(o_k(b'_k)-1) \cdot \\
\cdot \prod_{\substack{1\le i\le p\\ z_i=w_i}} (a_i+1) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k}} (b'_k+1)(b''_k+1) \cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!} \cdot \\
\cdot \frac{1}{\operatorname{Vol}(\mathcal{H}_1^{\phi}(\alpha))} \cdot \sum_{\substack{\phi'_1,\dots,\phi'_p\in\{even,\, odd\}\\ (\phi'_1+\dots+\phi'_p)\equiv\phi\,(mod\,2)}} \prod_{i=1}^{p}\Big(\operatorname{Vol}(\mathcal{H}_1^{\phi_i}(\alpha'_i)) + \delta(\alpha'_i,\phi'_i)\operatorname{Vol}(\mathcal{H}_1^{hyp}(\alpha'_i))\Big)
\end{aligned}
$$

— *If the assignment contains at least one parallelogram construction with odd* b'_i

$$
\begin{aligned}
c = \frac{1}{|\Gamma_-|} \cdot \frac{1}{|\Gamma|} \cdot \prod_{m\in\alpha}\left(\frac{o(m)!}{\prod_{i=1}^{p} o_i(m)!}\right) \cdot \\
\cdot \prod_{\substack{1\le i\le p\\ z_i=w_i\\ a_i\in\alpha}} o_i(a_i) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k\ne b''_k\\ b'_k\in\alpha}} o_k(b'_k) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k\ne b''_k\\ b''_k\in\alpha}} o_k(b''_k) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k\\ b'_k=b''_k\in\alpha}} o_k(b'_k)(o_k(b'_k)-1) \cdot \\
\cdot \prod_{\substack{1\le i\le p\\ z_i=w_i}} (a_i+1) \cdot \prod_{\substack{1\le k\le p\\ z_k\ne w_k}} (b'_k+1)(b''_k+1) \cdot \\
\cdot \frac{1}{2} \cdot \frac{1}{2^{p-1}} \cdot \frac{\prod_{i=1}^{p}(\frac{d_i}{2}-1)!}{(\frac{d}{2}-2)!} \cdot \frac{\prod_{i=1}^{p}\operatorname{Vol}(\mathcal{H}_1(\alpha'_i))}{\operatorname{Vol}(\mathcal{H}_1^{\phi}(\alpha))}
\end{aligned}
$$

Proof. The formula above is obtained by elementary adjustment of formula (23).

First suppose that the assignment under consideration does not involve any parallelogram constructions with odd zeroes. Then by Lemma 12.2 and Lemma 12.3 the spin structure of S is determined by the spin structures of S_i' and by the parities of a_i'. Thus the combinatorial constant is exactly the same as in equation (23). The numerator of the volume term is the sum over those products of $\mathrm{Vol}(\mathcal{H}^{even}(\alpha_i))$, $\mathrm{Vol}(\mathcal{H}^{odd}(\alpha_j))$, and $\mathrm{Vol}(\mathcal{H}^{hyp}(\alpha_i))$ (when appropriate) that satisfy the relevant relation between spin structures of component surfaces and the spin structure of the resulting surface, see Lemma 12.2 and Lemma 12.3 correspondingly. In the denominator of the volume term we have the volume of the component under consideration.

If there is a parallelogram construction with odd b_k' involved in the assignment, then by Lemma 12.4 we have to multiply c by $1/2$ to account for the fact that half of the constructions produce flat surfaces with even spin structure and another half of the constructions produce a flat surface with odd spin structure. Thus we need to divide by 2 the combinatorial factor in equation (23). In this case the numerator of the volume term is same as in (23): it is a product of $\mathrm{Vol}(\mathcal{H}_1(\alpha_i))$, where we take the total volume of all connected components in the stratum. In the denominator of the volume term we again have the volume of the component under consideration. $\square$

We complete this section considering the exceptional strata.

Formula 12.5. *Let $\alpha = (2g-2)$, or let $\alpha = (g-1, g-1)$ where g is odd. Let $g \geq 4$, or let $g = 3$ and $\phi = odd$.*

For almost all flat surfaces in the connected component $\mathcal{H}^{\phi}(\alpha)$ the constants in the quadratic asymptotics for the number of saddle connections joining some zero to itself are presented by the following formulae:

— If the assignment is different from any of the assignments listed in Lemmas 12.5, and 12.6, then the constant is the same as in Formula 12.4.

— For any of the assignments listed in one of the Lemmas 12.5, and 12.6 the constant is equal to

$$c = c_{regular} - c_{hyp} \cdot \frac{\mathrm{Vol}(\mathcal{H}^{hyp}(\alpha))}{\mathrm{Vol}(\mathcal{H}^{\phi}(\alpha))}.$$

Here $c_{regular}$ is given by Formula 12.4, and c_{hyp} is the constant for the corresponding assignment for the hyperelliptic component $\mathcal{H}^{hyp}(g-1, g-1)$, see Formulae 12.1 and 12.2.

Proof. For nonexceptional assignments the proof is exactly the same as for Formula 12.4. For the exceptional assignments, which may lead to the flat surfaces from hyperelliptic components one makes appropriate adjustment, see the proof of Formula 12.3 and Remark 12.11 after this proof. $\square$

Example 12.12. **Component $\mathcal{H}^{odd}(4)$**. Recall that the stratum $\mathcal{H}(4)$ has only one nonhyperelliptic connected component $\mathcal{H}^{odd}(4)$. Every flat surface in $\mathcal{H}(4)$ having even parity of the spin structure belongs to $\mathcal{H}^{hyp}(4)$. We have $d = \dim_{\mathbb{R}} \mathcal{H}(4) = 12$. Consider possible assignments for the flat surfaces from the component $\mathcal{H}^{odd}(4)$. We start with assignments of multiplicity one.

