RANDOM WALKS ON LOCALLY HOMOGENEOUS SPACES

ALEX ESKIN AND ELON LINDENSTRAUSS

Dedicated to the memory of Maryam Mirzakhani

Abstract. Let G be real Lie group, and Γ a discrete subgroup of G. Let μ be a measure on G. Under a certain condition on μ, we classify the finite μ-stationary measures on G/Γ. We give an alternative argument (which bypasses the Local Limit Theorem) for some of the breakthrough results of Benoist and Quint in this area.

Contents

1. Introduction

1.1. **Random walks and stationary measures.** Let G' be a real algebraic Lie Group, let $\mathfrak{g}'$ denote the Lie algebra of G', and let Γ' be a discrete subgroup of G'. Let μ be a probablity measure on G' with finite first moment. Let ν be an μ-stationary

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measure on G'/Γ', i.e.

$$\mu * \nu = \nu, \qquad \text{where } \mu * \nu = \int_{G'} g\nu \, d\mu(g).$$

We assume $\nu(G'/\Gamma') = 1$, and also that ν is ergodic (i.e. is extremal among the μ-stationary measures). Let $\mathcal{S}$ denote the support of μ, let $G_{\mathcal{S}} \subset G'$ denote the closure of the group generated by $\mathcal{S}$, and let $\overline{G}_{\mathcal{S}}^{Z} \subset GL(\mathfrak{g}')$ denote the Zariski closure of the adjoint group $\operatorname{Ad}(G_{\mathcal{S}})$.

We say that a measure ν on G'/Γ' is *homogeneous* if it is supported on a closed orbit of its stabilizer $\{g \in G' \; : \; g_* \nu = \nu\}$.

We recall the following breakthrough theorem of Benoist-Quint [BQ1], [BQ2]:

Theorem 1.1 (Benoist-Quint)**.** *Suppose μ is a compactly supported measure on G', $\overline{G}_{\mathcal{S}}^{Z}$ is semisimple, Zariski connected and has no compact factors. Then, any ergodic μ-stationary measure ν on G'/Γ' is homogeneous.*

In this paper, we prove some generalizations and extensions of Theorem 1.1. For a discussion of related results, see §1.5.

Recall that the measure μ has finite first moment if

$$\int_{G'} \log \max(1, \|g\|, \|g^{-1}\|) \, d\mu(g) < \infty.$$

One easy to state consequence of our results is the following:

Theorem 1.2. *Let G' be a Lie group and let μ be a probability measure on G' with finite first moment. Suppose $\overline{G}_{\mathcal{S}}^{Z}$ is generated by unipotents over $\mathbb{C}$. Let Γ' be a discrete subgroup of G', and let ν be any μ-stationary measure on G'/Γ'. Then, ν is $G_{\mathcal{S}}$-invariant.*

Another consequence is an alternative proof of an extension of Theorem 1.1 where the assumption that μ is compactly supported is replaced by the weaker assumption that μ has finite first moment. Thus, we prove the following:

Theorem 1.3. *Suppose μ is a measure on G' with finite first moment, $\overline{G}_{\mathcal{S}}^{Z}$ is semisimple, Zariski connected and has no compact factors. Then, any ergodic μ-stationary measure ν on G'/Γ' is homogeneous.*

1.2. **The main theorems.** Let $\mu^{(n)} = \mu * \mu \cdots * \mu$ (n times). If H is a Lie group, we denote the Lie algebra of H by $\operatorname{Lie}(H)$.

G' acts on $\mathfrak{g}'$ by the adjoint representation. For $G' = SL(n, \mathbb{R})$, $g \in G'$, $\mathbf{v} \in \mathfrak{g}$,

$$\operatorname{Ad}(g)\mathbf{v} = g\mathbf{v}g^{-1}.$$

Notation. We will often use the shorthand $(g)_* \mathbf{v}$ for $Ad(g)\mathbf{v}$.

Let V be a vector space on which G' acts.

Definition 1.4. The measure μ is uniformly expanding on V if there exist $C > 0$ and $N \in \mathbb{N}$ such that for all $\mathbf{v} \in V$,

$$\int_G \log \frac{\|g \cdot \mathbf{v}\|}{\|\mathbf{v}\|} \, d\mu^{(N)}(g) > C > 0. \tag{1.1}$$

Essentially, μ is uniformly expanding on V if every vector in V grows on average under the random walk.

Lemma 1.5. *μ is uniformly expanding on V if and only if for every $\mathbf{v} \in V$, for $\mu^{\mathbb{N}}$-a.e. $\vec{g} = (g_1, g_2, \dots, g_n, \dots) \in (G')^{\mathbb{N}}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \|(g_n \dots g_1) \cdot \mathbf{v}\| > 0$$

Thus, if μ is uniformly expanding, then with probability 1 every vector grows exponentially. The proof of this lemma is postponed to §3.

Remark. We can consider the Adjoint action of G' on its Lie algebra $\mathfrak{g}'$. If $\overline{G}_{\mathcal{S}}^{Z}$ is semisimple with no centralizer and no compact factors, then μ is uniformly expanding on $\mathfrak{g}'$, see e.g. [EMar, Lemma 4.1]. However, there are many examples with $\overline{G}_{\mathcal{S}}^{Z}$ not semisimple for which uniform expansion on $\mathfrak{g}'$ still holds, see e.g. (1.13) below. □

Instead of assuming that μ is uniformly expanding on $\mathfrak{g}'$ we consider the following somewhat more general setup, to accomodate centralizers and compact factors.

Definition 1.6. Let Z be a connected Lie subgroup of G'. We say that μ is uniformly expanding mod Z if the following hold:

(a) Z is normalized by $G_{\mathcal{S}}$.
(b) The conjugation action of $G_{\mathcal{S}}$ on Z factors through the action of a compact subgroup of $\operatorname{Aut}(Z)$.
(c) We have a $G_{\mathcal{S}}$-invariant direct sum decomposition

$$\mathfrak{g}' = \operatorname{Lie}(Z) \oplus V,$$

and μ is uniformly expanding on V. (Note that V need not be a subalgebra).

Our result is the following:

Theorem 1.7. *Let G' be a real Lie group, and let Γ' be a discrete subgroup of G'. Suppose μ is a probability measure on G' with finite first moment, and suppose there exists a connected subgroup Z such that μ is uniformly expanding mod Z.*

Let ν be any ergodic μ-stationary probability measure on G'/Γ'. Then one of the following holds:

(a) *There exists a closed subgroup $H \subset G'$ with $\dim(H) > 0$ and an H-homogeneous probability measure ν_0 on G'/Γ' such that the unipotent elements of H act ergodically on ν_0, and there exists a finite μ-stationary measure λ on G'/H such* *that*

$$\nu = \int_G g\nu_0 \, d\lambda(g). \tag{1.2}$$

Let H^0 denote the connected component of H containing the identity. If $\dim H^0$ is maximal then H^0 and ν_0 are unique up to conjugation of H^0 and the obvious corresponding modification of ν_0.

(b) *The measure ν is $G_{\mathcal{S}}$-invariant and is supported on a finite union of compact subsets of Z-orbits.*

In partucular, if μ is uniformly expanding on $\mathfrak{g}'$ we may take $Z = \{e\}$, and alternative (b) says that ν is $G_{\mathcal{S}}$-invariant and finitely supported.

Note that the subgroup H of Theorem 1.7(a) may not be connected.

Definition 1.8. A real-algebraic Lie subgroup $L \subset G'$ is called an H-envelope if the following hold:

(i) $L \supset H$ and H^0 is normal in L.
(ii) The image of H in L/H^0 is discrete.
(iii) There exists a representation $\rho' : G' \to GL(W)$ and a vector $v_L \in W$ such that the stabilizer of v_L is L.

Suppose H is as in Theorem 1.7. Let $N_{G'}(H^0)$ denote the normalizer of H^0 in G'. Let $\rho_H \in \bigwedge^{\dim H}(\mathfrak{g}')$ denote the vector $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n$, where $\mathbf{v}_1, \dots \mathbf{v}_n$ is a basis of $\operatorname{Lie}(H^0)$. Then the stabilizer of ρ_H in G' is an H-envelope. Also, if G' is an algebraic group and Γ' is an arithmetic lattice, the Zariski closure in G' of $\Gamma' \cap N_{G'}(H^0)$ is an H-envelope.

The following is an easy consequence of Theorem 1.7:

Theorem 1.9. *Let G' be a real Lie group, and let Γ' be a discrete subgroup of G'. Suppose μ is a probability measure on G' with finite first moment. Let ν be any ergodic μ-stationary probability measure on G'/Γ' such that (a) of Theorem 1.7 holds, let H, ν_0, λ be as in Theorem 1.7. Suppose $L \subset G'$ is an H-envelope. Then*

(a) *There exists a finite μ-stationary measure $\tilde{\lambda}$ on G'/L (the image of λ under the natural projection $G'/H \to G'/L$).*
(b) *Suppose $G_{\mathcal{S}} \subset L$. Then either ν is supported on finitely many H^0-orbits, or there exists a vector $\mathbf{v} \in \operatorname{Lie}(L)$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \int_{G'} \log \|(\operatorname{Ad}(g)\mathbf{v}) \wedge \rho_H\| \, d\mu^{(n)}(g) \le 0. \tag{1.3}$$

Here, as above, $\rho_H \in \bigwedge^{\dim H}(\mathfrak{g})$ is the vector $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n$, where $\mathbf{v}_1, \dots \mathbf{v}_n$ is a basis of $\operatorname{Lie}(H^0)$.

Proof of Theorem 1.9. Since the projection $G'/H \to G'/L$ commutes with the left action of G', the image of any μ-stationary measure on G'/H is a μ-stationary measure on G'/L. Thus (a) holds. Now (b) follows by applying Theorem 1.7 to the random walk (with measure μ) on $L/H = (L/H^0)/(H/H^0)$. $\square$

Remark. Under the assumptions of Theorem 1.9, there exists a representation $G' \to GL(W)$ and a vector $v_L \in W$ such that the stabilizer of v_L is L. Thus, if $L \neq G'$, then there exists a μ-stationary measure on the vector space W.

Remark. Loosely speaking, the assumption that there exists a finite stationary measure on G'/L means that on average, the random products given by μ do not get very far from L (and thus the random walk on G can be in some sense approximated by a random process on L).

Remark. The equation (1.3) means that there exists $\mathbf{v} \in \mathrm{Lie}(L)$ such that on average $\mathrm{Ad}(g)\mathbf{v}$ does not grow modulo $\mathrm{Lie}(H)$. (The component of $\mathrm{Ad}(g)\mathbf{v}$ in $\mathrm{Lie}(H)$ may grow enough to satisfy uniform expansion).

Example. We now present an example with uniform expansion where the stationary measure ν is not invariant and not homogeneous. Recall that for $\lambda \in (0,1)$ the λ-Bernoulli convolution is the measure on $\mathbb{R}$ given by the distribution of the random series $\sum_{n=0}^{\infty} \pm\lambda^n$.

Let $G' = SL(4,\mathbb{R})$, $\Gamma' = SL(4,\mathbb{Z})$. Pick two elements $\gamma_1, \gamma_2 \in SL(2,\mathbb{Z})$ each with trace greater than 2. Let $g_i \in SL(2,\mathbb{R})$ be such that $g_i\gamma_i g_i^{-1}$ is diagonal. Let $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in SL(4,\mathbb{R})$. Let μ be supported on the two matrices (each with weight 1/2)

$$g^{-1}\begin{pmatrix} 4 & 1 & 1 & 1 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1/2 \end{pmatrix} g \qquad \text{and} \qquad g^{-1}\begin{pmatrix} 4 & -1 & -1 & -1 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1/2 \end{pmatrix} g.$$

It is easy to check that uniform expansion holds.

Let $H^0 = \begin{pmatrix} * & * \\ 0 & I \end{pmatrix} \subset G$. Let D denote the diagonal subgroup of $SL(2,\mathbb{R})$, and let $A_2 = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \subset G'$. Let $U_{ij} \subset G'$ denote the one-parameter unipotent subgroup whose only non-diagonal entry is in the ij-entry of G'. Let ν_0 denote the H^0-invariant measure on G'/Γ' which is supported on $H^0 g\Gamma'$, let ν_1 denote the A_2H^0-invariant measure on G'/Γ' supported on $A_2H^0g\Gamma'$, and let $\nu = \int_{U_{43}} u\nu_1 \, d\eta(u)$, where η is the 1/4-Bernoulli convolution measure on $U_{43} \cong \mathbb{R}$. In view of the fact that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & \pm 2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & 1 \end{pmatrix} g\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x/4 \pm 1 & 1 \end{pmatrix} g\Gamma,$$

it is easy to check that ν is μ-stationary.

In the context of Theorem 1.7, $H = \begin{pmatrix} SL(2,\mathbb{R}) & * \\ 0 & SL(2,\mathbb{Z}) \end{pmatrix} \subset G$. In the context of Theorem 1.9, $L = \begin{pmatrix} SL(2,\mathbb{R}) & * \\ 0 & SL(2,\mathbb{R}) \end{pmatrix} \subset G$. We have $G_{\mathcal{S}} \subset L$. Theorem 1.9 does not fail, since any vector in $\mathrm{Lie}(U_{34})$ satisfies (1.3).

1.3. **Compact Extensions.** To accomodate the proof of Theorem 1.2 we will need the following somewhat more general setup.

Suppose M' is a compact Lie group, and suppose we have a homomorphism $\rho : \overline{G}_{\mathcal{S}}^{Z} \to M'$. Suppose M_0 is a closed subgroup of M' and let $M = M'/M_0$. Then $\overline{G}_{\mathcal{S}}^{Z}$ acts on M by $g \cdot m = \rho(g)m$.

Suppose G is a connected Lie group, and let $\mathfrak{g}$ denote the Lie algebra of G. Suppose $\vartheta : \overline{G}_{\mathcal{S}}^{Z} \times M \to G$ is a cocycle, i.e. $\vartheta(g_1 g_2, m) = \vartheta(g_1, g_2 \cdot m)\vartheta(g_2, m)$. Now $\overline{G}_{\mathcal{S}}^{Z}$ acts on $M \times G$ by

$$g' \cdot (m, g) = (g' \cdot m, \vartheta(g', m)g). \tag{1.4}$$

Furthermore, we assume that there exists $C > 0$ such that for all $g' \in \overline{G}_{\mathcal{S}}^{\mathbb{Z}}$ and $m \in M$,

$$\|\vartheta(g', m)\| \le C\|g'\| \text{ and } \|\vartheta(g', m)^{-1}\| \le C\|(g')^{-1}\|. \tag{1.5}$$

Suppose Γ is a discrete subgroup of G. We will be given a probability measure μ on $\overline{G}_{\mathcal{S}}^{Z} \subset G'$ and will need to understand μ-stationary measures ν on $M \times G/\Gamma$ which project to the Haar measure on M.

We say that M and ϑ are *trivial* if $G = G'$, $\Gamma = \Gamma'$, M is a point, and $\vartheta(g, m) = g$. This is the setup for Theorem 1.7. On first reading the reader is urged to think of this case.

The bounceback condition. In addition to uniform expansion mod Z as in Theorem 1.7, we will also use the following weaker condition (which in particular allows for compact factors and many cases where $\overline{G}_{\mathcal{S}}^{Z}$ is solvable).

Definition 1.10. A probability measure μ on G' satisfies the bounceback condition if for every compact set $K \subset G'$ there exists $k \in \mathbb{N}$ and $C \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, all $\mathbf{v} \in \mathfrak{g}$ and all $g_1', \dots, g_n' \in \operatorname{supp}\mu \cap K$ and all $m \in M$,

$$\int_G \log \frac{\|\operatorname{Ad}(\vartheta(g' g_n' \dots g_1', m))\mathbf{v}\|}{\|\mathbf{v}\|}\, d\mu^{(kn)}(g') > C.$$

Note that in this definition, C may be negative. This can be interpreted as follows: suppose we pick a vector $\mathbf{v} \in \mathfrak{g}$, and then try to contract it as much as we can for n steps (by picking $g_1', \dots, g_n'$). One can presumably make it very short. But then, if one takes a random word of length kn, it will "bounce back" and become long again. This means that any vector which can get contracted, gets expanded again on average.

Lemma 1.11. *Suppose M and ϑ is trivial, and there exists $Z \subset G$ such that μ is uniformly expanding mod Z. Then, μ satisfies the bounceback condition.*

Proof. Note that Definition 1.10 does not depend on the choice of the norm on $\mathfrak{g}$. We can thus choose $\|\cdot\|$ so that if $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in V$ and $\mathbf{v}_2 \in \mathrm{Lie}(Z)$ we have $\|\mathbf{v}\| = \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$, and $\|\operatorname{Ad}(g)\mathbf{v}_2\| = \|\mathbf{v}_2\|$ for all $g \in G_{\mathcal{S}}$. Then it is clear that the Lemma holds if $\mathbf{v}_1 = 0$ (with $C = 0$), and furthermore, it is enough to prove the Lemma assuming $\mathbf{v} \in V$. Then we have,

$$\log \frac{\|\operatorname{Ad}(g_n \dots g_1)\mathbf{v}\|}{\|\mathbf{v}\|} \geq -nC(K),$$

and by uniform expansion on V, iterated kn times and applied to the vector $\operatorname{Ad}(g_n \dots g_1)\mathbf{v}$ we have

$$\int_G \log \frac{\|\operatorname{Ad}(gg_n \dots g_1)\mathbf{v}\|}{\|\operatorname{Ad}(g_n \dots g_1)\mathbf{v}\|} \, d\mu^{(kn)}(g) \geq knC',$$

where $C' > 0$. Then we can choose $k \in \mathbb{Z}$ so that $kC' > C(K)$. □

In fact, we will really use a slightly weaker version of the bounceback condition, see Lemma 3.9.

1.4. **Skew Products.** Let $\mathcal{S} \subset G'$ denote the support of μ. We consider the two sided shift space $\mathcal{S}^{\mathbb{Z}}$. For $\omega \in \mathcal{S}^{\mathbb{Z}}$, we have $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$. We write $\omega = (\omega^-, \omega^+)$ where $\omega^- = (\dots, \omega_{-1})$ is the "past", and $\omega^+ = (\omega_0, \omega_1, \dots)$ is the "future". Let $T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ denote the left shift $(T\omega)_i = \omega_{i+1}$ (which we are thinking of as "taking one step into the future"). We also use the letter T to denote the "skew product" map $\mathcal{S}^{\mathbb{Z}} \times M \to \mathcal{S}^{\mathbb{Z}} \times M$ given by

$$T(\omega, m) = (T\omega, \omega_0 \cdot m). \tag{1.6}$$

We also have the skew product map $\hat{T} : \mathcal{S}^{\mathbb{Z}} \times M \times G \to \mathcal{S}^{\mathbb{Z}} \times M \times G$ given by

$$\hat{T}(\omega, m, g) = (T\omega, \omega_0 \cdot m, \vartheta(\omega_0, m)g), \qquad \text{where } \omega = (\dots, \omega_0, \dots). \tag{1.7}$$

For $x = (\omega, m) \in \mathcal{S}^{\mathbb{Z}} \times M$, let

$$T_x^n = \vartheta(\omega_{n-1} \dots \omega_0, m). \tag{1.8}$$

We can view $\hat{T}$ as a skew-product over the map $T : \mathcal{S}^{\mathbb{Z}} \times M \to \mathcal{S}^{\mathbb{Z}} \times M$. Then, for $n \in \mathbb{N}$ and $x = (\omega, m) \in \mathcal{S}^{\mathbb{Z}} \times M$,

$$\hat{T}^n(x, g) = (T^n x, T_x^n g).$$

We will often consider $\hat{T}$ to be a map from $\mathcal{S}^{\mathbb{Z}} \times (M \times G/\Gamma)$ to $\mathcal{S}^{\mathbb{Z}} \times (M \times G/\Gamma)$.

Measures on skew-products. Suppose we are given an ergodic μ-stationary measure ν on $M \times G/\Gamma$. As in [BQ1], for $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$, let

$$\nu_{\omega^-} = \lim_{n \to \infty} \omega_{-1} \dots \omega_{-n} \, \nu.$$

Here the action of $G_{\mathcal{S}}$ on $M \times G/\Gamma$ is as in (1.4). The fact that the limit exists follows from the Martingale convergence theorem. Then ν_{ω^-} is a measure on $M \times G/\Gamma$.

Basic Fact: Given a μ-stationary measure ν on $M \times G/\Gamma$, we get a $\hat{T}$-invariant measure $\hat{\nu}$ on $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$ given by

$$d\hat{\nu}(\omega^-, \omega^+, g\Gamma) = d\mu^{\mathbb{Z}}(\omega^-, \omega^+)\, d\nu_{\omega^-}(m, g\Gamma) \tag{1.9}$$

It is important that the measure $\hat{\nu}$ defined in (1.9) is a product of a measure depending on $(\omega^-, m, g\Gamma)$ and a measure depending on ω^+. (If instead of the two-sided shift space we use the one-sided shift $G^{\mathbb{N}} \times (M \times G/\Gamma)$, then $\mu^{\mathbb{Z}} \times \nu$ would be an invariant measure for $\hat{T}$.)

Proposition 1.12. *If ν is an ergodic stationary measure on $M \times G/\Gamma$, then the $\hat{T}$-invariant measure $\hat{\nu}$ is $\hat{T}$-ergodic.*

Proof. This follows from [Kif, Lemma I.2.4, Theorem I.2.1] □

The "group" $\mathcal{U}_1^+$. We would like to express the fact that for $\omega = (\omega^-, \omega^+)$ the conditional measures of $\hat{\nu}$ along the fiber $\{\omega\} \times M \times G$ do not depend on the ω^+ coordinate, as invariance under the action of a group. The group will be a bit artificial.

Let $P(\mathcal{S})$ denote the permutation group on $\mathcal{S}$, i.e. the set of bijections from $\mathcal{S}$ to $\mathcal{S}$. We do not put a topology on $P(\mathcal{S})$. Let

$$\begin{aligned} \mathcal{U}_1^+ &= P(\mathcal{S}) \times P(\mathcal{S}) \times P(\mathcal{S}) \dots \\ &\in \\ u &= (\sigma_0, \sigma_1, \dots, \sigma_n, \dots) \end{aligned}$$

The way $u = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots) \in \mathcal{U}_1^+$ acts on $\mathcal{S}^{\mathbb{Z}}$ is given by

$$u \cdot (\dots, \omega_{-n}, \dots, \omega_{-1}, \omega_0, \omega_1, \dots) = (\dots, \omega_{-n}, \dots, \omega_{-1}, \sigma_0(\omega_0), \sigma_1(\omega_1), \dots)$$

We then extend the action of $\mathcal{U}_1^+$ to $\mathcal{S}^{\mathbb{Z}} \times M$ and $\mathcal{S}^{\mathbb{Z}} \times M \times G$ by:

$$u \cdot (\omega, m) = (u\omega, m), \qquad u \cdot (\omega, m, g) = (u\omega, m, g).$$

(Thus $\mathcal{U}_1^+$ acts by "changing the combinatorial future". $\mathcal{U}_1^+$ fixes ω^-, m and g and changes ω^+.) We do not attempt to define a topology or a Haar measure on $\mathcal{U}_1^+$, as it is a formal construction.

We now refer to (1.9) as the *$\mathcal{U}_1^+$-invariance property* of $\hat{\nu}$. In fact $\hat{T}$-invariant measures on the skew-product which come from stationary measures are exactly the $\hat{T}$-invariant measures which are also have the $\mathcal{U}_1^+$-invariance property.

We have a similar group $\mathcal{U}_1^-$ which is changing the past. However, in general $\hat{\nu}$ does not have the $\mathcal{U}_1^-$-invariance property.

Stable and unstable manifolds. For $x = (\omega, m) \in \mathcal{S}^{\mathbb{Z}} \times M$, let

$$W^-[x] = \{(\omega', m') \in \mathcal{S}^{\mathbb{Z}} \times M \;\;:\;\; m' = m \text{ and for } n \in \mathbb{N} \text{ sufficiently large, } \omega'_n = \omega_n\}.$$

Then $W^-[x]$ consists of sequences y which eventually agree with x. We call $W^-[x]$ the "stable leaf through x". We also have the subset

$$W_1^-[x] = \{(\omega', m') \in \mathcal{S}^{\mathbb{Z}} \times M \;\;:\;\; m' = m \text{ and } (\omega')^+ = \omega^+\} \subset W^-[x].$$

Similarly, we define

$$W^+[x] = \{(\omega', m') \in \mathcal{S}^{\mathbb{Z}} \times M \;\;:\;\; m' = m \text{ and for } n \in \mathbb{N} \text{ sufficiently large, } \omega'_{-n} = \omega_{-n}\},$$

and we also have the subset

$$W_1^+[x] = \{(\omega', m) \in \mathcal{S}^{\mathbb{Z}} \times M \;\;:\;\; (\omega')^- = \omega^-\} \subset W^+[x].$$

For $x = (\omega, m) \in \mathcal{S}^{\mathbb{Z}} \times M$, we define $x^+ = (\omega^+, m)$ and $x^- = (\omega^-, m)$. Then $W_1^+[x]$ depends only on x^- and $W_1^-[x]$ depends only on x^+.

Let $d_G(\cdot,\cdot)$ be a right invariant Riemannian metric on G. For $\hat{x} = (x, g) \in (\mathcal{S}^{\mathbb{Z}} \times M) \times G$, let

$$\hat{W}_1^-[\hat{x}] = \{(y, g') \in (\mathcal{S}^{\mathbb{Z}} \times M) \times G \;\;:\;\; y \in W_1^-[x], \quad \limsup_{n\to\infty} \frac{1}{n} \log d_G(T_x^n g, T_x^n g') < 0\}.$$

Thus, $\hat{W}_1^-[\hat{x}]$ consists of the points $\hat{y}$ which have the same combinatorial future as $\hat{x}$ and such that at $n \to \infty$, $\hat{T}^n \hat{x}$ and $\hat{T}^n \hat{y}$ converge exponentially fast. Similarly, we have a subset

$$\hat{W}_1^+[\hat{x}] = \{(y, g') \in (\mathcal{S}^{\mathbb{Z}} \times M) \times G \;:\; y \in W_1^+[x], \quad \limsup_{n\to\infty} \frac{1}{n} \log d_G(T_x^{-n} g, T_x^{-n} g') < 0\},$$

consisting of the points $\hat{y}$ which have the same combinatorial past as $\hat{x}$ and such that at $n \to \infty$, $\hat{T}^{-n} \hat{x}$ and $\hat{T}^{-n} \hat{y}$ converge exponentially fast.

We will show below that that for almost all $x \in \mathcal{S}^{\mathbb{Z}} \times M$ there exist unipotent subgroups $N^+(x)$ and $N^-(x)$ so that $N^+(x) = N^+(x^-)$, $N^-(x) = N^-(x^+)$ and

$$\hat{W}_1^+[(x, g)] = W_1^+[x] \times N^+(x) g,$$

and

$$\hat{W}_1^-[(x, g)] = W_1^-[x] \times N^-(x) g, \tag{1.10}$$

Thus,

$$\hat{W}_1^+[(\omega^-, \omega^+, m, g)] = \{(\eta^-, \eta^+, m', h) \;\;:\;\; m' = m, \eta^- = \omega^-, \\ \eta^+ \text{ is arbitrary, } h \in N^+(\omega^-, m) g\}.$$

and

$$\hat{W}_1^-[(\omega^-, \omega^+, g)] = \{(\eta^-, \eta^+, h) \;\;:\;\; \eta^+ = \omega^+, \\ \eta^- \text{ is arbitrary, } h \in N^-(\omega^+, m) g\}.$$

The two cases. Note that $\mathcal{U}^+$ acts on $\hat{W}_1^+[(x,g)]$. Since $\hat{\nu}$ has the $\mathcal{U}_1^+$-invariance property (i.e. (1.9) holds), for almost all (x,g), the conditional measure $\hat{\nu}|_{\hat{W}_1^+[(x,g)]}$ is the product of the Bernoulli measure $\mu^{\mathbb{N}}$ on $W_1^+[x] \cong \mathcal{S}^{[0,\infty)}$ and an unknown measure on $N^+(x)g$. However, we have no such information on the conditional measures $\hat{\nu}|_{\hat{W}_1^-[(x,g)]}$. We distinguish two cases:

- Case I: Not Case II.
- Case II: For almost all $(x,g) \in \mathcal{S}^{\mathbb{Z}} \times M \times G$, the conditional measure $\hat{\nu}|_{\hat{W}_1^-[(x,g)]}$ is supported on a single $\mathcal{U}^-$ orbit.

Our proof breaks up into the following statements:

Theorem 1.13. *Suppose μ is a probability measure on G' with finite first moment and satisfying the bounceback condition Definition 1.10. Let M and ϑ be as in §1.3. Suppose ν is an ergodic μ-stationary measure on $M \times G/\Gamma$ which projects to the Haar measure on M, and suppose Case I holds. Then*

(a) *There exists a Lie subgroup $H \subset G$ with $\dim H > 0$, an H-homogeneous probability measure ν_0 on G/Γ such that the unipotent elements of H act ergodically on ν_0, and there exists a finite μ-stationary measure λ on $M \times G/H$ such that*

$$\nu = \lambda * \nu_0, \tag{1.11}$$

where for a measure λ on $M \times G/H$ and an H-invariant measure ν_0 on G/Γ, the measure $\lambda * \nu_0$ on $M \times G/\Gamma$ is defined by

$$(\lambda * \nu_0)(f) = \int_{M\times G/\Gamma} \int_G f(m, gg'\Gamma)\, d\lambda(m,g)\, d\nu_0(g'\Gamma)$$

(b) *Let H^0 denote the connected component of H containing the identity. If $\dim H^0$ is maximal then H^0 and ν_0 are unique up to conjugation of H^0 and the obvious corresponding modification of ν_0.*

(In fact, the bounceback condition in Theorem 1.13 can be replaced by a weaker assumption, see Remark 3.10).

Theorem 1.14. *Suppose μ is a probability measure on G' with finite first moment, M and ϑ are as in §1.3, ν is an ergodic μ-stationary measure on $M \times G/\Gamma$ which projects to the Haar measure on M, and suppose Case II holds.*

(a) *Let $\hat{\nu}$ be as in (1.9). Then, ν is $G_{\mathcal{S}}$ invariant, where the action of G' on $M \times G/\Gamma$ is as in (1.4). Furthermore, $\hat{\nu} = \mu^{\mathbb{Z}} \times \nu$ is the product of the Bernoulli measure $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ and the measure ν on $M \times G/\Gamma$.*

(b) *Suppose M and ϑ are trivial, and there exists a connected subgroup $Z \subset G$ such that μ satisfies uniform expansion mod Z (see Definition 1.6). Then ν is supported on a finite union of compact subsets of Z orbits.*

Clearly Theorem 1.7 follows from Lemma 1.11, Theorem 1.13 and Theorem 1.14. We will prove Theorem 1.13 in §2-§10, and we will prove Theorem 1.14 in §11. Finally, Theorem 1.2 will be derived from Theorem 1.13 and Theorem 1.14 in §12.

1.5. **Outline of the proofs, and discussion of related results.** Our proof of Theorem 1.14 (duplicated in a simpler setting in [EsL]) is new, and relies on a construction of a Margulis function, see §11.

Our proof of Theorem 1.13 follows roughly the same outline as the beginning of [EMi]. A simpler version of the argument has been presented in [EsL]. Also, for simplicity we assume in this subsection that M and ϑ is trivial, so $G = G'$, $\Gamma = \Gamma'$.

Suppose the following hold:

(A1) For almost all $x \in \mathcal{S}^{\mathbb{Z}}$, there exists a nilpotent subalgebra $\mathbf{E}(x)$ of $\mathfrak{g}$ such that with probability tending to 1, "any two points diverge along $\mathbf{E}(x)$, i.e. on the Lie algebra level, for any vector $\mathbf{v} \in \mathfrak{g}$, for almost all $x \in \mathcal{S}^{\mathbb{Z}}$,

$$\lim_{n\to\infty} d\left(\frac{(T_x^n)_*\mathbf{v}}{\|(T_x^n)_*\mathbf{v}\|}, \mathbf{E}(T^n x)\right) = 0.$$

(A2) There exists a cocycle $\lambda : \mathcal{S}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ such that for all $\mathbf{v} \in \mathbf{E}(x)$,

$$\|(T_x^n)_*\mathbf{v}\| = e^{\lambda(x,n)}\|\mathbf{v}\|.$$

(In general, the norm used in (A2) on $\mathbf{E}(x)$ depends on x.)

In the setting of [EsL] (i.e. $G_{\mathcal{S}}$ is Zariski dense in a simple Lie group G), (A1) and (A2) hold, with $\mathbf{E}(x) = \mathrm{Lie}(N_1)(x)$ where $N_1(x)$ is the subgroup corresponding to the top Lyapunov exponent λ_1 of the random walk, i.e.

$$\mathrm{Lie}(N_1)(x) = \{\mathbf{v} \in \mathfrak{g} \;\;:\;\; \lim_{n\to\pm\infty} \frac{1}{n} \log \frac{(T_x^n)_*\mathbf{v}}{\|\mathbf{v}\|} = \lambda_1\}.$$

(A2) holds since in the case $G_{\mathcal{S}}$ is Zariski dense in a simple Lie group G, the action on $\mathrm{Lie}(N_1)(x)$ is conformal, so we can let the $\mathbb{R}$-valued cocycle $\lambda(x,n) = \lambda_1(x,n)$ measure the growth of all vectors in $\mathrm{Lie}(N_1)(x)$.

The proof in the setting of [EsL] has been outlined in [EsL, §1.2]. We reproduce it here for completeness. The assumption of Case I implies that we can find points $\hat{q} = (q,g)$ and $\hat{q}' = (q',g')$ in the support of $\hat{\nu}$, with $\hat{q}' \in \hat{W}_1^-[\hat{q}]$ and $g \neq g'$. (Since $\hat{q}' \in \hat{W}_1^-[\hat{q}]$ we must have $q^+ = (q')^+$, but q^- need not be equal to $(q')^-$). Furthermore, we can find such $\hat{q}$ in a set of large measure, and also choose $\hat{q}'$ so that $d_G(g,g') \approx 1$.

(In the rest of the outline, we use a suspension flow construction which will allow us to make sense of expressions like $\hat{T}^t$ where $t \in \mathbb{R}$. This construction is defined in the beginning of §2.)

We now choose an arbitrary large parameter $\ell \in \mathbb{R}^+$, and let $\hat{q}_1 = \hat{T}^\ell \hat{q}$, $\hat{q}_1' = \hat{T}^\ell \hat{q}'$. Since $\hat{q}' \in \hat{W}_1^-[\hat{q}]$, $d(\hat{q}_1, \hat{q}_1')$ is exponentially small in ℓ.

Suppose $u \in \mathcal{U}_1^+$. For most choices of u, $u\hat{q}_1$ and $u\hat{q}_1'$ are no longer in the same stable for $\hat{T}$, and thus we expect $\hat{T}^t u\hat{q}_1$ and $\hat{T}^t u\hat{q}_1'$ to diverge as $t \to \infty$. Fix $0 < \epsilon < 1$ and

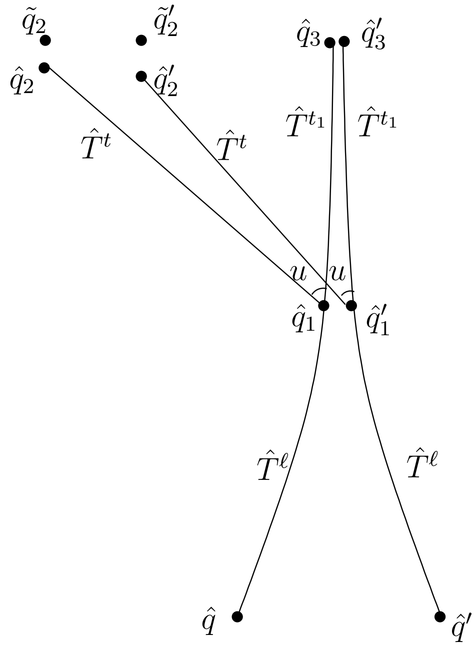

Figure 1. Outline of the proof of Theorem 1.13

choose t so that $\hat{q}_2 \equiv \hat{T}^t u \hat{q}_1$ and $\hat{q}_2' \equiv \hat{T}^t u \hat{q}_1'$ satisfy $d(\hat{q}_2, \hat{q}_2') \approx \epsilon$. Write $\hat{q}_2 = (q_2, g_2)$, $\hat{q}_2' = (q_2', g_2')$.

By (A1), for a.e. $x \in \mathcal{S}^{\mathbb{Z}}$ for most choices of u, $\hat{q}_2'$ and $\hat{q}_2$ 'diverge essentially along N_1, i.e. g_2' is very close to $N_1(q_2)g_2$, with the distance tending to 0 as $\ell \to \infty$.

Let the cocycle $\lambda_1(\cdot, \cdot)$ be as in (A2). Now choose $t_1 > 0$ such that $\lambda_1(q_1, t_1) = \lambda_1(uq_1, t)$, and let $\hat{q}_3 = \hat{T}^{t_1}\hat{q}_1$, $\hat{q}_3' = \hat{T}^{t_1} q_1'$. Then $\hat{q}_3$ and $\hat{q}_3'$ are even closer than $\hat{q}_1$ and $\hat{q}_1'$.