$\rightarrow (\overline{2}, \overline{0}) \rightarrow$

We have a single saddle connection that does not bound a cylinder. It returns at angle 3π. After degeneration we get a surface of genus 2 with a zero of order 2 and with a marked point. We apply the parallelogram construction gluing the circles directly. Since $b_1' = 2$, and $b_1'' = 0$ are even, by lemma 12.3 the parity of the spin structure of the resulting surface coincides with the parity of the spin structure of a surface from $\mathcal{H}(2)$, so it is really odd, see (7). Since $b_1' \neq b_2''$ we do not have a $\gamma \to -\gamma$ symmetry. Thus,

$$c = 3 \cdot \frac{(5-1)!}{(6-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}(2))}{\mathrm{Vol}(\mathcal{H}^{odd}(4))} = \frac{81}{8} \cdot \frac{1}{\zeta(2)} \approx 6.155$$

$\to (\overline{\mathbf{1}}, \overline{\mathbf{1}}) \to$

Here we again have a single saddle connection that does not bound a cylinder, but now it returns at angle 5π. After degeneration we get a surface of genus 2 with a pair of simple zeroes. We apply the parallelogram construction gluing the circles directly. Since $b_1' = 1$, and $b_1'' = 1$ we get $N = (b_1' + 1)(b_1'' + 1) = 4$ surfaces, but only half of them have odd parity of the spin structure, see lemma 12.4. Now we have a $\gamma \to -\gamma$ symmetry, $|\Gamma_-| = 2$.

$$c = 1 \cdot \frac{(5-1)!}{(6-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}(1,1))}{\mathrm{Vol}(\mathcal{H}^{odd}(4))} = 3 \cdot \frac{1}{\zeta(2)} \approx 1.824$$

$\Rightarrow (\overline{\mathbf{0} + \mathbf{2}}) \Rightarrow$

Suppose now that the saddle connection returns at angle π. Then it bounds a cylinder filled with homologous closed geodesics. In particular, there is another saddle connection returning at the angle π homologous to the initial one. Since the surface is not hyperelliptic, the angle between the saddle connections on the opposite sides of the cylinder is 6π. After degeneration we get a single zero of order 2 on a surface of genus 2. We apply the figure eight construction, breaking the zero of order 2 into a zero of order 2 and one of order 0. This corresponds to spacings of 2π and 6π, or, in our notation to the choice $a_1' = 0$, $a_1'' = 2$. We glue in an intervening cylinder. By lemma 12.2 the parity of the spin structure of the resulting flat surface equals $1 + \phi(S_1') + (0+1) = 1$ since $\phi(S_1') = 1$, see (7). Since $a_1' \neq a_1''$ we do not have the $\gamma \to -\gamma$ symmetry.

$$c = 3 \cdot \frac{(4-1)!}{(6-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}(2))}{\mathrm{Vol}(\mathcal{H}^{odd}(4))} = \frac{81}{32} \cdot \frac{1}{\zeta(2)} \approx 1.539$$

$\Rightarrow (\overline{\mathbf{0} + \mathbf{0}}) \to (\overline{\mathbf{0} + \mathbf{0}}) \Rightarrow$

We now consider multiplicity 2. We have a cylinder and a homologous curve that does not bound a cylinder. The degenerated surfaces S_1', S_2' are a pair of tori, each with a marked point. We apply the figure eight construction giving a pair of circles on each torus. We glue one pair of circles directly and put an intervening cylinder between the other pair. Though $\alpha_1' = \alpha_2'$ we do not have a stratum interchange, $|\Gamma| = 1$, but we have the $\gamma \to -\gamma$ symmetry.

$$c = \frac{1}{2} \cdot \frac{1}{2^{2-1}} \cdot \frac{(2-1)! \cdot (2-1)!}{(6-2)!} \cdot \frac{\mathrm{Vol}(\mathcal{H}(\emptyset))^2}{\mathrm{Vol}(\mathcal{H}^{odd}(4))} = \frac{15}{32} \cdot \frac{1}{\zeta(2)} \approx 0.2850$$

Example 12.13. **Component $\boldsymbol{\mathcal{H}^{odd}(2,2)}$.**

$\to (\mathbf{2}, \overline{\mathbf{0}}, \overline{\mathbf{0}}) \to$

In the multiplicity 1 case with no cylinder, the degenerating surface is genus 2 with a pair of marked points and a zero of order 2. We perform the parallelogram construction on the marked points, gluing the circles directly. Since $b_1' = b_1'' = 0$ is even, by lemma 12.3 the parity of the spin structure equals $\phi(S')$, which is odd for $S' \in \mathcal{H}(2)$.

$$c = 1 \cdot \frac{(4+2-1)!}{(7-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))} = 6 \cdot \frac{1}{\zeta(2)} \approx 3.648$$

$\Rightarrow \mathbf{(2, \overline{0+0})} \Rightarrow$

In the case of a cylinder with multiplicity 1, if the other side returns to the same zero, we get a genus 2 surface with a single marked point and a double zero. We perform the figure eight construction at the marked point, gluing in a cylinder to the two circles. We have $a'_1 = a''_1 = 0$. Thus, by lemma 12.2 $\phi(S) = 1 + \phi(S') + (0+1) = 1 + 1 + 1 = 1 \mod 2$. Now $|\Gamma_|| = 2$.

$$c = 1 \cdot \frac{(4+1-1)!}{(7-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(2))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))} = \frac{6}{5} \cdot \frac{1}{\zeta(2)} \approx 0.7295$$

$\Rightarrow \mathbf{(\bar{1}, \bar{1})} \Rightarrow$

If the other side of the cylinder returns to the other zero, after degeneration we get 2 simple zeroes. We perform the parallelogram construction gluing in a cylinder. Now $b'_1 = b''_1 = 1$ is odd. Thus we get $N = (b'_1 + 1)(b''_1 + 1) = 4$ surfaces, but by lemma 12.4 only $N/2 = 2$ of them have odd parity of the spin structure. There is also the $\gamma \to -\gamma$ symmetry.

$$c = 2 \cdot \frac{(5-1)!}{(7-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(1,1))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))} = \frac{32}{15} \cdot \frac{1}{\zeta(2)} \approx 1.297$$

$\Rightarrow \mathbf{(\overline{0+0})} \Rightarrow \mathbf{(\overline{0+0})} \Rightarrow$

In the multiplicity 2 case we have a pair of cylinders. The two sides of each cylinder return to different zeroes. The result is a pair of tori, each with a single marked point. We perform the figure eight construction creating two pairs of circles to which we glue intervening cylinders. We have $|\Gamma| = |\Gamma_-| = 2$.

$$c = \frac{1}{2} \cdot \frac{1}{2^{2-1}} \cdot \frac{(2-1)!\,(2-1)!}{(7-2)!} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1(\emptyset))\operatorname{Vol}(\mathcal{H}_1(\emptyset))}{\operatorname{Vol}(\mathcal{H}_1^{odd}(2,2))} = \frac{1}{6} \cdot \frac{1}{\zeta(2)} \approx 0.1013$$

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Appendix. Values of the Constants

Appendix A. Saddle Connections Joining Distinct Zeroes

A.1. **Connected Strata in Genus 4.** In the tables below we consider all connected strata in genus $g = 4$. For each stratum we present the complete list of all possible topological configurations of geodesic saddle connections. To make calculations traceable we present for every configuration the orders of possible symmetries Γ and Γ_-, the combinatorial constant M, and the result: the Siegel–Veech constant in quadratic asymptotics for the number of saddle connections (closed geodesics) of that type. The notations were introduced in Section 7.2 and Section 10.3

Stratum $\mathcal{H}(5,1)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$(1+5) \succ$	1	1	7	$\frac{4311167}{373248}$	11.5504
$(0+0) \succ (0+4) \succ$	1	1	5	$\frac{38125}{93312}$	0.408576
$(0+2) \succ (0+2) \succ$	1	2	$\frac{9}{2}$	$\frac{21}{512}$	0.0410156

Stratum $\mathcal{H}(4,1,1)$.