The rest of the setup follows Benoist-Quint [BQ1] (which only uses the "top half" of Figure 1). For $\hat{x} = (x, g) \in \hat{\Omega}$, let $f_1(\hat{x})$ denote the conditional measure (or more precisely the leafwise measure in the sense of [EiL2]) of $\hat{\nu}$ along $\{x\} \times N_1(x)g$. These measures are only defined up to normalization. Then, since $\hat{\nu}$ is $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant and since $\lambda_1(q_1, t_1) = \lambda_1(uq_1, t)$, we have,

$$f_1(\hat{q}_2) = f_1(\hat{q}_3).$$

Also, since one can show $\lambda_1(uq_1', t) \approx \lambda_1(q_1', t_1)$ we have,

$$f_1(\hat{q}_2') \approx f_1(\hat{q}_3').$$

Since $\hat{q}_3$ and $\hat{q}_3'$ are very close, we can ensure that, $f_1(\hat{q}_3') \approx f_1(\hat{q}_3)$. Then, we get, up to normalization,

$$f_1(\hat{q}_2) \approx f_1(\hat{q}_2').$$

Applying the argument with a sequence of ℓ's going to infinity, and passing to a limit along a subsequence, we obtain points $\tilde{q}_2 = (z, \tilde{g}_2)$ and $\tilde{q}'_2 = (z, \tilde{g}'_2)$ with $\tilde{g}'_2 \in N_1(z)\tilde{g}_2$, $d_G(\tilde{g}_2, \tilde{g}'_2) \approx \epsilon$ and, up to normalization, $f_1(\tilde{q}_2) = f_1(\tilde{q}'_2)$. Thus, $f_1(\tilde{q}_2)$ is invariant by a translation of size approximately ϵ. By repeating this argument with a sequence of ϵ's converging to 0, we show that for almost all $\hat{x} = (x, g) \in \mathcal{S}^{\mathbb{Z}} \times G/\Gamma$, $f_1(\hat{x})$ is invariant under arbitrarily small translations, which implies that there exists a connected non-trivial unipotent subgroup $U^+_{new}(\hat{x}) \subset N_1(x)$ so that $\hat{\nu}$ is "U^+_{new}-invariant" or more precisely, for almost all $\hat{x}$, the conditional measure of $\hat{\nu}$ along $\{x\} \times U^+_{new}(\hat{x})$ is Haar. The rest of the argument follows closely [BQ1, §8].

To make this scheme work, we need to make sure that all eight points $\hat{q}$,$\hat{q}'$, $\hat{q}_1$, $\hat{q}'_1$, $\hat{q}_2$, $\hat{q}'_2$, $\hat{q}_3$, $\hat{q}'_3$, are in some "good subset" $K_0 \subset \hat{\Omega}$ of almost full measure. (For instance we want the function f_1 to be uniformly continuous on K_0). Showing that this is possible is the heart of the proof. Our strategy for accomplishing this goal is substantially different from that of Benoist-Quint in [BQ1], where a time changed Martingale Convergence argument was used, and from that of Benoist-Quint in [BQ2], where a Local Limit Theorem (proved in [BQ3]) is used. Our strategy is outlined further in §10.1. This completes the outline of the proof of [EsL].

More generally, it is possible that (A1) and (A2) hold even if the Zariski closure of $G_{\mathcal{S}}$ is not semisimple; this is the situation in the case considered in Simmons-Weiss [SW]. However, even if the Zariski closure of $G_{\mathcal{S}}$ is semisimple, (A2) fails in general. For example, suppose the Zariski closure of $G_{\mathcal{S}}$ is $G_1 \times G_2$ where G_1 and G_2 are simple. Then, we have $\mathbf{E}(x) = \mathcal{V}_1^{G_1}(x) \oplus \mathcal{V}_1^{G_2}(x)$, where $\mathcal{V}_1^{G_i}(x)$ is the subspace corresponding to the top Lyapunov exponent in Lie(G_i). Then (A2) as stated fails, since vectors in $\mathcal{V}_1^{G_1}(x)$ and $\mathcal{V}_1^{G_2}(x)$ grow at different rates under a typical $x \in \mathcal{S}^{\mathbb{Z}}$.

In fact, if the Zariski closure of $G_{\mathcal{S}}$ is semisimple, then (A1) and (A2)$'$ hold, where

(A2)$'$ $\mathbf{E}(x) = \bigoplus_{i=1}^n \mathbf{E}_i(x)$ is a T-equivariant splitting, and for each $1 \le i \le n$ there exists a cocycle $\lambda_i : \mathcal{S}^{\mathbb{Z}} \times \mathbb{Z} \to \mathbb{R}$ such that for all $\mathbf{v} \in \mathbf{E}_i$,
$$\|(T^n_x)_* \mathbf{v}\| = e^{\lambda_i(x,n)} \|\mathbf{v}\|.$$

At first glance, it seems that one can follow the above outline (i.e. that of of [EsL, §1.5]), with t_1 replaced by some t_i, $1 \le i \le n$, and $\hat{q}_3$ replaced by some the $\hat{q}_{3,i}$ (see Figure 3, where for reasons which will become clear later, we write t_{ij} instead of t_i and $\hat{q}_{3,ij}$ instead of $\hat{q}_{3,i}$). However, there is a difficulty, since if one follows the outline, when one writes

$$\hat{q}'_2 = (q'_2, g'_2) \quad \hat{q}_2 = (q_2, g_2) \quad g'_2 = \exp(\mathbf{v}) g_2, \tag{1.12}$$

one get $\mathbf{v}$ close to $\bigoplus_{i=1}^n \mathbf{E}_i(q_2)$ but in order to proceed with the arguments involving the conditional measures, one needs $\mathbf{v}$ close to $\bigcup_{i=1}^n \mathbf{E}_i(q_2)$. Thus, a "tie-breaking" process is needed, i.e. one has to prove that if one changes u while keeping $\hat{q}$, $\hat{q}'$, and ℓ fixed, one can (for some fixed fraction of u make the vector $\mathbf{v}$ in (1.12) approach $\bigcup_{i=1}^n \mathbf{E}_i(\hat{q}_2)$. This tie-breaking process is done in §6. It is quite lengthy, and requires a lot of infrastructure, developed in §2-§5.

The case where the Zariski closure of $G_{\mathcal{S}}$ is semisimple has been previously handled in [BQ2], using the local limit theorem proved in [BQ3]. This paper gives an alternative proof of the main result of [BQ2]. Another potential approach to the case where (A1) and (A2)$'$ hold is the "floating" variant of the method of [BQ1], used in [EMi, §16] and in the setting of smooth dynamics on surfaces in Brown-Rodriguez-Hertz[B-RH]. We do not pursue this approach here, since it fails in most non-semisimple cases.

In the non-semisimple case, there might not be an analogue of (A2) or (A2)$'$ at all. For example, consider the random walk on $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ generated by the two matrices

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & (1/4) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & (1/4) \end{pmatrix}, \tag{1.13}$$

where each matrix has weight 1/2. In this example (A1) holds with $\mathbf{E}(x)$ not depending on x and equal to $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$. However, the action of the cocycle on $\mathbf{E}$ has a nilpotent part, and not all vectors in $\mathbf{E}$ grow at the same rate. This causes all the methods based on a time-changed martingale convergence theorem as in [BQ1] to fail, since one can not define a time change with the needed properties.

The main technical advantage of the method presented in this paper is that it can handle cases where (A2)$'$ fails (but uniform expansion holds). For instance, the example given by (1.13) is covered by Theorem 1.7. This is done by taking more care during the "tie-breaking" procedure: we show that by changing u (and keeping $\hat{q}$, $\hat{q}'$, and ℓ fixed) one can make the vector $\mathbf{v}$ in (1.12) approach one of finitely many subspaces (called $\mathbf{E}_{[ij],bdd}$ in §6), each of which satisfies (up to a bounded amount) some analogue of (A2). See Proposition 6.2 for an exact statement.

In the setting of Teichmüller dynamics, the need to handle non block-conformal situations analogous to (1.13) is the main difficulty in [EMi]. Ironically, it is proved near the end of [EMi] (see also [EMat]) that such cases do not actually occur in Teichmüller dynamics, even though they can not be ruled out a-priori. In the homogeneous dynamics setting, non block-conformal situations like (1.13) are ubiquitous.

Finally, the setting in Sargent-Shapira [SS] is not covered by our setup, since it does not satisfy uniform exansion (or even uniform exansion mod Z, for any Z).

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2. General cocycle lemmas

2.1. **Setup and notation.** Let $\Omega_0 = (\mathcal{S}^{\mathbb{Z}} \times M) \times [0,1]$. Let T^t denote the suspension flow on on Ω_0, i.e. T^t is obtained as a quotient of the flow $(x,s) \to (x, t+s)$ on $(\mathcal{S}^{\mathbb{Z}} \times M) \times \mathbb{R}$ by the equivalence relation $(x, s+1) \sim (Tx, s)$, where T is as in (1.6).

Measures on Ω_0. Let m denote the Haar measure on M. Let the measure $\tilde{\mu}$ on Ω_0 be the product of the measure $\mu^{\mathbb{Z}} \times m$ on $\mathcal{S}^{\mathbb{Z}} \times M$ and the Lebesgue measure on $[0,1]$.

Let $EBP \subset \mathcal{S}^{\mathbb{Z}}$ denote the set of sequences which are "eventually backwards periodic", i.e. $\omega \in EBP$ if an only if there exists $n > 0$ and $s > 0$ such that $\omega_{j+s} = \omega_j$ for $j < -n$. Let $\Omega_{ebp} = EBP \times M \times [0,1) \subset \Omega_0$. Since $\mu^{\mathbb{Z}}(EBP) = 0$, we have $\tilde{\mu}(\Omega_{ebp}) = 0$.

Suppose $x \in \Omega_0$. We will often denote the conditional measure measure $\tilde{\mu}|_{W_1^+[x]}$ by $|\cdot|$. Under the identification between $W_1^+[x]$ and $\mathcal{S}^{[0,\infty)}$, $|\cdot|$ becomes $\mu^{\mathbb{N}}$.

Cocycles and skew-products. For $x \in \mathcal{S}^{\mathbb{Z}} \times M$ and $n \in \mathbb{N}$, let $T_x^n \in G$ be as in (1.8). We also set $T_{T^n x}^{-n} = (T_x^n)^{-1}$, so that for $n \in \mathbb{Z}$, $\hat{T}^n(x,g) = (T^n x, T_x^n g)$. We then define

$$T_x^t = T_x^n, \qquad \text{where } n \text{ is the greatest integer smaller than or equal to } t.$$

We define $\hat{\Omega}_0 = \Omega_0 \times G$. We then have a skew-product flow $\hat{T}^t$ on Ω_0, defined by

$$\hat{T}^t(x,g) = (T^t x, T_x^t g).$$

Also Γ acts on $\hat{\Omega}_0$ on the right (by right multiplication on the second factor). We also use $\hat{T}$ to denote the induced map on $\hat{\Omega}_0/\Gamma$. We have an action on the trivial bundle $\Omega_0 \times \mathfrak{g}$ given by

$$\hat{T}^t(x,\mathbf{v}) = (T^t x, (T_x^t)_* \mathbf{v}).$$

We will also consider a certain finite set Σ and a lift of the flow T^t to $\Omega \equiv \Omega_0 \times \Sigma$, and of the flow $\hat{T}^t$ to $\hat{\Omega} \equiv \Omega_0 \times \Sigma \times G/\Gamma$. These will be defined in §2.3.

Lyapunov exponents. We fix some norm $\|\cdot\|_0$ on $\mathfrak{g}$, and apply the Osceledets multiplicative ergodic theorem to the cocycle $(T^t)_*$. Let λ_i denote the i-th Lyapunov exponent of this cocycle. We always number the exponents so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n.$$

Let

$$\{0\} = \mathcal{V}_{\leq 0}(x) \subset \mathcal{V}_{\leq 1}(x) \subset \cdots \subset \mathcal{V}_{\leq n}(x) = \mathfrak{g} \tag{2.1}$$

denote the backward flag, and let

$$\{0\} = \mathcal{V}_{\geq n+1}(x) \subset \mathcal{V}_{\geq n}(x) \subset \cdots \subset \mathcal{V}_{\geq 1}(x) = \mathfrak{g}$$

denote the forward flag. This means that for almost all $x \in \Omega_0$ and for $\mathbf{v} \in \mathcal{V}_{\leq i}(x)$ such that $\mathbf{v} \notin \mathcal{V}_{\leq i-1}(x)$,

$$\lim_{t \to -\infty} \frac{1}{t} \log \frac{\|(T_x^t)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i, \tag{2.2}$$

and for $\mathbf{v} \in \mathcal{V}_{\geq i}(x)$ such that $\mathbf{v} \notin \mathcal{V}_{\geq i+1}(x)$,

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|(T_x^t)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i. \tag{2.3}$$

It follows from (2.2) that for $y \in W_1^+[x]$, we have

$$\mathcal{V}_{\leq i}(y) = \mathcal{V}_{\leq i}(x). \tag{2.4}$$

Similarly, for $y \in W_1^-[x]$,

$$\mathcal{V}_{\geq i}(y) = \mathcal{V}_{\geq i}(x). \tag{2.5}$$

By e.g. [GM, Lemma 1.5], we have for a.e. $x \in \Omega$,

$$\mathfrak{g} = \mathcal{V}_{\leq i}(x) \oplus \mathcal{V}_{\geq i+1}(x). \tag{2.6}$$

Let

$$\mathcal{V}_i(x) = \mathcal{V}_{\leq i}(x) \cap \mathcal{V}_{\geq i}(x).$$

Then, in view of (2.6), for almost all x, we have

$$\mathcal{V}_{\leq i}(x) = \bigoplus_{j \leq i} \mathcal{V}_j(x), \qquad \mathcal{V}_{<i}(x) = \bigoplus_{j<i} \mathcal{V}_j(x),$$

$$\mathcal{V}_{\geq i}(x) = \bigoplus_{j \geq i} \mathcal{V}_j(x), \qquad \mathcal{V}_{>i}(x) = \bigoplus_{j>i} \mathcal{V}_j(x).$$

We have $\mathbf{v} \in \mathcal{V}_j(x)$ if and only if

$$\lim_{|t| \to \infty} \frac{1}{t} \log \frac{\|(T_x^t)_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \lambda_i.$$

The Lyapunov exponents λ_j and the Lyapunov subspaces $\mathcal{V}_j(x)$ do not depend on the choice of the norm $\| \cdot \|_0$.

It it easy to see that the subspaces

$$\bigoplus_{\lambda_j > 0} \mathcal{V}_j(x) \text{ and } \bigoplus_{\lambda_j < 0} \mathcal{V}_j(x)$$

are both nilpotent subalgebras of $\mathfrak{g}$. We thus define the unipotent subgroups $N^+(x)$ and $N^-(x)$ of G by

$$\mathrm{Lie}(N^+)(x) = \bigoplus_{\lambda_j > 0} \mathcal{V}_j(x), \qquad \mathrm{Lie}(N^-)(x) = \bigoplus_{\lambda_j < 0} \mathcal{V}_j(x).$$

There are the subgroups which appeared in §1.4.

2.2. Equivariant measurable flat connections. The maps $P^+(x,y)$ and $P^-(x,y)$.

For almost all $x \in \Omega$ and almost all $y \in W_1^+[x]$, any vector $\mathbf{v} \in \mathcal{V}_i(x)$ can be written uniquely as

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}'' \qquad \mathbf{v}' \in \mathcal{V}_i(y), \quad \mathbf{v}'' \in \mathcal{V}_{<i}(y).$$

Let $P_i^+(x,y) : \mathcal{V}_i(x) \to \mathcal{V}_i(y)$ be the linear map sending $\mathbf{v}$ to $\mathbf{v}'$. Let $P^+(x,y) : \mathfrak{g} \to \mathfrak{g}$ be the unique linear map which restricts to $P_i^+(x,y)$ on each of the subspaces $\mathcal{V}_i(x)$. (We think of the domain of $P^+(x,y)$ as the tangent space T_xG and of the co-domain of $P^+(x,y)$ as T_yG). We call $P^+(x,y)$ the "parallel transport" from x to y. The following is immediate from the definition:

Lemma 2.1. *Suppose $x, y \in W_1^+[z]$. Then*
- (a) $P^+(x,y)\mathcal{V}_i(x) = \mathcal{V}_i(y)$.
- (b) $P^+(T^t x, T^t y) = (T_y^t)_* \circ P^+(x,y) \circ (T_x^{-t})_*$.
- (c) $P^+(x,y)\mathcal{V}_{\le i}(x) = \mathcal{V}_{\le i}(y) = \mathcal{V}_{\le i}(x)$. *Thus, the map $P^+(x,y) : \mathfrak{g} \to \mathfrak{g}$ is unipotent.*
- (d) $P^+(x,z) = P^+(y,z) \circ P^+(x,y)$.

If $y \in W_1^-[x]$, then we can define a similar map which we denote by $P^-(x,y)$.

Recall that $\|\cdot\|_0$ denotes some fixed norm on $\mathfrak{g}$.

Lemma 2.2. *There exists $\alpha > 0$ depending only on the Lyapunov spectrum, and for every $\delta > 0$ there exists a subset $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and a constant $C(\delta) > 0$ such that for all $x \in K$, all $y \in W_1^-[x] \cap K$, and all $t > 0$,*

$$\sup_{\mathbf{v} \in \mathfrak{g} - \{0\}} \frac{\|P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \le C(\delta) e^{-\alpha t}. \tag{2.7}$$

Proof. Pick $\epsilon > 0$ smaller than $\frac{1}{3}\min_{i \ne j} |\lambda_i - \lambda_j|$. By the Osceledets multiplicative ergodic theorem, [KH, Theorem S.2.9 (2)] for $\tilde{\mu}$-a.e. $x \in \Omega$,

$$\lim_{t \to \infty} \frac{1}{t} \log |\sin \angle(\bigoplus_{i \in S} \mathcal{V}_i(T^{-t}x), \bigoplus_{j \notin S} \mathcal{V}_j(T^{-t}x))| = 0.$$

Therefore, there exists $K_1 \subset \Omega$ with $\tilde{\mu}(K_1) > 1 - \delta/2$ and $\sigma_0 = \sigma_0(\delta) > 0$ such that for $x \in K_1$, and any subset S of the Lyapunov exponents and any $t \ge 0$,

$$d_0(\bigoplus_{i \in S} \mathcal{V}_i(T^t x), \bigoplus_{j \notin S} \mathcal{V}_j(T^t x)) \ge \sigma_0 e^{-\epsilon t}. \tag{2.8}$$

(Here $d_0(\cdot,\cdot)$ is a distance on $\mathfrak{g}$ derived from the norm $\|\cdot\|_0$.) Then, (letting $t = 0$ in (2.8)), for all $x \in K_1$, all $y \in W_1^-[x] \cap K_1$, and all $\mathbf{w} \in \mathfrak{g}$,

$$\|P^-(x,y)\mathbf{w}\|_0 \le C(\delta)\|\mathbf{w}\|_0. \tag{2.9}$$

By the multiplicative ergodic theorem, there exists $K_2 \subset \Omega$ with $\tilde{\mu}(K_2) > 1 - \delta/2$ and $\rho = \rho(\delta) > 0$ such that for $x \in K_2$, any i, any $t > 0$ and any $\mathbf{w}_i \in \mathcal{V}_i(x)$,

$$\rho e^{(\lambda_i - \epsilon)t} \|\mathbf{w}_i\|_0 \le \|(T^t_x)_* \mathbf{w}_i\|_0 \le \rho^{-1} e^{(\lambda_i + \epsilon)t} \|\mathbf{w}_i\|_0. \tag{2.10}$$

Now let $K = K_1 \cap K_2$, and suppose $x \in K$, $y \in K$. Let $\mathbf{v}$ be such that the supremum in (2.7) is attained at $\mathbf{v}$. By (2.8) we may assume without loss of generality that $\mathbf{v} \in \mathcal{V}_i(T^t x)$ for some i. Let $\mathbf{w} \in \mathcal{V}_i(x)$ be such that $(T^t_x)_* \mathbf{w} = \mathbf{v}$. By (2.10),

$$\|\mathbf{v}\|_0 \ge \rho e^{(\lambda_i - \epsilon)t} \|\mathbf{w}\|_0. \tag{2.11}$$

Note that

$$P^-(T^t x, T^t y)\mathbf{v} = (T^t_y)_* P^-(x,y)\mathbf{w}.$$

Note that since $y \in W_1^-[x]$ and $t > 0$, $(T^t_x)_* = (T^t_y)_*$. By the definition of $P^-(x,y)$ we have

$$P^-(x,y)\mathbf{w} = \mathbf{w} + \sum_{j>i} \mathbf{w}_j, \qquad \mathbf{w}_j \in \mathcal{V}_j(x).$$

Thus,

$$P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v} = \sum_{j>i} (T^t_x)_* \mathbf{w}_j. \tag{2.12}$$

By (2.9), for all $j > i$,

$$\|\mathbf{w}_j\|_0 \le C_1(\delta)\|\mathbf{w}\|_0,$$

and then, by (2.10),

$$\|(T^t_x)_* \mathbf{w}_j\|_0 \le \rho^{-1} e^{(\lambda_j + \epsilon)t} \|\mathbf{w}_j\|_0 \le C_1(\delta) \rho^{-1} e^{(\lambda_j+\epsilon)t} \|\mathbf{w}\|_0.$$

Now, from (2.12) and (2.11),

$$\|P^-(T^t x, T^t y)\mathbf{v} - \mathbf{v}\|_0 \le \sum_{j>i} C_1(\delta) \rho^{-2} e^{(\lambda_j - \lambda_i + 2\epsilon)t} \|\mathbf{v}\|_0,$$

which immediately implies (2.7) since $\lambda_j < \lambda_i$ for $j > i$. □

2.3. **The Jordan Canonical Form of a cocycle.** Recall that the Lyapunov exponents of T^t are denoted λ_i, $1 \le i \le n$.

Zimmer's Amenable reduction. The following is a general fact about linear cocycles over an action of $\mathbb{R}$ or $\mathbb{Z}$. It is a special case of what is often called Zimmer's Amenable Reduction Theorem, see [Zi1].

Lemma 2.3. *There exists a finite set Σ and an extension of the flow T^t to $\Omega = \Omega_0 \times \Sigma$ such that the following holds: For each i, for almost all $x \in \Omega$, there there exists an invariant flag*

$$\{0\} = \mathcal{V}_{i,0}(x) \subset \mathcal{V}_{i,1}(x) \subset \cdots \subset \mathcal{V}_{i,n_i}(x) = \mathcal{V}_i(x), \tag{2.13}$$

and on each $\mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x)$ *there exists a nondegenerate quadratic form* $\langle\cdot,\cdot\rangle_{ij,x}$ *and a cocycle* $\lambda_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ *such that for all* $\mathbf{u}, \mathbf{v} \in \mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x)$*,*

$$\langle (T^t)_* \mathbf{u}, (T^t)_* \mathbf{v}\rangle_{ij,T^t x} = e^{\lambda_{ij}(x,t)} \langle \mathbf{u}, \mathbf{v}\rangle_{ij,x}.$$

Remark. The statement of Lemma 2.3 is the assertion that on the "finite cover" $\Omega \equiv \Omega_0 \times \Sigma$ of Ω_0 one can make a change of basis at each $x \in \Omega$ so that in the new basis, the matrix of the cocycle restricted to $\mathcal{V}_i$ is of the form

$$\begin{pmatrix} C_{i,1} & * & \dots & * \\ 0 & C_{i,2} & \dots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \dots & C_{i,n_i} \end{pmatrix}, \tag{2.14}$$

where each $C_{i,j}$ is a conformal matrix (i.e. is the composition of an orthogonal matrix and a scaling factor λ_{ij}).

Proof of Lemma 2.3. See [ACO]. Recall that a cocycle is *block-conformal* if in a suitable basis it can be written in the form (2.14) with all the off-diagonal entries labeled $*$ in (2.14) 0, and in addition, the cocycle is allowed to permute the blocks. The statement of Lemma 2.3 differs slightly from that of [ACO, Theorem 5.6] in that we want the cocycle in each block to be conformal (and not just block-conformal). However, our statement is in fact equivalent because we are willing to replace the original space Ω_0 by $\Omega \equiv \Omega_0 \times \Sigma$. □

Lifting $\tilde{\mu}$ to Ω and $\hat{\nu}$ to $\hat{\Omega}/\Gamma$. Let the measure $\tilde{\mu}$ on Ω denote the product of the measure $\tilde{\mu}$ on Ω_0 and the counting measure on Σ. Without loss of generality, we may assume that $\tilde{\mu}$ is $\hat{T}^t$-ergodic on Ω (or else we can make Σ smaller). Similarly, let the measure $\hat{\nu}$ on $\hat{\Omega}/\Gamma$ denote the product of the measure $\hat{\nu}$ on $\hat{\Omega}_0/\Gamma$ and the counting measure on Σ. Then, $\hat{\nu}$ is $\hat{T}$-ergodic in view of Proposition 1.12.

2.4. Covariantly constant subspaces.

Lemma 2.4. *Suppose* $\bar{M}(\cdot) \subset \mathfrak{g}$ *is a* T^t*-equivariant subbundle of the trivial bundle* $\Omega \times \mathfrak{g}$ *over the base* Ω*. Suppose also for a.e* $x \in \Omega$*,* $\mathcal{V}_{<i}(x) \subseteq \bar{M}(x) \subseteq \mathcal{V}_{\le i}(x)$*. Then, (up to a set of measure* 0*),* $\bar{M}(x)$ *depends only on* x^-*, and is thus constant along sets of the form* $W_1^+[z]$*.*

Proof of Lemma 2.4. Note that the quotient bundle $\mathcal{V}_{\le i}(x)/\mathcal{V}_{<i}(x)$ is constant along $W^+[x]$ and has a single Lyapunov exponent. Now the result follows immediately from [L, Theorem 1]. □

Lemma 2.5. *Suppose* $M(\cdot) \subset \mathfrak{g}$ *is a* T^t*-equivariant subbundle over the base* Ω*. Then, (up to a set of measure* 0*), for* $y \in W^+[x]$*,* $M(y) = P^+(x,y)M(x)$*.*

Proof. Since $M(\cdot)$ is T^t-equivariant, we have, for a.e. $x \in \Omega$,

$$M(x) = \bigoplus_{i=1}^{n} M_i(x) \qquad \text{where } M_i(x) = M(x) \cap \mathcal{V}_i(x).$$

Let $\bar{M}_i(x) = \mathcal{V}_{<i}(x) + M_i(x)$. Then, $\mathcal{V}_{<i}(x) \subseteq \bar{M}_i(x) \subseteq \mathcal{V}_{\le i}(x)$, and thus by Lemma 2.4, $\bar{M}_i(y) = \bar{M}_i(x)$ for almost all $y \in W^+[x]$. Suppose $\mathbf{v} \in M_i(x)$, and write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ where $\mathbf{v}' \in \mathcal{V}_i(y)$ and $\mathbf{v}'' \in \mathcal{V}_{<i}(y) = \mathcal{V}_{<i}(x)$. Since $\mathbf{v} \in \bar{M}_i(x)$ and $\bar{M}_i(y) = \bar{M}_i(x)$, $\mathbf{v} \in \bar{M}_i(y)$. Since $\bar{M}_i(y) \supset \mathcal{V}_{<i}(y)$, this implies that $\mathbf{v}' \in \bar{M}_i(y) \cap \mathcal{V}_i(y) = M_i(y)$. But, by the definition of $P^+(x,y)$, $P^+(x,y)\mathbf{v} = \mathbf{v}'$. Hence, $M_i(y) = P^+(x,y)M_i(x)$, and therefore $M(y) = P^+(x,y)M(x)$. $\square$

Lemma 2.6. *Suppose $x \in \Omega$ and $y \in W^+[x]$. Then, (outside of a set of measure 0), we have*

$$\langle \cdot, \cdot \rangle_{ij,x} = c(x,y)\langle \cdot, \cdot \rangle_{ij,y} \tag{2.15}$$

where $\langle \cdot, \cdot \rangle_{ij,x}$ is the inner product on $\mathcal{V}_{i,j}(x)/\mathcal{V}_{i,j-1}(x)$ of Lemma 2.3, and $c(x,y) \in \mathbb{R}^+$. In other words, the inner products $\langle \cdot, \cdot \rangle_{ij,x}$ are, up to a scaling factor, independent of x^+.

Proof. By Lemma 2.4 the $\mathcal{V}_{ij}(y)$ are independent of y for $y \in W^+[x]$. Let $K \subset \Omega$ denote a compact subset with $\tilde{\mu}(K) > 0.9$ where the function $x \to \langle \cdot, \cdot \rangle_{ij,x}$ is uniformly continuous. Consider the points $T^t x$ and $T^t y$, as $t \to -\infty$. Then, $d(T^t x, T^t y) \to 0$. Suppose $\mathbf{v}, \mathbf{w} \in \mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x) = \mathcal{V}_{ij}(y)/\mathcal{V}_{i,j-1}(y)$. Let

$$\mathbf{v}_t = e^{-\lambda_{ij}(x,t)}(T^t_x)_* \mathbf{v}, \quad \mathbf{w}_t = e^{-\lambda_{ij}(x,t)}(T^t_x)_* \mathbf{w},$$

where $\lambda_{ij}(x,t)$ is as in Lemma 2.3. Then, by Lemma 2.3, we have

$$\langle \mathbf{v}_t, \mathbf{w}_t \rangle_{ij,T^t x} = \langle \mathbf{v}, \mathbf{w} \rangle_{ij,x}, \quad \langle \mathbf{v}_t, \mathbf{w}_t \rangle_{ij,T^t y} = c(x,y,t)\langle \mathbf{v}, \mathbf{w} \rangle_{ij,y}. \tag{2.16}$$

where $c(x,y,t) = e^{\lambda_{ij}(x,t) - \lambda_{ij}(y,t)}$.

Now take a sequence $t_k \to -\infty$ with $T^{t_k} x \in K$, $T^{t_k} y \in K$ (such a sequence exists for $\tilde{\mu}$-a.e. x and y with $y \in W^+[x]$). After passing to a further subsequence, we may assume that as $t_k \to -\infty$, $c(x,y,t_k) \to c(x,y)$, where $c(x,y) \in \mathbb{R}^+ \cup \{\infty\}$. Also,

$$\langle \mathbf{v}_{t_k}, \mathbf{w}_{t_k} \rangle_{ij,T^{t_k} x} - \langle \mathbf{v}_{t_k}, \mathbf{w}_{t_k} \rangle_{ij,T^{t_k} y} \to 0.$$

Now the equation (2.15) follows from (2.16). Since both the inner products in (2.15) are non-degenerate, we have that outside of a set of measure 0, $c(x,y)$ is neither 0 nor ∞. $\square$

The function $\Xi(x)$. For $x \in \Omega$, let

$$\Xi^+(x) = \max_{ij} \sup \left\{ \langle \mathbf{v}, \mathbf{v} \rangle_{ij,x}^{1/2}, \ : \ \mathbf{v} \in \mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x), \ \|\mathbf{v}\|_0 = 1 \right\},$$

and let

$$\Xi^{-}(x) = \min_{ij} \inf \left\{ \langle \mathbf{v}, \mathbf{v} \rangle_{ij,x}^{1/2}, \ : \ \mathbf{v} \in \mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x), \ \|\mathbf{v}\|_0 = 1 \right\}.$$

Let

$$\Xi(x) = \Xi^{+}(x)/\Xi^{-}(x).$$

We have $\Xi(x) \geq 1$ for all $x \in \Omega$. For $x \in \Omega_0$, we define $\Xi(x)$ to be the maximum of Ξ over all the preimages of x under the projection $\Omega \to \Omega_0$.

Distance between subspaces. For a subspace V of $\mathfrak{g}$, let SV denote the intersection of V with the unit ball in the $\|\cdot\|_0$ norm. For subspaces V_1, V_2 of $\mathfrak{g}$, we define

$$d_0(V_1, V_2) = \text{The Hausdorff distance between } SV_1 \text{ and } SV_2 \tag{2.17}$$

measured with respect to the distance induced from the norm $\|\cdot\|_0$.

Lemma 2.7. *Fix $\epsilon > 0$ sufficiently small depending on the dimension and on the Lyapunov exponents. There exists a compact subset $\mathcal{C} \subset \Omega$ with $\tilde{\mu}(\mathcal{C}) > 0$ and a function $T_0 : \mathcal{C} \to \mathbb{N} \cup \{\infty\}$ with $T_0(c) < \infty$ for $\tilde{\mu}$-a.e. $c \in \mathcal{C}$ such that the following hold:*

(a) *There exists $\sigma_0 > 0$ such that for all $c \in \mathcal{C}$, and any subset S of the Lyapunov exponents,*

$$d_0\Big(\bigoplus_{i \in S} \mathcal{V}_i(c), \bigoplus_{j \notin S} \mathcal{V}_j(c)\Big) \geq \sigma_0.$$

(b) *There exists $M' > 1$ such that for all $c \in \mathcal{C}$, $\Xi(c) \leq M'$.*

(c) *For all $c \in \mathcal{C}$, for all $t > T_0(c)$ and for any subset S of the Lyapunov spectrum,*

$$d_0\Big(\bigoplus_{i \in S} \mathcal{V}_i(T^{-t}c), \bigoplus_{j \notin S} \mathcal{V}_j(T^{-t}c)\Big) \geq e^{-k\epsilon t}, \tag{2.18}$$

where k is a constant depending only on the dimension. The constant k is chosen so that (2.18) implies that for all $c \in \mathcal{C}$ and all $t > T_0(c)$ and all $c' \in \mathcal{C} \cap W_1^{+}[g_{-t}c]$, we have that

$$\|P^{+}(T^{-t}c, c')\|_0 \equiv \sup_{\mathbf{v} \neq 0} \frac{\|P^{+}(T^{-t}c, c')\mathbf{v}\|_0}{\|\mathbf{v}\|_0}$$

satisfies

$$\rho_1 e^{-\epsilon t} \leq \|P^{+}(T^{-t}c, c')\|_0 \leq \rho_1^{-1} e^{\epsilon t}. \tag{2.19}$$

(d) *There exists $\rho > 0$ such that for all $c \in \mathcal{C}$, for all $t > T_0(c)$, for all i and all $\mathbf{v} \in \mathcal{V}_i(c)$,*

$$\rho e^{-(\lambda_i + \epsilon)t} \|\mathbf{v}\|_0 \leq \|(T_c^{-t})_* \mathbf{v}\|_0 \leq \rho^{-1} e^{-(\lambda_i - \epsilon)t} \|\mathbf{v}\|_0.$$

Proof. Parts (a) and (b) hold since the inverse of the angle between Lyapunov subspaces and the ratio of the norms are finite a.e., therefore bounded on a set of almost full measure. Part (c) was already established as part of the proof of Lemma 2.2, see (2.8). Also, (d) follows immediately from the multiplicative ergodic theorem. $\square$

2.5. A Markov Partition.

Proposition 2.8. *Suppose $\mathcal{C} \subset \Omega$ is a set with $\tilde{\mu}(\mathcal{C}) > 0$, and $T_0 : \mathcal{C} \to \mathbb{R}^+$ is a measurable function which is finite a.e. Then we can find $x_0 \in \Omega$, a subset $\mathcal{C}_1 \subset W_1^-[x_0] \cap \mathcal{C}$ and for each $c \in \mathcal{C}_1$ a subset $B^+[c] \subset W_1^+[c]$ depending measurably on c, and a number $t(c) > 0$ such that if we let*

$$J_c = \bigcup_{0 \le t < t(c)} T^{-t} B^+[c],$$

then the following holds:

(a) *$B^+[c]$ is relatively open in $W_1^+[c]$, and $\tilde{\mu}|_{W_1^+[c]}(B^+[c]) > 0$.*
(b) *$J_c \cap J_{c'} = \emptyset$ if $c \neq c'$.*
(c) *$\bigcup_{c \in \mathcal{C}_1} J_c$ is conull in Ω.*
(d) *For every $c \in \mathcal{C}_1$ there exists $c' \in \mathcal{C}_1$ such that $T^{-t(c)} B^+[c] \subset B^+[c']$.*
(e) *$t(c) > T_0(c)$ for all $c \in \mathcal{C}_1$.*

Proof. This proof is essentially identical to the proof of [MaT, Lemma 9.1], except that we need to take care that (e) is satisfied.

We choose a metric $d(\cdot, \cdot)$ on Ω so that for all $c \in \Omega$ and all $x \in W_1^+[c]$,

$$d(Tx, Tc) \le \frac{1}{10} d(x, c).$$

For $a > 0$ and $c \in \Omega$, let $V_a^+[c]$ denote the intersection of $W_1^+[c]$ and the open ball of radius c.

Lemma 2.9. *Let $\mathcal{C} \subset \Omega$ be such that $\tilde{\mu}(\mathcal{C}) > 0$, and let $T_1 : \mathcal{C} \to \mathbb{R}^+$ is a measurable function which is finite a.e. Then there exists $a > 0$, $x_0 \in \mathcal{C}$ and $\mathcal{C}_1 \subset W_1^-[x_0] \cap \mathcal{C}$ such that the following hold:*

(a) *$\tilde{\mu}|_{W_1^-[x_0]}(\mathcal{C}_1) > 0$.*
(b) *For all $c \in \mathcal{C}_1$, every neighborhood of c in $W_1^+[c]$ has positive $\tilde{\mu}|_{W_1^+[c]}$ measure.*
(c) *For any $\epsilon > 0$, $\tilde{\mu}\left(\bigcup_{|t|<\epsilon} T^t \bigcup_{c \in \mathcal{C}_1} V_{a_0}^+[c]\right) > 0$, where $a_0 = a/10$.*
(d) *For all $x \in \bigcup_{c \in \mathcal{C}_1} V_a^+[c]$ and all $0 < t < T_1(c)$, $T^t x \notin \bigcup_{c \in \mathcal{C}_1} V_a^+[c]$.*

Proof. Choose $T_2 > 0$ so that if we let $\mathcal{C}_2 = \{x \in \mathcal{C} \;:\; T_1(x) < \frac{1}{2} T_2 - 1\}$ then $\tilde{\mu}(\mathcal{C}_2) > 0$. Let $\mathcal{C}_3 \subset \mathcal{C}_2$ be such that $\tilde{\mu}(\mathcal{C}_3) > 0$ and for $c \in \mathcal{C}_3$, every neighborhood of c in $W_1^+[c]$ has positive $\tilde{\mu}|_{W_1^+[c]}$ measure. Let μ' be the restriction of μ to $\mathcal{C}_3$. By

Lusin's theorem, we can choose $\mathcal{C}_4 \subset \mathcal{C}_3$ with $\mu'(\mathcal{C}_4) > 0$ such that the conditional measure $\mu'|_{W_1^-[c]}$ depends continuously on c as c varies over $\mathcal{C}_4$.

Let $\Omega_{per} \subset \Omega$ denote the set of periodic orbits of T^t. Since $\Omega_{per} \subset \Omega_{ebp}$ and $\tilde{\mu}(\Omega_{ebp}) = 0$ (see §2.1) we can find $x_0 \in \mathcal{C}_4$ such that $x_0 \notin \Omega_{per}$, for every neighborhood V' of x_0, $\tilde{\mu}(V' \cap \mathcal{C}_4) > 0$, and also every neighborhood of x_0 in $W_1^-[x_0]$ has positive $\mu'|_{W_1^-[x_0]}$-measure.