$(1+1,4) \succ$	2	1	3	$\frac{2403}{616}$	3.90097
$(1+4,1) \succ$	1	1	12	$\frac{186624}{9625}$	19.3895
$(0+0) \succ (0+0,4) \succ$	2	1	1	$\frac{61}{616}$	0.099026
$(0+0) \succ (0+3,1) \succ$	1	1	8	$\frac{1024}{1925}$	0.531948
$(0+2) \succ (0+1,1) \succ$	1	1	12	$\frac{108}{1375}$	0.0785455

Stratum $\mathcal{H}(3,2,1)$.

$(1+2,3) \succ$	1	1	4	$\frac{368}{63}$	5.84127
$(1+3,2) \succ$	1	1	5	$\frac{55625}{7168}$	7.76018
$(2+3,1) \succ$	1	1	6	$\frac{81}{7}$	11.5714
$(0+0) \succ (0+2,2) \succ$	1	1	3	$\frac{765}{3584}$	0.213449
$(0+0) \succ (0+1,3) \succ$	1	1	2	$\frac{10}{63}$	0.15873
$(0+0) \succ (1+2,1) \succ$	1	1	4	$\frac{20}{63}$	0.31746
$(0+0,2) \succ (0+2) \succ$	1	1	3	$\frac{27}{1024}$	0.0263672
$(0+2) \succ (1+0,1) \succ$	1	1	6	$\frac{3}{64}$	0.046875
$(1+1) \succ (0+1,1) \succ$	1	1	6	$\frac{3}{64}$	0.046875
$(0+0) \succ (0+0) \succ (0+1,1) \succ$	1	1	2	$\frac{5}{288}$	0.0173611

Configurations of geodesic saddle connections.
Stratum $\mathcal{H}(3,1,1,1)$.

Degeneration pattern	$\|\Gamma_-\|$	$\|\Gamma\|$	M	c	c approx.
$(1+1,3,1)\succ$	2	1	9	$\frac{729}{62}$	11.7581
$(1+3,1,1)\succ$	1	1	15	$\frac{185625}{7936}$	23.3902
$(0+0)\succ(0+0,3,1)\succ$	2	1	3	$\frac{15}{62}$	0.241935
$(0+0)\succ(0+2,1,1)\succ$	1	1	9	$\frac{2025}{3968}$	0.510333
$(0+2)\succ(0+0,1,1)\succ$	1	1	9	$\frac{405}{7936}$	0.0510333
$(0+1,1)\succ(0+1,1)\succ$	1	2	12	$\frac{3}{62}$	0.0483871

Stratum $\mathcal{H}(2,2,1,1)$.

$(1+1,2,2)\succ$	2	1	3	$\frac{4101}{1048}$	3.91317
$(1+2,2,1)\succ$	1	1	16	$\frac{3072}{131}$	23.4504
$(2+2,1,1)\succ$	2	1	5	$\frac{6875}{786}$	8.74682
$(0+0)\succ(0+0,2,2)\succ$	2	1	1	$\frac{85}{1048}$	0.0811069
$(0+0)\succ(0+1,2,1)\succ$	1	1	8	$\frac{200}{393}$	0.508906
$(0+0)\succ(1+1,1,1)\succ$	2	1	3	$\frac{25}{131}$	0.19084
$(0+0,2)\succ(0+0,2)\succ$	2	2	1	$\frac{3}{524}$	0.00572519
$(0+0,2)\succ(0+1,1)\succ$	1	1	8	$\frac{16}{393}$	0.0407125
$(1+1)\succ(0+0,1,1)\succ$	2	1	3	$\frac{5}{262}$	0.019084
$(0+1,1)\succ(1+0,1)\succ$	2	1	8	$\frac{128}{3537}$	0.0361889
$(0+0)\succ(0+0)\succ(0+0,1,1)\succ$	2	1	1	$\frac{25}{3537}$	0.00706814

Stratum $\mathcal{H}(2,1,1,1,1)$.

$(1+1,2,1,1)\succ$	2	1	18	$\frac{1179}{50}$	23.58
$(1+2,1,1,1)\succ$	1	1	16	$\frac{15872}{675}$	23.5141
$(0+0)\succ(0+0,2,1,1)\succ$	2	1	6	$\frac{2}{5}$	0.4
$(0+0)\succ(0+1,1,1,1)\succ$	1	1	8	$\frac{56}{135}$	0.414815
$(0+0,2)\succ(0+0,1,1)\succ$	2	1	6	$\frac{1}{50}$	0.02
$(0+0,1,1)\succ(0+1,1)\succ$	1	1	24	$\frac{16}{225}$	0.0711111

Table 8. Principal stratum $\mathcal{H}(1,\dots,1)$. Approximate values of the constants c for saddle connections of multiplicity two joining a pair of distinct zeroes.

$$\mathcal{H}(\overbrace{1,\dots,1}^{2g_1-2},0+0) \succ \mathcal{H}(\overbrace{1,\dots,1}^{2g_2-2},0+0) \succ \qquad g_1+g_2=g$$

	$g_2=1$	$g_2=2$	$g_2=3$	$g_2=4$	$g_2=5$	$g_2=6$
$g_1=1$	0.6250	0.8571	0.8355	0.8393	0.8426	0.8435
$g_1=2$	0.8571	0.07958	0.07193	0.04160	0.02708	0.01896
$g_1=3$	0.8355	0.07193	0.009358	0.007018	0.003195	0.001651
$g_1=4$	0.8393	0.04160	0.007018	$0.9204\cdot10^{-3}$	$0.6185\cdot10^{-3}$	$0.2454\cdot10^{-3}$
$g_1=5$	0.8426	0.02708	0.003195	$0.6185\cdot10^{-3}$	$0.7978\cdot10^{-4}$	$0.5011\cdot10^{-4}$
$g_1=6$	0.8435	0.01896	0.001651	$0.2454\cdot10^{-3}$	$0.5011\cdot10^{-4}$	$0.6383\cdot10^{-5}$
$g_1=7$	0.8430	0.01398	$0.9350\cdot10^{-3}$	$0.1100\cdot10^{-3}$	$0.1822\cdot10^{-4}$	$0.3840\cdot10^{-5}$
$g_1=8$	0.8418	0.01072	$0.5675\cdot10^{-3}$	$0.5415\cdot10^{-4}$	$0.7420\cdot10^{-5}$	$0.1315\cdot10^{-5}$
$g_1=9$	0.8406	0.008475	$0.3638\cdot10^{-3}$	$0.2872\cdot10^{-4}$	$0.3309\cdot10^{-5}$	$0.5001\cdot10^{-6}$
$g_1=10$	0.8393	0.006864	$0.2438\cdot10^{-3}$	$0.1618\cdot10^{-4}$	$0.1590\cdot10^{-5}$	$0.2072\cdot10^{-6}$
$g_1=11$	0.8382	0.005670	$0.1694\cdot10^{-3}$	$0.9585\cdot10^{-5}$	$0.8122\cdot10^{-6}$	$0.9226\cdot10^{-7}$
$g_1=12$	0.8372	0.004762	$0.1213\cdot10^{-3}$	$0.5921\cdot10^{-5}$	$0.4372\cdot10^{-6}$	$0.4366\cdot10^{-7}$
$g_1=13$	0.8363	0.004056	$0.8909\cdot10^{-4}$	$0.3790\cdot10^{-5}$	$0.2461\cdot10^{-6}$	$0.2177\cdot10^{-7}$
$g_1=14$	0.8354	0.003495	$0.6691\cdot10^{-4}$	$0.2503\cdot10^{-5}$	$0.1440\cdot10^{-6}$	$0.1136\cdot10^{-7}$
$g_1=15$	0.8347	0.003043	$0.5122\cdot10^{-4}$	$0.1697\cdot10^{-5}$	$0.8712\cdot10^{-7}$	$0.6173\cdot10^{-8}$
$g_1=16$	0.8340	0.002673	$0.3986\cdot10^{-4}$	$0.1179\cdot10^{-5}$	$0.5431\cdot10^{-7}$	$0.3474\cdot10^{-8}$
$g_1=17$	0.8334	0.002367	$0.3149\cdot10^{-4}$	$0.8358\cdot10^{-6}$	$0.3476\cdot10^{-7}$	$0.2017\cdot10^{-8}$
$g_1=18$	0.8329	0.002110	$0.2520\cdot10^{-4}$	$0.6038\cdot10^{-6}$	$0.2278\cdot10^{-7}$	$0.1204\cdot10^{-8}$
$g_1=19$	0.8324	0.001893	$0.2041\cdot10^{-4}$	$0.4436\cdot10^{-6}$	$0.1525\cdot10^{-7}$	$0.7378\cdot10^{-9}$
$g_1=20$	0.8319	0.001708	$0.1670\cdot10^{-4}$	$0.3308\cdot10^{-6}$	$0.1041\cdot10^{-7}$	$0.4624\cdot10^{-9}$
$g_1=21$	0.8315	0.001549	$0.1380\cdot10^{-4}$	$0.2501\cdot10^{-6}$	$0.7226\cdot10^{-8}$	$0.2960\cdot10^{-9}$
$g_1=22$	0.8312	0.001411	$0.1150\cdot10^{-4}$	$0.1915\cdot10^{-6}$	$0.5098\cdot10^{-8}$	$0.1931\cdot10^{-9}$
$g_1=23$	0.8308	0.001290	$0.9663\cdot10^{-5}$	$0.1482\cdot10^{-6}$	$0.3650\cdot10^{-8}$	$0.1282\cdot10^{-9}$
$g_1=24$	0.8305	0.001185	$0.8177\cdot10^{-5}$	$0.1160\cdot10^{-6}$	$0.2648\cdot10^{-8}$	$0.8649\cdot10^{-10}$
$g_1=25$	0.8302	0.001091	$0.6966\cdot10^{-5}$	$0.9163\cdot10^{-7}$	$0.1946\cdot10^{-8}$	$0.5923\cdot10^{-10}$
$g_1=26$	0.8299	0.001009	$0.5971\cdot10^{-5}$	$0.7304\cdot10^{-7}$	$0.1446\cdot10^{-8}$	$0.4113\cdot10^{-10}$
$g_1=27$	0.8297	$0.9353\cdot10^{-3}$	$0.5147\cdot10^{-5}$	$0.5870\cdot10^{-7}$	$0.1086\cdot10^{-8}$	$0.2893\cdot10^{-10}$
$g_1=28$	0.8294	$0.8695\cdot10^{-3}$	$0.4461\cdot10^{-5}$	$0.4754\cdot10^{-7}$	$0.8236\cdot10^{-9}$	$0.2060\cdot10^{-10}$
$g_1=29$	0.8292	$0.8104\cdot10^{-3}$	$0.3885\cdot10^{-5}$	$0.3878\cdot10^{-7}$	$0.6305\cdot10^{-9}$	$0.1483\cdot10^{-10}$
$g_1=30$	0.8290	$0.7571\cdot10^{-3}$	$0.3400\cdot10^{-5}$	$0.3184\cdot10^{-7}$	$0.4869\cdot10^{-9}$	$0.1079\cdot10^{-10}$

TABLE 9. Connected component $\mathcal{H}^{hyp}(g-1, g-1)$; approximate values of the constants for saddle connections of multiplicity two joining distinct zeroes; $g_1 + g_2 = g$

$$\mathcal{H}_1^{hyp}\big((\overline{g_1-1})+(\overline{g_1-1})\big) \succ \mathcal{H}_1^{hyp}\big(\overline{(g_2-1)}+\overline{(g_2-1)}\big) \succ$$