Since x_0 is not periodic and $x \to T^t x$ is continuous, we can find a neighborhood V^- of x_0 in $W_1^-[x_0]$ and $a > 0$ such that for any $x \in \bigcup_{c \in V^-} V_a^+[c]$, $T^t x \notin \bigcup_{c \in V^-} V_a^+[c]$ for $0 < t < T_1$.

Now let $\mathcal{C}_1 = V^- \cap \mathcal{C}_4$. Then (a), (b) and (d) follow by construction. Let E_0 be the set in (c). We will show that $\mu'(E_0) = \tilde{\mu}(E_0 \cap \mathcal{C}_3) > 0$. To do that we disintegrate μ' along the partition whose atoms are of the form $W_1^-[x]$. Let Y denote the quotient space, and μ'_Y the quotient measure. Let $\bar{\mathcal{C}}_4 \subset Y$ denote the image of $\mathcal{C}_4$ under the quotient map. Then,

$$\mu'(E_0) = \int_Y \mu'|_{W_1^-[\bar{c}]}(E_0 \cap W_1^-[\bar{c}])\, d\mu'_Y(\bar{c}) \geq \int_{\bar{\mathcal{C}}_4} \mu'|_{W_1^-[\bar{c}]}(E_0 \cap W_1^-[\bar{c}])\, d\mu'_Y(\bar{c}).$$

Let $\bar{x}_0 \in Y$ denote the image of x_0 under the quotient map. Then the intersection with $\bar{\mathcal{C}}_4$ of every neighborhood of $\bar{x}_0 \in Y$ has positive μ'_Y-measure (or else the intersection with $\mathcal{C}_4$ of some neighborhood of x_0 would have 0 μ'-measure). Also since E_0 is a product set, $\mu'|_{W_1^-[\bar{x}_0]}(E_0 \cap W_1^-[\bar{x}_0]) > 0$ and $\mu'|_{W_1^-[c]}(E_0 \cap W_1^-[c])$ depends continuously on $c \in \mathcal{C}_4$, we have $\mu'|_{W_1^-[\bar{c}]}(E_0 \cap W_1^-[\bar{c}]) > 0$ for all $\bar{c}$ in the intersection with $\bar{\mathcal{C}}_4$ with some neighborhood of $\bar{x}_0$. Therefore $\mu'(E_0) > 0$. □

We now fix a, $a_0 = a/10$ and $\mathcal{C}_1$ as in Lemma 2.9. Recall that $V_a^+[c]$ is the intersection with $W^+[c]$ of a ball in the $d(\cdot, \cdot)$ metric of radius a and centered at c.

Most of the proof of Proposition 2.8 consists of the following:

Lemma 2.10. *(cf. [MaT, Lemma 9.1]) For each $c \in \mathcal{C}_1$ there exists a subset $B^+[c] \subset W^+[c]$ such that*

(1) $V_{a_0}^+[c] \subset B^+[c] \subset V_a^+[c]$.
(2) *$B^+[c]$ is open in $W_1^+[c]$, and for any $\epsilon > 0$ the subset $E \equiv \bigcup_{c \in C_1} B^+[c]$ satisfies* $\tilde{\mu}\left(\bigcup_{|t|<\epsilon} T^t E\right) > 0$.
(3) *Whenever*

$$T^{-n} B^+[c] \cap E \neq 0, \qquad c \in \mathcal{C}_1, \qquad n > 0,$$

we have $T^{-n} B^+[c] \subset E$.

Proof. Let $B^{(0)}[c] = V_{a_0}^+[c]$, and for $j > 0$ let

$$B^{(j)}[c] = B^{(j-1)}[c] \cup \{T^{-n} V_{a_0}^+[c'] \ : \ c' \in \mathcal{C}_1, n > 0 \text{ and } T^{-n} V_{a_0}^+[c'] \cap B^{(j-1)}[c] \neq 0\}.$$

Let

$$B^+[c] = \bigcup_{j \ge 0} B^{(j)}[c], \qquad \text{and } E = \bigcup_{c \in C_1} B^+[c].$$

It easily follows from the above definition that property (3) holds, $B^+[c]$ is open in $W_1^+[c]$ and that $B^+[c] \supset V_{a_0}^+[c]$. Now (2) follows from Lemma 2.9(c). It remains to show that $B^+[c] \subset V_a[c]$. It is enough to show that for each j,

$$d(x, c) < a/2, \quad \text{for all } x \in B^{(j)}[c]. \tag{2.20}$$

This is done by induction on j. The case $j = 0$ holds since $a_0 = a/10 < a/2$. Suppose (2.20) holds for $j-1$, and suppose $x \in B^{(j)}[c] \smallsetminus B^{(j-1)}[c]$. Then there exist $c_0 = c, c_1, \dots, c_j = x$ in C_1 and non-negative integers $n_0 = 0, \dots, n_j$ such that for all $1 \le k \le n$,

$$T^{-n_k}(V_{a_0}^+[c_k]) \cap T^{-n_{k-1}}(V_{a_0}^+[c_{k-1}]) \neq \emptyset. \tag{2.21}$$

Let $1 \le k \le j$ be such that n_k is minimal. Recall that $V_{a_0}^+[y] \cap V_{a_0}^+[z] = \emptyset$ if $y \neq z$, $y \in \mathcal{C}_1$, $z \in \mathcal{C}_1$. Therefore, in view of the inductive assumption, $n_k \ge 1$. Applying T^{n_k} to (2.21) we get

$$\left(\bigcup_{i=1}^{k-1} T^{-n_i+n_k} V_{a_0}^+[c_i] \right) \cap V_{a_0}^+[c_k] \neq \emptyset, \quad \text{and} \quad \left(\bigcup_{i=k+1}^{j} T^{-n_i+n_k} V_{a_0}^+[c_i] \right) \cap V_{a_0}^+[c_k] \neq \emptyset.$$

Therefore, in view of (2.21), and the definition of the sets $B^{(j)}[c]$,

$$\left(\bigcup_{i=1}^{k} T^{-n_i+n_k} V_{a_0}^+[c_i] \right) \subset B^{(k)}[c_k], \quad \text{and} \quad \left(\bigcup_{i=k}^{j} T^{-n_i+n_k} V_{a_0}^+[c_i] \right) \subset B^{(j-k)}[c_k]$$

By the induction hypothesis, $\operatorname{diam}(B^{(k)}[c_k]) < a/2$, and $\operatorname{diam}(B^{(j-k)}[c_k]) < a/2$. Therefore,

$$\operatorname{diam}\left(\bigcup_{i=1}^{j} T^{-n_i+n_k} V_{a_0}^+[c_i] \right) \le a.$$

Then, applying T^{-n_k} we get,

$$\operatorname{diam}\left(\bigcup_{i=1}^{j} T^{-n_i} V_{a_0}^+[c_i] \right) \le \frac{a}{10}$$

Since $\operatorname{diam}(V_{a_0}^+[c]) \le a/10$, we get

$$\operatorname{diam}\left(\bigcup_{i=0}^{j} T^{-n_i} V_{a_0}^+[c_i] \right) \le \operatorname{diam}(V_{a_0}^+[c_0]) + \operatorname{diam}\left(\bigcup_{i=1}^{j} T^{-n_i} V_{a_0}^+[c_i] \right) \le \frac{a}{10} + \frac{a}{10} < \frac{a}{2}.$$

But the set on the left-hand-side of the above equation contains both $c = c_0$ and $x = c_j$. Therefore $d(c, x) < a/2$, proving (2.20). Thus (1) holds. $\square$

Proof of Proposition 2.8. Let $E = \bigcup_{c \in \mathcal{C}_1} B^+[c]$. For $x \in E$, let $t(x) \in \mathbb{R}^+$ be the smallest such that $T^{-t(x)}x \in E$. By property (3), the function $t(x)$ is constant on each set of the form $B^+[c]$. Let $F_t = \{x \in E \ : \ t(x) = t\}$. By property (2) and the ergodicity of T^{-t}, up to a null set,

$$\Omega = \bigsqcup_{t>0} \bigsqcup_{s<t} T^{-s} F_t.$$

Then properties (a)-(f) are easily verified. □

Warning. We lift the partition defined in Proposition 2.8 from Ω_0 to Ω and denote the resulting sets by the same letters. Also, we sometimes use $\tilde{\mu}$ as a measure on Ω as well as Ω_0.

Notation. For $x \in \Omega_0$, let $J[x]$ denote the set J_c containing x.

Lemma 2.11. *Suppose $x \in \Omega_0$, $y \in W^+[x] \cap J[x]$. Then for any $t > 0$,*

$$T^{-t}y \in J[T^{-t}x] \cap W^+[T^{-t}x].$$

Proof. This follows immediately from property (e) of Proposition 2.8. □

Notation. For $x \in \Omega_0$, let

$$\mathfrak{B}_t[x] = T^{-t}(J[T^t x] \cap W_1^+[T^t x]).$$

Lemma 2.12.

(a) *For $t' > t \geq 0$, $\mathfrak{B}_{t'}[x] \subseteq \mathfrak{B}_t[x]$.*
(b) *Suppose $t \geq 0, t' \geq 0$, $x \in \Omega_0$ and $x' \in \Omega_0$ are such that $\mathfrak{B}_t[x] \cap \mathfrak{B}_{t'}[x'] \neq \emptyset$. Then either $\mathfrak{B}_t[x] \supseteq \mathfrak{B}_{t'}[x']$ or $\mathfrak{B}_{t'}[x'] \supseteq \mathfrak{B}_t[x]$ (or both).*

Proof. Part (a) is a restatement of Lemma 2.11. For (b), without loss of generality, we may assume that $t' \geq t$. Then, by (a), we have $\mathfrak{B}_t[x] \cap \mathfrak{B}_t[x'] \neq \emptyset$.

Suppose $y \in \mathfrak{B}_t[x] \cap \mathfrak{B}_t[x']$. Then $T^t y \in \mathfrak{B}_0[T^t x]$ and $T^t y \in \mathfrak{B}_0[T^t x']$. Since the sets $\mathfrak{B}_0[z]$, $z \in \Omega_0$ form a partition, we must have $\mathfrak{B}_0[T^t x] = \mathfrak{B}_0[T^t x']$. Therefore, $\mathfrak{B}_t[x] = \mathfrak{B}_t[x']$, and thus, by (a),

$$\mathfrak{B}_{t'}[x'] \subseteq \mathfrak{B}_t[x'] = \mathfrak{B}_t[x].$$

□

Recall the notation $|\cdot|$ from §2.1. Then, in view of Proposition 2.8(a), for almost all $x \in \Omega$ and all $t \geq 0$, $|\mathfrak{B}_t[x]| > 0$.

Lemma 2.13. *Suppose $\delta > 0$ and $K \subset \Omega_0$ is such that $\tilde{\mu}(K) > 1 - \delta$. Then for any $\eta > \delta$ there exists a subset $K^* \subset K$ with $\tilde{\mu}(K^*) > 1 - \eta$ such that for any $x \in K^*$, and any $t > 0$,*

$$|K \cap \mathfrak{B}_t[x]| \geq (1 - (\delta/\eta))|\mathfrak{B}_t[x]|.$$

Proof. Let $E = K^c$, so $\tilde{\mu}(E) \le \delta$. Let E^* denote the set of $x \in \Omega_0$ such that there exists some $\tau \ge 0$ with

$$|E \cap \mathfrak{B}_\tau[x]| \ge (\delta/\eta)|\mathfrak{B}_\tau[x]|. \tag{2.22}$$

It is enough to show that $\tilde{\mu}(E^*) \le \eta$. Let $\tau(x)$ be the smallest $\tau > 0$ so that (2.22) holds for x. Then the (distinct) sets $\{\mathfrak{B}_{\tau(x)}[x]\}_{x \in E^*}$ cover E^* and are pairwise disjoint by Lemma 2.12(b). Let

$$F = \bigcup_{x \in E^*} \mathfrak{B}_{\tau(x)}[x].$$

Then $E^* \subset F$. For every set of the form $\mathfrak{B}_0[y]$, let $\Delta(y)$ denote the set of distinct sets $\mathfrak{B}_{\tau(x)}[x]$ where x varies over $\mathfrak{B}_0[y]$. Then, by (2.22)

$$|F \cap \mathfrak{B}_0[y]| = \sum_{\Delta(y)} |\mathfrak{B}_{\tau(x)}[x]| \le (\eta/\delta) \sum_{\Delta(y)} |E \cap \mathfrak{B}_{\tau(x)}[x]| \le (\eta/\delta)|E \cap \mathfrak{B}_0[y]|.$$

Integrating over y, we get $\tilde{\mu}(F) \le (\eta/\delta)\tilde{\mu}(E)$. Hence,

$$\tilde{\mu}(E^*) \le \tilde{\mu}(F) \le (\eta/\delta)\tilde{\mu}(E) \le \eta.$$

□

2.6. **Dynamically defined norms.** For almost all $x \in \Omega$, we will define a cetain dynamical norm $\|\cdot\|_x$ on $\mathfrak{g}$, which has some advantages over the fixed norm $\|\cdot\|_0$. If it is clear from the context, we sometimes drom the subscript and write $\|\cdot\|$ instead of $\|\cdot\|_x$.

The main result of this subsection is the following:

Proposition 2.14. *There exists a T^t-invariant subset $H \subset \Omega$ with $\tilde{\mu}(H) = 1$ and for all $x \in H$ there exists an inner product $\langle \cdot, \cdot \rangle_x$ on $\mathfrak{g}$ and cocycles $\lambda_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ with the following properties:*

(a) *For all $x \in H$, the distinct eigenspaces $\mathcal{V}_i(x)$ are orthogonal.*
(b) *Let $\mathcal{V}'_{ij}(x)$ denote the orthogonal complement, relative to the inner product $\langle \cdot, \cdot \rangle_x$ of $\mathcal{V}_{i,j-1}(x)$ in $\mathcal{V}_{ij}(x)$. Then, for all $x \in H$, all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathcal{V}'_{ij}(x) \subset \mathfrak{g}$,*
$$(T^t_x)_* \mathbf{v} = e^{\lambda_{ij}(x,t)} \mathbf{v}' + \mathbf{v}'',$$
where $\lambda_{ij}(x,t) \in \mathbb{R}$, $\mathbf{v}' \in \mathcal{V}'_{ij}(T^t x)$, $\mathbf{v}'' \in \mathcal{V}_{i,j-1}(T^t x)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. Hence (since $\mathbf{v}'$ and $\mathbf{v}''$ are orthogonal),
$$\|(T^t_x)_* \mathbf{v}\| \ge e^{\lambda_{ij}(x,t)} \|\mathbf{v}\|.$$
(c) *There exists a constant $\kappa > 1$ such that for all $x \in H$ and for all $t > 0$, and all i such that $\lambda_i > 0$,*
$$\kappa^{-1} t \le \lambda_{ij}(x,t) \le \kappa t.$$

(d) *There exists a constant $\kappa > 1$ such that for all $x \in H$ and for all $\mathbf{v} \in \mathrm{Lie}(N^+)(x)$, and all $t \geq 0$,*

$$e^{\kappa^{-1}t}\|\mathbf{v}\| \leq \|(T_x^t)_*\mathbf{v}\| \leq e^{\kappa t}\|\mathbf{v}\|,$$

and for all $x \in H$ and for all $\mathbf{v} \in \mathrm{Lie}(N^-)(x)$, and all $t \geq 0$,

$$e^{-\kappa t}\|\mathbf{v}\| \leq \|(T_x^t)_*\mathbf{v}\| \leq e^{-\kappa^{-1}t}\|\mathbf{v}\|.$$

Also, for all $\mathbf{v} \in \mathfrak{g}$ and all $t \in \mathbb{R}$,

$$e^{-\kappa|t|}\|\mathbf{v}\| \leq \|(T_x^t)_*\mathbf{v}\| \leq e^{\kappa|t|}\|\mathbf{v}\|.$$

In particular, the map $t \to \|(T_x^t)_*\mathbf{v}\|$ is continuous.

(e) *For all $x \in H$, all $y \in \mathfrak{B}_0[x] \cap H$ and all $t \leq 0$,*

$$\lambda_{ij}(x,t) = \lambda_{ij}(y,t).$$

(f) *For a.e. $x \in H$, a.e. $y \in \mathfrak{B}_0[x] \cap H$, and any $\mathbf{v}, \mathbf{w} \in \mathfrak{g}$,*

$$\langle P^+(x,y)\mathbf{v}, P^+(x,y)\mathbf{w}\rangle_y = \langle \mathbf{v}, \mathbf{w}\rangle_x.$$

We often omit the subscript from $\langle\cdot,\cdot\rangle_x$ and from the associated norm $\|\cdot\|_x$.

Proof strategy. Let $\mathcal{C}$ and T_0 be as in Lemma 2.7. Let $T_1 : \mathcal{C} \to \mathbb{R}^+$ be a finite a.e. measurable function to be chosen later. We will choose T_1 so that in particular $T_1(c) > T_0(c)$ for a.e. $c \in \mathcal{C}$. Let $\mathcal{C}_1$ and $B^+[c]$ for $c \in \mathcal{C}_1$ be as in Proposition 2.8. For $c \in \mathcal{C}_1$, let $t(c)$ be the smallest $t > 0$ such that $T^t c \in \mathcal{C}_1$.

The inner product $\langle\cdot,\cdot\rangle_x$ is first defined for $x \in \mathcal{C}_1$, and then extended to any $x \in B^+[c]$, using $P^+(c,x)$. We then interpolate between $x \in B^+[c]$ and $T^{-t(c)}x$.

The inner products $\langle\cdot,\cdot\rangle_{ij}$ on $B^+[c]$. Note that the inner products $\langle\cdot,\cdot\rangle_{ij}$ and the $\mathbb{R}$-valued cocycles λ_{ij} of Lemma 2.3 are not unique, since we can always multiply $\langle\cdot,\cdot\rangle_{ij,x}$ by a scalar factor $c(x)$, and then replace $\lambda_{ij}(x,t)$ by $\lambda_{ij}(x,t) + \log c(T^t x) - \log c(x)$. In view of Lemma 2.4 and (2.15) we may (and will) use this freedom to make $\langle\cdot,\cdot\rangle_{ij,x}$ constant on each set $B^+[c]$.

$$\{0\} = \mathcal{V}_{\leq 0}(c) \subset \mathcal{V}_{\leq 1}(c) \subset \dots \tag{2.23}$$

be the Lyapunov flag for the cocycle $(T_c^t)_*$, and for each i, let

$$\mathcal{V}_{\leq i-1}(c) = \mathcal{V}_{\leq i,0}(c) \subset \mathcal{V}_{i,1}(c) \subset \dots \mathcal{V}_{\leq i,n_i}(c) = \mathcal{V}_{\leq i}(c) \tag{2.24}$$

be a maximal invariant refinement.

Suppose $c \in \mathcal{C}_1$. By Lemma 2.7 (b), we can (and do) rescale the inner products $\langle\cdot,\cdot\rangle_{ij,c}$ so that after the rescaling, for all $\mathbf{v} \in \mathcal{V}_{ij}(c)/\mathcal{V}_{i,j-1}(c)$,

$$(M')^{-1}\|\mathbf{v}\|_0 \leq \langle \mathbf{v}, \mathbf{v}\rangle_{ij,c}^{1/2} \leq M'\|\mathbf{v}\|_0,$$

where $M' > 1$ is as in Lemma 2.7.

Now for $c \in \mathcal{C}_1$ we can choose $\mathcal{V}'_{ij}(c) \subset \mathcal{V}_{ij}(c)$ to be a complementary subspace to $\mathcal{V}_{i,j-1}(c)$ in $\mathcal{V}_{ij}(c)$, so that for all $\mathbf{v} \in \mathcal{V}_{i,j-1}(c)$ and all $\mathbf{v}' \in \mathcal{V}'_{ij}(c)$,

$$\|\mathbf{v} + \mathbf{v}'\|_0 \geq \rho'' \max(\|\mathbf{v}\|_0, \|\mathbf{v}'\|_0),$$

and $\rho'' > 0$ depends only on the dimension.

Then,

$$\mathcal{V}'_{ij}(c) \cong \mathcal{V}_{ij}(c)/\mathcal{V}_{i,j-1}(c).$$

Let $\pi_{ij} : \mathcal{V}_{\leq i,j} \to \mathcal{V}_{\leq i,j}/\mathcal{V}_{\leq i,j-1}$ be the natural quotient map. Then the restriction of π_{ij} to $\mathcal{V}'_{ij}(c)$ is an isomorphism onto $\mathcal{V}_{\leq i,j}(c)/\mathcal{V}_{\leq i,j-1}(c)$.

We can now define for $\mathbf{u}, \mathbf{v} \in \mathfrak{g}$,

$$\langle \mathbf{u}, \mathbf{v} \rangle_c \equiv \sum_{ij} \langle \pi_{ij}(\mathbf{u}_{ij}), \pi_{ij}(\mathbf{v}_{ij}) \rangle_{ij,c},$$

$$\text{where } \mathbf{u} = \sum_{ij} \mathbf{u}_{ij},\ \mathbf{v} = \sum_{ij} \mathbf{v}_{ij},\ \mathbf{u}_{ij} \in \mathcal{V}'_{ij}(c),\ \mathbf{v}_{ij} \in \mathcal{V}'_{ij}(c).$$

In other words, the distinct $\mathcal{V}'_{ij}(c)$ are orthogonal, and the inner product on each $\mathcal{V}'_{ij}(c)$ coincides with $\langle \cdot, \cdot \rangle_{ij,c}$ under the identification π_{ij} of $\mathcal{V}'_{ij}(c)$ with $\mathcal{V}_{\leq i,j}(c)/\mathcal{V}_{\leq i,j-1}(c)$.

We now define, for $x \in B^+[c]$, and $\mathbf{u}, \mathbf{v} \in \mathfrak{g}$

$$\langle \mathbf{u}, \mathbf{v} \rangle_x \equiv \langle P^+(x,c)\mathbf{u}, P^+(x,c)\mathbf{v} \rangle_c,$$

where $P^+(\cdot, \cdot)$ is the connection defined in §2.2. Then for $x \in B^+[c]$, the inner product $\langle \cdot, \cdot \rangle_x$ induces the inner product $\langle \cdot, \cdot \rangle_{ij,x}$ on $\mathcal{V}_{\leq i,j}(x)/\mathcal{V}_{\leq i,j-1}(x)$.

Symmetric space interpretation. We want to define the inner product $\langle \cdot, \cdot \rangle_x$ for any $x \in J[c_0]$. We may write $x = T^{-t}c$ where $0 \leq t < t(c)$ and $c \in B^+[c_0]$. We then define $\langle \cdot, \cdot \rangle_x$ by interpolating between $\langle \cdot, \cdot \rangle_c$ and $\langle \cdot, \cdot \rangle_{c'}$, where $c' = T^{-t(c)}c$. To define this interpolation, we recall that the set of inner products on a vector space V is canonically isomorphic to $SO(V)\backslash GL(V)$, where $GL(V)$ is the general linear group of V and $SO(V)$ is the subgroup preserving the inner product on V. In our case, $V = \mathfrak{g}$ with the inner product $\langle \cdot, \cdot \rangle_c$.

Let K_c denote the subgroup of $GL(\mathfrak{g})$ which preserves the inner product $\langle \cdot, \cdot \rangle_c$. Let $\mathcal{Q}$ denote the parabolic subgroup of $GL(\mathfrak{g})$ which preserves the flags (2.23) and (2.24), and on each successive quotient $\mathcal{V}_{\leq i,j}(c)/\mathcal{V}_{\leq i,j-1}(c)$ preserves $\langle \cdot, \cdot \rangle_{ij,c}$. Let $K_c A'$ denote the point in $K_c\backslash GL(\mathfrak{g})$ which represents the inner product $\langle \cdot, \cdot \rangle_{c'}$, i.e.

$$\langle \mathbf{u}, \mathbf{v} \rangle_{c'} = \langle A'\mathbf{u}, A'\mathbf{v} \rangle_c.$$

Then, since $\langle \cdot, \cdot \rangle_{c'}$ induces the inner products $\langle \cdot, \cdot \rangle_{ij,c'}$ on the space $\mathcal{V}_{\leq i,j}(c')/\mathcal{V}_{\leq i,j-1}(c')$ and $(T_c^{-t(c)})_* \mathcal{V}_{\leq i,j}(c') = \mathcal{V}_{\leq i,j}(c)$, we may assume that $A'(T_c^{-t(c)})_* \in \mathcal{Q}$.

Let $N_{\mathcal{Q}}$ be the normal subgroup of $\mathcal{Q}$ in which all diagonal blocks are the identity, and let $\mathcal{Q}' = \mathcal{Q}/N_{\mathcal{Q}}$. (We may consider $\mathcal{Q}'$ to be the subgroup of $\mathcal{Q}$ in which all off-diagonal blocks are 0). Let π' denote the natural map $\mathcal{Q} \to \mathcal{Q}'$.

Claim 2.15. *We may write*

$$A'(T_c^{-t(c)})_* = \Lambda A'',$$

where $\Lambda \in \mathcal{Q}'$ *is the diagonal matrix which is scaling by* $e^{-\lambda_i t(c)}$ *on* $\mathcal{V}_i(c)$*,* $A'' \in \mathcal{Q}$ *and* $\|A''\| = O(e^{\epsilon t(c)})$*, where the implied constant depends only on the constants* $\sigma_0, M, \rho, \rho_1$ *of Lemma 2.7.*

Proof of claim. Suppose $x \in B^+[c]$ and $t = -t(c) < 0$ where $c \in \mathcal{C}_1$ and $t(c)$ is as in Proposition 2.8. By construction, $t(c) > T_0(c)$, where $T_0(c)$ is as in Lemma 2.7. Then, the claim follows from (2.19) and Lemma 2.7 (d). □

Interpolation. We may write $A'' = DA_1$, where D is diagonal, and $\det A_1 = 1$. In view of Claim 2.15, $\|D\| = O(e^{\epsilon t})$ and $\|A_1\| = O(e^{\epsilon t})$.

We now connect $K_c \backslash A_1$ to the identity by the shortest possible path $\Gamma : [-t(c), 0] \to K_c\backslash K_c\mathcal{Q}$, which stays in the subset $K_c\backslash K_c\mathcal{Q}$ of the symmetric space $K_c\backslash SL(V)$. (We parametrize the path so it has constant speed). This path has length $O(\epsilon t)$ where the implied constant depends only on the symmetric space and the constants $\sigma_0, M, \rho, \rho_1$ of Lemma 2.7.

Now for $-t(c) \le t \le 0$, let

$$A(t) = (\Lambda D)^{-t/t(c)}\Gamma(t). \tag{2.25}$$

Then $A(0)$ is the identity map, and $A(-t(c)) = A'(T_c^{-t(c)})_*$. Suppose $x \in J(c_0)$. Then, $x = T^t c$, where $-t(c) \le t < 0$ and $c \in B^+[c_0]$. Then, we define,

$$\langle \mathbf{u}, \mathbf{v}\rangle_x = \langle A(t)(T_x^{-t})_*\mathbf{u}, A(t)(T_x^{-t})_*\mathbf{v}\rangle_c. \tag{2.26}$$

In particular, since $c' = T^{-t(c)}c$ and $(T_{c'}^{t(c)})_* = (T_c^{-t(c)})_*^{-1}$, we have, letting $t = -t(c)$ in (2.26),

$$\langle \mathbf{u}, \mathbf{v}\rangle_{c'} = \langle A(-t(c))(T_{c'}^{t(c)})_*\mathbf{u}, A(-t(c))(T_{c'}^{t(c)})_*\mathbf{v}\rangle_c = \langle A'\mathbf{u}, A'\mathbf{v}\rangle_c,$$

as required.

Proof of Proposition 2.14. Suppose first that $x \in \mathcal{C}_1$. Then, by construction, (a) and (b) hold. Also, from the construction, it is clear that the inner product $\langle \cdot,\cdot\rangle_c$ induces the inner product $\langle\cdot,\cdot\rangle_{ij,c}$ on $\mathcal{V}_{ij}(c)/\mathcal{V}_{i,j-1}(c)$.

Now by Lemma 2.5, for $x \in B^+[c]$, $P^+(x,c)\mathcal{V}_{ij}(x) = \mathcal{V}_{ij}(c)$, and for $\bar{\mathbf{u}}, \bar{\mathbf{v}} \in \mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x)$, $\langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle_{ij,x} = \langle P^+(x,c)\bar{\mathbf{u}}, P^+(x,c)\bar{\mathbf{v}}\rangle_{ij,c}$. Therefore, (a), (b), (e) and (f) hold for $x \in B^+[c]$, and also for $x \in B^+[c]$, the inner product $\langle\cdot,\cdot\rangle_x$ induces the inner product $\langle\cdot,\cdot\rangle_{ij,x}$ on $\mathcal{V}_{ij}(x)/\mathcal{V}_{i,j-1}(x)$. Now, (a),(b),(e) and (f) hold for arbitrary $x \in J[c]$ since $A(t) \in \mathcal{Q}$.

Let $\psi_{ij} : \mathcal{Q}' \to \mathbb{R}_+$ denote the homomorphism taking the block-conformal matrix $\mathcal{Q}'$ to the scaling part of block corresponding to $\mathcal{V}_{ij}/\mathcal{V}_{i,j-1}$. Let $\varphi_{ij} = \psi_{ij} \circ \pi'$; then $\varphi_{ij} : \mathcal{Q} \to \mathbb{R}_+$ is a homomorphism.

From (2.25), we have, for $x \in B^+[c]$ and $-t(c) \le t \le 0$,

$$\lambda_{ij}(x,t) = \log \varphi_{ij}(A(t)) = t\lambda_i + \gamma_{ij}(x,t),$$

where $t\lambda_i$ is the contribution of $\Lambda^{t/t(c)}$ and $\gamma_{ij}(x,t)$ is the contribution of $D^{t/t(c)}\Gamma(t)$. By Claim 2.15, for all $-t(c) \le t \le 0$,

$$|\frac{\partial}{\partial t}\gamma_{ij}(x,t)| = k'\epsilon + O(1/t), \tag{2.27}$$

where k' depends only on the symmetric space, and the implied constant depends only on the symmetric space and the constants $\sigma_0, M, \rho, \rho_1$ of Lemma 2.7. Therefore, if $\epsilon > 0$ in Lemma 2.7 is chosen small enough and $T_1(c)$ in Lemma 2.9 is chosen large enough, $|\gamma_{ij}(x,t)| < \lambda_i/2$ and (c) holds.

The lower bound in (d) now follows immediately from (b) and (c). The upper bound in (d) follows from (2.27). $\square$

Lemma 2.16. *For every $\delta > 0$ and every $\epsilon > 0$ there exists a compact subset $K(\delta) \subset \Omega$ with $\tilde{\mu}(K(\delta)) > 1 - \delta$ and a number $C_1(\delta,\epsilon) < \infty$ such that for all $x \in K(\delta)$ and all $\mathbf{v} \in \mathfrak{g}$ and all $t \in \mathbb{R}$,*

$$C_1(\delta)^{-1} e^{-\epsilon|t|} \le \frac{\|\mathbf{v}\|_{T^t x}}{\|\mathbf{v}\|_0} \le C_1(\delta) e^{\epsilon|t|} \tag{2.28}$$

where $\|\cdot\|_x$ is the dynamical norm defined in this subsection and $\|\cdot\|_0$ is some fixed norm on $\mathfrak{g}$.

Proof. Since any two norms on a finite dimensional vector space are equivalent, there exists a function $\Xi_0 : \Omega \to \mathbb{R}^+$ finite a.e. such that for all $x \in \Omega$ and all $\mathbf{v} \in \mathfrak{g}$,

$$\Xi_0(x)^{-1}\|\mathbf{v}\|_0 \le \|\mathbf{v}\|_x \le \Xi_0(x)\|\mathbf{v}\|_0.$$

Since $\bigcup_{N\in\mathbb{N}}\{x \ : \ \Xi_0(x) < N\}$ is conull in Ω, we can choose $K(\delta) \subset \Omega$ and $C_1 = C_1(\delta)$ so that $\Xi_0(x) < C_1(\delta)$ for $x \in K(\delta)$ and $\tilde{\mu}(K(\delta)) \ge (1-\delta)$. This implies (2.28) for the case $t = 0$. The general case follows from the case $t = 0$ and the ergodic theorem. $\square$

3. The inert subspaces $\mathbf{E}_j(x)$

For $x \in \Omega_0$, let

$$\mathbf{F}_{\ge j}(x) = \{\mathbf{v} \in \mathfrak{g} \ : \ \text{for almost all } ux \in \mathcal{U}_1^+ x,\ \mathbf{v} \in \mathcal{V}_{\ge j}(ux)\}. \tag{3.1}$$

In other words, if $\mathbf{v} \in \mathbf{F}_{\ge j}(x)$, then for almost all $ux \in \mathcal{U}_1^+ x$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(T^t_{ux})_* \mathbf{v}\|_0 \le \lambda_j. \tag{3.2}$$

It follows from (3.2) that if M and ϑ are trivial, $\mathbf{F}_{\ge j}(x)$ does not in fact depend on $x \in \Omega_0$. In the general case, $\mathbf{F}_{\ge j}(x)$ depends on $x \in \mathcal{S}^{\mathbb{Z}} \times M \times [0,1)$ only through the M component.

From the definition of $\mathbf{F}_{\geq j}(x)$, we have

$$\{0\} = \mathbf{F}_{\geq n+1}(x) \subseteq \mathbf{F}_n(x) \subseteq \mathbf{F}_{\geq n-1}(x) \subseteq \dots \mathbf{F}_2(x) \subseteq \mathbf{F}_1(x) = \mathfrak{g}. \tag{3.3}$$

Let

$$\mathbf{E}_j(x) = \mathbf{F}_{\geq j}(x) \cap \mathcal{V}_{\leq j}(x).$$

In particular, $\mathbf{E}_1(x) = \mathcal{V}_{\leq 1}(x) = \mathcal{V}_1(x)$. We may have $\mathbf{E}_j(x) = \{0\}$ if $j \neq 1$.

Lemma 3.1. *For almost all $x \in \Omega_0$ the following holds: suppose $\mathbf{v} \in \mathbf{E}_j(x) \smallsetminus \{0\}$. Then for almost all $ux \in \mathcal{U}_1^+ x$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \|(T^t_{ux})_*(\mathbf{v})\|_0 = \lambda_j. \tag{3.4}$$

Thus we have

$$\mathbf{E}_j(x) \subset \mathcal{V}_j(x).$$

In particular, if $i \neq j$, $\mathbf{E}_i(x) \cap \mathbf{E}_j(x) = \{0\}$ for almost all $x \in \Omega_0$.

Proof. Suppose $\mathbf{v} \in \mathbf{E}_j(x)$. Then $\mathbf{v} \in \mathcal{V}_{\leq j}(x)$. Since in view of (2.2), $\mathcal{V}_{\leq j}(ux) = \mathcal{V}_{\leq j}(x)$ for all $u \in \mathcal{U}_1^+$, we have for almost all $ux \in \mathcal{U}_1^+ x$, $\mathbf{v} \in \mathcal{V}_{\leq j}(ux)$. It follows from (2.6) that (outside of a set of measure 0), $\mathbf{v} \notin \mathcal{V}_{>j}(ux)$. Now (3.4) follows from (2.3). □

Lemma 3.2. *There exists a subset $\Psi \subset \Omega_0$ of full measure such that for any $x \in \Psi$ and any $t \in \mathbb{R}$,*

$$(T^t_x)_* \mathbf{E}_j(x) = \mathbf{E}_j(T^t x), \text{ and } (T^t_x)_* \mathbf{F}_{\geq j}(x) = \mathbf{F}_{\geq j}(T^t x).$$

Proof. From (3.1), for $x \in \Omega_0$,

$$(T^{-t}_x)_* \mathbf{F}_j(x) = \{\mathbf{v} \in \mathfrak{g} \ : \ \text{for a.e. } ux \in T^{-t}(\mathcal{U}_1^+ x), \mathbf{v} \in \mathcal{V}_{\geq j}(ux)\ \}.$$

Note that for $t > 0$, $T^{-t}(\mathcal{U}_1^+ x) \subset \mathcal{U}_1^+ T^{-t} x$. Therefore,

$$\mathbf{F}_{\geq j}(T^{-t} x) \subset (T^{-t}_x)_* \mathbf{F}_{\geq j}(x).$$

Let $\phi(x) = \dim \mathbf{F}_{\geq j}(x)$. Then, ϕ is bounded, integral valued and is decreasing under the flow T^{-t}. Therefore, ϕ is constant a.e., and $\mathbf{F}_{\geq j}(T^{-t} x) = \mathbf{F}_{\geq j}(x)$ almost everywhere. Then the corresponding statement about $\mathbf{E}_j(x)$ also holds since $\mathbf{E}_j(x) = \mathbf{F}_{\geq j}(x) \cap \mathcal{V}_{\leq j}(x)$.

By considering a transversal for the flow T^t, it is easy to check that it is possible to modify $\mathbf{E}_j(x)$ and $\mathbf{F}_{\geq j}(x)$ on a subset of measure 0 of Ω_0 in such a way that the lemma holds for x in a subset of full measure and all $t \in \mathbb{R}$. □

Remark 3.3. Suppose that M and ϑ are trivial in the sense of §1.3. Then, since $\mathbf{F}_{\geq j}(x)$ does not depend on x, Lemma 3.2 implies that the subspaces $\mathbf{F}_{\geq j}$ are $G_{\mathcal{S}}$-invariant. Thus, if the Zariski closure of $G_{\mathcal{S}}$ is simple, then $\mathbf{F}_{\geq 1} = \mathfrak{g}$ and $\mathbf{F}_{\geq j} = 0$ for $j \geq 2$.

Recall that $|\cdot|$ denotes the conditional measure of $\tilde{\mu}$ on $W_1^+[x] = \mathcal{U}_1^+ x$ (see §2.1).

Lemma 3.4.

(a) *For almost all $x \in \Omega_0$ and almost all $ux \in W^+[x]$, we have $\mathbf{E}_j(ux) = \mathbf{E}_j(x)$, and $\mathbf{F}_{\geq j}(ux) = \mathbf{F}_{\geq j}(x)$.*

(b) *For $x \in \Omega_0$ and $\mathbf{v} \in \mathfrak{g}$, let*
$$Q(\mathbf{v}) = \{ux \in \mathcal{U}_1^+ x \ \ : \ \ \mathbf{v} \in \mathcal{V}_{\geq j}(ux)\}.$$
Then for almost all x, either $|Q(\mathbf{v})| = 0$, or $|Q(\mathbf{v})| = |\mathcal{U}_1^+ x|$ (and thus $\mathbf{v} \in \mathbf{F}_{\geq j}(x)$).