	$g_2=1$	$g_2=2$	$g_2=3$	$g_2=4$	$g_2=5$	$g_2=6$	$g_2=7$	$g_2=8$	$g_2=9$	$g_2=10$
$g_1=1$	0.6250	1.9687	2.9297	4.1122	5.5112	7.1245	8.9513	10.991	13.244	15.709
$g_1=2$	1.9687	1.1074	2.7191	3.3679	4.1334	5.0053	5.9792	7.0526	8.2244	9.4935
$g_1=3$	2.9297	2.7191	1.4729	3.3412	3.8411	4.4225	5.0749	5.7932	6.5745	7.4168
$g_1=4$	4.1122	3.3679	3.3412	1.7750	3.8803	4.2917	4.7662	5.2948	5.8725	6.4961
$g_1=5$	5.5112	4.1334	3.8411	3.8803	2.0372	4.3611	4.7134	5.1173	5.5653	6.0526
$g_1=6$	7.1245	5.0053	4.4225	4.2917	4.3611	2.2714	4.7983	5.1083	5.4619	5.8527
$g_1=7$	8.9513	5.9792	5.0749	4.7662	4.7134	4.7983	2.4849	5.2016	5.4798	5.7955
$g_1=8$	10.991	7.0526	5.7932	5.2948	5.1173	5.1083	5.2016	2.6821	5.5776	5.8309
$g_1=9$	13.244	8.2244	6.5745	5.8725	5.5653	5.4619	5.4798	5.5776	2.8663	5.9309
$g_1=10$	15.709	9.4935	7.4168	6.4961	6.0526	5.8527	5.7955	5.8309	5.9309	3.0396
$g_1=11$	18.387	10.860	8.3192	7.1637	6.5763	6.2764	6.1434	6.1170	6.1642	6.2652
$g_1=12$	21.277	12.322	9.2807	7.8740	7.1344	6.7305	6.5198	6.4315	6.4266	6.4821
$g_1=13$	24.379	13.881	10.301	8.6259	7.7255	7.2128	6.9221	6.7710	6.7143	6.7250
$g_1=14$	27.694	15.536	11.379	9.4190	8.3487	7.7223	7.3487	7.1334	7.0243	6.9906
$g_1=15$	31.221	17.288	12.516	10.253	9.0032	8.2577	7.7983	7.5170	7.3547	7.2764
$g_1=16$	34.961	19.135	13.710	11.127	9.6885	8.8184	8.2697	7.9205	7.7040	7.5806
$g_1=17$	38.913	21.078	14.962	12.041	10.404	9.4038	8.7624	8.3431	8.0710	7.9019
$g_1=18$	43.077	23.116	16.271	12.994	11.150	10.014	9.2757	8.7839	8.4548	8.2390
$g_1=19$	47.453	25.251	17.638	13.987	11.926	10.647	9.8091	9.2425	8.8547	8.5913
$g_1=20$	52.042	27.481	19.062	15.020	12.731	11.304	10.362	9.7183	9.2702	8.9580
$g_1=21$	56.842	29.807	20.543	16.092	13.566	11.985	10.935	10.211	9.7007	9.3385
$g_1=22$	61.856	32.229	22.081	17.203	14.430	12.689	11.527	10.720	10.146	9.7325
$g_1=23$	67.081	34.746	23.677	18.354	15.323	13.416	12.138	11.246	10.606	10.140
$g_1=24$	72.518	37.359	25.329	19.544	16.245	14.166	12.768	11.788	11.079	10.559
$g_1=25$	78.168	40.067	27.039	20.772	17.197	14.939	13.417	12.345	11.567	10.992
$g_1=26$	84.030	42.871	28.806	22.040	18.177	15.735	14.085	12.919	12.069	11.436
$g_1=27$	90.104	45.770	30.630	23.347	19.186	16.553	14.771	13.508	12.584	11.893
$g_1=28$	96.391	48.766	32.510	24.692	20.224	17.394	15.475	14.113	13.113	12.361
$g_1=29$	102.89	51.856	34.448	26.077	21.291	18.257	16.198	14.733	13.655	12.842
$g_1=30$	109.60	55.042	36.443	27.500	22.387	19.143	16.939	15.369	14.210	13.334

Appendix B. Saddle Connections Joining a Zero to Itself

B.1. **Connected Strata in Genus 4.** In the tables below we consider all connected strata in genus $g = 4$. For each stratum we present the complete list of admissible constructions (= all possible topological configurations of homologous closed geodesics). To make calculations traceable we present for every configuration the orders of possible symmetries Γ and Γ_-, the combinatorial constant M, and the result: the constant in quadratic asymptotics for the number of saddle connections (closed geodesics) of this type.

Stratum $\mathcal{H}(5,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{4}}) \Rightarrow$	1	1	5	$\frac{38125}{15552}$	2.45145
$\rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{3}}; 1) \rightarrow$	1	1	4	$\frac{2240}{243}$	9.21811
$\Rightarrow (\overline{\mathbf{0}+\mathbf{3}}; 1) \Rightarrow$	1	1	4	$\frac{320}{243}$	1.31687
$\Rightarrow (\overline{\mathbf{1}+\mathbf{2}}; 1) \Rightarrow$	1	1	4	$\frac{320}{243}$	1.31687
$\rightarrow (\overline{\mathbf{1}}, \overline{\mathbf{2}}; 1) \rightarrow$	1	1	6	$\frac{175}{18}$	9.72222
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{2}}) \rightarrow$	1	1	3	$\frac{35}{288}$	0.121528
$\rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{2}}) \rightarrow$	1	1	3	$\frac{35}{576}$	0.0607639
$\rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}+\mathbf{1}}) \rightarrow$	1	1	3	$\frac{35}{576}$	0.0607639
$\rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{2}+\mathbf{0}}) \rightarrow$	1	1	3	$\frac{35}{576}$	0.0607639
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{1}}; 1) \rightarrow$	1	1	2	$\frac{175}{486}$	0.360082
$\rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}+\mathbf{1}}; 1) \rightarrow$	1	1	2	$\frac{35}{243}$	0.144033
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{1}}; 1) \rightarrow$	1	1	2	$\frac{35}{486}$	0.0720165
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}+\mathbf{0}}; 1) \rightarrow$	1	1	2	$\frac{35}{486}$	0.0720165
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \rightarrow$	1	1	1	$\frac{175}{7776}$	0.0225051

Stratum $\mathcal{H}(4,1,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};4)\Rightarrow$	2	1	1	$\frac{61}{88}$	0.693182
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{3}};1)\Rightarrow$	1	1	8	$\frac{1024}{275}$	3.72364
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{2}};1,1)\rightarrow$	1	1	3	$\frac{432}{55}$	7.85455
$\Rightarrow(\overline{\mathbf{0}+\mathbf{2}};1,1)\Rightarrow$	1	1	3	$\frac{54}{55}$	0.981818
$\Rightarrow(\overline{\mathbf{1}+\mathbf{1}};1,1)\Rightarrow$	2	1	$\frac{3}{2}$	$\frac{27}{55}$	0.490909
$\rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}};1,1)\rightarrow$	2	1	2	$\frac{224}{55}$	4.07273
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{2}})\rightarrow$	1	1	6	$\frac{27}{275}$	0.0981818
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{2}})\Rightarrow$	1	1	6	$\frac{27}{1100}$	0.0245455
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{1}+\mathbf{1}})\Rightarrow$	2	1	3	$\frac{27}{2200}$	0.0122727
$\rightarrow(\overline{\mathbf{0}+\mathbf{0}})\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};1,1)\rightarrow$	2	1	$\frac{1}{2}$	$\frac{6}{55}$	0.109091
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow(\overline{\mathbf{0}+\mathbf{0}};1,1)\rightarrow$	2	1	$\frac{1}{2}$	$\frac{2}{55}$	0.0363636
$\rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};1,1)\rightarrow$	1	1	1	$\frac{2}{55}$	0.0363636
$\rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{1}};1)\rightarrow$	1	1	4	$\frac{8}{55}$	0.145455
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{1}};1)\rightarrow$	1	1	4	$\frac{16}{275}$	0.0581818
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{1}+\mathbf{0}};1)\rightarrow$	1	1	4	$\frac{16}{275}$	0.0581818
$\rightarrow(\overline{\mathbf{0}+\mathbf{0}})\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow$	2	1	1	$\frac{1}{110}$	0.00909091
$\rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\rightarrow$	2	1	1	$\frac{1}{220}$	0.00454545

Stratum $\mathcal{H}(3,2,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{2}};2) \Rightarrow$	1	1	3	$\frac{765}{512}$	1.49414
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}};3,1) \rightarrow$	2	1	$\frac{1}{2}$	$\frac{20}{9}$	2.22222
$\Rightarrow (\overline{\mathbf{0}+\mathbf{0}};3,1) \Rightarrow$	2	1	$\frac{1}{2}$	$\frac{5}{18}$	0.277778
$\Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{1}};3) \Rightarrow$	1	1	2	$\frac{10}{9}$	1.11111
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{1}};2,1) \rightarrow$	1	1	2	$\frac{25}{4}$	6.25
$\Rightarrow (\overline{\mathbf{0}+\mathbf{1}};2,1) \Rightarrow$	1	1	2	$\frac{25}{32}$	0.78125
$\Rightarrow (\overline{\mathbf{1}},\overline{\mathbf{2}};1) \Rightarrow$	1	1	6	$\frac{75}{32}$	2.34375
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}};2) \rightarrow$	1	1	1	$\frac{25}{512}$	0.0488281
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{0}};2) \rightarrow$	1	1	1	$\frac{5}{256}$	0.0195313
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{2}},\overline{\mathbf{0}}) \rightarrow$	1	1	3	$\frac{15}{256}$	0.0585938
$\Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{2}}) \Rightarrow$	1	1	3	$\frac{15}{512}$	0.0292969
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}},\overline{\mathbf{1}};1) \rightarrow$	1	1	2	$\frac{25}{144}$	0.173611
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}},\overline{\mathbf{0}};1) \rightarrow$	1	1	2	$\frac{25}{288}$	0.0868056
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}},\overline{\mathbf{1}}) \rightarrow$	1	1	4	$\frac{5}{72}$	0.0694444
$\Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{1}};1) \Rightarrow$	1	1	2	$\frac{5}{288}$	0.0173611
$\rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \rightarrow$	1	1	1	$\frac{25}{2304}$	0.0108507
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}},\overline{\mathbf{0}}) \rightarrow$	1	1	1	$\frac{25}{4608}$	0.00542535

Stratum $\mathcal{H}(3,1,1,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\|\Gamma_-\|$	$\|\Gamma\|$	M	c	c approx.
$\Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}};3,1) \Rightarrow$	2	1	3	$\frac{60}{31}$	1.93548
$\Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{2}};1,1) \Rightarrow$	1	1	9	$\frac{2025}{496}$	4.08266
$\rightarrow (\bar{\mathbf{0}},\bar{\mathbf{1}};1,1,1) \rightarrow$	1	1	2	$\frac{1575}{248}$	6.35081
$\Rightarrow (\overline{\mathbf{0}+\mathbf{1}};1,1,1) \Rightarrow$	1	1	2	$\frac{175}{248}$	0.705645
$\Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{2}}) \Rightarrow$	1	1	18	$\frac{405}{7936}$	0.0510333
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}};1,1) \rightarrow$	1	1	3	$\frac{225}{1984}$	0.113407
$\rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{0}};1,1) \rightarrow$	1	1	3	$\frac{75}{1984}$	0.0378024
$\rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{1}};1) \rightarrow$	1	1	12	$\frac{75}{496}$	0.15121
$\Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{1}};1) \Rightarrow$	1	1	12	$\frac{15}{496}$	0.0302419
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \Rightarrow (\bar{\mathbf{0}},\bar{\mathbf{0}}) \rightarrow$	1	1	6	$\frac{75}{7936}$	0.0094506

Stratum $\mathcal{H}(2,2,1,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};2,2)\Rightarrow$	2	1	1	$\frac{85}{131}$	0.648855
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};2,1,1)\rightarrow$	2	1	1	$\frac{600}{131}$	4.58015
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2,1,1)\Rightarrow$	2	1	1	$\frac{200}{393}$	0.508906
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{1}};2,1)\Rightarrow$	1	1	8	$\frac{1600}{393}$	4.07125
$\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}};1,1)\Rightarrow$	2	1	4	$\frac{5600}{3537}$	1.58326
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};2)\rightarrow$	1	1	4	$\frac{25}{393}$	0.0636132
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};2)\Rightarrow$	2	1	2	$\frac{25}{1572}$	0.0159033
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2)\Rightarrow$	2	1	2	$\frac{5}{786}$	0.00636132
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}};1,1)\rightarrow$	2	1	1	$\frac{100}{1179}$	0.0848176
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{0}};1)\rightarrow$	1	1	8	$\frac{400}{3537}$	0.11309
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};1,1)\Rightarrow$	2	1	1	$\frac{25}{3537}$	0.00706814
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{1}};1)\Rightarrow$	1	1	8	$\frac{200}{3537}$	0.0565451
$\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}})\Rightarrow$	2	1	8	$\frac{80}{3537}$	0.022618
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow$	2	1	2	$\frac{25}{3537}$	0.00706814
$\rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\rightarrow$	2	1	2	$\frac{25}{7074}$	0.00353407
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}},\overline{\mathbf{0}})\Rightarrow$	2	1	2	$\frac{25}{14148}$	0.00176703

Stratum $\mathcal{H}(2,1,1,1,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};2,1,1)\Rightarrow$	2	1	6	$\frac{18}{5}$	3.6
$\rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};1,1,1,1)\rightarrow$	2	1	$\frac{1}{2}$	$\frac{7}{3}$	2.33333
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};1,1,1,1)\Rightarrow$	2	1	$\frac{1}{2}$	$\frac{7}{30}$	0.233333
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{1}};1,1,1)\Rightarrow$	1	1	8	$\frac{56}{15}$	3.73333
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};2)\Rightarrow$	2	1	12	$\frac{1}{40}$	0.025
$\rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};1,1)\rightarrow$	1	1	12	$\frac{2}{15}$	0.133333
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};1,1)\Rightarrow$	2	1	6	$\frac{1}{30}$	0.0333333
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};1,1)\Rightarrow$	2	1	6	$\frac{1}{90}$	0.0111111
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{1}};1)\Rightarrow$	1	1	48	$\frac{4}{45}$	0.0888889
$\rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\rightarrow$	2	1	12	$\frac{1}{180}$	0.00555556
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow$	2	1	12	$\frac{1}{360}$	0.00277778