Proof. For (a) note that since for $ux \in \mathcal{U}_1^+ x$, $\mathcal{U}_1^+ ux = \mathcal{U}_1^+ x$, $\mathbf{F}_{\geq j}(x) = \mathbf{F}_{\geq j}(ux)$ by the definition (3.1). By (2.4), $\mathcal{V}_{\leq j}(ux) = \mathcal{V}_{\leq j}(x)$. Since $\mathbf{E}_j(x) = \mathbf{F}_{\geq j}(x) \cap \mathcal{V}_{\leq j}(x)$, $\mathbf{E}_j(ux) = \mathbf{E}_j(x)$.

We now start the proof of (b). For a subspace $\mathbf{V} \subset \mathfrak{g}$, let
$$Q(\mathbf{V}) = \{ux \in \mathfrak{B}_0[x] \ \ : \ \ \mathbf{V} \subset \mathcal{V}_{\geq j}(ux)\}.$$
Let d be the maximal number such that there exists $E' \subset \Omega_0$ with $\nu(E') > 0$ such that for $x \in E'$ there exists a subspace $\mathbf{V} \subset \mathfrak{g}$ of dimension d with $|Q(\mathbf{V})| > 0$. For a fixed $x \in E'$, let $\mathcal{W}(x)$ denote the set of subspaces $\mathbf{V}$ of dimension d for which $|Q(\mathbf{V})| > 0$. Then, by the maximality of d, if $\mathbf{V}$ and $\mathbf{V}'$ are distinct elements of $\mathcal{W}(x)$ then $Q(\mathbf{V}) \cap Q(\mathbf{V}')$ has measure 0. Let $\mathbf{V}_x \in \mathcal{W}(x)$ be such that $|Q(\mathbf{V}_x)|$ is maximal (among elements of $\mathcal{W}(x)$).

Let $\epsilon > 0$ be arbitrary. By the same Vitali-type argument as in the proof of Lemma 2.13, there exists $t_0 > 0$, a positive measure subset $E'' \subset E'$ and for each $x \in E''$ a subset $Q(\mathbf{V}_x)^* \subset Q(\mathbf{V}_x)$ with $|Q(\mathbf{V}_x)^*| > 0$ such that for all $ux \in Q(\mathbf{V}_x)^*$ and all $t > t_0$,
$$|\mathfrak{B}_t[ux] \cap Q(\mathbf{V}_x)| \geq (1-\epsilon)|\mathfrak{B}_t[ux]|. \tag{3.5}$$
(In other words, $Q(\mathbf{V}_x)^*$ are "points of density" for $Q(\mathbf{V}_x)$, relative to the "balls" $\mathfrak{B}_t$.) Let
$$E^* = \{ux \ \ : \ \ x \in E'', \quad ux \in Q(\mathbf{V}_x)^*\}.$$
Then, $\tilde{\mu}(E^*) > 0$. Let $\Theta = \{x \in \Omega_0 \ \ : \ \ T^{-t}x \in E^* \text{ for an unbounded set of } t > 0\ \}$. Then $\tilde{\mu}(\Theta) = 1$. Suppose $x \in \Theta$. We can choose $t > t_0$ such that $T^{-t}x \in E^*$. Note that
$$\mathfrak{B}_0[x] = T^t \mathfrak{B}_t[T^{-t}x]. \tag{3.6}$$
Let $x' = T^{-t}x$, and let $\mathbf{V}_{t,x} = (T^t)_* \mathbf{V}_{x'}$. Then in view of (3.5) and (3.6),
$$|Q(\mathbf{V}_{t,x})| \geq (1-\epsilon)|\mathfrak{B}(x)|.$$
By the maximality of d (and assuming $\epsilon < 1/2$), $\mathbf{V}_{t,x}$ does not depend on t. Hence, for every $x \in \Theta$, there exists $\mathbf{V} \subset \mathfrak{g}$ such that $\dim \mathbf{V} = d$ and $|Q(\mathbf{V})| \geq (1-\epsilon)|\mathfrak{B}_0[x]|$. Since $\epsilon > 0$ is arbitrary, for each $x \in \Theta$, there exists $\mathbf{V} \subset \mathfrak{g}$ with $\dim \mathbf{V} = d$, and

$|Q(\mathbf{V})| = |\mathfrak{B}_0[x]|$. Now the maximality of d implies that if $\mathbf{v} \notin \mathbf{V}$ then $|Q(\mathbf{v}) \cap \mathfrak{B}_0[x]| = 0$. Then, equivariance by T^t implies that $|Q(\mathbf{v})| = 0$. □

By Lemma 3.1, $\mathbf{E}_j(x) \cap \mathbf{E}_k(x) = \{0\}$ if $j \neq k$. Let

$$\Lambda_0' = \{i \;:\; \mathbf{E}_i(x) \neq \{0\} \text{ for a.e. } x\}.$$

and let

$$\Lambda' = \{i \in \Lambda_0' \;:\; \lambda_i > 0\}.$$

In view of (3.2), (3.3) and Lemma 3.1, we have $\mathbf{F}_{\geq j}(x) = \mathbf{F}_{\geq j+1}(x)$ unless $j \in \Lambda_0'$. Therefore if we write the elements of Λ_0' in decreasing order as $i_1, \dots, i_m$ we have the flag (consisting of distinct subspaces)

$$\{0\} = \mathbf{F}_{\geq i_{m+1}} \subset \mathbf{F}_{\geq i_m}(x) \subset \mathbf{F}_{\geq i_{m-1}}(x) \subset \dots \mathbf{F}_{\geq i_2}(x) \subset \mathbf{F}_{\geq i_1}(x) = \mathfrak{g}. \tag{3.7}$$

For $x \in \Omega$ and any subspace $V \subset \mathfrak{g}$, we write $V^\perp$ for its orthogonal complement using the inner product $\langle \cdot, \cdot \rangle_x$ defined in §2.6. For a.e. $x \in \Omega_0$, and $1 \leq r \leq m$, let $\mathbf{F}'_{i_r}(x) = (\mathbf{F}_{\geq i_{r+1}})^\perp(x) \cap \mathbf{F}_{\geq i_r}(x)$. Then, $\mathbf{F}'_j(x)$ is only defined for $j \in \Lambda_0'$, and for $j \in \Lambda_0'$,

$$\mathbf{F}'_j(x) = (\mathbf{F}_{\geq j+1})^\perp(x) \cap \mathbf{F}_{\geq j}(x).$$

Lemma 3.5. *Given $\delta > 0$ there exists a compact $K_{01} \subset \Omega$ with $\nu(K_{01}) > 1 - \delta$, $\beta(\delta) > 0$, $\beta'(\delta) > 0$, and for every $x \in K_{01}$ any $j \in \Lambda_0'$ any $0 \neq \mathbf{v}' \in (\mathbf{F}_{\geq j+1})^\perp(x)$ a subset $Q_{01} = Q_{01}(x, \mathbf{v}'/\|\mathbf{v}'\|) \subset \mathcal{U}_1^+$ with $|Q_{01}x| > (1-\delta)|\mathcal{U}_1^+ x|$ such that for any $u \in Q_{01}$, we can write*

$$\mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \quad \mathbf{v}_u \in \mathcal{V}_{\leq j}(ux), \quad \mathbf{w}_u \in \mathcal{V}_{>j}(ux),$$

with $\|\mathbf{v}_u\| \geq \beta(\delta)\|\mathbf{v}'\|$, and $\|\mathbf{v}_u\| > \beta'(\delta)\|\mathbf{w}_u\|$. Furthermore, if $j \in \Lambda_0'$ and $\mathbf{v}' \in \mathbf{F}'_j(x)$, then $\mathbf{v}_u \in \mathbf{E}_j(ux)$.

Proof. This is a corollary of Lemma 3.4(b). Let $\Phi \subset \Omega$ be the conull set where (2.6) holds and where $\mathbf{F}_{\geq i}(x) = \mathbf{F}_{\geq i+1}(x)$ for all $i \notin \Lambda_0'$. Suppose $x \in \Phi$. By Lemma 3.4(b), since $\mathbf{v}' \notin \mathbf{F}_{\geq j+1}(x)$, for almost all u if we decompose using (2.6)

$$\mathbf{v}' = \mathbf{v}_u + \mathbf{w}_u, \quad \mathbf{v}_u \in \mathcal{V}_{\leq j}(ux) \quad \mathbf{w}_u \in \mathcal{V}_{>j}(ux),$$

then $\mathbf{v}_u \neq 0$. Let

$$\mathcal{E}_n(x) = \{\mathbf{v}' \in \mathbb{P}(\mathbf{F}_{\geq j+1})^\perp(x) \;:\; |\{ux \in \mathcal{U}_1^+ x \;:\; \|\mathbf{v}_u\| \geq \tfrac{1}{n}\|\mathbf{v}'\|\}| > (1-\delta)|\mathcal{U}_1^+ x|\}.$$

Then the $\mathcal{E}_n(x)$ are an increasing family of open sets, and $\bigcup_{n=1}^\infty \mathcal{E}_n(x) = \mathbb{P}(\mathbf{F}_{\geq j+1})^\perp(x)$. Since $\mathbb{P}(\mathbf{F}_{\geq j+1})^\perp(x)$ is compact, there exists $n(x)$ such that $\mathcal{E}_{n(x)}(x) = \mathbb{P}(\mathbf{F}_{\geq j+1})^\perp(x)$. We can now choose $K_{01} \subset \Phi$ with $\tilde{\mu}(K_{01}) > 1-\delta$ such that for $x \in K_{01}$, $n(x) < 1/\beta(\delta)$. This shows that for $x \in K_{01}$ and any $\mathbf{v}' \in (\mathbf{F}_{\geq j+1})^\perp(x)$, for $(1-\delta)$-fraction of $ux \in \mathcal{U}_1^+ x$ we have $\|\mathbf{v}_u\| > \beta(\delta)\|\mathbf{v}'\|$.

It remains to prove the final assertion. Suppose $j \in \Lambda_0'$ and $\mathbf{v}' \in \mathbf{F}'_j(x) \subset \mathbf{F}_{\geq j}(x)$. By Lemma 3.2, $\mathbf{F}_{\geq j}$ is T^t-equivariant, and therefore, by the multiplicative ergodic theorem applied to $\mathbf{F}_{\geq j}$, $\mathbf{F}_{\geq j}$ is the direct sum of its Lyapunov subspaces. Therefore, in view of Lemma 3.4(a), for almost all $ux \in \mathcal{U}_1^+ x$,

$$\mathbf{F}_{\geq j}(x) = \mathbf{F}_{\geq j}(ux) = (\mathbf{F}_{\geq j}(ux) \cap \mathcal{V}_{\leq j}(ux)) \oplus (\mathbf{F}_{\geq j}(ux) \cap \mathcal{V}_{>j}(ux)). \tag{3.8}$$

Therefore, $\mathbf{v}_u \in \mathbf{F}_{\geq j}(ux) \cap \mathcal{V}_{\leq j}(ux) \equiv \mathbf{E}_j(ux)$ for almost all $ux \in \mathcal{U}_1^+ x$. □

Lemma 3.6. *For any $\delta > 0$ and any $\epsilon > 0$ there exists $K(\delta) \subset \Omega$ with $\tilde{\mu}(K(\delta)) > 1-\delta$ and for every $x \in K(\delta)$ and every $\mathbf{v} \in \mathfrak{g}$ and every $1 \leq j \leq n$, there exists a subset $Q = Q(x, \mathbf{v}) \subset \mathcal{U}_1^+$ with $|Qx| \geq (1-\delta)|\mathcal{U}_1^+ x|$ such that for $x \in K(\delta)$, $u \in Q$ and any $t > 0$ we have*

$$\|(T^t_{ux})_* \mathbf{v}\|_{T^t ux} \geq C(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} d(\mathbf{v}, \mathbf{F}_{\geq j+1}(x)),$$

where $d(\cdot, \cdot)$ is distance using the norm $\|\cdot\|_x$. Also

$$\|(T^t_{ux})_* \mathbf{v}\|_0 \geq C_1(\epsilon, \delta) e^{(\lambda_j - \epsilon)t} d_0(\mathbf{v}, \mathbf{F}_{\geq j+1}(x)). \tag{3.9}$$

Proof. Let K_{01} be as in Lemma 3.5. Write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ where $\mathbf{v}' \in (\mathbf{F}_{\geq j+1})^\perp(x)$, $\mathbf{v}'' \in \mathbf{F}_{\geq j+1}(x)$. Then, $\|\mathbf{v}'\| = d(\mathbf{v}, \mathbf{F}_{\geq j+1}(x))$.

Let $Q_{01} = Q_{01}(x, \mathbf{v}')$ be as in Lemma 3.5. Also, by the multiplicative ergodic theorem, for any $\delta > 0$ there exists $K_2 \subset \Omega$ with $\tilde{\mu}(K_2) > 1-\delta$ and for $x \in K_2$ a subset $Q_2(x) \subset \mathcal{U}_1^+$ with $|Q_2|x \geq (1-\delta)|\mathcal{U}_1^+ x|$ such that for any $u \in Q_2(x)$ and any $\mathbf{v}_j \in \mathcal{V}_{\geq j}(ux)$ and any $t > 0$, $\|(T^t_{ux})_* \mathbf{v}_j\| \geq C_2(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} \|\mathbf{v}_j\|$. Now let $K = K_{01} \cap K_2$ and $Q = Q_{01}(x, \mathbf{v}') \cap Q_2(x)$. Let $\mathbf{v}_u$ and $\mathbf{w}_u$ be as in Lemma 3.5. Then, by Lemma 3.5, for $u \in Q$,

$$\|(T^t_{ux})_* \mathbf{v}_u\| \geq C_2(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} \|\mathbf{v}_u\| \geq \beta(\delta) C_2(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} \|\mathbf{v}'\|.$$

Since $(T^t_{ux})_* \mathbf{w}_u \in \mathcal{V}_{>j}(T^t ux)$ and also by Lemma 3.2 and Lemma 3.4(a), $(T^t_{ux})_* \mathbf{v}'' \in \mathcal{V}_{>j}(T^t ux)$, we have, in view of Proposition 2.14(a),

$$\|(T^t_{ux})_* \mathbf{v}\| \geq \|(T^t_{ux})_* \mathbf{v}_u\| \geq C(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} \|\mathbf{v}'\| = C(\epsilon, \delta) e^{(\lambda_j - \epsilon/2)t} d(\mathbf{v}, \mathbf{F}_{\geq j+1}(x)).$$

The final assertion follows from Lemma 2.16. □

Remark. In the case when ϑ is trivial, the assertion (3.9) does not depend on x. Therefore, if we are interested in (3.9), we may take $K(\delta) = \Omega$.

Lemma 3.7. *For all $\delta > 0$ there exists $\eta(\delta) > 0$ with $\eta(\delta) \to 0$ as $\delta \to 0$ such that the following holds: Suppose $x \in \Omega$, $Q^c \subset \mathcal{U}_1^+ x = W_1^+[x]$ is such that $|Q^c| = \tilde{\mu}|_{W_1^+[x]}(Q^c) \leq \delta$. Then, for any $n \in \mathbb{N}$ and any $\mathbf{v} \in \mathfrak{g}$,*

$$\frac{1}{n} \int_{Q^c} \log \frac{\|(T^n_{ux})_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \, d\tilde{\mu}|_{W_1^+[x]}(ux) \leq \eta(\delta). \tag{3.10}$$

Proof. Let $\sigma : \Omega \times \mathfrak{g} \to \mathbb{R}$ be given by the formula $\sigma(\omega, m, s, \mathbf{v}) = \log \frac{\|\vartheta(\omega_0,m)\mathbf{v}\|_0}{\|\mathbf{v}\|_0}$. Let $\|g\|_0$ denote the operator norm on G (or G') derived from $\|\cdot\|_0$, and let $\|g\|_+ = \max(1, \|g\|_0, \|g^{-1}\|_0)$. Let $\sigma_+ : \Omega \to \mathbb{R}$ be defined by $\sigma_+(x) = \log\|\omega_0\|_+$, where $x = (\omega, m, s)$. Then, by (1.5),

$$\sigma(x, \mathbf{v}) \le \sigma_+(x) + C,$$

where C is a constant (depending only on ϑ).

Let χ_δ denote the characteristic function of the set

$$\{(\omega, m, s) \in \Omega \ : \ \log\|\omega_0\|_+ > \delta^{-1/2}\}.$$

Then, let $\eta_1(\delta) = \int_\Omega \chi_\delta(\sigma_+ + C)\, d\tilde{\mu}$. We have $\eta_1(\delta) \to 0$ as $\delta \to 0$.

Let $\sigma_1 = (1-\chi_\delta)\sigma$, $\sigma_2 = \chi_\delta\sigma$. Then, we have $\sigma(x,\mathbf{v}) = \sigma_1(x,\mathbf{v}) + \sigma_2(x,\mathbf{v})$ where for all $(x,\mathbf{v}) \in \Omega \times \mathfrak{g}$, $|\sigma_1(x,\mathbf{v})| \le \delta^{-1/2}$, $|\sigma_2(x,\mathbf{v})| \le (\sigma_+(x) + C)\chi_\delta(x)$, and for all $x \in \Omega$,

$$\int_{W_1^+[x]} \chi_\delta(ux)(\sigma_+(ux) + C)\, d\tilde{\mu}|_{W_1^+[x]}(ux) = \int_\Omega \chi_\delta(\sigma_+ + C)\, d\tilde{\mu} < \eta_1(\delta). \tag{3.11}$$

Now we write

$$\log\frac{\|(T^n_{ux})_*\mathbf{v}\|_0}{\|\mathbf{v}\|_0} = \frac{1}{n}\sum_{j=1}^n \sigma(\hat{T}^j(ux,\mathbf{v})) = \frac{1}{n}\sum_{j=1}^n \sigma_1(\hat{T}^j(ux,\mathbf{v})) + \frac{1}{n}\sum_{j=1}^n \sigma_2(\hat{T}^j(ux,\mathbf{v})),$$

and integrate both sides over $ux \in Q^c$. The contribution of the first sum is at most $\delta^{1/2}$, since $|\sigma_1(x,\mathbf{v})| \le \delta^{-1/2}$. The contribution of the second sum is bounded by $\eta_1(\delta)$ by (3.11) and the L^1 ergodic theorem, applied to the shift $T : \mathcal{S}^{\mathbb{Z}} \to \mathcal{S}^{\mathbb{Z}}$ and the L^1-function $\sigma_+ : \mathcal{S}^{\mathbb{Z}} \to \mathbb{R}$. Thus we can set $\eta(\delta) = \delta^{1/2} + \eta_1(\delta)$. □

3.1. **Inert subspaces, uniform expansion and the bounceback condition.** Let j_1 be the index of smallest positive Lyapunov exponent of T^t and let $j_2 > j_1$ be the index of the largest negative exponent. Then,

$$\operatorname{Lie}(N^+)(x) = \bigoplus_{j\le j_1} \mathcal{V}_j(x).$$

and

$$\operatorname{Lie}(N^-)(x) = \bigoplus_{j\ge j_2} \mathcal{V}_j(x)$$

If there is a 0-exponent, its index is $j_1+1 = j_2-1$. If not, then $j_2 = j_1+1$, and j_1+1 is a negative exponent. In any case $\lambda_{j_1+1} \le 0$. Then, $\mathbf{F}_{\ge j_1+1}(x)$ is the set of vectors in $\mathfrak{g}$ which fail to grow exponentially fast under almost all possible futures ux.

Lemma 3.8. *Suppose V is a $G_{\mathcal{S}}$-invariant subspace of $\mathfrak{g}$ and that M and ϑ are trivial in the sense of §1.3. Then the following are equivalent:*

(i) *μ is uniformly expanding on V.*
(ii) *For a.e. $x \in \Omega$, $\mathbf{F}_{\ge j_1+1}(x) \cap V = \{0\}$.*

(ii)′ *For any non-zero* $\mathbf{v} \in V$*, for almost all* $ux \in W_1^+[x]$, $\mathbf{v} \not\in \mathcal{V}_{\geq j_1+1}(x)$.

(iii) *There exists* $\lambda > 0$ *and for any* $\delta > 0$ *each* $x \in \Omega$ *and each* $\mathbf{v} \in V$ *there exists* $Q(x, \mathbf{v}) \subset \mathcal{U}_1^+$ *with* $|Q(x,\mathbf{v})x| > (1-\delta)|\mathcal{U}_1^+ x|$ *such that for* $u \in Q(x,\mathbf{v})$ *and* $t > 0$,

$$\|(T_{ux}^t)_* \mathbf{v}\|_0 \geq C(\delta) e^{\lambda t} \|\mathbf{v}\|_0.$$

Note that Lemma 1.5 is the equivalence of (i) and (ii) in Lemma 3.8. Also (ii) is equivalent to (ii)′ in view of Lemma 3.4(b).

Proof of (i) $\implies$ (ii)**.** Suppose that (i) holds. By iterating (1.1), we get for a.e. x, and some $N \in \mathbb{N}$ and any $\mathbf{v} \in V$,

$$\lim_{k\to\infty} \int_{W_1^+[x]} \frac{1}{kN} \log \frac{\|(T_{ux}^{kN})_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \, d\tilde{\mu}|_{W_1^+[x]}(ux) \geq C > 0. \tag{3.12}$$

Now suppose $\mathbf{F}_{\geq j_1+1}(x) \cap V \neq \{0\}$. Pick $\mathbf{v} \in \mathbf{F}_{\geq j+1}(x) \cap V$. Then, by the multiplicative ergodic theorem and the definition of $\mathbf{F}_{\geq j_1+1}$, for any $\epsilon > 0$ and any $\delta > 0$ there exists $Q = Q(x, \mathbf{v}, \delta)$ with $|Q| \geq (1-\delta)|W_1^+[x]|$ such that

$$\limsup_{n\to\infty} \int_Q \log \frac{\|(T_{ux}^{jN})_* \mathbf{v}\|_0}{\|\mathbf{v}\|_0} \, d\tilde{\mu}|_{W_1^+[x]}(ux) \leq \epsilon.$$

This contradicts (3.12), in view of Lemma 3.7.

Proof of (ii) $\implies$ (iii)**.** This follows immediately from Lemma 3.6.

Proof of (iii) $\implies$ (i)**.** This follows immediately from Lemma 3.7. □

Recall that $\mathbf{F}_{\geq j_1+1}(x)$ is the set of vectors in $\mathfrak{g}$ which fail to grow exponentially fast under almost all possible futures ux.

Lemma 3.9. *Suppose* μ *satisfies the bounceback condition (see Definition 1.10). Then for almost all* $x \in \Omega_0$,

$$\mathbf{F}_{\geq j_1+1}(x) \cap \mathrm{Lie}(N^-)(x) = \{0\}. \tag{3.13}$$

Proof. Suppose that there exists a set E of positive measure such that for $x \in E$, $\mathbf{F}_{\geq j_1+1}(x) \cap \mathrm{Lie}(N^-)(x) \neq \{0\}$. Pick $x \in E$, and $\mathbf{v} \in \mathbf{F}_{\geq j_1+1}(x) \cap \mathrm{Lie}(N^-)(x)$. Then, arguing as in the proof of the assertion (i) $\implies$ (ii) of Lemma 3.8, we see that the bounceback condition fails for $(x, \mathbf{v})$. □

Remark 3.10. In fact we prove Theorem 1.13 with the bounce-back condition replaced by the weaker assumption that (3.13) holds.

4. Preliminary divergence estimates

Standing Assumption. In §4-§10 we assume that (3.13) holds.

Let the inert subbundle $\mathbf{E}$ be defined by

$$\mathbf{E}(x) = \bigoplus_{i\in\Lambda'} \mathbf{E}_i(x).$$

Then $\mathbf{E}(x) \subset \mathrm{Lie}(N^+)(x)$.

The map $\mathcal{A}(q_1, u, \ell, t)$. For $q_1 \in \Omega$, $u \in \mathcal{U}_1^+$, $\ell > 0$ and $t > 0$, let $\mathcal{A}(q_1, u, \ell, t) : \mathfrak{g} \to \mathfrak{g}$ denote the map

$$\mathcal{A}(q_1, u, \ell, t)\mathbf{v} = (T^t_{uq_1})_*(T^\ell_{T^{-\ell}q_1})_*\mathbf{v}. \tag{4.1}$$

Proposition 4.1. *For every $\delta > 0$ there exists a subset $K \subset \Omega_0$ of measure at least $1-\delta$ and for $q_1 \in K$ and any $\mathbf{v} \in \mathrm{Lie}(N^-)(T^{-\ell}q_1)$, there exists a subset $Q = Q(q_1, \mathbf{v}) \subset \mathcal{U}_1^+$ with $|Qq_1| \geq (1-\delta)|\mathcal{U}_1^+ q_1|$ such that for any $u \in Q(q_1)$ and any $t > 0$,*

$$\|\mathcal{A}(q_1, u, \ell, t)\mathbf{v}\| \geq C(\delta)e^{-\kappa\ell+\lambda t}\|\mathbf{v}\|, \tag{4.2}$$

where κ is as in Proposition 2.14, and $\lambda > 0$ depends only on the Lyapunov spectrum of T^t. Also for $t > 0$,

$$d\left(\frac{\mathcal{A}(q_1, u, \ell, t)\mathbf{v}}{\|\mathcal{A}(q_1, u, \ell, t)\mathbf{v}\|}, \mathbf{E}(T^t u q_1)\right) \leq C(\delta)e^{-\alpha t}, \tag{4.3}$$

where $d(\cdot,\cdot)$ is the distance on $\mathfrak{g}$ defined by the dynamical norm $\|\cdot\|_{T^t uq_1}$ and α depends only on the Lyapunov spectrum of T^t.

Proof. Suppose $\mathbf{v} \in \mathrm{Lie}(N^-)(T^{-\ell}q_1)$ and let $\mathbf{w} = (T^\ell_{T^{-\ell}q})_*\mathbf{v}$. Then, $\mathbf{w} \in \mathrm{Lie}(N^-)(q_1)$, and by Proposition 2.14, $\|\mathbf{w}\| \geq e^{-\kappa\ell}\|\mathbf{v}\|$. Now (4.2) follows immediately from Lemma 3.6 and Lemma 3.9.

We now begin the proof of (4.3). Let $\epsilon > 0$ be smaller than one third of the difference between any two Lyapunov exponents for the cocycle $(T^t_x)_*$. By the Osceledets multiplicative ergodic theorem, there exists a compact subset $K_1 \subset \Omega_0$ with $\tilde{\mu}(K_1) > 1-\delta^2$ and $L > 0$ such that for $x \in K_1$ and all j and all $t > L$,

$$\|(T^t_x)_*\mathbf{v}\| \leq e^{(\lambda_j+\epsilon)t}\|\mathbf{v}\|, \qquad \mathbf{v} \in \mathcal{V}_{\geq j}(x)$$

and

$$\|(T^t_x)_*\mathbf{v}\| \geq e^{(\lambda_j-\epsilon)t}\|\mathbf{v}\|, \qquad \mathbf{v} \in \mathcal{V}_{\leq j}(x).$$

By Fubini's theorem there exists $K_1^* \subset \Omega_0$ with $\tilde{\mu}(K_1^*) > 1-2\delta$ such that for $x \in K_1^*$,

$$|\{ux \in \mathcal{U}_1^+ x \ : \ ux \in K_1\}| \geq (1-\delta/2)|\mathcal{U}_1^+ x|.$$

Let $K = K_{01} \cap K_1^*$, where K_{01} is as in Lemma 3.5 (with δ replaced by $\delta/2$).

Let $q = T^{-\ell}q_1$, and suppose $\mathbf{v} \in \mathrm{Lie}(N^-)(q)$. We can write

$$(T_q^\ell)_* \mathbf{v} = \sum_{j \in \Lambda'_0} \mathbf{v}'_j, \qquad \mathbf{v}'_j \in \mathbf{F}'_j(q_1). \tag{4.4}$$

Since

$$(T_q^\ell)_* \mathbf{v} \in \mathrm{Lie}(N^-)(q_1),$$

by Lemma 3.9, in the decomposition (4.4), the sum goes over $j \in \Lambda'$ (and not over $j \in \Lambda'_0$). Thus, for every j with $\mathbf{v}'_j \neq 0$, we have $\lambda_j > 0$.

Suppose $q_1 \in K$, $u \in Q_{01}(q_1, (T_q^\ell)_*\mathbf{v})$ and $uq_1 \in K_1^*$, where Q_{01} is as in Lemma 3.5. Then, by Lemma 3.5, we have

$$(T_q^\ell)_* \mathbf{v} = \sum_{j \in \Lambda'} (\mathbf{v}_j + \mathbf{w}_j), \tag{4.5}$$

where $\mathbf{v}_j \in \mathbf{E}_j(uq_1)$, $\mathbf{w}_j \in \mathcal{V}_{>j}(uq_1)$, and for all $j \in \Lambda'$,

$$\|\mathbf{v}_j\| \geq \beta'(\delta)\|\mathbf{w}_j\|. \tag{4.6}$$

Then,

$$\|(T_{uq_1}^t)_* \mathbf{w}_j\| \leq e^{(\lambda_{j+1}+\epsilon)t}\|\mathbf{w}_j\|,$$

and,

$$\|(T_{uq_1}^t)_* \mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)t}\|\mathbf{v}_j\| \geq e^{(\lambda_j-\epsilon)t}\beta'(\delta)\|\mathbf{w}_j\|. \tag{4.7}$$

Thus, for all $j \in \Lambda'$,

$$\|(T_{uq_1}^t)_* \mathbf{w}_j\| \leq e^{-(\lambda_j-\lambda_{j+1}+2\epsilon)t}\beta'(\delta)^{-1}\|(T_{uq_1}^t)_* \mathbf{v}_j\|.$$

Since $(T_{uq_1}^s)_* \mathbf{v}_j \in \mathbf{E}$ and using part (a) of Proposition 2.14, we get Proposition 4.1. $\square$

The following variant of Proposition 4.1 will be used in §6.

Proposition 4.2. *There exists $\epsilon' > 0$ (depending only on the Lyapunov spectrum) and for every $\delta > 0$ compact sets K, K'' with $\tilde{\mu}(K) > 1 - \delta$, $\tilde{\mu}(K'') > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$ such that the following holds: Suppose $x \in K$, $\mathbf{v} \in \mathrm{Lie}(N^+)(x)$ and that there exist arbitrarily large $t > 0$ with $T^{-t}x \in K''$ so that for at least $(1-\delta)$-fraction of $z \in \mathfrak{B}_0[T^{-t}x]$, the number $s > 0$ satisfying*

$$\|(T_z^s)_*(T_x^{-t})_*\mathbf{v}\| = \|\mathbf{v}\|, \tag{4.8}$$

also satisfies

$$s \geq (1 - \epsilon')t. \tag{4.9}$$

Then $\mathbf{v} \in \mathbf{E}(x)$.

Proof. Let $K_0 = \{x \in \Omega \ : \ |\mathfrak{B}_0[x]| \geq \delta^{-1}|\mathcal{U}_1^+ x|\}$. Let $K(\delta)$ be as in Lemma 3.6, and let $K'' = K_0 \cap K(\delta^2)$.

Suppose $\mathbf{v} \notin \mathbf{E}(x)$. We may write

$$\mathbf{v} = \sum_{i \in \Lambda'} \hat{\mathbf{v}}_i, \qquad \hat{\mathbf{v}}_i \in \mathbf{F}'_i(x)$$

Let j be minimal such that $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$. Let $k > j$ be such that $\mathbf{F}_{\geq k}(x) \subset \mathbf{F}_{\geq j}(x)$ is the subspace preceding $\mathbf{F}_{\geq j}(x)$ in (3.7). Then, $\mathbf{F}_{\geq i}(x) = \mathbf{F}_{\geq j}(x)$ for $j+1 \leq i \leq k$. In particular, $\mathbf{F}_{\geq j+1}(x) = \mathbf{F}_k(x)$.

Since $\hat{\mathbf{v}}_j \notin \mathbf{E}_j(x)$, $\hat{\mathbf{v}}_j$ must have a component in $\mathcal{V}_i(x)$ for some $i \geq j+1$. Therefore, by looking only at the component in $\mathcal{V}_i$, we get

$$\|(T_x^{-t})_* \mathbf{v}\| \geq C(\mathbf{v}) e^{-(\lambda_{j+1}+\epsilon)t},$$

Also since $\mathbf{F}_{\geq j+1}$ is T^t-equivariant, we have $\mathbf{F}_{\geq j+1}(x) = \bigoplus_m \mathbf{F}_{\geq k}(x) \cap \mathcal{V}_m(x)$. Note that by the multiplicative ergodic theorem, the restriction of g_{-t} to $\mathcal{V}_i$ is of the form $e^{-\lambda_i t} h_t$, where $\|h_t\| = O(e^{\epsilon t})$. Therefore, (again by looking only at the component in $\mathcal{V}_i$ and using Proposition 2.14 (a)), we get

$$d((T_x^{-t})_* \mathbf{v}, \mathbf{F}_{\geq j+1}(T^{-t}x)) \geq C(\mathbf{v}) e^{-(\lambda_{j+1}+2\epsilon)t}.$$

(Here and below, $d(\cdot,\cdot)$ denotes the distance on $\mathfrak{g}$ given by the dynamical norm $\|\cdot\|_x$.) Suppose $T^{-t}x \in K''$. Now, in view of Lemma 3.6, there exists a subset $Q \subset \mathfrak{B}_0[T^{-t}x]$ with $|Q| \geq (1-\delta)|\mathfrak{B}_0[T^{-t}x]|$, such that for $u \in Q$,

$$\|(T^s_{uT^{-t}x})_* \mathbf{v}\| \geq e^{(\lambda_j-\epsilon)s} C(\delta,\epsilon) C(\mathbf{v}) e^{-(\lambda_{j+1}+2\epsilon)t}. \tag{4.10}$$

If s satisfies (4.8), then $\|(T^s \circ u)_* \mathbf{v}_j\| = O(1)$. Therefore, in view of (4.10),

$$e^{(\lambda_j-\epsilon)s} e^{-(\lambda_{j+1}+2\epsilon)t} \leq c = c(\mathbf{v},\delta,\epsilon).$$

Therefore,

$$s \leq \frac{(\lambda_{j+1}+2\epsilon)t + \log c(\mathbf{v},\delta,\epsilon)}{(\lambda_j - \epsilon)}.$$

Since $\lambda_j > \lambda_{j+1}$, this contradicts (4.9) if ϵ is sufficiently small and t is sufficiently large. □

5. The action of the cocycle on E

In this section, we work on $\Omega = \Omega_0 \times \Sigma$. Recall that if $f(\cdot)$ is an object defined on Ω_0, then for $x \in \Omega$ we write $f(x)$ instead of $f(\sigma_0(x))$ (where $\sigma_0 : \Omega \to \Omega_0$ is the forgetful map).

In this section and in §6, assertions will hold at best for a.e $x \in \Omega$, and never for all $x \in \Omega$. This will be sometimes suppressed from the statements of the lemmas.

5.1. **The Jordan canonical form of the cocycle on $\mathbf{E}(x)$.** We consider the action of the cocycle $(T_x^t)_*$ on $\mathbf{E}$. The Lyapunov exponents are λ_i, $i \in \Lambda'$, so in particular $\lambda_i > 0$. For each $i \in \Lambda'$, we intersect $\mathbf{E}_i(x) \subset \mathcal{V}_i(x)$ with the maximal flag as in Lemma 2.3, to get a T-invariant flag

$$\{0\} \subset \mathbf{E}_{i1}(x) \subset \cdots \subset \mathbf{E}_{i,n_i}(x) = \mathbf{E}_i(x). \tag{5.1}$$

(The second index in (5.1) has been renumbered if needed to take all integer values from 1 to n_i with all the subspaces in (5.1) distinct.) Let Λ'' denote the set of pairs ij which appear in (5.1). By Lemma 3.4(a) and Lemma 2.4, we have for a.e. $ux \in \mathfrak{B}_0[x]$,

$$\mathbf{E}_{ij}(ux) = \mathbf{E}_{ij}(x).$$

Let $\|\cdot\|_x$ and $\langle\cdot,\cdot\rangle_x$ denote the restriction to $\mathbf{E}(x)$ of the norm and inner product on $\mathfrak{g}$ defined in §2.6. (We will often omit the subscript from $\langle\cdot,\cdot\rangle_x$ and $\|\cdot\|_x$.) Then, the distinct $\mathbf{E}_i(x)$ are orthogonal. For each $ij \in \Lambda''$ let $\mathbf{E}'_{ij}(x)$ be the orthogonal complement (relative to the inner product $\langle\cdot,\cdot\rangle_x$) to $\mathbf{E}_{i,j-1}(x)$ in $\mathbf{E}_{ij}(x)$.

Then, by Proposition 2.14, we can write, for $\mathbf{v} \in \mathbf{E}'_{ij}(x)$,

$$(T_x^t)_*\mathbf{v} = e^{\lambda_{ij}(x,t)}\mathbf{v}' + \mathbf{v}'', \tag{5.2}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(T^t x)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(T^t x)$, and $\|\mathbf{v}'\| = \|\mathbf{v}\|$. Hence (since $\mathbf{v}'$ and $\mathbf{v}''$ are orthogonal),

$$\|(T_x^t)_*\mathbf{v}\| \ge e^{\lambda_{ij}(x,t)}\|\mathbf{v}\|.$$

In view of Proposition 2.14 for a.e. $x \in \Omega$ the map $t \to \lambda_i(x,t)$ is monotone increasing and there exists a constant $\kappa > 1$ such that for a.e. $x \in \Omega$ and for all $\mathbf{v} \in \mathbf{E}(x)$ and all $t \ge 0$,

$$e^{\kappa^{-1}t}\|\mathbf{v}\| \le \|(T_x^t)_*\mathbf{v}\| \le e^{\kappa t}\|\mathbf{v}\|. \tag{5.3}$$

Lemma 5.1. *For a.e. $x \in \Omega$ and for a.e. $ux \in \mathcal{U}_1^+ x$,*

$$P^+(x, ux)\mathbf{E}(x) = \mathbf{E}(ux).$$

Proof. This follows immediately from Lemma 2.5. □

5.2. **Time changes.**

The flows $T^{ij,t}$ and the time changes $\tilde{\tau}_{ij}(x,t)$. We define the time changed flow $T^{ij,t}$ so that (after the time change) the cocycle $\lambda_{ij}(x,t)$ of (5.2) becomes $\lambda_i t$. We write $T^{ij,t}x = T^{\tilde{\tau}_{ij}(x,t)}x$. Then, by construction, $\lambda_{ij}(x, \tilde{\tau}_{ij}(x,t)) = \lambda_i t$. In view of Proposition 2.14, we note the following:

Lemma 5.2. *Suppose $y \in \mathfrak{B}_0[x]$. Then for any $ij \in \Lambda''$ and any $t > 0$,*

$$T^{ij,-t}y \in \mathfrak{B}_0[T^{ij,-t}x].$$

Proof. In view of Proposition 2.14(e), for all $t > 0$, and for all $y \in \mathfrak{B}_0[x]$, $\lambda_{ij}(x,-t) = \lambda_{ij}(y,-t)$. Now the lemma follows immediately from Lemma 2.12(a) and the definition of the flow $T^{ij,-t}$. □

In view of Proposition 2.14, we have for $t > t'$,

$$\frac{1}{\kappa}(t-t') \le \tilde{\tau}_{ij}(x,t) - \tilde{\tau}_{ij}(x,t') \le \kappa(t-t'), \tag{5.4}$$

where κ depends only on the Lyapunov spectrum.