Stratum $\mathcal{H}(1,1,1,1,1,1)$.
Configurations of closed geodesics.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	M	c	c approx.
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};1,1,1,1)\Rightarrow$	2	1	15	$\frac{3150}{377}$	8.35544
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}};1,1)\Rightarrow$	2	1	180	$\frac{90}{377}$	0.238727
$\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow(\bar{\mathbf{0}},\bar{\mathbf{0}})\Rightarrow$	2	3	120	$\frac{5}{754}$	0.0066313

B.2. **Closed Geodesics for Nonconnected Strata in Genus 4**

Component $\mathcal{H}^{hyp}(6)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	$c \cdot \zeta(2)$	$c \cdot \zeta(2)$ approx.
$\to (\overline{\mathbf{2}}, \overline{\mathbf{2}}) \to$	2	1	$\frac{384}{25}$	15.36
$\Rightarrow (\overline{\mathbf{2}+\mathbf{2}}) \Rightarrow$	2	1	$\frac{6}{1}$	6.
$\to (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \to (\overline{\mathbf{1}+\mathbf{1}}) \to$	2	1	$\frac{84}{25}$	3.36
$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{1}}, \overline{\mathbf{1}}) \to$	2	1	$\frac{896}{225}$	3.98222

Component $\mathcal{H}^{even}(6)$.

$\to (\overline{\mathbf{0}}, \overline{\mathbf{4}}) \to$	1	1	$\frac{30375}{4096}$	7.41577
$\Rightarrow (\overline{\mathbf{0}+\mathbf{4}}) \Rightarrow$	1	1	$\frac{10125}{8192}$	1.23596
$\Rightarrow (\overline{\mathbf{1}+\mathbf{3}}) \Rightarrow$	1	1	$\frac{875}{256}$	3.41797
$\to (\overline{\mathbf{2}}, \overline{\mathbf{2}}) \to$	2	1	$\frac{405}{128}$	3.16406
$\to (\overline{\mathbf{1}}, \overline{\mathbf{3}}) \to$	1	1	$\frac{15}{1}$	15.
$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{0}}, \overline{\mathbf{2}}) \to$	1	1	$\frac{2835}{2048}$	1.38428
$\to (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \to (\overline{\mathbf{0}+\mathbf{2}}) \to$	1	1	$\frac{2835}{4096}$	0.692139
$\to (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}+\mathbf{1}}) \to$	1	1	$\frac{2835}{8192}$	0.346069

Component $\mathcal{H}^{odd}(6)$.

$\to (\overline{\mathbf{0}}, \overline{\mathbf{4}}) \to$	1	1	$\frac{350}{27}$	12.963
$\Rightarrow (\overline{\mathbf{0}+\mathbf{4}}) \Rightarrow$	1	1	$\frac{175}{81}$	2.16049
$\Rightarrow (\overline{\mathbf{1}+\mathbf{3}}) \Rightarrow$	1	1	$\frac{25}{32}$	0.78125
$\Rightarrow (\overline{\mathbf{2}+\mathbf{2}}) \Rightarrow$	2	1	$\frac{175}{162}$	1.08025
$\to (\overline{\mathbf{2}}, \overline{\mathbf{2}}) \to$	2	1	$\frac{105}{16}$	6.5625
$\to (\overline{\mathbf{1}}, \overline{\mathbf{3}}) \to$	1	1	$\frac{256}{27}$	9.48148
$\to (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{2}}) \to$	1	1	$\frac{7}{32}$	0.21875
$\to (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{2}+\mathbf{0}}) \to$	1	1	$\frac{7}{32}$	0.21875
$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{1}}, \overline{\mathbf{1}}) \to$	2	1	$\frac{7}{27}$	0.259259
$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{0}}, \overline{\mathbf{0}}) \to$	2	1	$\frac{35}{432}$	0.0810185
$\to (\overline{\mathbf{0}+\mathbf{0}}) \to (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \to$	2	1	$\frac{35}{864}$	0.0405093

Configurations of closed geodesics.

Component $\mathcal{H}^{hyp}(3,3)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	$c \cdot \zeta(2)$	$c \cdot \zeta(2)$ approx.
$\Rightarrow (\bar{\mathbf{2}}, \bar{\mathbf{2}}) \Rightarrow$	2	1	$\frac{15}{2}$	7.5
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}) \rightarrow (\bar{\mathbf{1}}, \bar{\mathbf{1}}) \rightarrow$	2	1	$\frac{35}{9}$	3.88889

Component $\mathcal{H}^{nonhyp}(3,3)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	$c \cdot \zeta(2)$	$c \cdot \zeta(2)$ approx.
$\Rightarrow (\bar{\mathbf{2}}, \bar{\mathbf{2}}) \Rightarrow$	2	1	$\frac{3699}{1120}$	3.30268
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{1}}; 3) \rightarrow$	1	1	$\frac{64}{5}$	12.8
$\Rightarrow (\overline{\mathbf{0}+\mathbf{1}}; 3) \Rightarrow$	1	1	$\frac{64}{35}$	1.82857
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\bar{\mathbf{2}}, \bar{\mathbf{0}}) \rightarrow$	1	1	$\frac{27}{80}$	0.3375
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}) \rightarrow (\bar{\mathbf{1}}, \bar{\mathbf{1}}) \rightarrow$	2	1	$\frac{1}{5}$	0.2
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \rightarrow$	2	1	$\frac{1}{32}$	0.03125

Component $\mathcal{H}^{even}(4,2)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	$c \cdot \zeta(2)$	$c \cdot \zeta(2)$ approx.
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}; 4) \rightarrow$	2	1	$\frac{4}{3}$	1.33333
$\Rightarrow (\overline{\mathbf{0}+\mathbf{0}}; 4) \Rightarrow$	2	1	$\frac{4}{21}$	0.190476
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{2}}; 2) \rightarrow$	1	1	$\frac{256}{45}$	5.68889
$\Rightarrow (\overline{\mathbf{0}+\mathbf{2}}; 2) \Rightarrow$	1	1	$\frac{256}{315}$	0.812698
$\Rightarrow (\overline{\mathbf{1}+\mathbf{1}}; 2) \Rightarrow$	2	1	$\frac{8}{9}$	0.888889
$\Rightarrow (\bar{\mathbf{1}}, \bar{\mathbf{3}}) \Rightarrow$	1	1	$\frac{32768}{8505}$	3.85279
$\rightarrow (\bar{\mathbf{1}}, \bar{\mathbf{1}}; 2) \rightarrow$	2	1	$\frac{128}{27}$	4.74074
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}; 2) \rightarrow$	2	1	$\frac{8}{27}$	0.296296
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}) \rightarrow (\overline{\mathbf{0}+\mathbf{0}}; 2) \rightarrow$	2	1	$\frac{16}{135}$	0.118519
$\rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{0}}) \rightarrow (\bar{\mathbf{0}}, \bar{\mathbf{2}}) \rightarrow$	1	1	$\frac{32}{45}$	0.711111
$\Rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\overline{\mathbf{1}+\mathbf{1}}) \Rightarrow$	2	1	$\frac{2}{45}$	0.0444444
$\rightarrow (\overline{\mathbf{0}+\mathbf{0}}) \Rightarrow (\bar{\mathbf{1}}, \bar{\mathbf{1}}) \rightarrow$	1	1	$\frac{256}{1215}$	0.2107

Component $\mathcal{H}^{odd}(4,2)$.

Degeneration pattern	$\lvert\Gamma_-\rvert$	$\lvert\Gamma\rvert$	$c\cdot\zeta(2)$	$c\cdot\zeta(2)$ approx.
$\to(\overline{\mathbf{0}},\overline{\mathbf{0}};4)\to$	2	1	$\frac{49}{18}$	2.72222
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};4)\Rightarrow$	2	1	$\frac{7}{18}$	0.388889
$\to(\overline{\mathbf{0}},\overline{\mathbf{2}};2)\to$	1	1	$\frac{147}{16}$	9.1875
$\Rightarrow(\overline{\mathbf{0}+\mathbf{2}};2)\Rightarrow$	1	1	$\frac{21}{16}$	1.3125
$\Rightarrow(\overline{\mathbf{1}+\mathbf{1}};2)\Rightarrow$	2	1	$\frac{3}{10}$	0.3
$\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{3}})\Rightarrow$	1	1	$\frac{128}{45}$	2.84444
$\to(\overline{\mathbf{1}},\overline{\mathbf{1}};2)\to$	2	1	$\frac{7}{2}$	3.5
$\to(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2)\to$	1	1	$\frac{7}{80}$	0.0875
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{2}})\Rightarrow$	1	1	$\frac{21}{320}$	0.065625
$\to(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}})\to$	1	1	$\frac{7}{45}$	0.155556
$\to(\overline{\mathbf{0}+\mathbf{0}})\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to$	2	1	$\frac{7}{144}$	0.0486111
$\to(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\to$	2	1	$\frac{7}{576}$	0.0121528

Component $\mathcal{H}^{even}(2,2,2)$.

$\to(\overline{\mathbf{0}},\overline{\mathbf{0}};2,2)\to$	2	1	$\frac{180}{37}$	4.86486
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2,2)\Rightarrow$	2	1	$\frac{45}{74}$	0.608108
$\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}};2)\Rightarrow$	2	1	$\frac{225}{37}$	6.08108
$\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to(\overline{\mathbf{0}},\overline{\mathbf{0}};2)\to$	2	1	$\frac{225}{296}$	0.760135
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}})\Rightarrow$	2	1	$\frac{5}{37}$	0.135135

Component $\mathcal{H}^{odd}(2,2,2)$.

$\to(\overline{\mathbf{0}},\overline{\mathbf{0}};2,2)\to$	2	1	$\frac{252}{31}$	8.12903
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2,2)\Rightarrow$	2	1	$\frac{63}{62}$	1.01613
$\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}};2)\Rightarrow$	2	1	$\frac{144}{31}$	4.64516
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}};2)\Rightarrow$	2	1	$\frac{9}{155}$	0.0580645
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{1}},\overline{\mathbf{1}})\Rightarrow$	2	1	$\frac{16}{155}$	0.103226
$\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to(\overline{\mathbf{0}},\overline{\mathbf{0}})\to$	2	3	$\frac{4}{93}$	0.0430108
$\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow(\overline{\mathbf{0}+\mathbf{0}})\Rightarrow$	2	3	$\frac{1}{186}$	0.00537634

References

[At] M. Atiyah, *Riemann surfaces and spin structures*, Ann. scient. ÉNS 4^e série **4** (1971), 47–62.

[C] E. Calabi, *An intrinsic characterization of harmonic 1-forms*, Global Analysis, Papers in Honor of K.Kodaira, (D.C.Spencer and S.Iyanaga, ed.), 1969, pp. 101–117.

[EMa] A. Eskin, H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory and Dynamical Systems, **21:2** (2001), 443–478.

[EZo] A. Eskin, A. Zorich, *Billiards in rectangular polygons*, to appear.

[EOk] A. Eskin, A. Okounkov. *Asymptotics of number of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Inventiones Mathematicae, **145:1** (2001), 59–104.

[G] E. Gutkin. *Billiards in polygons.* Physica D, **19**, (1986), 311-333.

[GJ] E. Gutkin, C. Judge. *Affine mappings of translation surfaces: geometry and arithmetic.* Duke Math. J., in print.

[HM] J. Hubbard, H. Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), 221–274.

[J] D. Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. (2) **22** (1980), 365–373.

[KaZe] A. Katok, A. Zemlyakov. *Topological transitivity of billiards in polygons.* Math. Notes, **18**, (1975), 760-764.

[KMS] S. Kerckhoff, H. Masur, J. Smillie. *Ergodicity of Billiard Flows and Quadratic Differentials.* Ann. of Math., **124**, (1986), 293-311.

[Ko] M.Kontsevich, Lyapunov exponents and Hodge theory. "The mathematical beauty of physics" (Saclay, 1996), (in Honor of C. Itzykson) 318–332, Adv. Ser. Math. Phys., 24, World Sci. Publishing, River Edge, NJ, 1997.

[KonZo] M. Kontsevich, A. Zorich, *Connected components of the moduli spaces of Abelian differentials*, to appear; electronic version in `math.GT/0201292`.

[M] H. Masur, *Interval exchange transformations and measured foliations*, Ann of Math. **115** (1982) 169-200.

[MS] H. Masur, J. Smillie, *Hausdorff dimension of sets of nonergodic foliations*, Ann. of Math. **134** (1991) 455-543.

[MT] H. Masur, S. Tabachnikov, Flat structures and rational billiards, to appear, Handbook on Dynamical systems

[S] K. Strebel *Quadratic differentials.* Springer-Verlag, 1984.

[Ve1] W. Veech, *Teichmuller geodesic flow*, Ann. Math. **124** (1986) 441-530

[Ve2] W. Veech. *Moduli spaces of quadratic differentials.* J. D'Analyse Math., **55**, (1990), 117-171.

[Ve3] W. Veech. *Teichmuller curves in moduli space. Eisenstein series and an application to triangular billiards.* Invent. Math., **97**, (1990), 117-171.

[Ve4] W. Veech, *Siegel measures*, Annals of Math. **148** (1998) 895-944

[Zo] A. Zorich, *Square tiled surfaces and Teichmüller volumes of the moduli spaces of Abelian differentials*, in collection "Rigidity in Dynamics and Geometry", M. Burger, A. Iozzi (Editors), Springer Verlag, 2002, 459–471.