5.3. **The foliations $\mathcal{F}_{ij}$, $\mathcal{F}_{\mathbf{v}}$ and the parallel transport $R(x,y)$.**

The sets Ω_{ebp} and Ω'. Let Ω_{ebp} be as in §2.1. We have $\tilde{\mu}(\Omega_{ebp}) = 0$ (see §2.1). Let $\Omega' = \Omega_{ebp}^c$, so $\tilde{\mu}(\Omega') = 1$.

For $x \in \Omega'$, let

$$\mathcal{H}[x] = \{T^s \circ u \circ T^{-t}x \ \ : \ \ t \ge 0, s \ge 0, u \in \mathcal{U}_1^+\} \subset \Omega.$$

For $y = (T^s \circ u \circ T^{-t})x \in \mathcal{H}[x]$, let

$$R(x,y) = (T^s_{uT^{-t}x})_*(T^{-t}_x)_*.$$

It is easy to see that for $x \in \Omega'$, given $y \in \mathcal{H}[x]$ and $t > 0$ there exist unique $u \in \mathcal{U}_1^+$ and $s \in \mathbb{R}$ such that $y = (T^s \circ u \circ T^{-t})x$. Therefore, for $x \in \Omega'$, $R(x,y) : \mathfrak{g} \to \mathfrak{g}$ depends only on x, y and not on the choices of t, u, s. Note that $R(x,y)\mathbf{E}(x) = \mathbf{E}(y)$.

In view of (5.2), Lemma 5.1, and Proposition 2.14 (e) and (f), for any $x \in \Omega'$, $\mathbf{v} \in \mathbf{E}'_{ij}(x)$, and any $y = T^s u T^{-t} x \in \mathcal{H}[x]$, we have

$$R(x,y)\mathbf{v} = e^{\lambda_{ij}(x,y)}\mathbf{v}' + \mathbf{v}'' \tag{5.5}$$

where $\mathbf{v}' \in \mathbf{E}'_{ij}(y)$, $\mathbf{v}'' \in \mathbf{E}_{i,j-1}(y)$, $\|\mathbf{v}'\| = \|\mathbf{v}\|$, and $\lambda_{ij}(x,y)$ is defined by

$$\lambda_{ij}(x,y) = \lambda_{ij}(x,-t) + \lambda_{ij}(uT^{-t}x, s). \tag{5.6}$$

For $x \in \Omega$ and $ij \in \Lambda''$, let $\mathcal{F}_{ij}[x]$ denote the set of $y \in \mathcal{H}[x]$ such that there exists $\ell \ge 0$ so that

$$T^{ij,-\ell}y \in \mathfrak{B}_0[T^{ij,-\ell}x]. \tag{5.7}$$

By Lemma 5.2, if (5.7) holds for some ℓ, it also holds for any bigger ℓ. Alternatively,

$$\mathcal{F}_{ij}[x] = \{T^{ij,\ell}uT^{ij,-\ell}x \ \ : \ \ \ell \ge 0, \quad uT^{ij,-\ell}x \in \mathfrak{B}_0[T^{ij,-\ell}x]\}.$$

In view of (5.6), it follows that for $y \in \mathcal{H}[x]$,

$$\lambda_{ij}(x,y) = 0 \qquad \text{if and only if } y \in \mathcal{F}_{ij}[x]. \tag{5.8}$$

We refer to the sets $\mathcal{F}_{ij}[x]$ as *leaves* of a foliation corresponding to the index ij.

For any compact subset $A \subset \mathcal{F}_{ij}[x]$ there exists ℓ large enough so that $T^{ij,-\ell}(A)$ is contained in a set of the form $\mathfrak{B}_0[z] \subset W_1^+[z]$. Then the same holds for $T^{ij,-t}(A)$, for any $t > \ell$.

Recall that the sets $\mathfrak{B}_0[x]$ support the conditional measure $\tilde{\mu}|_{W_1^+[x]}$ which we sometimes denote by $|\cdot|$. We have, for a.e. x, $|\mathfrak{B}_0[x]| > 0$. As a consequence, the leaves

$\mathcal{F}_{ij}[x]$ also support a measure (defined up to normalization), which we also denote by $|\cdot|$. More precisely, if $A \subset \mathcal{F}_{ij}[x]$ and $B \subset \mathcal{F}_{ij}[x]$ are compact subsets, we define

$$\frac{|A|}{|B|} \equiv \frac{|T^{ij,-\ell}(A)|}{|T^{ij,-\ell}(B)|}, \tag{5.9}$$

where ℓ is chosen large enough so that both $T^{ij,-\ell}(A)$ and $T^{ij,-\ell}(B)$ are contained in a set of the form $\mathfrak{B}_0[z]$, $z \in \Omega$. It is clear that if we replace ℓ by a larger number ℓ', the right-hand-side of (5.9) remains the same.

We define the "balls" $\mathcal{F}_{ij}[x,\ell] \subset \mathcal{F}_{ij}[x]$ by

$$\mathcal{F}_{ij}[x,\ell] = \{y \in \mathcal{F}_{ij}[x] \;\;:\;\; T^{ij,-\ell}y \in \mathfrak{B}_0[T^{ij,-\ell}x]\}. \tag{5.10}$$

Lemma 5.3. *Suppose $x \in \Omega$ and $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough,*

$$\mathcal{F}_{ij}[x,\ell] = \mathcal{F}_{ij}[y,\ell].$$

Proof. Suppose $y \in \mathcal{F}_{ij}[x]$. Then, for ℓ large enough, $T^{ij,-\ell}y \in \mathfrak{B}_0[T^{ij,-\ell}x]$, and then $\mathfrak{B}_0[T^{ij,-\ell}y] = \mathfrak{B}_0[T^{ij,-\ell}x]$. □

The flows $\tilde{T}^t$. Suppose $x \in \Omega$ and $\mathbf{v} \in \mathfrak{g}$. Let $\tilde{T}^t : \Omega \times \mathfrak{g} \to \Omega \times \mathfrak{g}$ be the flow defined by $\tilde{T}^t(x,\mathbf{v}) = \hat{T}^{\tilde{\tau}_{\mathbf{v}}(x,t)}(x,\mathbf{v})$, where the time change $\tilde{\tau}_{\mathbf{v}}(x,t)$ is chosen so that

$$\|(T_x^{\tilde{\tau}_{\mathbf{v}}(x,t)})_* \mathbf{v}\|_{T^{\tilde{\tau}_{\mathbf{v}}(x,t)}x} = e^t \|\mathbf{v}\|_x.$$

We have, for $x \in \Omega$,

$$\tilde{T}^{t+s}(x,\mathbf{v}) = \tilde{T}^t \tilde{T}^s(x,\mathbf{v}).$$

By (5.3), (5.4) holds for $\tilde{\tau}_{\mathbf{v}}$ instead of $\tilde{\tau}_{ij}$.

Let $\pi_\Omega : \Omega \times \mathfrak{g} \to \Omega$ denote projection onto the first factor. For $x \in \Omega$, $\mathbf{v} \in \mathfrak{g}$ and $\ell \in \mathbb{R}$, let $\tilde{g}_{-\ell}^{\mathbf{v},x} : \mathcal{H}[x] \to \mathcal{H}$ be defined by

$$\tilde{g}_{-\ell}^{\mathbf{v},x}(y) = \pi_\Omega(\tilde{T}^{-\ell}(y,\mathbf{w})), \qquad \text{where } \mathbf{w} = R(x,y)\mathbf{v}. \tag{5.11}$$

(When there is no potential for confusion about the point x and the vector $\mathbf{v}$ used, we denote $\tilde{g}_{-\ell}^{\mathbf{v},x}$ by $\tilde{g}_{-\ell}$.) Note that Lemma 5.2 still holds if $T^{ij,-t}$ is replaced by $\tilde{g}_{-t}^{\mathbf{v},x}$.

The "system of curves" $\mathcal{F}_{\mathbf{v}}$. For $\mathbf{v} \in \mathbf{E}(x)$ we can define the "leaves" $\mathcal{F}_{\mathbf{v}}[x]$ and the "balls" $\mathcal{F}_{\mathbf{v}}[x,\ell]$ as in (5.7) and (5.10), with $\tilde{g}_{-t}^{\mathbf{v},x}$ replacing the role of $T^{ij,-t}$.

For $y \in \mathcal{F}_{\mathbf{v}}[x]$, we have

$$\mathcal{F}_{\mathbf{v}}[x] = \mathcal{F}_{\mathbf{w}}[y], \qquad \text{where } \mathbf{w} = R(x,y)\mathbf{v}.$$

We can define the measure (up to normalization) $|\cdot|$ on $\mathcal{F}_{\mathbf{v}}[x,\ell]$ as in (5.9). Lemma 5.3 holds for $\mathcal{F}_{\mathbf{v}}[x]$ without modifications.

The following follows immediately from the construction:

Lemma 5.4. *For a.e. $x \in \Omega$, any $\mathbf{v} \in \mathbf{E}(x)$, and a.e. $y \in \mathcal{F}_{\mathbf{v}}[x]$, we have*

$$\|R(x,y)\mathbf{v}\|_y = \|\mathbf{v}\|_x.$$

Remark 5.5. We will need to consider the somewhat artificial object $\mathcal{F}_{\mathbf{v}}$ for the following reason. Suppose we follow the outline of [EsL, §1.2] and have picked $\hat{q}$, $\hat{q}'$, ℓ and u. Then we get points $\hat{q}_2 = (q_2, g_2)$ and $\hat{q}_2' = (q_2, g_2')$ with $d_G(g_2, g_2') \approx \epsilon$, and write $g_2' = \exp(\mathbf{v})g_2$, where $\mathbf{v} \in \mathfrak{g}$, and $\|\mathbf{v}\| \approx \epsilon$. (In fact, we will have $\mathbf{v}$ very near $\mathbf{E}(q_2)$).

Now suppose in the scheme of [EsL, §1.2] we replace u by u_1 near u, but keep $\hat{q}$, $\hat{q}'$, ℓ the same. Let $\hat{q}_2(u_1)$, $\hat{q}_2'(u_1)$ be the analogues of $\hat{q}_2$ and $\hat{q}_2'$ but with u replaced by u_1, and write $\hat{q}_2(u_1) = (q_2(u_1), g_2(u_1))$, $\hat{q}_2'(u_1) = (q_2'(u_1), \exp(\mathbf{v}(u_1)g_2(u_1))$. Then, in view of Lemma 5.4, to a high degree of approximation, $q_2(u_1) \in \mathcal{F}_{\mathbf{v}}[q_2]$, and $\mathbf{v}(u_1) = R(q_2, q_2(u_1))\mathbf{v}$. (The error comes from the fact that $\mathbf{v}$ is not exactly in $\mathbf{E}$ and we are defining $\mathcal{F}_{\mathbf{v}}$ only for $\mathbf{v} \in \mathbf{E}$).

5.4. **Time changes for nearby points.**

Lemma 5.6. *For every $\delta > 0$ there exists a compact set $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ such that the following holds: Suppose $t > 0$, $x \in K$, $y \in W_1^-[x] \cap K$, and $T^t x \in K$ and $T^t y \in T^{[-a,a]}K$. Then,*

$$|\lambda_{ij}(x,t) - \lambda_{ij}(y,t)| \le C_1, \tag{5.12}$$

where C_1 depends only on a and δ.

Proof of Lemma 5.6. Let K be as Lemma 2.2. Suppose $\mathbf{v} \in \mathbf{E}_{ij}(x)$, and that $\mathbf{v}$ is orthogonal to $\mathbf{E}_{i,j-1}(x) \subset \mathbf{E}_{ij}(x)$. Let

$$\mathbf{v}' = P^-(x,y)\mathbf{v}.$$

Then, $\mathbf{v}' \in \mathbf{E}_{ij}(y)$ and $\mathbf{v}'$ is orthogonal to $\mathbf{E}_{i,j-1}(y) \subset \mathbf{E}_{ij}(y)$. For an invertible linear operator $A : \mathfrak{g} \to \mathfrak{g}$, let $\|A\|_x^y = |A|_x^y + |A^{-1}|_y^x$, where for a linear operator $B : \mathfrak{g} \to \mathfrak{g}$, $|B|_x^y$ denotes operator norm of B relative to the norms $\|\cdot\|_x$ on the domain and $\|\cdot\|_y$ on the range. By Lemma 2.2 and Lemma 2.16, there exist $C = C(\delta)$ and $C_1 = C_1(a,\delta)$ such that

$$\|P^-(x,y)\|_x^y \le C(\delta), \text{ and } \|P^-(T^t x, T^t y)\|_{T^t x}^{T^t y} \le C_1(a,\delta). \tag{5.13}$$

Therefore,

$$C(\delta)^{-1} \le \frac{\|\mathbf{v}'\|_y}{\|\mathbf{v}\|_x} \le C(\delta). \tag{5.14}$$

Note that

$$(T_y^t)_*\mathbf{v}' = P^-(T^t x, T^t y)(T_x^t)_*\mathbf{v}, \text{ and } P^-(T^t x, T^t y)\mathbf{E}_{i,j-1}(T^t x) = \mathbf{E}_{i,j-1}(T^t y).$$

Then, in view of (5.13), there exists $C_2 = C_2(a,\delta)$ such that

$$C_2(a,\delta)^{-1} \le \frac{\|(T_y^t)_*\mathbf{v}' + \mathbf{E}_{i,j-1}(T^t y)\|_{T^t y}}{\|(T_x^t)_*\mathbf{v} + \mathbf{E}_{i,j-1}(T^t x)\|_{T^t x}} \le C_2(a,\delta). \tag{5.15}$$

By the definition of $\lambda_{ij}(\cdot,\cdot)$,

$$\lambda_{ij}(x,t) = \log \frac{\|(T^t_x)_* \mathbf{v} + \mathbf{E}_{i,j-1}(T^t x)\|_{T^t x}}{\|\mathbf{v}\|_x}, \quad \lambda_{ij}(y,t) = \log \frac{\|(T^t_y)_* \mathbf{v}' + \mathbf{E}_{i,j-1}(T^t y)\|_{T^t y}}{\|\mathbf{v}'\|_y}.$$

Now (5.12) follows from (5.14) and (5.15). □

5.5. **A maximal inequality.** Let κ be as in Proposition 2.14.

Lemma 5.7. *Suppose $K \subset \Omega$ with $\tilde{\mu}(K) > 1-\delta$. Then, for any $\theta' > 0$ there exists a subset $K^* \subset \Omega$ with $\tilde{\mu}(K^*) > 1 - 2\kappa^2\delta/\theta'$ such that for any $x \in K^*$ and any $\ell > 0$,*

$$|\mathcal{F}_{ij}[x,\ell] \cap K| > (1-\theta')|\mathcal{F}_{ij}[x,\ell]|. \tag{5.16}$$

Proof. For $t > 0$ let

$$\mathfrak{B}^{ij}_t[x] = T^{ij,-t}(\mathfrak{B}_0[T^{ij,t}x]) = \mathfrak{B}_\tau[x],$$

where τ is such that $T^{ij,t}x = T^\tau x$. Let $s > 0$ be arbitrary. Let $K_s = T^{ij,-s}K$. By Proposition 2.14(c),

$$\tilde{\mu}(T^{ij,-s}(K^c)) \le \kappa\, \tilde{\mu}(K^c) < \kappa\delta.$$

Therefore, $\tilde{\mu}(K_s) > 1 - \kappa\delta$. Then, by Lemma 2.13, there exists a subset K'_s with $\tilde{\mu}(K'_s) \ge (1 - 2\kappa\delta/\theta')$ such that for $x \in K'_s$ and all $t > 0$,

$$|K_s \cap \mathfrak{B}^{ij}_t[x]| \ge (1-\theta'/2)|\mathfrak{B}^{ij}_t[x]|.$$

Let $K^*_s = T^{ij,s}K'_s$, and note that $T^{ij,s}\mathfrak{B}^{ij}_t[x] = \mathcal{F}_{ij}[T^{ij,s}x, s-t]$. Then, for all $x \in K^*_s$ and all $0 < s-t < s$,

$$|\mathcal{F}_{ij}[x,s-t] \cap K| \ge (1-\theta'/2)|\mathcal{F}_{ij}[x,s-t]|.$$

We have $\tilde{\mu}(K^*_s) \ge (1 - 2\kappa^2\delta/\theta')$. Now take a sequence $s_n \to \infty$, and let K^* be the set of points which are in infinitely many $K^*_{s_n}$. □

6. Bounded subspaces and synchronized exponents

Recall that Λ'' indexes the "fine Lyapunov spectrum" on $\mathbf{E} \subset N^+$. In this section we define an equivalence relation called "synchronization" on Λ''; the equivalence class of ij is denoted $[ij]$, and the set of equivalence classes is denoted by $\tilde{\Lambda}$. For each $ij \in \Lambda''$ we define a T^t-equivariant subbundle $\mathbf{E}_{ij,bdd}$ of the bundle $\mathbf{E}_i = \mathcal{V}_i \cap \mathbf{E}$ so that $\mathbf{E}_{ij,bdd}(ux) = \mathbf{E}_{ij,bdd}(x)$ a.e., and let

$$\mathbf{E}_{[ij],bdd}(x) = \sum_{kr \in [ij]} \mathbf{E}_{kr,bdd}(x).$$

In fact we will show that there exists an subset $[ij]' \subset [ij]$ such that

$$\mathbf{E}_{[ij],bdd}(x) = \bigoplus_{kr \in [ij]'} \mathbf{E}_{kr,bdd}(x). \tag{6.1}$$

Then, we claim that the following three propositions hold:

Proposition 6.1. *There exist* $0 < \theta_1 < 1$ *depending only on* $\hat{\nu}$ *such that the following holds: for every* $\delta > 0$ *and every* $\eta > 0$*, there exists a subset* $K = K(\delta, \eta)$ *of measure at least* $1 - \delta$ *and* $L_0 = L_0(\delta, \eta) > 0$ *such that the following holds: Suppose* $x \in \Omega$*,* $\mathbf{v} \in \mathbf{E}(x)$*,* $L \geq L_0$*, and*

$$|T^{[-1,1]}K \cap \mathcal{F}_{\mathbf{v}}[x, L]| \geq (1 - \theta_1)|\mathcal{F}_{\mathbf{v}}[x, L]|.$$

Then, for at least θ_1*-fraction of* $y \in \mathcal{F}_{\mathbf{v}}[x, L]$*,*

$$d\left(\frac{R(x, y)\mathbf{v}}{\|R(x, y)\mathbf{v}\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < \eta.$$

Recall that under the condition of the Proposition, $\|R(x, y)\mathbf{v}\| = \|\mathbf{v}\|$. In view of Remark 5.5, we can (and will) use this Proposition for "tie-breaking", see §1.5.

The following two Propositions will allow us to carry out the argument outlined in §1.5 using conditional measures on $\exp(\mathbf{E}_{[ij],bdd})$:

Proposition 6.2. *There exists a function* $C_3 : \Omega \to \mathbb{R}^+$ *finite almost everywhere so that for all* $x \in \Omega$*, for all* $y \in \mathcal{F}_{ij}[x]$*, for all* $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$*,*

$$C_3(x)^{-1}C_3(y)^{-1}\|\mathbf{v}\| \leq \|R(x, y)\mathbf{v}\| \leq C_3(x)C_3(y)\|\mathbf{v}\|.$$

In particular, for every $\delta > 0$ *there exist* $C > 1$ *and a compact set* $K \subset \Omega$ *with* $\tilde{\mu}(K) > 1 - \delta$ *such that if* $x \in K$ *and* $y \in \mathcal{F}_{ij}[x] \cap K$ *then for all* $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$*,*

$$C^{-1}\|\mathbf{v}\| \leq \|R(x, y)\mathbf{v}\| \leq C\|\mathbf{v}\|.$$

The following Proposition will be used to show that for a.e. x, $\mathbf{E}_{[ij],bdd}(x)$ is a subalgebra of $\mathfrak{g}$.

Proposition 6.3. *There exists* $\theta > 0$ *(depending only on* $\hat{\nu}$*) and a subset* $\Psi \subset \Omega$ *with* $\tilde{\mu}(\Psi) = 1$ *such that the following holds:*

Suppose $x \in \Psi$*,* $\mathbf{v} \in Lie(N^+)(x)$*, and there exists* $C > 0$ *such that for all* $\ell > 0$*, and at least* $(1 - \theta)$*-fraction of* $y \in \mathcal{F}_{ij}[x, \ell]$*,*

$$\|R(x, y)\mathbf{v}\| \leq C\|\mathbf{v}\|.$$

Then, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$*.*

The numbers $\theta > 0$ and $\theta_1 > 0$, the synchronization relation and the subspaces $\mathbf{E}_{ij,bdd}$ are defined in §6.1. Also Proposition 6.1 is proved in §6.1. Proposition 6.2 and Proposition 6.3 are proved in §6.3.

6.1. **Bounded subspaces and synchronized exponents.** For $x \in \Omega$, $y \in \Omega$, let

$$\rho(x, y) = \begin{cases} |t| & \text{if } y = T^t x, \\ \infty & \text{otherwise.} \end{cases}$$

If $x \in \Omega$ and $E \subset \Omega$, we let $\rho(x, E) = \inf_{y \in E} \rho(x, y)$.

Lemma 6.4. *For every $\eta > 0$ and $\eta' > 0$ there exists $h = h(\eta', \eta)$ such that the following holds: Suppose $\mathbf{v} \in \mathbf{E}_{ij}(x)$ and*

$$d\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}, \mathbf{E}_{i,j-1}(x)\right) > \eta'.$$

Then if $y \in \mathcal{F}_{\mathbf{v}}[x]$ and

$$\rho(y, \mathcal{F}_{ij}[x]) > h$$

then

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) \leq \eta\|\mathbf{v}\|.$$

Proof. There exists $t \in \mathbb{R}$ such that $y' = T^t y \in \mathcal{F}_{ij}[x]$. Then

$$\rho(y, \mathcal{F}_{ij}[x]) = \rho(y, y') = |t| > h.$$

We have the orthogonal decomposition $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{w}$, where $\hat{\mathbf{v}} \in \mathbf{E}'_{ij}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$. Then by (5.5) we have the orthogonal decomposition.

$$R(x,y')\hat{\mathbf{v}} = e^{\lambda_{ij}(x,y')}\mathbf{v}' + \mathbf{w}', \qquad \text{where } \mathbf{v}' \in \mathbf{E}'_{ij}(y'),\ \mathbf{w}' \in \mathbf{E}_{i,j-1}(y'),\ \|\hat{\mathbf{v}}\| = \|\mathbf{v}'\|.$$

Since $R(x,y')\mathbf{w} \in \mathbf{E}_{i,j-1}(y')$, we have

$$\|R(x,y')\mathbf{v}\|^2 = e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2 + \|\mathbf{w}' + R(x,y')\mathbf{w}\|^2 \geq e^{2\lambda_{ij}(x,y')}\|\hat{\mathbf{v}}\|^2.$$

By (5.8), we have $\lambda_{ij}(x,y') = 0$. Hence,

$$\|R(x,y')\mathbf{v}\| \geq \|\hat{\mathbf{v}}\| \geq \eta'\|\mathbf{v}\|.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, by Lemma 5.4, $\|R(x,y)\mathbf{v}\| = \|\mathbf{v}\|$. Since $|t| > h$, we have either $t > h$ or $t < -h$. If $t < -h$, then by (5.3),

$$\|\mathbf{v}\| = \|R(x,y)\mathbf{v}\| = \|(T_{y'}^{-t})_* R(x,y')\mathbf{v}\| \geq e^{\kappa^{-1}h}\|R(x,y')\mathbf{v}\| \geq e^{\kappa^{-1}h}\eta'\|\mathbf{v}\|,$$

which is a contradiction if $h > \kappa \log(1/\eta')$. Hence we may assume that $t > h$. We have,

$$R(x,y)\mathbf{v} = e^{\lambda_{ij}(x,y)}\mathbf{v}'' + \mathbf{w}''$$

where $\mathbf{v}'' \in \mathbf{E}'_{ij}(y)$ with $\|\mathbf{v}''\| = \|\hat{\mathbf{v}}\|$, and $\mathbf{w}'' \in \mathbf{E}_{i,j-1}(y)$. Hence,

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) = e^{\lambda_{ij}(x,y)}\|\hat{\mathbf{v}}\| \leq e^{\lambda_{ij}(x,y)}\|\mathbf{v}\|.$$

But,

$$\lambda_{ij}(x,y) = \lambda_{ij}(x,y') + \lambda_{ij}(y', -t) \leq -\kappa^{-1}t$$

by (5.8) and Proposition 2.14. Therefore,

$$d(R(x,y)\mathbf{v}, \mathbf{E}_{i,j-1}(y)) \leq e^{-\kappa^{-1}t}\|\mathbf{v}\| \leq e^{-\kappa^{-1}h}\|\mathbf{v}\|.$$

□

The bounded subspace. Fix $\theta > 0$. (We will eventually choose θ sufficiently small depending only on the dimension).

Definition 6.5. Suppose $x \in \Omega$. A vector $\mathbf{v} \in \mathbf{E}_{ij}(x)$ is called (θ, ij)-*bounded* if there exists $C < \infty$ such that for all $\ell > 0$ and for $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x, \ell]$,

$$\|R(x,y)\mathbf{v}\| \le C\|\mathbf{v}\|. \tag{6.2}$$

Remark. From the definition and (5.5), it is clear that every vector in $\mathbf{E}_{i1}(x)$ is $(\theta, i1)$-bounded for every θ. Indeed, we have $\mathbf{E}'_{i1} = \mathbf{E}_{i1}$, and $\lambda_{i1}(x,y) = 0$ for $y \in \mathcal{F}_{i1}[x]$, thus for $y \in \mathcal{F}_{i1}[x]$ and $\mathbf{v} \in \mathbf{E}_{i1}(x)$, $\|R(x,y)\mathbf{v}\| = \|\mathbf{v}\|$.

Lemma 6.6. *Let $n = \dim \mathbf{E}_{ij}(x)$ (for a.e x). If there exists no non-zero θ/n-bounded vector in $\mathbf{E}_{ij}(x) \smallsetminus \mathbf{E}_{i,j-1}(x)$, we set $\mathbf{E}_{ij,bdd} = \{0\}$. Otherwise, we define $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ to be the linear span of the θ/n-bounded vectors in $\mathbf{E}_{ij}(x)$. This is a subspace of $\mathbf{E}_{ij}(x)$, and any vector in this subspace is θ-bounded. Also,*

(a) *$\mathbf{E}_{ij,bdd}(x)$ is T^t-equivariant, i.e. $(T^t_x)_*\mathbf{E}_{ij,bdd}(x) = \mathbf{E}_{ij,bdd}(T^t x)$.*
(b) *For almost all $ux \in \mathfrak{B}_0[x]$, $\mathbf{E}_{ij,bdd}(ux) = \mathbf{E}_{ij,bdd}(x)$.*

Proof. Let $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ denote the linear span of all $(\theta/n, ij)$-bounded vectors. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are any n $(\theta/n, ij)$-bounded vectors, then there exists $C > 1$ such that for $1-\theta$ fraction of y in $\mathcal{F}_{ij}[x, L]$, (6.2) holds. But then (6.2) holds (with a different C) for any linear combination of the $\mathbf{v}_i$. This shows that any vector in $\mathbf{E}_{ij,bdd}(x)$ is (θ, ij)-bounded. To show that (a) holds, suppose that $\mathbf{v} \in \mathbf{E}_{ij}(x)$ is $(\theta/n, ij)$-bounded, and $t < 0$. In view of Lemma 3.2, it is enough to show that $\mathbf{v}' \equiv (T^{ij,t}_x)_*\mathbf{v} \in \mathbf{E}_{ij}(T^{ij,t}x)$ is $(\theta/n, ij)$-bounded. (This would show that for $t < 0$, $(T^{ij,t}_x)_*\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij,bdd}(T^{ij,t}x)$ which, in view of the ergodicity of the action of T^t, would imply (a).)

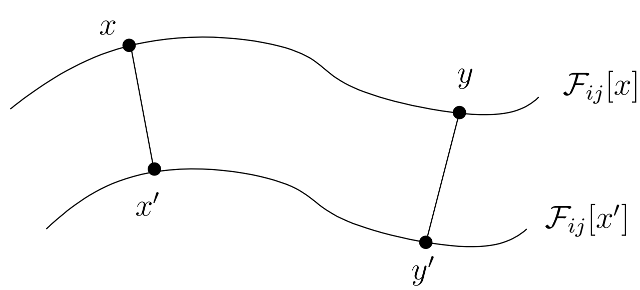

Figure 2. Proof of Lemma 6.6 (a).

Let $x' = T^{ij,t}x$. By (5.3), there exists $C_1 = C_1(t)$ such that for all $z \in \Omega$ and all $\mathbf{w} \in \mathbf{E}(z)$,

$$C_1^{-1}\|\mathbf{w}\| \le \|(T^{ij,t}_z)_*\mathbf{w}\| \le C_1\|\mathbf{w}\|. \tag{6.3}$$

Suppose $y \in \mathcal{F}_{ij}[x, L]$ satisfies (6.2). Let $y' = T^{ij,t}y$. Then $y' \in \mathcal{F}_{ij}[x']$. Let $\mathbf{v}' = (T^{ij,t})_*\mathbf{v}$. (See Figure 2). Note that

$$R(x', y')\mathbf{v}' = R(y, y')R(x, y)R(x', x)\mathbf{v}' = R(y, y')R(x, y)\mathbf{v}$$

hence by (6.3), (6.2), and again (6.3),

$$\|R(x',y')\mathbf{v}'\| \le C_1\|R(x,y)\mathbf{v}\| \le C_1C\|\mathbf{v}\| \le C_1^2C\|\mathbf{v}'\|.$$

Hence, for $y \in \mathcal{F}_{ij}[x,L]$ satisfying (6.2), $y' = T^{ij,t}y \in \mathcal{F}_{ij}[x']$ satisfies

$$\|R(x',y')\mathbf{v}'\| < CC_1^2\|\mathbf{v}'\|. \tag{6.4}$$

Therefore, since $\mathcal{F}_{ij}[T^{ij,t}x, L+t] = T^{ij,t}\mathcal{F}_{ij}[x,L]$, we have that for $1-\theta/n$ fraction of $y' \in \mathcal{F}_{ij}[x', L+t]$, (6.4) holds. Therefore, $\mathbf{v}'$ is $(\theta/n, ij)$-bounded. Thus, $\mathbf{E}_{ij,bdd}(x)$ is T^t-equivariant. This completes the proof of (a). Then (b) follows immediately from (a) since Lemma 5.3 implies that $\mathcal{F}_{ij}[ux,L] = \mathcal{F}_{ij}[x,L]$ for L large enough. □

Remark 6.7. Formally, from its definition, the subspace $\mathbf{E}_{ij,bdd}(x)$ depends on the choice of θ. It is clear that as we decrease θ, the subspace $\mathbf{E}_{ij,bdd}(x)$ decreases. Therefore there exists $\theta_0 > 0$ and $m \ge 0$ such that for all $\theta < \theta_0$ and almost all $x \in \Omega$, the dimension of $\mathbf{E}_{ij,bdd}(x)$ is m. We will always choose $\theta \ll \theta_0$.

Synchronized Exponents.

Definition 6.8. Suppose $\theta > 0$, and $E \subset \Omega$ with $\tilde{\mu}(E) > 0$. We say that $ij \in \Lambda''$ and $kr \in \Lambda''$ are (E,θ)-*synchronized* if there exists $C < \infty$, such that for all $x \in E$, for all $\ell > 0$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,\ell]$, we have

$$\rho(y, \mathcal{F}_{kr}[x]) < C.$$

We say that $ij \in \Lambda''$ and $kr \in \Lambda''$ are θ-*synchronized* if there exists $E \subset \Omega$ with $\tilde{\mu}(E) > 0$ such that ij and kr are (E,θ)-synchronized.

Remark 6.9. By the same argument as in the proof of Lemma 6.6 (a), if ij and kr are (E,θ)-synchronized then they are also $(\bigcup_{|s|<t} T^sE, \theta)$-synchronized (with C depending on t). Therefore, we can take the set E in Definition 6.8 to have measure arbitrarily close to 1.

Remark 6.10. Clearly if ij and kr are not θ-synchronized, then they are also not θ'-synchronized for any $\theta' < \theta$. Therefore there exists $\theta_0' > 0$ such that if any pairs ij and kr are not θ-synchronized for some $\theta > 0$ then they are also not θ_0'-synchronized. We will always consider $\theta \ll \theta_0'$, and will sometimes use the term "synchronized" with no modifier to mean θ-synchronized for $\theta \ll \theta_0'$.

By the definition, if ij and kr are $(E_1, \theta/2)$-synchronized and kr and mn are $(E_2,\theta/2)$-synchronized, then ij and mn are $(E_1 \cap E_2, \theta)$-synchronized. Then in view of Remark 6.9, as long as $\theta \ll \theta_0'$, synchronization (with no modifier) is an equivalence relation.

We now fix $\theta \ll \min(\theta_0, \theta_0')$.

If $\mathbf{v} \in \mathbf{E}(x)$, we can write

$$\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}, \qquad \text{where } \mathbf{v}_{ij} \in \mathbf{E}_{ij}(x), \text{ but } \mathbf{v}_{ij} \notin \mathbf{E}_{i,j-1}(x). \tag{6.5}$$

In the sum, $I_{\mathbf{v}}$ is a finite set of pairs $ij \in \Lambda''$. Since for a fixed i the $\mathbf{E}_{ij}(x)$ form a flag, without loss of generality we may (and always will) assume that $I_{\mathbf{v}}$ contains at most one pair ij for each $i \in \Lambda'$. (Recall that Λ' denotes positive the Lyapunov spectrum of $\mathbf{E}$).

For $\mathbf{v} \in \mathbf{E}(x)$, and $y \in \mathcal{F}_{\mathbf{v}}[x]$, let

$$H_{\mathbf{v}}(x,y) = \max_{ij \in I_{\mathbf{v}}} \rho(y, \mathcal{F}_{ij}[x]).$$

Lemma 6.11. *There exists a set $\Psi \subset \Omega$ with $\tilde{\mu}(\Psi) = 1$ such that the following holds: Suppose $x \in \Psi$, $C < \infty$, and there exists $\mathbf{v} \in \mathbf{E}(x)$ so that for each $L > 0$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$*

$$H_{\mathbf{v}}(x,y) < C.$$

Then, if we write $\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$ as in (6.5), then all $\{ij\}_{ij \in I_{\mathbf{v}}}$ are synchronized, and also for all $ij \in I_{\mathbf{v}}$, $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$.

Proof. Let $\Psi = \bigcup_{t \in \mathbb{R}} T^t E$, where E is as in Definition 6.8. (In view of Remark 6.9, we may assume that the same E works for all synchronized pairs). Suppose $ij \in I_{\mathbf{v}}$ and $kr \in I_{\mathbf{v}}$. We have for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L]$,

$$\rho(y, \mathcal{F}_{ij}[x]) < C, \qquad \rho(y, \mathcal{F}_{kr}[x]) < C.$$

Let $y_{ij} \in \mathcal{F}_{ij}[x]$ be such that $\rho(y, \mathcal{F}_{ij}[x]) = \rho(y, y_{ij})$. Similarly, let $y_{kr} \in \mathcal{F}_{kr}[x]$ be such that $\rho(y, \mathcal{F}_{kr}[x]) = \rho(y, y_{kr})$. We have

$$\rho(y_{ij}, y_{kr}) \leq \rho(y_{ij}, y) + \rho(y, y_{kr}) \leq 2C. \tag{6.6}$$

Note that $\tilde{g}_{-L}^{\mathbf{v},x}(\mathcal{F}_{\mathbf{v}}[x, L]) = T^{ij,-L'}(\mathcal{F}_{ij}[x, L'])$, where L' is chosen so that $\pi_\Omega(\tilde{T}^{-L}(x, \mathbf{v})) = T^{ij,-L'}x$, where the notation $\tilde{g}$ is as in (5.11). Hence, in view of (6.6) and (5.9), for any $L' > 0$, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, $\rho(y_{ij}, \mathcal{F}_{kr}[x]) \leq 2C$. Then, for any $t \in \mathbb{R}$, for any $L'' > 0$, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[T^t x, L'']$, $\rho(y_{ij}, \mathcal{F}_{kr}[T^t x]) \leq C(t)$. Since $x \in \Psi$, we can choose t so that $T^t x \in E$ where E is as in Definition 6.8. This implies that ij and kr are synchronized.

Recall that $I_{\mathbf{v}}$ contains at most one j for each $i \in \Lambda'$. Since $R(x,y)$ preserves each $\mathbf{E}_i$, and the distinct $\mathbf{E}_i$ are orthogonal, for all $y'' \in \mathcal{H}[x]$,

$$\|R(x, y'')\mathbf{v}\|^2 = \sum_{ij \in I_{\mathbf{v}}} \|R(x, y'')\mathbf{v}_{ij}\|^2.$$

Therefore, for each $ij \in I_{\mathbf{v}}$, and all $y'' \in \mathcal{H}[x]$,

$$\|R(x, y'')\mathbf{v}_{ij}\| \leq \|R(x, y'')\mathbf{v}\|.$$

In particular,

$$\|R(x,y_{ij})\mathbf{v}_{ij}\| \le \|R(x,y_{ij})\mathbf{v}\|.$$

By the assumption of the Lemma, and by the definitions of the measures on $\mathcal{F}_{ij}[x]$ and $\mathcal{F}_{\mathbf{v}}[x]$, we have for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$, there exists $y \in \mathcal{F}_{\mathbf{v}}[x]$ with $\rho(y_{ij}, y) < C$. We have, by Lemma 5.4, $\|R(x,y)\mathbf{v}\| = \|\mathbf{v}\|$, and hence, by (5.3), for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L]$,

$$\|R(x,y_{ij})\mathbf{v}\| \le C_2\|\mathbf{v}\|.$$

Hence, for $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, L']$,

$$\|R(x,y_{ij})\mathbf{v}_{ij}\| \le C_2\|\mathbf{v}\|.$$

This implies that $\mathbf{v}_{ij} \in \mathbf{E}_{ij,bdd}(x)$. □

We write $ij \sim kr$ if ij and kr are synchronized. With our choice of $\theta > 0$, synchronization is an equivalence relation, see Remark 6.10. We write $[ij] = \{kr \; : \; kr \sim ij\}$. Let

$$\mathbf{E}_{[ij],bdd}(x) = \sum_{kr \in [ij]} \mathbf{E}_{kr,bdd}(x).$$

Lemma 6.12. *There exists a $\Psi \subset \Omega$ with $\tilde{\mu}(\Psi) = 1$ such that the following holds: Suppose $ij \sim ik$, $j < k$, $x \in \Psi$ and $\mathbf{E}_{ik,bdd}(x) \neq \{0\}$ (see Definition 6.5). Then $\mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ik,bdd}(x)$. Thus, (6.1) holds.*

Proof. In view of Remark 6.9, we may assume that the same $E \subset \Omega$ works in Definition 6.8 for all synchronized pairs. Choose a conull subset $\Psi \subset \bigcup_{t\in\mathbb{R}} T^t E$ such that for all $ij \in \Lambda''$, $\dim E_{ij}$ is constant on Ψ. Suppose $x \in \Psi$. Then, it follows from Definition 6.8 (see also Remark 6.9) that there exists $C = C(x) > 0$ such that for all $\ell > 0$ and at least $(1-\theta)$-fraction of $y_{ik} \in \mathcal{F}_{ik}[x, \ell]$,

$$\rho(y_{ik}, \mathcal{F}_{ij}[x]) < C.$$

Suppose $\mathbf{v} \in \mathbf{E}_{ij,bdd}(x)$. By Definition 6.5, for any $\ell' > 0$ and at least $(1-\theta)$-fraction of $y_{ij} \in \mathcal{F}_{ij}[x, \ell']$, $\|R(x,y_{ij})\mathbf{v}\| \le C_1 = C_1(x)$. Note that $T^{ij,-\ell'}\mathcal{F}_{ij}[x,\ell'] = T^{ik,-\ell}\mathcal{F}_{ik}[x,\ell]$ where ℓ is chosen so that $T^{ik,-\ell}x = T^{ij,-\ell'}x$. Therefore, in view of (5.9), for any $\ell > 0$, for at least $(1-2\theta)$-fraction of $y_{ik} \in \mathcal{F}_{ik}[x,\ell]$,

$$\|R(x,y_{ik})\mathbf{v}\| \le C_2(x)\|\mathbf{v}\|.$$

Thus, (as long as $\mathbf{E}_{ik} \neq \{0\}$, see Definition 6.5), we have $\mathbf{v} \in \mathbf{E}_{ik,bdd}(x)$. □

For $\mathbf{v} \in \mathbf{E}(x)$, write $\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$, as in (6.5). Define

$$\text{height}(\mathbf{v}) = \sum_{ij \in I_{\mathbf{v}}} (\dim \mathbf{E} + j)$$

The height is defined so it would have the following properties:

- If $\mathbf{v} \in \mathbf{E}_{ij}(x) \setminus \mathbf{E}_{i,j-1}(x)$ and $\mathbf{w} \in \mathbf{E}_{i,j-1}(x)$ then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.

- If $\mathbf{v} = \sum_{i \in I_{\mathbf{v}}} \mathbf{v}_i$, $\mathbf{v}_i \in \mathbf{E}_i$, $\mathbf{v}_i \neq 0$, and $\mathbf{w} = \sum_{j \in J} \mathbf{w}_j$, $\mathbf{w}_j \in \mathbf{E}_j$, $\mathbf{w}_j \neq 0$, and also the cardinality of J is smaller than the cardinality of $I_{\mathbf{v}}$, then $\text{height}(\mathbf{w}) < \text{height}(\mathbf{v})$.

Let $\mathcal{P}_k(x) \subset \mathbf{E}(x)$ denote the set of vectors of height at most k. This is a closed subset of $\mathbf{E}(x)$.

Lemma 6.13. *For every $\delta > 0$ and every $\eta > 0$ there exists a subset $K \subset \Omega$ of measure at least $1 - \delta$ and $L'' > 0$ such that for any $x \in K$ and any unit vector $\mathbf{v} \in \mathcal{P}_k(x)$ with $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ and $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) > \eta$, there exists $0 < L' < L''$ so that for at least θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,*

$$d\left(\frac{R(x,y)\mathbf{v}}{\|R(x,y)\mathbf{v}\|}, \mathcal{P}_{k-1}(y)\right) < \eta.$$

Proof. Suppose $C > 1$ (we will later choose C depending on η). We first claim that we can choose K with $\tilde{\mu}(K) > 1 - \delta$ and $L'' > 0$ so that for every $x \in T^{[-1,1]}K$ and every $\mathbf{v} \in \mathcal{P}_k(x)$ such that $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$ there exists $0 < L' < L''$ so that for θ-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$,

$$H_{\mathbf{v}}(x,y) \geq C. \tag{6.7}$$

(Essentially, this follows from Lemma 6.11, but the argument given below is a bit more elaborate since we want to choose L'' uniformly over all $\mathbf{v} \in \mathcal{P}_k(x)$ satisfying $d(\mathbf{v}, \bigcup_{ij} \mathbf{E}_{[ij],bdd}) > \eta$). Indeed, let $E_L \subset \mathcal{P}_k(x)$ denote the set of unit vectors $\mathbf{v} \in \mathcal{P}_k(x)$ such that for all $0 < L' < L$, for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$, $H_{\mathbf{v}}(x,y) \leq C$. Then, the E_L are closed sets which are decreasing as L increases, and by Lemma 6.11,

$$\bigcap_{L=1}^{\infty} E_L \subset \left(\bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(x)\right) \cap \mathcal{P}_k(x).$$

Let F denote the subset of the unit sphere in $\mathcal{P}_k(x)$ which is the complement of the η-neighborhood of $\bigcup_{ij} \mathbf{E}_{[ij],bdd}(x)$. Then the E_L^c are an open cover of F, and since F is compact, there exists $L = L_x$ such that $F \subset E_L^c$. Now for any $\delta > 0$ we can choose L'' so that $L'' > L_x$ for all x in a set K of measure at least $(1-\delta)$.

Now suppose $\mathbf{v} \in F$. Since $F \subset E_{L''}^c$, $\mathbf{v} \notin E_{L''}$, hence there exists $0 < L' < L''$ (possibly depending on $\mathbf{v}$) such that the fraction of $y \in \mathcal{F}_{\mathbf{v}}[x, L']$ which satisfies $H_{\mathbf{v}}(x,y) \geq C$ is greater than θ. Then, (6.7) holds.

Now suppose (6.7) holds (with a yet to be chosen $C = C(\eta)$). Write

$$\mathbf{v} = \sum_{ij \in I_{\mathbf{v}}} \mathbf{v}_{ij}$$

as in (6.5). Let

$$\mathbf{w} = R(x,y)\mathbf{v}, \qquad \mathbf{w}_{ij} = R(x,y)\mathbf{v}_{ij}.$$

Since $y \in \mathcal{F}_{\mathbf{v}}[x]$, by Lemma 5.4, $\|\mathbf{w}\| = \|\mathbf{v}\| = 1$. Let $ij \in I_{\mathbf{v}}$ be such that the supremum in the definition of $H_{\mathbf{v}}(x,y)$ is achieved for ij. If $\|\mathbf{w}_{ij}\| < \eta/2$ we are done, since $\mathbf{w}' = \sum_{kr \neq ij} \mathbf{w}_{kr}$ has smaller height than $\mathbf{v}$, and $d(\mathbf{w}, \frac{\mathbf{w}'}{\|\mathbf{w}'\|}) < \eta$. Hence we may assume that $1 \geq \|\mathbf{w}_{ij}\| \geq \eta/2$.

Since $d(\mathbf{v}, \mathcal{P}_{k-1}(x)) \geq \eta$, we have

$$d(\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(x)) \geq \eta \geq \eta \|\mathbf{v}_{ij}\|.$$

where the last inequality follows from the fact that $\|\mathbf{v}_{ij}\| \leq 1$. In particular, we have $1 \geq \|\mathbf{v}_{ij}\| \geq \eta$.

Let $y' = T^t y$ be such that $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$. Note that

$$\|\mathbf{v}_{ij}\| = \|R(x,y')\mathbf{v}_{ij}\| = \|R(y,y')\mathbf{w}_{ij}\| = \|(T^t)_* \mathbf{w}_{ij}\|$$

Hence, we have

$$1 \geq \|(T^t)_* \mathbf{w}_{ij}\| \geq \eta \qquad \text{and} \qquad 1 \geq \|\mathbf{w}_{ij}\| \geq \eta/2.$$

Then, in view of (5.3), $|t| \leq C_0(\eta)$, and hence $\|R(y',y)\| \leq C_0'(\eta)$.

Let $C_1 = C_0(\eta) + h(\eta, \frac{1}{2}\eta/C_0'(\eta))$, where $h(\cdot,\cdot)$ is as in Lemma 6.4. We now choose the constant C in (6.7) to be C_1. If $H_{\mathbf{v}}(x,y) > C_1$ then, by the choice of ij, $\rho(y, \mathcal{F}_{ij}[x]) > C_1$. Since $y' = T^t y$ and $|t| \leq C_0(\eta)$, we have

$$\rho(y', \mathcal{F}_{ij}[x]) > C_1 - C_0(\eta) = h(\eta, \tfrac{1}{2}\eta/C_0'(\eta)).$$

Then, by Lemma 6.4 applied to $\mathbf{v}_{ij}$ and $y' \in \mathcal{F}_{\mathbf{v}_{ij}}[x]$,

$$d(R(x,y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')) \leq \tfrac{1}{2}(\eta/C_0'(\eta))\|\mathbf{v}_{ij}\| \leq \tfrac{1}{2}\eta/C_0'(\eta).$$

Then, since $\mathbf{w}_{ij} = R(y',y)R(x,y')\mathbf{v}_{ij}$,

$$\|d(\mathbf{w}_{ij}, \mathbf{E}_{i,j-1}(y))\| \leq \|R(y',y)\| d(R(x,y')\mathbf{v}_{ij}, \mathbf{E}_{i,j-1}(y')) \leq \|R(y',y)\|(\eta/C_0'(\eta)) \leq \frac{\eta}{2}.$$

Let $\mathbf{w}'_{ij}$ be the closest vector to $\mathbf{w}_{ij}$ in $\mathbf{E}_{i,j-1}(y)$, and let $\mathbf{w}' = \mathbf{w}'_{ij} + \sum_{kr \neq ij} \mathbf{w}_{kr}$. Then $d(\mathbf{w}, \frac{\mathbf{w}'}{\|\mathbf{w}'\|}) < \eta$ and $\mathbf{w}' \in \mathcal{P}_{k-1}$. □

Proof of Proposition 6.1. Let n denote the maximal possible height of a vector. We claim that Proposition 6.1 holds with $\theta_1 = (\theta/2)^{n+1}$.

Let $\delta' = \delta/n$. We choose $\eta_j > 0$, $K_j \subset \Omega$ and $L_j > 0$ inductively as follows: Let $\eta_1 = \eta/n$. If $\eta_j > 0$ was already chosen, let $L_j = L_j(\delta', \eta_j)$ and $K_j = K_j(\delta', \eta_j)$ be chosen so that Lemma 6.13 holds for $k = j$, $K = K_j$, $L'' = L_j$ and $\eta = \eta_j$. Then, let η_{j+1} be chosen so that $\exp(\kappa(L_j + 1))\eta_{j+1} \leq \eta_j$, where κ is as in Proposition 2.14. We repeat this process until we choose L_n, K_n, η_n. We then let $L_0 = L_n + 1$, and let $K = K_1 \cap \cdots \cap K_n$. Then $\tilde{\mu}(K) > 1 - \delta$.

Let

$$E'_k = \left\{ y \in \mathcal{F}_{\mathbf{v}}[x, L] \;\; : \;\; d\left(\frac{R(x,y)\mathbf{v}}{\|R(x,y)\mathbf{v}\|}, \mathcal{P}_k(y) \cup \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y) \right) < \eta_k \right\}.$$

and let

$$E_k = \tilde{g}^{\mathbf{v},x}_{-L}(E'_k),$$

so $E_k \subset \mathfrak{B}_0[z]$, where $z = \tilde{g}_{-L}x$. Since $E'_n = \mathcal{F}_{\mathbf{v}}[x, L]$, we have $E_n = \mathfrak{B}_0[z]$. Let $Q = \tilde{g}_{-L}(T^{[-1,1]}K \cap \mathcal{F}_{\mathbf{v}}[x, L])$. Then, by assumption,

$$|Q| \geq (1 - (\theta/2)^{n+1})|\mathfrak{B}_0[z]|. \tag{6.8}$$

By Lemma 6.13, for every point $uz \in (E_k \cap Q) \setminus E_{k-1}$ there exists a "ball" $\mathfrak{B}_t[uz]$ (where $t = L - L'$ and L' is as in Lemma 6.13 for $L'' = L_0$) such that

$$|E_{k-1} \cap \mathfrak{B}_t[uz]| \geq \theta|\mathfrak{B}_t[uz]|. \tag{6.9}$$

(When we are applying Lemma 6.13 we do not have $\mathbf{v} \in \mathcal{P}_k$ but rather $d(\mathbf{v}/\|\mathbf{v}\|, \mathcal{P}_k) < \eta_k$; however by the choice of the η's and the L's this does not matter). The collection of balls $\{\mathfrak{B}_t[uz]\}_{uz\in(E_k\cap Q)\setminus E_{k-1}}$ as in (6.9) are a cover of $(E_k \cap Q) \setminus E_{k-1}$. These balls satisfy the condition of Lemma 2.12 (b); hence we may choose a pairwise disjoint subcollection which still covers $(E_k \cap Q) \setminus E_{k-1}$. We get $|E_{k-1}| \geq \theta|E_k \cap Q|$, and hence by (6.8), $|E_{k-1}| \geq \theta|E_k| - (\theta/2)^{n+1}|\mathfrak{B}_0[z]|$. Hence, by induction over k, we have for all k,

$$|E_k| \geq (\theta/2)^{n-k}|\mathfrak{B}_0[z]|.$$

Hence, $|E_0| \geq (\theta/2)^n|\mathfrak{B}_0[z]|$. Therefore $|E'_0| \geq (\theta/2)^n|\mathcal{F}_{\mathbf{v}}[x, L]|$. Since $\mathcal{P}_0 = \emptyset$, the Proposition follows from the definition of E'_0. □

6.2. **Invariant measures on vector bundles over Ω.** Recall that any bundle is measurably trivial.

In this subsection, $\mathbf{L}$ is any finite dimensional vector bundle over Ω on which the cocycle $(T^t_x)_*$ acts. (We will only use the cases $\mathbf{L} = \mathbf{E}_{ij,bdd}$ and $\mathbf{L} = \mathbf{E}_{ij}/\mathbf{E}_{i,j-1} \oplus \mathbf{E}_{kr}/\mathbf{E}_{k,r-1}$). We fix once and for all a measure ρ_0 on $\mathbb{P}(\mathbf{L})$ in the measure class of Lebesgue measure and independent of x.

Lemma 6.14. *Let $\tilde{\mu}_\ell$ be the measure on $\Omega \times \mathbb{P}(\mathbf{L})$ defined by*

$$\tilde{\mu}_\ell(f) = \int_\Omega \int_{\mathbb{P}(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f(x, R(y, x)\mathbf{v})\, dy\, d\rho_0(\mathbf{v})\, d\tilde{\mu}(x). \tag{6.10}$$

Let $\hat{\mu}_\ell$ be the measure on $\Omega \times \mathbb{P}(\mathbf{L})$ defined by

$$\hat{\mu}_\ell(f) = \int_\Omega \int_{\mathbb{P}(\mathbf{L})} \frac{1}{|\mathcal{F}_{ij}[x, \ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} f(y, R(x, y)\mathbf{v})\, dy\, d\rho_0(\mathbf{v})\, d\tilde{\mu}(x). \tag{6.11}$$

Then $\hat{\mu}_\ell$ is in the same measure class as $\tilde{\mu}_\ell$, and

$$\kappa^{-2} \leq \frac{d\hat{\mu}_\ell}{d\tilde{\mu}_\ell} \leq \kappa^2, \tag{6.12}$$

where κ is as in Proposition 2.14.

Proof. Let

$$F(x,y) = \int_{\mathbb{P}(\mathbf{L})} f(x, R(y,x)\mathbf{v})\, d\rho_0(\mathbf{v}).$$

Then,

$$\tilde{\mu}_\ell(f) = \int_\Omega \frac{1}{|\mathcal{F}_{ij}[x,\ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(x,y)\, dy\, d\tilde{\mu}(x) \tag{6.13}$$

$$\hat{\mu}_\ell(f) = \int_\Omega \frac{1}{|\mathcal{F}_{ij}[x,\ell]|} \int_{\mathcal{F}_{ij}[x,\ell]} F(y,x)\, dy\, d\tilde{\mu}(x) \tag{6.14}$$

Let $x' = T^{ij,-\ell}x$. Then, in view of Proposition 2.14, $\kappa^{-1}\, d\tilde{\mu}(x) \le d\tilde{\mu}(x') \le \kappa\, d\tilde{\mu}(x)$. Then,

$$\frac{1}{\kappa}\tilde{\mu}_\ell(f) \le \int_\Omega \frac{1}{|\mathfrak{B}_0[x']|} \int_{\mathfrak{B}_0[x']} F(T^{ij,\ell}x', T^{ij,\ell}z)\, dz\, d\tilde{\mu}(x') \le \kappa\tilde{\mu}_\ell(f),$$

and

$$\frac{1}{\kappa}\hat{\mu}_\ell(f) \le \int_\Omega \frac{1}{|\mathfrak{B}_0[x']|} \int_{\mathfrak{B}_0[x']} F(T^{ij,\ell}z, T^{ij,\ell}x')\, dz\, d\tilde{\mu}(x') \le \kappa\hat{\mu}_\ell(f)$$

Let Ω'' consist of one point from each $\mathfrak{B}_0[x]$. We now disintegrate $d\tilde{\mu}(x') = d\beta(x'')dz'$ where $x'' \in \Omega''$, $z' \in \mathfrak{B}_0[x']$. Then,

$$\begin{aligned}
\int_\Omega \frac{1}{|\mathfrak{B}_0[x']|} \int_{\mathfrak{B}_0[x']} F(T^{ij,\ell}x', T^{ij,\ell}z)\, dz\, d\tilde{\mu}(x') &= \int_{\Omega''} \int_{\mathfrak{B}_0[x'']\times\mathfrak{B}_0[x'']} F(T^{ij,\ell}z', T^{ij,\ell}z)\, dz'\, dz\, d\beta(x'') \\
&= \int_{\Omega''} \int_{\mathfrak{B}_0[x'']\times\mathfrak{B}_0[x'']} F(T^{ij,\ell}z, T^{ij,\ell}z')\, dz'\, dz\, d\beta(x'') \\
&= \int_\Omega \frac{1}{|\mathfrak{B}_0[x']|} \int_{\mathfrak{B}_0[x']} F(T^{ij,\ell}z, T^{ij,\ell}x')\, dz\, d\tilde{\mu}(x').
\end{aligned}$$

Now (6.12) follows from (6.13) and (6.14). □

Lemma 6.15. *Let $\tilde{\mu}_\infty$ be any weak-star limit of the measures $\tilde{\mu}_\ell$. Then,*

(a) *We may disintegrate $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\tilde{\mu}(x)\, d\lambda_x(\mathbf{v})$, where for each $x \in \Omega$, λ_x is a measure on $\mathbb{P}(\mathbf{L})$.*

(b) *For $x \in \Omega$ and $y \in \mathcal{F}_{ij}[x]$,*

$$\lambda_y = R(x,y)_*\lambda_x,$$

(c) *Let $\mathbf{w} \in \mathbb{P}(\mathbf{L})$ be a point. For $\eta > 0$ let*

$$B(\mathbf{w},\eta) = \{\mathbf{v} \in \mathbb{P}(\mathbf{L}) \;:\; d(\mathbf{v},\mathbf{w}) \le \eta\}.$$

Then, for any $t < 0$ there exists $C_1 = C_1(t,\mathbf{w}) > 0$ and $c_2 = c_2(t,\mathbf{w}) > 0$ such that for $x \in \Omega$,

$$\lambda_{T^t x}(B((T^t_x)_*\mathbf{w}, C_1\eta)) \ge c_2\lambda_x(B(\mathbf{w},\eta)).$$

Consequently, for $t < 0$, the support of $\lambda_{T^t x}$ contains the support of $(T^t_x)_*\lambda_x$.

(d) *For almost all $x \in \Omega$ there exist a measure ψ_x on $\mathbb{P}(\mathbf{L})$ such that*

$$\lambda_x = h(x)\psi_x$$

for some $h(x) \in SL(\mathbf{L})$, and also for almost all $y \in \mathcal{F}_{ij}[x]$, $\psi_y = \psi_x$ (so that ψ is constant on the leaves $\mathcal{F}_{ij}$). The maps $x \to \psi_x$ and $x \to h(x)$ are both ν-measurable.

Proof. If $f(x, \mathbf{v})$ is independent of the second variable, then it is clear from the definition of $\tilde{\mu}_\ell$ that $\tilde{\mu}_\ell(f) = \int_\Omega f\, d\tilde{\mu}$. This implies (a). To prove (b), note that $R(y', y) = R(x, y)R(y', x)$. Then,

$$\begin{aligned}
\lambda_y &= \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y,\ell_k]} (R(y', y)_* \rho_0)\ dy' \\
&= R(x, y)_* \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[y, \ell_k]|} \int_{\mathcal{F}_{ij}[y,\ell_k]} (R(y', x)_* \rho_0)\ dy' \\
&= R(x, y)_* \lim_{k\to\infty} \frac{1}{|\mathcal{F}_{ij}[x, \ell_k]|} \int_{\mathcal{F}_{ij}[x,\ell_k]} (R(y', x)_* \rho_0)\ dy' \\
&= R(x, y)_* \lambda_x
\end{aligned}$$

where to pass from the second line to the third we used the fact that $\mathcal{F}_{ij}[x, \ell] = \mathcal{F}_{ij}[y, \ell]$ for ℓ large enough. This completes the proof of (b).

We now begin the proof of (c). Let $\mathbf{w}(x) = \mathbf{w}$. Working in the universal cover, we define for $y \in \mathcal{H}[x]$, $\mathbf{w}(y) = R(x, y)\mathbf{w}(x)$. We define

$$\mathbf{w}_\eta(x) = \{\mathbf{v} \in \mathbb{P}(\mathbf{L}(x)) \ : \ d(\mathbf{v}, \mathbf{w}(x)) \leq \eta\}.$$

(Here we are thinking of the space as $\Omega \times \mathbb{P}(\mathbf{L})$ and using the same metric on all the $\mathbb{P}(\mathbf{L})$ fibers).

Let $x' = T^{ij,t}x$, $y' = T^{ij,t}y$. We have

$$R(y', x') = R(x, x')R(y, x)R(y', y).$$

Since $\|R(x, x')^{-1}\| \leq c^{-1}$, where c depends on t, we have $R(x, x')^{-1}\mathbf{w}_{c\eta}(x') \subset \mathbf{w}_\eta(x)$. Then,

$$\begin{aligned}
\rho_0\{\mathbf{v} \ : \ R(y', x')\mathbf{v} \in \mathbf{w}_{c\eta}(x')\} &= \rho_0\{\mathbf{v} \ : \ R(y, x)R(y', y)\mathbf{v} \in R(x, x')^{-1}\mathbf{w}_{c\eta}(x')\} \\
&\geq \rho_0\{\mathbf{v} \ : \ R(y, x)R(y', y)\mathbf{v} \in \mathbf{w}_\eta(x)\} \\
&= \rho_0\{R(y, y')^{-1}\mathbf{u} \ : \ R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\} \\
&= R(y, y')_*^{-1}\rho_0\{\mathbf{u} \ : \ R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\} \\
&\geq c'\rho_0\{\mathbf{u} \ : \ R(y, x)\mathbf{u} \in \mathbf{w}_\eta(x)\}.
\end{aligned}$$

Note that for $t < 0$, $T^{ij,t}\mathcal{F}_{ij}[x, \ell] \subset \mathcal{F}_{ij}[T^{ij,t}x, \ell]$ and $|T^{ij,t}\mathcal{F}_{ij}[x, \ell]| \geq c(t)|\mathcal{F}_{ij}[T^{ij,t}x, \ell]|$. Substituting into (6.10) completes the proof of (c).

To prove part (d), let $\mathcal{M}$ denote the space of measures on $\mathbb{P}(\mathbf{L})$. Recall that by [Zi2, Theorem 3.2.6] the orbits of the special linear group $SL(\mathbf{L})$ on $\mathcal{M}$ are locally closed. Then, by [Ef, Theorem 2.9 (13), Theorem 2.6(5)] [1] there exists a Borel cross section $\phi : \mathcal{M}/SL(\mathbf{L}) \to \mathcal{M}$. Then, let $\psi_x = \phi(\pi(\lambda_x))$ where $\pi : \mathcal{M} \to \mathcal{M}/SL(\mathbf{L})$ is the quotient map. ☐

We also recall the following well known Lemma of Furstenberg (see e.g. [Zi2, Lemma 3.2.1]):

Lemma 6.16. *Let $\mathbf{V}$ be a vector space, and suppose μ and ν are two probability measures on $\mathbb{P}(\mathbf{V})$. Suppose $T^i \in SL(\mathbf{V})$ are such that $T^i \to \infty$ and $T^i\mu \to \nu$. Then the support of ν is contained in a union of two proper subspaces of $\mathbf{V}$.*

In particular, if the support of a measure ν on $\mathbb{P}(\mathbf{V})$ is not contained in a union of two proper subspaces, then the stabilizer of ν in $SL(\mathbf{V})$ is bounded.

Lemma 6.17. *Suppose that $\theta > 0$, and suppose that for all $\delta > 0$ there exists a set $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and a constant $C_1 < \infty$, such that for all $x \in K$, all $\ell > 0$ and at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x, \ell]$,*

$$\|R(x, y)\mathbf{v}\| \le C_1 \|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}. \tag{6.15}$$

Then for all $\delta > 0$ and for all $\ell > 0$ there exists a subset $K''(\ell) \subset \Omega$ with $\tilde{\mu}(K''(\ell)) > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and there exists $\theta'' = \theta''(\theta, \delta)$ with $\theta'' \to 0$ as $\theta \to 0$ and $\delta \to 0$ such that for all $x \in K''(\ell)$, for at least $(1-\theta'')$-fraction of $y \in \mathcal{F}_{ij}[x, \ell]$,

$$C_1^{-1}\|\mathbf{v}\| \le \|R(x, y)\mathbf{v}\| \le C_1 \|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{L}. \tag{6.16}$$

Proof. Let f be the characteristic function of $K \times \mathbb{P}(\mathbf{L})$. By (6.10), $\tilde{\mu}_\ell(f) \ge (1-\delta)$. By Lemma 6.14 we have $\hat{\mu}_\ell(f) \ge (1-\kappa^2\delta)$. Therefore, by (6.11), there exists a subset $K'(\ell) \subset \Omega$ with $\tilde{\mu}(K'(\ell)) \ge 1 - (\kappa^2\delta)^{1/2}$ such that such that for all $x \in K'(\ell)$,

$$|\mathcal{F}_{ij}[x, \ell] \cap K| \ge (1 - (\kappa^2\delta)^{1/2})|\mathcal{F}_{ij}[x, \ell]|.$$

For $x_0 \in \Omega$, let

$$Z_\ell[x_0] = \{(x, y) \in \mathcal{F}_{ij}[x_0, \ell] \times \mathcal{F}_{ij}[x_0, \ell] \ : \ x \in K, \quad y \in K, \quad \text{and (6.15) holds } \}.$$

Then, if $x_0 \in K'(\ell)$ and $\theta' = \theta + (\kappa^2\delta)^{1/2}$ then, by Fubini's theorem,

$$|Z_\ell[x_0]| \ge (1-\theta')|\mathcal{F}_{ij}[x_0, \ell] \times \mathcal{F}_{ij}[x_0, \ell]|.$$

Let

$$Z_\ell[x_0]^t = \{(x, y) \in \mathcal{F}_{ij}[x_0, \ell] \times \mathcal{F}_{ij}[x_0, \ell] \ : \ (y, x) \in Z_\ell[x_0]\}.$$

Then, for $x_0 \in K'(\ell)$,

$$|Z_\ell[x_0] \cap Z_\ell[x_0]^t| \ge (1-2\theta')|\mathcal{F}_{ij}[x_0, \ell] \times \mathcal{F}_{ij}[x_0, \ell]|.$$

For $x \in \mathcal{F}_{ij}[x_0, \ell]$, let

$$Y'_\ell(x) = \{y \in \mathcal{F}_{ij}[x, \ell] \ : \ (x, y) \in Z_\ell[x] \cap Z_\ell[x]^t\}.$$

[1] The "condition C" of [Ef] is satisfied since $SL(\mathbf{L})$ is locally compact and $\mathcal{M}$ is Hausdorff.

Therefore, by Fubini's theorem, for all $x_0 \in K'(\ell)$ and $\theta'' = (2\theta')^{1/2}$,

$$|\{x \in \mathcal{F}_{ij}[x_0,\ell] \ : \ |Y'_\ell(x)| \geq (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|\}| \geq (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|. \tag{6.17}$$

(Note that $\mathcal{F}_{ij}[x_0,\ell] = \mathcal{F}_{ij}[x,\ell]$.) Let

$$K''(\ell) = \{x \in \Omega \ : \ |Y'_\ell(x)| \geq (1-\theta'')|\mathcal{F}_{ij}[x,\ell]|\}.$$

Therefore, by (6.17), for all $x_0 \in K'(\ell)$,

$$|\mathcal{F}_{ij}[x_0,\ell] \cap K''(\ell)| \geq (1-\theta'')|\mathcal{F}_{ij}[x_0,\ell]|.$$

Then, by the definition of $\hat{\mu}_\ell$,

$$\hat{\mu}_\ell(K''(\ell) \times \mathbb{P}(\mathbf{L})) \geq (1-\theta'')\nu(K'(\ell)) \geq (1-2\theta''),$$

and therefore, by Lemma 6.14,

$$\tilde{\mu}(K''(\ell)) = \tilde{\mu}_\ell(K''(\ell) \times \mathbb{P}(\mathbf{L})) \geq (1-2\kappa^2\theta'').$$

Now, for $x \in K''(\ell)$, and $y \in Y'_\ell(x)$, (6.16) holds. □

6.3. **Proofs of Proposition 6.2 and Proposition 6.3.**

Lemma 6.18. *There exists a function $C : \Omega \to \mathbb{R}^+$ finite almost everywhere such that for all $x \in \Omega$, all $\mathbf{v} \in \mathbf{E}_{ij,bdd}(x)$, and all $y \in \mathcal{F}_{ij}[x]$,*

$$C(x)^{-1}C(y)^{-1}\|\mathbf{v}\| \leq \|R(x,y)\mathbf{v}\| \leq C(x)C(y)\|\mathbf{v}\|,$$

Proof. Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 6.14, with $\mathbf{L} = \mathbf{E}_{[ij],bdd}$. Take a sequence $\ell_k \to \infty$ such that $\tilde{\mu}_{\ell_k} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_k} \to \hat{\mu}_\infty$. Then decomposing as in Lemma 6.15 (a), we have $d\tilde{\mu}_\infty(x,\mathbf{v}) = d\tilde{\mu}(x)\, d\lambda_x(\mathbf{v})$ where λ_x is a measure on $\mathbb{P}(\mathbf{E}_{ij,bdd})$. Let $E \subset \Omega$ be such that for $x \in E$, λ_x is supported on at most two proper subspaces. We will show that $\tilde{\mu}(E) = 0$.

Suppose not; then $\tilde{\mu}(E) > 0$, and for $x \in E$, λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$, where $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ are subspaces of $\mathbf{E}_{ij,bdd}(x)$. We always choose $\mathbf{F}_1(x)$ and $\mathbf{F}_2(x)$ to be of minimal dimension, and if λ_x is supported on a single subspace $\mathbf{F}(x)$ (of minimal dimension), we let $\mathbf{F}_1(x) = \mathbf{F}_2(x) = \mathbf{F}(x)$. Then, for $x \in E$, $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is uniquely determined by x. After possibly replacing E by a smaller subset of positive measure, we may assume that $\dim \mathbf{F}_1(x)$ and $\dim \mathbf{F}_2(x)$ are independent of $x \in E$.

Let

$$\Psi = \{x \in \Omega \ : \ T^t x \in E \text{ and } T^{-s}x \in E \text{ for some } t > 0 \text{ and } s > 0.\}$$

Then, $\tilde{\mu}(\Psi) = 1$. If $x \in \Psi$, then, by Lemma 6.15 (c),

$$(T^s_{T^{-s}x})_*\mathbf{F}_1(T^{-s}x) \cup (T^s_{T^{-s}x})_*\mathbf{F}_2(T^{-s}x) \subset \operatorname{supp}\lambda_x \subset (T^{-t}_{T^t x})_*\mathbf{F}_1(T^t x) \cup (T^{-t}_{T^t x})_*\mathbf{F}_2(T^t x), \tag{6.18}$$

Since $\mathbf{F}_i(T^t x)$ and $\mathbf{F}_i(T^{-s}x)$ have the same dimension, the sets on the right and on the left of (6.18) coincide. Therefore, $E \supset \Psi$ (and so E has full measure) and the set $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is T^t-equivariant.

Fix $\delta > 0$ (which will be chosen sufficiently small later). Suppose $\ell > 0$ is arbitrary. By the definition of $\mathbf{E}_{ij,bdd}$, there exists a constant C_1 independent of ℓ and a compact subset $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and for each $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x,\ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x,\ell]|$, such that for $x \in K$ and $y \in Y_\ell(x) \cap K$ we have

$$\|R(x,y)\mathbf{v}\| \leq C_1 \|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in \mathbf{E}_{ij,bdd}(x).$$

Therefore by Lemma 6.17, there exists $0 < \theta'' < 1/2$, $K''(\ell) \subset \Omega$ with $\tilde{\mu}(K''(\ell)) > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and for each $x \in K''(\ell)$ a subset $Y'_\ell(x) \subset \mathcal{F}_{ij}[x,\ell]$ with $|Y'_\ell(x)| \geq (1-\theta'')|\mathcal{F}_{ij}[x,\ell]|$ such that for $x \in K''(\ell)$ and $y \in Y'_\ell(x)$, (6.16) holds.

Let

$$\mathbf{Z}(x,\eta) = \{\mathbf{v} \in \mathbb{P}(\mathbf{E}_{ij,bdd}) \ : \ d(\mathbf{v}, \mathbf{F}_1(x) \cup \mathbf{F}_2(x)) \geq \eta\}.$$

We may choose $\eta > 0$ small enough so that for all $x \in \Omega$,

$$\rho_0(\mathbf{Z}(x, C_1\eta)) > 1/2.$$

Let

$$S(\eta) = \{(x,\mathbf{v}) \ : \ x \in \Omega, \quad \mathbf{v} \in \mathbf{Z}(x,\eta)\}$$

Let f denote the characteristic function of the set $S(\eta)$. We now claim that for any ℓ,

$$\hat{\mu}_\ell(f) \geq \tilde{\mu}(K''(\ell))(1-\theta'')(1/2). \tag{6.19}$$

Indeed, let Ψ' denote the set of triples $(x,y,\mathbf{v})$ such that $x \in K''(\ell)$, $y \in Y'_\ell(x)$, $\mathbf{v} \in \mathbf{Z}(x, C_1\eta)$. Then by (6.16), if $(x,y,\mathbf{v}) \in \Psi'$, then $f(y, R(x,y)\mathbf{v}) = 1$. Therefore, in view of the definition (6.11), we can estimate $\hat{\mu}_\ell(f)$ from below by the $\tilde{\mu} \times |\cdot| \times \rho_0$-measure of Ψ'. This implies (6.19). Thus, (provided $\delta > 0$ and $\theta > 0$ in Definition 6.5 are sufficiently small), there exists $c_0 > 0$ such that for all ℓ, $\hat{\mu}_\ell(S(\eta)) \geq c_0 > 0$. Therefore, by Lemma 6.14, $\tilde{\mu}_\ell(S(\eta)) \geq c_0/\kappa^2$.

There exists compact $K_0 \subset \Omega$ with $\tilde{\mu}(K_0) > 1 - c_0/(2\kappa^2)$ such that the map $x \to \mathbf{F}_1(x) \cup \mathbf{F}_2(x)$ is continuous on K_0. Let $K'_0 = \{(x,\mathbf{v}) \ : \ x \in K_0\}$. Then $S(\eta) \cap K'_0$ is a closed set with $\tilde{\mu}_\ell(S(\eta) \cap K'_0) \geq c_0/(2\kappa^2)$. Therefore, $\tilde{\mu}_\infty(S(\eta) \cap K'_0) > c_0/(2\kappa^2) > 0$, which is a contradiction to the fact that λ_x is supported on $\mathbf{F}_1(x) \cup \mathbf{F}_2(x)$.

Thus, for almost all x, λ_x is not supported on a union of two subspaces. Thus the same holds for the measure ψ_x of Lemma 6.15 (d). By combining (b) and (d) of Lemma 6.15 we see that for almost all x and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x,y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x,y)h(x)$ stabilizes ψ_x. Hence by Lemma 6.16,

$$h(y)^{-1}\bar{R}(x,y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $SL(\mathbf{E}_{ij,bdd})$, and $\bar{R}(x,y)$ is the image of $R(x,y)$ under the natural map $GL(\mathbf{E}_{ij,bdd}) \to SL(\mathbf{E}_{ij,bdd})$. Thus, $\bar{R}(x,y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\|\bar{R}(x,y)\| \leq C(x)C(y). \tag{6.20}$$

Since $\bar{R}(x,y)^{-1} = \bar{R}(y,x)$, we get, by exchanging x and y,

$$\|\bar{R}(x,y)^{-1}\| \leq C(x)C(y). \tag{6.21}$$

Note that by Lemma 6.6, there exists $\mathbf{v} \in \mathbf{E}_{ij,bdd}(x) \subset \mathbf{E}_{ij}(x)$ such that $\mathbf{v} \notin \mathbf{E}_{i,j-1}(x)$. Then, (5.5) and the fact that $\lambda_{ij}(x,y) = 0$ for $y \in \mathcal{F}_{ij}[x]$ shows that (6.20) and (6.21) must hold for $R(x,y)$ in place of $\bar{R}(x,y)$. This implies the statement of the lemma. $\square$

Lemma 6.19. *Suppose that for all $\delta > 0$ there exists a constant $C > 0$ and a compact subset $K \subset \Omega$ with $\tilde{\mu}(K) > 1 - \delta$ and for each $\ell > 0$ and $x \in K$ a subset $Y_\ell(x)$ of $\mathcal{F}_{ij}[x,\ell]$ with $|Y_\ell(x)| \geq (1-\theta)|\mathcal{F}_{ij}[x,\ell]|$, such that for $x \in K$ and $y \in Y_\ell(x)$ we have*

$$\lambda_{kr}(x,y) \leq C. \tag{6.22}$$

Then, ij and kr are synchronized, and there exists a function $C : \Omega \to \mathbb{R}^+$ finite $\tilde{\mu}$-almost everywhere such that for all $x \in \Omega$, and all $y \in \mathcal{F}_{ij}[x]$,

$$\rho(y, \mathcal{F}_{kr}[x]) \leq C(x)C(y). \tag{6.23}$$

Remark. From the definitions, ij and kr are synchronized if the assumptions of Lemma 6.19 hold, with (6.22) replaced by $-C \leq \lambda_{kr}(x,y) \leq C$. In Lemma 6.19 we are only assuming the upper bound, so to prove synchronization of ij and kr an argument is needed.

Proof. The proof is a simplified version of the proof of Lemma 6.18. Let $\mathbf{L}_1 = \mathbf{E}_{ij}/\mathbf{E}_{i,j-1}$, $\mathbf{L}_2 = \mathbf{E}_{kr}/\mathbf{E}_{k,r-1}$, and $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$.

We have, in view of Proposition 2.14, for $y \in \mathcal{H}[x]$, and $(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}$,

$$R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}}) = (e^{\lambda_{ij}(x,y)}\bar{\mathbf{v}}', e^{\lambda_{kr}(x,y)}\bar{\mathbf{w}}'), \quad \text{where } \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\| \text{ and } \|\bar{\mathbf{w}}'\| = \|\bar{\mathbf{w}}\|. \tag{6.24}$$

Recall that $\lambda_{ij}(x,y) = 0$ for all $y \in \mathcal{F}_{ij}[x]$. Therefore, (6.22) implies that for all $x \in K$, all $\ell > 0$ and all $y \in Y_\ell(x)$,

$$\|R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1 \|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

Therefore, by Lemma 6.17, there exists a subset $K''(\ell) \subset \Omega$ with $\tilde{\mu}(K''(\ell)) > 1 - c(\delta)$ where $c(\delta) \to 0$ as $\delta \to 0$, and for each $x \in K''(\ell)$ a subset $Y'_\ell \subset \mathcal{F}_{ij}[x,\ell]$ with $|Y'_\ell| > (1-\theta'')|\mathcal{F}_{ij}[x,\ell]|$ such that for all $y \in Y'_\ell$,

$$C_1^{-1}\|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq \|R(x,y)(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| \leq C_1 \|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\|.$$

This implies that for $x \in K''(\ell)$, $y \in Y'_\ell(x)$,

$$|\lambda_{kr}(x,y)| = |\lambda_{ij}(x,y) - \lambda_{kr}(x,y)| \le C_1. \tag{6.25}$$

Let $\tilde{\mu}_\ell$ and $\hat{\mu}_\ell$ be as in Lemma 6.14. Take a sequence $\ell_m \to \infty$ such that $\tilde{\mu}_{\ell_m} \to \tilde{\mu}_\infty$, and $\hat{\mu}_{\ell_m} \to \hat{\nu}_\infty$. Then by Lemma 6.15 (a), we have $d\tilde{\mu}_\infty(x, \mathbf{v}) = d\tilde{\mu}(x)\, d\lambda_x(\bar{\mathbf{v}})$ where λ_x is a measure on $\mathbb{P}(\mathbf{L})$. We will show that for almost all $x \in \Omega$, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$.

Suppose that for a set of positive measure λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$. Then, in view of the ergodicity of T^t and Lemma 6.15 (c), λ_x is supported on $(\mathbf{L}_1 \times \{0\}) \cup (\{0\} \times \mathbf{L}_2)$ for almost all $x \in \Omega$. Let

$$\mathbf{Z}(x,\eta) = \{(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{L}(x), \quad \|(\bar{\mathbf{v}}, \bar{\mathbf{w}})\| = 1, \quad d(\bar{\mathbf{v}}, \mathbf{L}_1) \ge \eta, \quad d(\bar{\mathbf{w}}, \mathbf{L}_2) \ge \eta\}.$$

and let

$$S(\eta) = \{(x, (\bar{\mathbf{v}}, \bar{\mathbf{w}})) \ : \ x \in \Omega, (\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{Z}(x,\eta)\}.$$

Then we have $\tilde{\mu}_\infty(S(\eta)) = 0$. By Lemma 6.14, $\hat{\mu}_\infty(S(\eta)) = 0$.

By (6.24) and (6.25), for $x \in K''(\ell_m)$ and $y \in Y'_{\ell_m}(x)$,

$$R(x,y)\, \mathbf{Z}(x, C_1\eta) \subset \mathbf{Z}(y,\eta). \tag{6.26}$$

Choose $\eta > 0$ so that for all $x \in \Omega$, $\rho_0(\mathbf{Z}(x, C_1\eta)) > (1/2)$. Let f be the characteristic function of $S(\eta)$. Let Ψ denote the set of quadruples $(x, y, \bar{\mathbf{v}}, \bar{\mathbf{w}})$ such that $x \in K''(\ell_m)$, $y \in Y'_{\ell_m}(x)$, and $(\bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \mathbf{Z}(x, C_1\eta)$. Then by (6.26), for $(x, y, \bar{\mathbf{v}}, \bar{\mathbf{w}}) \in \Psi$, $f(y, R(x,y)\bar{\mathbf{v}}) = 1$. Therefore, in view of the definition (6.11), we can estimate $\hat{\mu}_{\ell_m}(f)$ from below by the $\tilde{\mu} \times |\cdot| \times \rho_0$-measure of Ψ. This implies that for all m,

$$\hat{\mu}_{\ell_m}(S(\eta)) \ge \tilde{\mu}(K''(\ell_m) \cap K')(1 - \theta'')(1/2).$$

Hence $\hat{\mu}_\infty(S(\eta)) > 0$ which is a contradiction. Therefore, for almost all x, λ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$. Thus the same holds for the measure ψ_x of Lemma 6.15 (d). By combining (b) and (d) of Lemma 6.15 we see that for almost all $x \in \Omega$ and almost all $y \in \mathcal{F}_{ij}[x]$,

$$R(x,y)h(x)\psi_x = h(y)\psi_x,$$

hence $h(y)^{-1}R(x,y)h(x)$ stabilizes ψ_x. Note that in view of (6.24), $h(x)$ and $h(y)$ are conformal, and hence

$$h(y)^{-1}R(x,y)h(x)(\bar{\mathbf{v}}, \bar{\mathbf{w}}) = (e^{\alpha(x,y)}\bar{\mathbf{v}}', e^{\alpha'(x,y)}\bar{\mathbf{w}}'),$$
$$\text{where } \alpha(x,y) \in \mathbb{R},\ \alpha'(x,y) \in \mathbb{R},\ \|\bar{\mathbf{v}}'\| = \|\bar{\mathbf{v}}\| \text{ and } \|\bar{\mathbf{w}}'\| = \|\bar{\mathbf{w}}\|.$$

For $i = 1, 2$ let $\mathrm{Conf}_x(\mathbf{L}_i)$ denote the subgroup of $GL(\mathbf{L}_i)$ which preserves the inner product $\langle \cdot, \cdot \rangle_x$ up to a scaling factor. Let $\mathrm{Conf}_x(\mathbf{L}) = \mathrm{Conf}_x(\mathbf{L}_1) \times \mathrm{Conf}_x(\mathbf{L}_2)$. Then, by an elementary variant of Lemma 6.16, since ψ_x is not supported on $\mathbf{L}_1 \times \{0\} \cup \{0\} \times \mathbf{L}_2$, we get

$$h(y)^{-1}R(x,y)h(x) \in K(x)$$

where $K(x)$ is a compact subset of $\mathrm{Conf}_x(\mathbf{L})$. Thus, $R(x,y) \in h(y)K(x)h(x)^{-1}$, and thus

$$\|R(x,y)\| \le C(x)C(y).$$

Note that by reversing x and y we get $\|R(x,y)^{-1}\| \le C(x)C(y)$. Therefore, by (6.24),

$$|\lambda_{ij}(x,y) - \lambda_{kr}(x,y)| \le C(x)C(y).$$

This completes the proof of (6.23).

For any $\delta > 0$ we can choose a compact $K \subset \Omega$ with $\tilde{\mu}(K) > 1-\delta$ and $N < \infty$ such that $C(x) < N$ for $x \in K$. Now, the fact that ij and kr are synchronized follows from applying Lemma 5.7 to K. □

Proof of Proposition 6.2. This follows immediately from combining Lemma 6.19 and Lemma 6.18. □

Proof of Proposition 6.3. Choose $\epsilon < \epsilon'/(10\lambda_{min})$, where $\lambda_{min} = \min\{\lambda_i \;:\; \lambda_i > 0\}$, and where ϵ' is as in Proposition 4.2. By the multiplicative ergodic theorem, there exists a set $K_1'' \subset \Omega$ with $\tilde{\mu}(K_1'') > 1-\theta$ and $T > 0$, such that for $x \in K_1''$ and $|t| > T$,

$$|\lambda_{ij}(x,t) - \lambda_i t| < \epsilon |t|, \tag{6.27}$$

where $\lambda_{ij}(x,t)$ is as in (5.2). Then, by Fubini's theorem there exists a set $K_2'' \subset K_1''$ with $\tilde{\mu}(K_2'') > 1-3\theta$ such that for $x \in K_2''$, for $(1-\theta)$-fraction of $ux \in \mathfrak{B}_0[x]$, $ux \in K_1''$.

Let K'' be as in Proposition 4.2 with $\delta = \theta$. We may assume that the conull set Ψ in Proposition 6.3 is such so that for $x \in \Psi$, $T^{-t}x \in K'' \cap K_2''$ for arbitrarily large $t > 0$. Suppose $T^{-t}x \in K'' \cap K_2''$ and $y \in \mathcal{F}_{ij}[x]$. We may write

$$y = T^{ij,t'} u T^{ij,-t'} x = T^{s'} u T^{-t} x.$$

Then, $\lambda_{ij}(x,-t) = -\lambda_i t'$. Hence,

$$|\lambda_i t - \lambda_i t'| = |\lambda_i t + \lambda_{ij}(x,-t)| \le \epsilon t,$$

where for the last estimate, we used (6.27) with $(-t)$ in place of t.

By the definition of $\mathcal{F}_{ij}[x,t']$, and since $T^{-t}x \in K_2''$, we have $T^{-t}x \in K_1''$ and for at least $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,t']$, we have $uT^{-t}x \in K_1''$, and thus, using (6.27) as above, we have

$$|s' - t'| \le (\epsilon/\lambda_i)t' \quad \text{and} \quad |t - t'| \le (\epsilon/\lambda_i)t.$$

Therefore for $(1-\theta)$-fraction of $y \in \mathcal{F}_{ij}[x,t']$ or equivalently for $(1-\theta)$-fraction of $uT^{-t}x \in \mathfrak{B}_0[T^{-t}x]$,

$$|s' - t| \le 4(\epsilon/\lambda_i)t. \tag{6.28}$$

Now suppose $\mathbf{v} \in \mathrm{Lie}(N^+)(x)$. Note that if $\|R(x,y)\mathbf{v}\| \le C\|\mathbf{v}\|$, and s is as in Proposition 4.2, then $s > s' - O(1)$ (where the implied constant depends on C.)

Therefore, in view of (6.28), for $(1-\theta)$-fraction of $uT^{-t}x \in \mathfrak{B}_0[T^{-t}x]$, (4.9) holds. Thus, by Proposition 4.2, we have $\mathbf{v} \in \mathbf{E}(x)$. Thus, we can write

$$\mathbf{v} = \sum_{kr \in I_{\mathbf{v}}} \mathbf{v}_{kr}$$

where the indexing set $I_{\mathbf{v}}$ contains at most one r for each $k \in \Lambda'$. Without loss of generality, Ψ is such that for $x \in \Psi$, $T^{-t}x$ satisfies the conclusions of Proposition 2.14 infinitely often. Note that for $y \in \mathcal{F}_{ij}[x]$,

$$\|R(x,y)\mathbf{v}\| \geq \|R(x,y)\mathbf{v}_{kr}\| \geq e^{\lambda_{kr}(x,y)}\|\mathbf{v}_{kr}\|.$$

By assumption, for all $\ell > 0$ and for at least $1-\theta$ fraction of $y \in \mathcal{F}_{ij}[x,\ell]$, $\|R(x,y)\mathbf{v}\| \leq C\|\mathbf{v}\|$. Therefore, for all $\ell > 0$ and for at least $(1-\theta)$ fraction of $y \in \mathcal{F}_{ij}[x,\ell]$, (6.22) holds. Then, by Lemma 6.19, for all $kr \in I_{\mathbf{v}}$, kr and ij are synchronized, i.e. $kr \in [ij]$. Therefore, for at least $(1-2\theta)$-fraction of $y' \in \mathcal{F}_{kr}[x,\ell]$,

$$\|R(x,y')\mathbf{v}_{kr}\| \leq \|R(x,y')\mathbf{v}\| \leq C'\|\mathbf{v}\| = C''\|\mathbf{v}_{kr}\|.$$

Now, by Definition 6.5, $\mathbf{v}_{kr}(x) \in \mathbf{E}_{kr,bdd}(x)$. Therefore, $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(x)$. □

7. Bilipshitz estimates

The subspace $\mathcal{L}^-(\hat{x})$. For $\hat{x} = (x,g) \in \hat{\Omega}$, let $\hat{W}^-_{loc}[\hat{x}] = \{(y,g') \in \hat{W}^-_1[\hat{x}] \ : \ d_G(g,g') < 1\}$. Let $\mathcal{L}^-(\hat{x}) \subset \mathrm{Lie}(N^-)(x) \subset \mathfrak{g}$ denote the smallest subspace of $\mathrm{Lie}(N^-)(x)$ such that the projection to G of the conditional measure $\hat{\nu}|_{\hat{W}^-_{loc}[\hat{x}]}$ is supported on $\exp(\mathcal{L}^-(\hat{x})g)$. The assumption that we are in Case I (see §1) implies $\dim(\mathcal{L}^-(\hat{x})) > 0$ for a.e. $\hat{x}$.

Lemma 7.1. *For almost all $\hat{x} \in \hat{\Omega}$ and all $t \in \mathbb{R}$,*

$$\mathcal{L}^-(\hat{T}^t\hat{x}) = (T^t_{\hat{x}})_*\mathcal{L}^-(\hat{x}). \tag{7.1}$$

Also, for almost all $\hat{x} = (x,g) \in \hat{\Omega}$, $\mathcal{L}^-(\hat{x})$ is a subalgebra of $\mathrm{Lie}(N^-)(x)$.

Proof. From the definition, for $t > 0$, $(\hat{T}^{-t}_{\hat{x}})_*\mathcal{L}^-(\hat{x}) \subset \mathcal{L}^-(\hat{T}^{-t}\hat{x})$. Let $\phi(\hat{x}) = \dim(\mathcal{L}^-(\hat{x}))$. Then, ϕ is a bounded integer valued function which is increasing under the flow $\hat{T}^{-t}$. Since the flow is ergodic on $\hat{\Omega}/\Gamma$, it follows that ϕ is constant, and therefore (7.1) holds.

For the second assertion, the proof of [EiL1, Proposition 6.2] goes through almost verbatim. □

The function $A(q_1,u,\ell,t)$. Suppose $\hat{q}_1 = (q_1,g)$, $u \in \mathcal{U}^+_1$, $\ell > 0$ and $t > 0$. For $x \in \Omega$, let $\pi_{\mathbf{E}} : \mathfrak{g} \to \mathbf{E}(x)$ denote the orthogonal projection using the inner product $\langle\cdot,\cdot\rangle_x$. We consider the restriction of $\mathcal{A}(q_1,u,\ell,t)$ to $\mathcal{L}^-(\hat{T}^{-t}\hat{q}_1)$, so we are considering $\mathcal{A}(q_1,u,\ell,t)$ as a linear map from $\mathcal{L}^-(\hat{T}^{-t}\hat{q}_1)$ to $\mathfrak{g}$. Let $A(\hat{q}_1,u,\ell,t) = \|\pi_{\mathbf{E}} \circ \mathcal{A}(q_1,u,\ell,t)\|$ (the norm of the restriction) where the operator norm is with respect to the dynamical norms $\|\cdot\|_{T^{-\ell}q_1}$ and $\|\cdot\|_{T^t uq_1}$. In view of (4.2), for almost all $\hat{q}_1$ and $u\hat{q}_1 \in \mathcal{U}_1^+\hat{q}_1$, $A(\hat{q}_1, u, \ell, t) \to \infty$ as $t \to \infty$.

The function $\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)$. For $\epsilon > 0$, almost all $\hat{q}_1 \in \hat{\Omega}$, almost all $uq_1 \in \mathcal{U}_1^+\hat{q}_1$ and $\ell > 0$, let

$$\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) = \sup\{t \;:\; t > 0 \text{ and } A(\hat{q}_1, u, \ell, t) \le \epsilon\}.$$

Note that $\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, 0)$ need not be 0. The following easy estimate plays a key role in our proof.

Proposition 7.2. *Suppose $0 < \epsilon < 1/100$. For almost all $q_1 \in \hat{\Omega}$, almost all $uq_1 \in \mathcal{U}_1^+q_1$, all $\ell > 0$ and all $s > 0$,*

$$\tilde{\tau}_{(\epsilon)}(q_1, u, \ell) + \kappa^{-2}s < \tilde{\tau}_{(\epsilon)}(q_1, u, \ell + s) < \tilde{\tau}_{(\epsilon)}(q_1, u, \ell) + \kappa^2 s, \tag{7.2}$$

where κ is as in Proposition 2.14(d).

Proof. For $\hat{x} = (x, g) \in \hat{\Omega}$ and $t > 0$ let $\mathcal{A}_+(\hat{x}, t) = \mathcal{A}_+(x, t)$ denote the restriction of $(T_x^t)_*$ to $\mathbf{E}(x) \subset \mathrm{Lie}(N^+)(x)$. For $\hat{x} = (x, g) \in \hat{\Omega}$, let $\mathcal{A}_-(\hat{x}, s) : \mathcal{L}^-(\hat{x}) \to \mathcal{L}^-(\hat{T}^s\hat{x})$ denote the restriction of $(T_x^s)_*$ to $\mathcal{L}^-(\hat{x})$. It follows immediately from Proposition 2.14(d) that for some $\kappa > 1$, almost all $\hat{x}$ and $t > 0$,

$$e^{-\kappa^{-1}t} \ge \|\mathcal{A}_-(\hat{x}, t)\| \ge e^{-\kappa t}, \qquad e^{\kappa^{-1}t} \le \|\mathcal{A}_+(\hat{x}, t)\| \le e^{\kappa t}. \tag{7.3}$$

and,

$$e^{\kappa t} \ge \|\mathcal{A}_-(\hat{x}, -t)\| \ge e^{\kappa^{-1}t}, \qquad e^{-\kappa t} \le \|\mathcal{A}_+(\hat{x}, -t)\| \le e^{-\kappa^{-1}t}. \tag{7.4}$$

Note that by (4.1)

$$(\pi_{\mathbf{E}} \circ \mathcal{A})(q_1, u, \ell + s, t + \tau) = (T^\tau_{T^t uq_1})_* \circ (\pi_{\mathbf{E}} \circ \mathcal{A})(q_1, u, \ell, t) \circ (T^s_{T^{-(\ell+s)}q_1})_*$$

Let $t = \tilde{\tau}_{(\epsilon)}(q_1, u, \ell)$, so that $A(q_1, u, \ell, t) = \epsilon$. Therefore, by (7.3) and (7.4),

$$\begin{aligned} A(\hat{q}_1, u, \ell + s, t + \tau) &\le \|\mathcal{A}_+(\hat{T}^t u\hat{q}_1, \tau)\| A(\hat{q}_1, u, \ell, t) \|\mathcal{A}_-(\hat{T}^{-(\ell+s)}\hat{q}_1, s)\| \le \\ &\qquad \epsilon \|\mathcal{A}_+(\hat{T}^t u\hat{q}_1, \tau)\| \|\mathcal{A}_-(\hat{T}^{-(\ell+s)}\hat{q}_1, s)\| \le \epsilon e^{\kappa\tau - \kappa^{-1}s}, \end{aligned}$$

where we have used the fact that $A(\hat{q}_1, u, \ell, t) = \epsilon$. If $t + \tau = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell + s)$ then $A(\hat{q}_1, u, \ell + s, t + \tau) = \epsilon$. It follows that $\kappa\tau - \kappa^{-1}s > 0$, i.e. $\tau > \kappa^{-2}s$. Hence, the lower bound in (7.2) holds.

The proof of the upper bound is similar. Note that we have

$$\mathcal{A}(q_1, u, \ell, t) = (T^{-\tau}_{T^{t+\tau}uq_1})_* \circ (\pi_{\mathbf{E}} \circ \mathcal{A})(q_1, u, \ell + s, t + \tau) \circ (T^{-s}_{T^{-\ell}q_1})_*.$$

Let $t + \tau = \tilde{\tau}_{(\epsilon)}(q_1, u, \ell + s)$. Then, by (7.3) and (7.4),

$$\begin{aligned} A(\hat{q}_1, u, \ell, t) &\le \|\mathcal{A}_+(\hat{T}^{t+\tau}u\hat{q}_1, -\tau)\| A(\hat{q}_1, u, \ell + s, t + \tau) \|\mathcal{A}_-(\hat{T}^{-\ell}\hat{q}_1, -s)\| \le \\ &\qquad \epsilon \|\mathcal{A}_+(\hat{T}^{t+\tau}u\hat{q}_1, -\tau)\| \|\mathcal{A}_-(\hat{T}^{-\ell}\hat{q}_1, -s)\| \le \epsilon e^{-\kappa^{-1}\tau + \kappa s}, \end{aligned}$$

where we have used the fact that $A(\hat{q}_1, u, \ell + s, t + \tau) = \epsilon$. Since $A(\hat{q}_1, u, \ell, t) = \epsilon$, it follows that $-\kappa^{-1}\tau + \kappa s > 0$, i.e. $\tau < \kappa^2 s$. It follows that the upper bound in (7.2) holds. □

8. Conditional measures.

8.1. **Conditional Measure Lemmas.** We note the following

Lemma 8.1. *For any $\rho > 0$ there is a constant $c(\rho)$ with the following property: Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between Euclidean spaces. Then there exists a proper subspace $\mathcal{M} \subset \mathcal{V}$ such that for any $v \in \mathcal{V}$ with $\|v\| = 1$ and $d(v, \mathcal{M}) > \rho$, we have*

$$\|A\| \geq \|Av\| \geq c(\rho)\|A\|.$$

Proof of Lemma 8.1. The matrix $A^t A$ is symmetric, so it has a complete orthogonal set of eigenspaces $W_1, \dots, W_m$ corresponding to eigenvalues $\mu_1 > \mu_2 > \dots \mu_m$. Let $\mathcal{M} = W_1^\perp$. □

Let $\mathcal{B}$ be an abstract finite measure space.

Proposition 8.2. *For every $\delta > 0$ there exist constants $c_1(\delta) > 0$, $\epsilon_1(\delta) > 0$ with $c_1(\delta) \to 0$ and $\epsilon_1(\delta) \to 0$ as $\delta \to 0$, and also constants $\rho(\delta) > 0$ and $\rho'(\delta) > 0$, such that the following holds:*

For any Γ-invariant subset $K' \subset \hat{\Omega}_0$ with $\hat{\nu}(K'/\Gamma) > 1-\delta$, there exists a Γ-invariant subset $K \subset K'$ with $\hat{\nu}(K/\Gamma) > 1 - c_1(\delta)$ such that the following holds: suppose for each $\hat{q} \in \hat{\Omega}_0$ we have a measurable map from $\mathcal{B}$ to proper subspaces of $\mathcal{L}^-(\hat{q})$, written as $u \to \mathcal{M}_u(\hat{q})$. Then, for any $\hat{q} = (q, g) \in K$ there exists $\hat{q}' = (q', \exp(\mathbf{w})g) \in K'$ with $q' \in W_1^-[q]$, $\mathbf{w} \in \mathcal{L}^-(\hat{q})$,

$$\rho'(\delta) \leq \|\mathbf{w}\|_0 \leq 1/100 \tag{8.1}$$

and

$$d_0(\mathbf{w}, \mathcal{M}_u(q)) > \rho(\delta) \qquad \text{for at least } (1 - \epsilon_1(\delta))\text{-fraction of } u \in \mathcal{B}. \tag{8.2}$$

In the rest of this subsection we will prove Proposition 8.2.

Notation. For $\hat{x} = (x, g) \in \hat{\Omega}$, let $\hat{\nu}_{\hat{x}}^-$ denote conditional measure of $\hat{\nu}$ on $\hat{W}_1^-[\hat{x}]$. Let $\tilde{\nu}_{\hat{x}}$ denote the projection of $\hat{\nu}_{\hat{x}}^-$ to the G factor. By abuse of notation, we think of $\tilde{\nu}_{\hat{x}}$ as a measure on $\mathfrak{g}$. Then, by the definition of $\mathcal{L}^-(\hat{x})$, $\tilde{\nu}_{\hat{x}}$ is supported on $\mathcal{L}^-(\hat{x})$. Recall that $\mathcal{L}^-(\hat{x})$ is a subalgebra of $\mathfrak{g}$, by Lemma 7.1.

Lemma 8.3. *For $\hat{\nu}$-almost all $\hat{x} = (x, g) \in \hat{\Omega}$, for any $\epsilon > 0$ (which is allowed to depend on $\hat{x}$), the restriction of the measure $\tilde{\nu}_{\hat{x}}$ to the ball $B(0, \epsilon) \subset \mathcal{L}^-(\hat{x})$ is not supported on a finite union of proper affine subspaces of $\mathcal{L}^-(\hat{x})$.*

Outline of proof. Suppose not. Let $N(\hat{x})$ be the minimal integer N such that for some $\epsilon = \epsilon(\hat{x}) > 0$, the restriction of $\tilde{\nu}_{\hat{x}}$ to $B(0,\epsilon)$ is supported on N affine subspaces. Since $\mathcal{L}^-(\hat{x}) \subset \mathrm{Lie}(N^-)(x)$, the induced action on on $\mathcal{L}^-$ of T^{-t} for $t \geq 0$ is expanding. Then $N(\hat{x})$ is invariant under T^{-t}, $t \geq 0$. This implies that $N(\hat{x})$ is constant for $\hat{\nu}$-almost all $\hat{x}$, and also that the only affine subspaces of $\mathcal{L}^-(\hat{x})$ which contribute to $N(\cdot)$ pass through the origin. Then, $N(\hat{x}) > 1$ almost everywhere is impossible. Indeed, suppose $N(\hat{x}) = k$ a.e., then for $\hat{x} = (x,g)$ pick $\hat{y} = (y, \exp(\mathbf{w})g) \in \hat{W}_1^-[\hat{x}]$ near $\hat{x}$ such that $\mathbf{w}$ is in one of the affine subspaces through 0; then there must be exactly k affine subspaces of non-zero measure passing though $\mathbf{w}$, but then at most one of them passes through 0. Thus, the measure restricted to a neighborhood of 0 gives positive weight to at least $k+1$ subspaces, contradicting our assumption. Thus, we must have $N(\hat{x}) = 1$ almost everywhere; but then (after flowing by $\hat{T}^{-t}$ for sufficiently large $t > 0$) we see that for almost all $\hat{x}$, $\tilde{\nu}_{\hat{x}}$ is supported on a proper subspace of $\mathcal{L}^-(\hat{x})$, which contradicts the definition of $\mathcal{L}^-(\hat{x})$. □

The partitions $\hat{\mathfrak{B}}^-$ and $\mathfrak{B}^-$. We may choose a Γ-invariant partition of $\hat{\mathfrak{B}}^-$ of $\hat{\Omega}$ subordinate to $\hat{W}_1^-$, so that for each $\hat{x} = (x^+, x^-, g)$ the atom $\hat{\mathfrak{B}}^-[\hat{x}]$ containing $\hat{x}$ is of the form $W_1^-[x^+] \times \mathfrak{B}^-[x^+, g]$, where $\mathfrak{B}^-[x^+, g] \subset N^-(x)g$. Following our conventions, we will write $\mathfrak{B}^-[x^+, g\Gamma]$ as $\mathfrak{B}^-[\hat{x}]$. We may also assume that the diameter of each $\mathfrak{B}^-[\hat{x}]$ is at most $1/100$.

The measure $\nu'_{\hat{x}}$. For $x \in \hat{\Omega}$, let $\nu'_{\hat{x}} = \tilde{\nu}_{\hat{x}}|_{\mathfrak{B}^-[\hat{x}]})$, i.e. $\nu'_{\hat{x}}$ is the restriction of $\tilde{\nu}_{\hat{x}}$ (which is a measure on $N^-(x)g$) to the subset $\mathfrak{B}^-[\hat{x}]$. Then, for $\hat{y} \in \hat{\mathfrak{B}}^-[\hat{x}]$, $\nu'_{\hat{y}} = \nu'_{\hat{x}}$.

Lemma 8.4. *For every $\eta > 0$ and every $N > 0$ there exists $\beta_1 = \beta_1(\eta, N) > 0$, $\rho_1 = \rho_1(\eta, N) > 0$ and a Γ-invariant subset $K_{\eta,N}$ with $K_{\eta,N}/\Gamma$ compact and of measure at least $1-\eta$ such that for all $\hat{x} \in K_{\eta,N}$, and any proper subspaces $\mathcal{M}_1(\hat{x}), \dots, \mathcal{M}_N(\hat{x}) \subset \mathcal{L}^-(\hat{x})$,*

$$\nu'_{\hat{x}}\Big(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}_k(\hat{x}), \rho_1)\Big) \geq \beta_1 \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{8.3}$$

Outline of Proof. By Lemma 8.3, there exist $\beta_{\hat{x}} = \beta_{\hat{x}}(N) > 0$ and $\rho_{\hat{x}} = \rho_{\hat{x}}(N) > 0$ such that for any subspaces $\mathcal{M}_1(\hat{x}), \dots \mathcal{M}_N(\hat{x}) \subset \mathcal{L}^-(\hat{x})$,

$$\nu'_{\hat{x}}\Big(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}(\hat{x}), \rho_{\hat{x}})\Big) \geq \beta_{\hat{x}} \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{8.4}$$

Let $E(\rho_1, \beta_1)$ be the set of $\hat{x}$ such that (8.3) holds. By (8.4),

$$\hat{\nu}\left(\bigcup_{\substack{\rho_1 > 0 \\ \beta_1 > 0}} E(\rho_1, \beta_1)\right) = 1.$$

Therefore, we can choose $\rho_1 > 0$ and $\beta_1 > 0$ such that $\hat{\nu}(E(\rho_1, \beta_1)) > 1 - \eta$. $\square$

Lemma 8.5. *For every $\eta > 0$ and every $\epsilon_1 > 0$ there exists $\beta = \beta(\eta, \epsilon_1) > 0$, a Γ-invariant $K_\eta = K_\eta(\epsilon_1) \subset \hat{\Omega}$ with K_η/Γ compact and of measure at least $1 - \eta$, and $\rho = \rho(\eta, \epsilon_1) > 0$ such that the following holds: Suppose for each $u \in \mathcal{B}$ let $\mathcal{M}_u(\hat{x})$ be a proper subspace of $\mathcal{L}^-(\hat{x})$. Let*

$$E_{good}(\hat{x}) = \{v \in \mathfrak{B}^-[\hat{x}] \ : \ \text{for at least } (1-\epsilon_1)\text{-fraction of } u \text{ in } \mathcal{B}, \\ d_0(v, \mathcal{M}_u(\hat{x})) > \rho/2\}.$$

Then, for $\hat{x} \in K_\eta$,

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq \beta \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{8.5}$$

Proof. Let $n = \dim \mathcal{L}^-[\hat{x}]$. By considering determinants, it is easy to show that for any $C > 0$ there exists a constant $c_n = c_n(C) > 0$ depending on n and C such that for any $\eta > 0$ and any points $v_1, \dots, v_n$ in a ball of radius C with the property that $\|v_1\| \geq \eta$ and for all $1 < i \leq n$, v_i is not within η of the subspace spanned by $v_1, \dots, v_{i-1}$, then $v_1, \dots, v_n$ are not within $c_n \eta^n$ of any $n-1$ dimensional subspace. Let $k_{max} \in \mathbb{N}$ denote the smallest integer greater then $1 + n/\epsilon_1$, and let $N = N(\epsilon_1) = \binom{k_{max}}{n-1}$. Let β_1, ρ_1 and $K_{\eta,N}$ be as in Lemma 8.4. Let $\beta = \beta(\eta, \epsilon_1) = \beta_1(\eta, N(\epsilon_1))$, $\rho = \rho(\eta, \epsilon_1) = c_n \rho_1(\eta, N(\epsilon_1))^n$, $K_\eta(\epsilon_1) = K_{\eta, N(\epsilon_1)}$. Let $E_{bad}(\hat{x}) = \mathfrak{B}^-[\hat{x}] \smallsetminus E_{good}(\hat{x})$. To simplify notation, we choose coordinates so that $\hat{x} = 0$. We claim that $E_{bad}(\hat{x})$ is contained in the union of the ρ_1-neighborhoods of at most N subspaces. Suppose this is not true. Then, for $1 \leq k \leq k_{max}$ we can inductively pick points $v_1, \dots, v_k \in E_{bad}(\hat{x})$ such that v_j is not within ρ_1 of any of the subspaces spanned by $v_{i_1}, \dots, v_{i_{n-1}}$ where $i_1 \leq \dots \leq i_{n-1} < j$. Then, any n-tuple of points $v_{i_1}, \dots, v_{i_n}$ is not contained within $\rho = c_n \rho_1$ of a single subspace. Now, since $v_i \in E_{bad}(\hat{x})$, there exists $U_i \subset \mathcal{B}$ with $|U_i| \geq \epsilon_1 |\mathcal{B}|$ such that for all $u \in U_i$, $d_0(v_i, \mathcal{M}_u) < \rho/2$. We now claim that for any $1 \leq i_1 < i_2 < \dots < i_n \leq k$,

$$U_{i_1} \cap \dots \cap U_{i_n} = \emptyset. \tag{8.6}$$

Indeed, suppose u belongs to the intersection. Then each of the $v_{i_1}, \dots v_{i_n}$ is within $\rho/2$ of the single subspace $\mathcal{M}_u$, but this contradicts the choice of the v_i. This proves (8.6). Now,

$$\epsilon_1 k_{max} |\mathcal{B}| \leq \sum_{i=1}^{k_{max}} |U_i| \leq n \left| \bigcup_{i=1}^{k_{max}} U_i \right| \leq n |\mathcal{B}|.$$

This is a contradiction, since $k_{max} > 1 + n/\epsilon_1$. This proves the claim. Now (8.3) implies that

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}] \smallsetminus \bigcup_{k=1}^{N} \mathrm{Nbhd}(\mathcal{M}_k(\hat{x}), \rho_1)) \geq \beta \nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]).$$

□

Proof of Proposition 8.2. Let $\hat{\nu}|_{\hat{W}^-[\hat{x}]}$ denote the conditional measure of $\hat{\nu}$ on the stable leaf $\hat{W}^-[\hat{x}]$. Let

$$K'' = \{\hat{x} \in \hat{\Omega} \;\;:\;\; \hat{\nu}|_{W^-[\hat{x}]}(K' \cap \hat{\mathfrak{B}}^-[\hat{x}]) \geq (1-\delta^{1/2})\hat{\nu}|_{W^-[\hat{x}]}(\hat{\mathfrak{B}}^-[\hat{x}])\}.$$

Since $\hat{\mathfrak{B}}^-$ is a partition, we have $\hat{\nu}(K''/\Gamma) \geq 1-\delta^{1/2}$.

Let π_G denote the projection $\hat{\Omega} \to G$. We have, for $\hat{x} \in K''$,

$$\nu'_{\hat{x}}(\pi_G(K') \cap \mathfrak{B}^-[\hat{x}]) \geq (1-\delta^{1/2})\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{8.7}$$

Let $\beta(\eta, \epsilon_1)$ be as in Lemma 8.5. Let

$$c(\delta) = \delta + \inf\{(\eta^2+\epsilon_1^2)^{1/2} \;\;:\;\; \beta(\eta,\epsilon_1) \geq 8\delta^{1/2}\}.$$

We have $c(\delta) \to 0$ as $\delta \to 0$. By the definition of $c(\delta)$ we can choose $\eta = \eta(\delta) < c(\delta)$ and $\epsilon_1 = \epsilon_1(\delta) < c(\delta)$ so that $\beta(\eta, \epsilon_1) \geq 8\delta^{1/2}$.

By (8.5), for $\hat{x} \in K_\eta$,

$$\nu'_{\hat{x}}(E_{good}(\hat{x})) \geq 8\delta^{1/2}\nu'_{\hat{x}}(\mathfrak{B}^-[\hat{x}]). \tag{8.8}$$

Let $K = K' \cap K'' \cap K_\eta$. We have $\hat{\nu}(K/\Gamma) \geq 1-\delta-\delta^{1/2}-c(\delta)$, so $\hat{\nu}(K/\Gamma) \to 1$ as $\delta \to 0$. Also, if $\hat{q} \in K$, by (8.7) and (8.8),

$$\pi_G(K') \cap \mathfrak{B}^-[\hat{q}] \cap E_{good}(\hat{q}) \neq \emptyset.$$

Thus, we can choose $\hat{q}' \in K' \cap \hat{\mathfrak{B}}^-[\hat{q}]$ such that $\pi_G(\hat{q}') \in E_{good}(\hat{q})$. Then (8.2) holds with $\rho = \rho(\eta(\delta), \epsilon_1(\delta)) > 0$. Also the upper bound in (8.1) holds since $\mathfrak{B}^-[\hat{q}]$ has diameter at most $1/100$. Since all $\mathcal{M}_u(\hat{q})$ contain the origin, the lower bound in (8.1) follows from (8.2). □

9. Equivalence relations on W^+

Proposition 9.1. *For all $ij \in \tilde{\Lambda}$ and a.e $x \in \Omega$, the subspace $\mathbf{E}_{[ij],bdd}(x)$ is in fact a subalgebra of $\mathfrak{g}$.*

Proof. Suppose $\mathbf{v}, \mathbf{w} \in \mathbf{E}_{[ij],bdd}(x)$. Then, (since $R(x,y)$ acts by conjugation), $[\mathbf{v}, \mathbf{w}]$ satisfies all the conditions of Proposition 6.3. Thus, by Proposition 6.3, $[\mathbf{v}, \mathbf{w}] \in \mathbf{E}_{[ij],bdd}(x)$. □

For $ij \in \tilde{\Lambda}$ and $x \in \Omega$ let

$$\mathcal{E}_{ij}(x) = \exp(\mathbf{E}_{[ij],bdd}(x)).$$

In view of Proposition 9.1, this is a subgroup of G.

Equivalence relations. For $\hat{x} = (x, g)$, $\hat{x}' = (x', g') \in \hat{\Omega}$ we say that

$$\hat{x}' \sim_{ij} \hat{x} \text{ if } x' = x \text{ and } g' \in \mathcal{E}_{ij}(x)g.$$

The following is clear from the definitions:

Proposition 9.2. *The relation $\sim_{ij}$ is a (measurable) equivalence relation.*

The sets $\mathcal{E}_{ij}[\hat{x}]$. For $\hat{x} \in \hat{\Omega}$, we denote the equivalence class of $\hat{x}$ by $\mathcal{E}_{ij}[\hat{x}]$.

It is clear that the following equivariance properties hold:

Lemma 9.3. *Suppose $x \in \hat{\Omega}$, $t \in \mathbb{R}$ and $u \in \mathcal{U}_1^+$ is such that $ux \in \mathfrak{B}_0[x]$.*

(a) $\hat{T}^t \mathcal{E}_{ij}[x] = \mathcal{E}_{ij}[\hat{T}^t x]$.
(b) $u\mathcal{E}_{ij}[x] = \mathcal{E}_{ij}[ux]$.

Proof. Note that the sets $\mathbf{E}_{[ij],bdd}(x)$ are T^t-equivariant. Therefore, so are the $\mathcal{E}_{ij}(x)$, which implies (a). Part (b) is also clear, since by Lemma 3.4(a), $\mathbf{E}_{ij}(x) = \mathbf{E}_{ij}(ux)$. □

The measures f_{ij}. Write $\hat{x} = (x, g)$. Recall that $\mathcal{E}_{ij}(x)$ is a unipotent subgroup of G. We now apply the leafwise measure construction described in [EiL2] to get leafwise measures $f_{ij}(\hat{x})$ of $\hat{\nu}$ on $\mathcal{E}_{ij}(x)$. (Roughly speaking, $f_{ij}(\hat{x})$ is the pullback to $\mathcal{E}_{ij}(x)$ of the "conditional measure of $\hat{\nu}$ along $\mathcal{E}_{ij}(x)g$"). The measure $f_{ij}(\hat{x})$ is only defined up to normalization. We view $f_{ij}(\hat{x})$ as a measure on G which happens to be supported on the subgroup $\mathcal{E}_{ij}(x)$.

Lemma 9.4. *Suppose $\hat{y} = (y, g') \in \hat{\Omega}$, $\hat{x} = (x, g) \in \hat{\Omega}$ and $y \in \mathcal{H}[x]$. We have*

$$f_{ij}(\hat{y}) \propto R(x, y)_* f_{ij}(\hat{x}).$$

Proof. See [EiL2, Lemma 4.2(iv)]. □

10. The Eight Points

Let $\pi_\Omega : \hat{\Omega} \to \Omega$ denote the forgetful map. If $f(\cdot)$ is a function on Ω, and $\hat{x} \in \hat{\Omega}$, we will often write $f(\hat{x})$ instead of $f(\pi_\Omega(\hat{x}))$. Let $\pi_G : \hat{\Omega} \to G$ denote projection to the second factor.

We will derive Theorem 1.13 from the following:

Proposition 10.1. *Suppose μ satisfies the weak bounceback condition (3.13), and $\hat{\nu}$ is a $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant measure on $\hat{\Omega}/\Gamma$. Suppose also that Case I holds (see §1). Then for almost all $x \in \Omega/\Gamma$ there exists a unipotent subgroup $U_2^+(x) \subset N^+(x)$ and for almost all $\hat{x} \in \hat{\Omega}/\Gamma$ there exists a nontrivial unipotent subgroup $U_{new}^+(\hat{x}) \subset U_2^+(\hat{x})$ such that the following hold:*

(a) *For almost all $x \in \Omega$ and all $t \in \mathbb{R}$, $U_2^+(T^t x) = \operatorname{Ad}(T_x^t) U_2^+(x)$ and for almost all $u \in \mathcal{U}_1^+$, $U_2^+(ux) = U_2^+(x)$.*
(b) *For almost all $\hat{x} = (x, g) \in \hat{\Omega}$ and all $t \in \mathbb{R}$, $U_{new}^+(\hat{T}^t \hat{x}) = \operatorname{Ad}(T_x^t) U_{new}^+(\hat{x})$ and for almost all $u \in \mathcal{U}_1^+$, $U_{new}^+(u\hat{x}) = U_{new}^+(\hat{x})$.*
(c) *For almost all $\hat{x} = (x, g) \in \hat{\Omega}$, the leafwise measure of $\hat{\nu}$ along $U_2^+[\hat{x}] = \{x\} \times U_2^+(x)g$ (which is by definition a measure on $U_2^+(x)$) is right invariant by $U_{new}^+(\hat{x}) \subset U_2^+(x)$.*

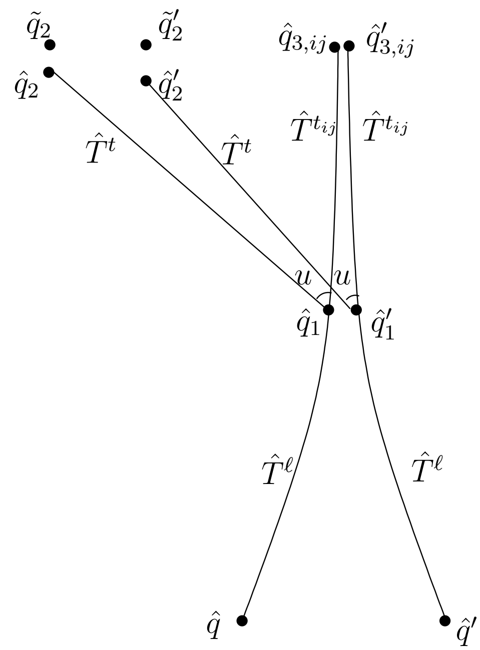

Figure 3. Outline of the proof of Theorem 1.13

Most of the rest of §10 will consist of the proof of Proposition 10.1. The argument has been outlined in [EsL, §1.2] and §1.5, and we have kept the same notation (in particular, see Figure 3).

For $\hat{x} = (x, g) \in \hat{\Omega}$, let $f_{ij}(\hat{x})$ be the measures on $\mathcal{E}_{ij}(x) \subset G$ defined in §9. Proposition 10.1 will be derived from the following:

Proposition 10.2. *Suppose μ and $\hat{\nu}$ are as in Proposition 10.1. Then there exists $0 < \delta_0 < 0.1$, a Γ-invariant subset $K_* \subset \hat{\Omega}$ with $\hat{\nu}(K_*/\Gamma) > 1 - \delta_0$ such that all the functions f_{ij}, $ij \in \tilde{\Lambda}$ are uniformly continuous on K_*, and $C > 1$ (depending on K_*) such that for every $0 < \epsilon < C^{-1}/100$ there exists a Γ-invariant subset $E \subset K_*$ with $\hat{\nu}(E/\Gamma) > \delta_0$, such that for every $\hat{x} \in E$ there exists $ij \in \tilde{\Lambda}$ and $\hat{y} \in \mathcal{E}_{ij}[\hat{x}] \cap K_*$ with*

$$C^{-1}\epsilon \leq d_G(\pi_G(\hat{x}), \pi_G(\hat{y})) \leq C\epsilon \tag{10.1}$$

and

$$f_{ij}(\hat{y}) \propto f_{ij}(\hat{x}). \tag{10.2}$$

10.1. **Outline of the proof of Proposition 10.2.** We use the same notation as in §1.5. Recall that for $x \in \Omega$, $\pi_{\mathbf{E}} : \mathfrak{g} \to \mathbf{E}(x)$ denotes the orthogonal projection, using the inner product $\langle \cdot, \cdot \rangle_x$.

Similarly to [EsL], a simplified scheme for choosing the eight points is as follows:

(i) Choose $\hat{q}_1$ in some good set, so that in particular, for most t, $\hat{T}^t \hat{q}_1 \in K_*$ and $\hat{T}^{-t}\hat{q}_1 \in K_*$ and for most u and most t, $\hat{T}^t u \hat{q}_1 \in K_*$.

(ii) Let $\mathcal{A}(\hat{q}_1, u, \ell, t)$ be as in §4, so that if $\pi_G(\hat{q}') = \exp(\mathbf{w})\pi_G(\hat{q})$ then we have $\pi_G(\hat{q}_2') = \exp(\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w})\pi_G(\hat{q}_2)$. Let $\hat{q} = \hat{T}^{-\ell}\hat{q}_1$ and let $\hat{q}_2 = \hat{T}^t u\hat{q}_1$, where $t = \tau_{(\epsilon)}(\hat{q}_1, u, \ell)$ is the solution to the equation $\|\pi_{\mathbf{E}}\mathcal{A}(\hat{q}_1, u, \ell, t)\| = \epsilon$. Since by Proposition 7.2, for fixed $\hat{q}_1$, u, ϵ, $\tau_{(\epsilon)}(\hat{q}_1, u, \ell)$ is bilipshitz in ℓ, for most choices of ℓ, $\hat{q} \in K_*$ and $\hat{q}_2 \in K_*$.

(iii) For all $ij \in \tilde{\Lambda}$, let $t_{ij} = t_{ij}(\hat{q}_1, u, \ell)$ be defined by the equation $\lambda_{ij}(u\hat{q}_1, t) = \lambda_{ij}(\hat{q}_1, t_{ij})$. Since $\lambda_{ij}(x, t)$ is bilipshitz in t, the same argument shows that for most choices of ℓ, $\hat{q}_{3,ij} \equiv \hat{T}^{t_{ij}}\hat{q}_1 \in K_*$.

(iv) Let $M_u \subset \mathcal{L}^-(\hat{q})$ be the subspace of Lemma 8.1 for the linear map $(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)$ restricted to $\mathcal{L}^-(\hat{q})$. By Proposition 8.2, we can choose $\hat{q}' \in K_*$ with $\pi_G(\hat{q}') = \exp(\mathbf{w})\pi_G(\hat{q})$ with $\|\mathbf{w}\| \approx 1$ and so that $\mathbf{w}$ avoids most of the subspaces M_u as u varies over $\mathcal{U}_1^+$. Then, for most u, (provided $\pi_{\mathbf{E}}\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}$ is a good approximation to $\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}$),

$$d_G(\pi_G(\hat{q}_2), \pi_G(\hat{q}_2')) \approx \|\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}\| \approx \|\mathcal{A}(\hat{q}_1, u, \ell, t)\|\|\mathbf{w}\| \approx \epsilon,$$

as required.

(v) In view of Proposition 4.1, for most choices of u, $\hat{q}_2'$ is close to $\exp(\mathbf{E}(\hat{q}_2))\hat{q}_2$, justifying the assumption that $\pi_{\mathbf{E}}\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}$ is a good approximation to $\mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w}$. Then, in view of Proposition 6.1, we can then choose u so that $\hat{q}_2'$ is close to $\exp(\mathbf{E}_{[ij],bdd}(\hat{q}_2))\hat{q}_2$ for some $ij \in \tilde{\Lambda}$.

(vi) We now proceed as in §1.5. Let $\hat{q}_1' = \hat{T}^\ell\hat{q}'$, $\hat{q}_2' = \hat{T}^t u\hat{q}_1'$ where $t = \tau_{(\epsilon)}(\hat{q}_1, u, \ell)$, and let $\hat{q}_{3,ij}' = \hat{T}^{t_{ij}}\hat{q}_1'$. Since $\hat{\nu}$ is $\hat{T}$-invariant and $\mathcal{U}_1^+$-invariant and since $\lambda_{ij}(\hat{q}_1, t_{ij}) = \lambda_{ij}(u\hat{q}_1, t)$, we have,

$$f_{ij}(\hat{q}_2) \propto f_{ij}(\hat{q}_{3,ij}).$$

Also, since one can show $\lambda_{ij}(uq_1', t) \approx \lambda_{ij}(q_1', t_{ij})$ we have,

$$f_{ij}(\hat{q}_2') \approx f_{ij}(\hat{q}_{3,ij}').$$

Since $\hat{q}_{3,ij}$ and $\hat{q}_{3,ij}'$ are very close, we can ensure that, $f_{ij}(\hat{q}_{3,ij}') \approx f_{ij}(\hat{q}_{3,ij})$. Then, we get, up to normalization,

$$f_{ij}(\hat{q}_2) \approx f_{ij}(\hat{q}_2').$$

Applying the argument with a sequence of ℓ's going to infinity, and passing to a limit along a subsequence, we obtain points $\hat{x}$, $\hat{y}$ satisfying (10.1) and (10.2).

The formal proof uses the same ideas, but we need to take a bit more care, mostly because we also need to make sure that $\hat{q}_2'$ and $\hat{q}_{3,ij}'$ belong to K_*. We now give a slightly more precise outline of the strategy.

Suppose $ij \in \tilde{\Lambda}$. We define a *Y-configuration* $Y_{ij} = Y_{ij}(\hat{q}_1, u, \ell)$ depending on the parameters $\hat{q}_1 \in \hat{\Omega}$, $u \in \mathcal{U}_1^+$, $\ell > 0$ to be a quadruple of points $\hat{q}$, $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_{3,ij}$ such that $\hat{q}$, $\hat{q}_2$, $\hat{q}_{3,ij}$ are chosen as in (ii) and (iii) (depending on $\hat{q}_1$, u, ℓ). Given a Y-configuration Y, we refer to its points as $\hat{q}(Y)$, $\hat{q}_1(Y)$, etc. A Y-configuration Y_{ij} is *good* if $\hat{q}(Y_{ij})$, $\hat{q}_1(Y_{ij})$, $\hat{q}_2(Y_{ij})$, and $\hat{q}_{3,ij}(Y_{ij})$ all belong to some good set K_*. The argument of (i),(ii), (iii) and Fubini's theorem show that for an almost full density set of ℓ, there are very many good Y-configurations with that value of ℓ. See Claim 10.4 below for the exact statement.

We say that two Y-configurations $Y = Y_{ij}(\hat{q}_1, u, \ell)$ and $Y' = Y_{ij}(\hat{q}'_1, u', \ell')$ with the same ij are *coupled* if $\ell = \ell'$, $u = u'$, $\hat{q}(Y') \in \hat{W}_1^-[\hat{q}(Y)]$, and also if we write $\hat{q}(Y) = (q, g)$ with $q \in \Omega$ and $g \in G$, $\hat{q}(Y') = (q', \exp(\mathbf{w})g)$ then $\|\mathbf{w}\| \approx 1$ and also $\mathbf{w}$ avoids the subspace M_u of (iv). Then the argument of (iv) shows that we can (for most values of ℓ) choose points $\hat{q}_1$, $\hat{q}'_1$ such that for most u and all ij, the Y-configurations $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}'_1, u, \ell)$ are both good and also are coupled. (see "Choice of parameters #2" below for the precise statement).

We then choose u as in (v). (See Claim 10.7, Claim 10.11 and "Choice of parameters #5"). We are now almost done, except for the fact that the lengths of the legs of $Y_{ij} = Y_{ij}(\hat{q}_1, u, \ell)$ and $Y'_{ij} = Y_{ij}(\hat{q}'_1, u, \ell)$ are not same. (The bottom leg of Y_{ij} has length ℓ, and so does the bottom leg of Y'_{ij}, but the two top legs of Y_{ij} can potentially have different lengths than the corresponding legs of Y'_{ij}). We show that the lengths of the corresponding legs are close (see Claim 10.8 and (10.24)) then make some corrections using (10.4). Then we proceed to part (vi).

10.2. **Choosing the eight points.** We now begin the formal proof of Proposition 10.2.

Choice of parameters #1. Fix $\theta_1 > 0$ as in Proposition 6.1 We then choose $\delta > 0$ sufficiently small; the exact value of δ will be chosen at the end of this section. All subsequent constants will depend on δ. (In particular, $\delta \ll \theta_1$; we will make this more precise below). Let $\epsilon > 0$ be arbitrary and $\eta > 0$ be arbitrary; however, we will always assume that ϵ and η are sufficiently small depending on δ.

We will show that Proposition 10.2 holds with $\delta_0 = \delta/10$. Let $K_* \subset \hat{\Omega}$ be any Γ-invariant subset with $\hat{\nu}(K_*/\Gamma) > 1 - \delta_0$ on which all the functions f_{ij} are uniformly continuous. It is enough to show that there exists $C = C(\delta)$ such that for any $\epsilon > 0$ and for an arbitrary Γ-invariant set $K_{00} \subset \hat{\Omega}$ with K_{00}/Γ compact and $\hat{\nu}(K_{00}/\Gamma) \geq (1 - 2\delta_0)$, there exists $\hat{x} \in K_{00} \cap K_*$ $ij \in \tilde{\Lambda}$ and $\hat{y} \in \mathcal{E}_{ij}[\hat{x}] \cap K_*$ satisfying (10.1) and (10.2). Thus, let $K_{00} \subset \hat{\Omega}$ be an arbitrary Γ-invariant set with K_{00}/Γ compact and $\nu(K_{00}/\Gamma) > 1 - 2\delta_0$.

Let $\epsilon' > 0$ be a constant which will be chosen later depending only on the Lyapunov exponents. Then, by the multiplicative ergodic theorem, for any $\delta > 0$ there exists a Γ-invariant set $K'_0 \subset \hat{\Omega}$ with K'_0/Γ compact and $\hat{\nu}(K'_0/\Gamma) > 1 - \delta$ and $T'_0 = T'_0(\delta) > 0$ such that for $t > T'_0$, $\hat{x} \in K'_0$ and any $\mathbf{v} \in \mathcal{V}_i(\hat{x})$,

$$e^{-(\lambda_i + \epsilon')t}\|\mathbf{v}\| \leq \|(\hat{T}_{\hat{x}}^{-t})_* \mathbf{v}\| \leq e^{-(\lambda_i - \epsilon')t}\|\mathbf{v}\|. \tag{10.3}$$

We can choose a Γ-invariant set $K_0 \subset K_{00} \cap K_* \cap K'_0$ with K_0/Γ compact and $\hat{\nu}(K_0/\Gamma) > 1 - 5\delta_0 = 1 - \delta/2$ so that Proposition 6.2 holds.

Let $\kappa > 1$ be as in Proposition 7.2, and so that (5.4) holds. Without loss of generality, assume $\delta < 0.01$. We now choose a Γ-invariant subset $K \subset \hat{\Omega}$ with K/Γ compact and $\hat{\nu}(K/\Gamma) > 1 - \delta$ such that the following hold:

- There exists a number $T_0(\delta)$ such that for any $\hat{x} \in K$ and any $T > T_0(\delta)$,
$$\{t \in [-T/2, T/2] \;:\; \hat{T}^t \hat{x} \in K_0\} \geq 0.9T. \tag{10.4}$$
(This can be done by the Birkhoff ergodic theorem).
- Proposition 4.1 holds.
- Proposition 6.1 holds.
- There exists a constant $C = C(\delta)$ such that for $\hat{x} = (x, g) \in K$, $C_3(x)^2 < C(\delta)$ where C_3 is as in Proposition 6.2.
- Lemma 2.16 holds for $\epsilon = \epsilon'$, $K(\delta) = \pi_\Omega(K)$ and $C_1 = C_1(\delta)$.

For $u \in \mathcal{U}_1^+$, $ij \in \tilde{\Lambda}$ and $\hat{q}_1 \in \hat{\Omega}$ and $t > 0$, let $t_{ij} = t_{ij}(q_1, u, t)$ be the unique solution to
$$\lambda_{ij}(q_1, t_{ij}) = \lambda_{ij}(uq_1, t)$$
Then, in view of Proposition 2.14(c), for fixed q_1, u, $t_{ij}(q_1, u, t)$ is bilipshitz in t. Let $\tilde{\tau}_{(\epsilon)}(q_1, u, \ell)$ be as in §7. Let
$$E_2(\hat{q}_1, u) = E_2(\hat{q}_1, u, K_{00}, \delta, \epsilon, \eta) = \{\ell \;:\; \hat{T}^{\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)} u\hat{q}_1 \in K\},$$
$$E_3(\hat{q}_1, u) = E_3(\hat{q}_1, u, K_{00}, \delta, \epsilon, \eta) = $$
$$= \{\ell \in E_2(\hat{q}_1, u) \;:\; \forall ij \in \tilde{\Lambda}, \quad T^{t_{ij}(q_1, u, \tilde{\tau}_{(\epsilon)}(q_1, u, \ell))} q_1 \in K\}.$$
Note that if we make choices as in §10.1 (ii) and (iii), then if $\ell \in E_3(\hat{q}_1, u)$ then $\hat{q}_2 \in K$ and $\hat{q}_3 \in K$.

Claim 10.3. *There exists $\ell_3 = \ell_3(K_{00}, \delta, \epsilon, \eta) > 0$, a Γ-invariant set $K_3 = K_3(K_{00}, \delta, \epsilon, \eta)$ with $K_3 \subset K$ and K_3/Γ of measure at least $1 - c_3(\delta)$ and for each $\hat{q}_1 \in K_3$ a subset $Q_3 = Q_3(\hat{q}_1\Gamma, K_{00}, \delta, \epsilon, \eta) \subset \mathcal{U}_1^+$ with $|Q_3\hat{q}_1| \geq (1 - c_3'(\delta))|\mathcal{U}_1^+\hat{q}_1|$ such that for all $\hat{q}_1 \in K_3$, $u \in Q_3$, $u\hat{q}_1 \in K$, and for $\ell > \ell_3$, $|E_3(\hat{q}_1, u) \cap [0, \ell]| > (1 - c_3''(\delta))\ell$. Also, we have $c_3(\delta)$, $c_3'(\delta)$ and $c_3''(\delta) \to 0$ as $\delta \to 0$.*

Proof of claim. By the ergodic theorem, for any $\delta > 0$ there exists a Γ-invariant set $K_2(\delta) \subset \hat{\Omega}$ with K_2/Γ compact and $\hat{\nu}(K_2/\Gamma) > 1-\delta$ and $\ell_2 > 0$ such that for any $\hat{q}_1 \in K_2$, and $L > \ell_2$ the measure of $\{t \in [0, L] \;:\; \hat{T}^t\hat{q}_1 \in K\}$ is at least $(1-\delta)L$. We choose
$$K_3 = K_2 \cap \{\hat{x} \in \hat{\Omega} \;:\; |\mathcal{U}_1^+\hat{x} \cap K_2| > (1 - \delta)|\mathcal{U}_1^+\hat{x}|\}.$$
Suppose $\hat{q}_1 \in K_3$, and $u\hat{q}_1 \in K_2$.

Let
$$E_{bad} = \{t \;:\; \hat{T}^t u\hat{q}_1 \in K^c\}.$$
Then, since $u\hat{q}_1 \in K_2$, for $\ell > \ell_2$, the density of E_{bad} is at most δ. We have
$$E_2(\hat{q}_1, u)^c = \{\ell \;:\; \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) \in E_{bad}\}.$$

Then, by Proposition 7.2, for $\ell > \kappa\ell_2$, the density of $E_2(\hat{q}_1, u)$ is at least $1 - 4\kappa^2\delta$. Similarly, since for any $ij \in \tilde{\Lambda}$ the function $\ell \to t_{ij}(\hat{q}_1, u, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$ is κ^2-bilipshitz (since it is the compostion of two κ-bilipshitz functions), for $\ell > \kappa^2\ell_2$, the density of $E_3(\hat{q}_1, u)$ is at least $1 - 8\kappa^4\delta|\tilde{\Lambda}|$. □

The following claim states that good Y-configurations are plentiful for an almost full density set of ℓ.

Claim 10.4. *There exists a set $\mathcal{D}_4 = \mathcal{D}_4(K_{00}, \delta, \epsilon, \eta) \subset \mathbb{R}^+$ and a number $\ell_4 = \ell_4(K_{00}, \delta, \epsilon, \eta) > 0$ so that $\mathcal{D}_4$ has density at least $1 - c_4(\delta)$ for $\ell > \ell_4$, and for $\ell \in \mathcal{D}_4$ a Γ-invariant subset $K_4(\ell) = K_4(\ell, K_{00}, \delta, \epsilon, \eta) \subset \hat{\Omega}$ with $K_4 \subset K$ and $\hat{\nu}(K_4(\ell)/\Gamma) > 1 - c_4'(\delta)$, such that for any $\hat{q}_1 \in K_4(\ell)$ there exists a subset $Q_4 = Q_4(\hat{q}_1\Gamma, \ell) \subset Q_3 \subset \mathcal{U}_1^+$ with $|Q_4\hat{q}_1| \geq (1 - c_4''(\delta))|\mathcal{U}_1^+\hat{q}_1|$ so that for all $\ell \in \mathcal{D}_4$, for all $\hat{q}_1 \in K_4(\ell)$ and all $u \in Q_4$,*

$$\ell \in E_3(\hat{q}_1, u). \tag{10.5}$$

(We have $c_4(\delta)$, $c_4'(\delta)$ and $c_4''(\delta) \to 0$ as $\delta \to 0$).

Proof of Claim. This follows from Claim 10.3 by applying Fubini's theorem to $\Omega_{\mathcal{B}} \times [0, L]$, where $\Omega_{\mathcal{B}} = \{(\hat{x}, u\hat{x}) \ : \ \hat{x} \in \Omega, \ ux \in \mathcal{U}_1^+ x\}$, where $L \in \mathbb{R}$. □

Suppose $\ell \in \mathcal{D}_4$.

Choice of parameters #2: Choice of $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$ (depending on δ, ϵ, $\hat{q}_1$, ℓ). Suppose $\ell \in \mathcal{D}_4$, and let $\mathcal{A}(\hat{q}_1, u, \ell, t)$ be as in (4.1). (Note that following our conventions, we use the notation $\mathcal{A}(\hat{q}_1, u, \ell, t)$ for $\hat{q}_1 \in \hat{\Omega}$, even though $\mathcal{A}(q_1, u, \ell, t)$ was originally defined for $q_1 \in \Omega$). For $u \in Q_4(\hat{q}_1\Gamma, \ell)$ let $\mathcal{M}_u$ be the subspace of Lemma 8.1 applied to the restriction of the linear map $\pi_{\mathbf{E}}\mathcal{A}(\hat{q}_1, u, \ell, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$ to $\mathcal{L}^-(\hat{T}^{-\ell}\hat{q}_1)$. We now apply Proposition 8.2 with $K' = \hat{T}^{-\ell}K_4(\ell)$. We denote the resulting set K by $K_5(\ell) = K_5(\ell, K_{00}, \delta, \epsilon, \eta)$. We have $\nu(K_5(\ell)/\Gamma) \geq 1 - c_5(\delta)$, where $c_5(\delta) \to 0$ as $\delta \to 0$. Let $K_6(\ell) = \hat{T}^{\ell}K_5(\ell)$.

Suppose $\ell \in \mathcal{D}_4$ and $\hat{q}_1 \in K_6(\ell)$. Let $\hat{q} = \hat{T}^{-\ell}\hat{q}_1$. Then, $\hat{q} \in K_5(\ell)$. Write $\hat{q} = (q, g)$ where $q = \pi_\Omega(\hat{q}) \in \Omega$. By Proposition 8.2 and the definition of $K_5(\ell)$, we can choose

$$\hat{q}' = (q', \exp(\mathbf{w})g) \in \hat{T}^{-\ell}K_4(\ell) \tag{10.6}$$

so that $q' \in W_1^-[q]$, and $\mathbf{w} \in \mathcal{L}^-(\hat{q})$ with $\rho'(\delta) \leq \|\mathbf{w}\| \leq 1/100$ and so that (8.2) holds with $\epsilon_1(\delta) \to 0$ as $\delta \to 0$. Let $\hat{q}_1' = \hat{T}^{\ell}\hat{q}'$. Then $\hat{q}_1' \in K_4(\ell)$.

Standing Assumption. We assume $\ell \in \mathcal{D}_4$, $\hat{q}_1 \in K_6(\ell)$ and $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$ are as in Choice of parameters #2. (This means that in the language of §10.1, for all ij, for most u, the Y configurations $Y_{ij}(\hat{q}_1, u, \ell)$ and $Y_{ij}(\hat{q}_1', u, \ell)$ are both good and are coupled). We think of $\hat{q}_1'$ as a measurable function of $\hat{q}_1$.

Notation. For $u \in \mathcal{U}_1^+$, let

$$\tau(u) = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell), \qquad \tau'(u) = \tilde{\tau}_{(\epsilon)}(\hat{q}_1', u, \ell),$$

Claim 10.5. *For $u \in Q_4(\hat{q}_1\Gamma, \ell) \cap Q_4(\hat{q}_1'\Gamma, \ell)$,*

$$\hat{T}^{\tau(u)} u\hat{q}_1 \in K, \quad \text{and} \quad \hat{T}^{\tau'(u)} u\hat{q}_1' \in K. \tag{10.7}$$

Proof of Claim. Suppose $u \in Q_4(\hat{q}_1\Gamma, \ell)$. Since $\hat{q}_1 \in K_4$ and $\ell \in \mathcal{D}_4$, it follows from (10.5) that $\ell \in E_2(\hat{q}_1, u)$, and then from the definition of $E_2(\hat{q}_1, u)$ is follows that $\hat{T}^{\tau(u)} u\hat{q}_1 \in K$. Similarly, since $\hat{q}_1' \in K_4$, we have for $u \in Q_4(\hat{q}_1'\Gamma, \ell)$ we have $\hat{T}^{\tau'(u)} u\hat{q}_1' \subset K$. This completes the proof of (10.7). □

The numbers t_{ij} and t'_{ij}. Suppose $u \in Q_4(\hat{q}_1\Gamma, \ell)$, and suppose $ij \in \tilde{\Lambda}$. Let $t_{ij} = t_{ij}(\hat{q}_1, \ell, u)$ be defined by the equation

$$\lambda_{ij}(u\hat{q}_1, \tau(u)) = \lambda_{ij}(\hat{q}_1, t_{ij}). \tag{10.8}$$

Then, since $\ell \in \mathcal{D}_4$ and in view of (10.5), we have $\ell \in E_3(\hat{q}_1, u)$. In view of the definition of E_3, it follows that

$$\hat{T}^{t_{ij}} \hat{q}_1 \in K. \tag{10.9}$$

Similarly, suppose $u \in Q_4(\hat{q}_1'\Gamma, \ell)$ and $ij \in \tilde{\Lambda}$. Let t'_{ij} be defined by the equation

$$\lambda_{ij}(u\hat{q}_1', \tau'(u)) = \lambda_{ij}(\hat{q}_1', t'_{ij}). \tag{10.10}$$

By the same argument,

$$\hat{T}^{t'_{ij}} \hat{q}_1' \in K. \tag{10.11}$$

The map $\mathbf{v}(u)$. For $u \in \mathcal{U}_1^+$, let

$$\mathbf{v}(u) = \mathbf{v}(\hat{q}, \hat{q}', u, \ell, t) = \mathcal{A}(\hat{q}_1, u, \ell, t)\mathbf{w} \tag{10.12}$$

where $t = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)$ and $\mathbf{w}$ is as in (10.6).

Claim 10.6. *There exists a subset $Q_5 = Q_5(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_4(\hat{q}_1\Gamma, \ell) \subset \mathcal{U}_1^+$ with $|Q_5\hat{q}_1| \geq (1 - c_5''(\delta))|\mathcal{U}_1^+\hat{q}_1|$ (with $c_5''(\delta) \to 0$ as $\delta \to 0$), and a number $\ell_5 = \ell_5(\delta, \epsilon)$ such that if $\ell > \ell_5$, for all $u \in Q_5$,*

$$C'(\delta)^{-1}\epsilon \leq \|\pi_{\mathbf{E}}(\mathbf{v}(u))\| \leq C'(\delta)\epsilon. \tag{10.13}$$

Proof of claim. Let $\mathcal{M}_u \subset \mathcal{L}^-(\hat{q})$ be the subspace of Lemma 8.1 applied to the restriction to $\mathcal{L}^-(\hat{q})$ of the linear map $(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell))$, where $\mathcal{A}(,,,)$ is as in (4.1). Let $Q_5 \subset Q_4(\hat{q}_1\Gamma) \cap Q_4(\hat{q}_1'\Gamma)$ be the set of $u \in Q_4(\hat{q}_1\Gamma) \cap Q_4(\hat{q}_1'\Gamma)$ such that

$$d(\mathbf{w}, \mathcal{M}_u) \geq \rho(\delta)$$

Then, by (8.2),

$$|Q_5\hat{q}_1| \geq |(Q_4(\hat{q}_1\Gamma) \cap Q_4(\hat{q}_1'\Gamma))\hat{q}_1| - \epsilon_1(\delta)|\mathcal{U}_1^+\hat{q}_1| \geq (1 - \epsilon_1(\delta) - c_4''(\delta))|\mathcal{U}_1^+\hat{q}_1|.$$

We now apply Lemma 8.1 to the linear map $(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)$. Then, for all $u \in Q_5$,

$$c(\delta)\|(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\| \leq \|(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\mathbf{w}\| \leq \|(\pi_{\mathbf{E}} \circ \mathcal{A})(\hat{q}_1, u, \ell, t)\|.$$

Therefore, since $t = \tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell)$, (10.13) holds. $\square$

Standing assumption: We assume $C(\delta)\epsilon < 1/100$ for any constant $C(\delta)$ arising in the course of the proof. In particular, this applies to $C_2(\delta)$ and $C_2'(\delta)$ in the next claim.

Claim 10.7. *There exists a number $\ell_6 = \ell_6(\delta)$ and constants $c_6(\delta)$ and $c_6'(\delta) > 0$ with $c_6(\delta)$ and $c_6'(\delta) \to 0$ as $\delta \to 0$, a Γ-invariant subset $K_6' = K_6'(\ell, K_{00}, \delta, \epsilon) \subset K_5$ with $\hat{\nu}(K_6'/\Gamma) > 1 - c_6(\delta)$, for each $\hat{q}_1 \in K_6'$ a subset $Q_6 = Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon) \subset \mathcal{U}_1^+$ with $|Q_6\hat{q}_1| \geq (1 - c_6'(\delta))|\mathcal{U}_1^+\hat{q}_1|$ such that for $\ell > \ell_6$, $\hat{q}_1 \in K_6'$, $u \in Q_6$,*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}(\hat{T}^{\tau(u)}u\hat{q}_1)\right) \leq C_8(\delta)e^{-\alpha'\ell}, \tag{10.14}$$

where α' depends only on the Lyapunov spectrum. In addition,

$$C_1(\delta)\epsilon \leq d(\hat{T}^{\tau(u)}u\hat{q}_1, \hat{T}^{\tau(u)}u\hat{q}_1') \leq C_2(\delta)\epsilon, \tag{10.15}$$

$$C_1'(\delta)\epsilon \leq \|\mathbf{v}(u)\| \leq C_2'(\delta)\epsilon, \tag{10.16}$$

and

$$\alpha_3^{-1}\ell \leq \tau(u) \leq \alpha_3\ell. \tag{10.17}$$

where $\alpha_3 > 1$ depends on the Lyapunov spectrum.

Proof. Let Q be as in Proposition 4.1 for $\mathbf{v} = \mathbf{w}$, and let $Q_6 = Q_5 \cap Q$. Then, (10.14) follows immediately from Proposition 4.1 and the definition of $\mathbf{v}(u)$. This immediately implies (10.15) and (10.16), in view of (10.13). Now the upper bound in (10.17) follows easily from (4.2). The lower bound in (10.17) follows from Proposition 2.14(d). $\square$

Standing Assumption. We assume $\hat{q}_1 \in K_6'$ and $\ell > \ell_6$.

Claim 10.8. *Suppose $u \in Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell)$. Then, there exists $C_0 = C_0(\delta)$ such that*

$$|\tau(u) - \tau'(u)| \leq C_0(\delta). \tag{10.18}$$

Proof of claim. Note that $\hat{q} = (q, g)$, $\hat{q}' = (q', g')$ where $q' \in W_1^-[q]$ and $g' \in \exp(\mathcal{L}^-[q'])g$. This implies in particular that $N^-(q') = N^-(q)$, and

$$\mathcal{A}(\hat{q}_1, u, \ell, t) = \mathcal{A}(\hat{q}_1', u, \ell, t).$$

By Lemma 7.1, we have $\mathcal{L}^-(\hat{q}') = \mathcal{L}^-(\hat{q})$. Thus, in view of Lemma 2.16 and (10.14)

$$|\tilde{\tau}_{(\epsilon)}(\hat{q}_1, u, \ell) - \tilde{\tau}_{(\epsilon)}(\hat{q}_1', u, \ell)| \leq C_0(\delta)$$

i.e. (10.18) holds. $\square$

The next few claims will help us choose u (once the other parameters have been chosen). Recall that $\mathfrak{B}_0[x] \subset W_1^+[x]$ is defined in §2.5.

Claim 10.9. *There exists a constants $c_7(\delta) > 0$ and $c_7'(\delta)$ with $c_7(\delta) \to 0$ and $c_7'(\delta) \to 0$ as $\delta \to 0$ and a Γ-invariant subset $K_7(\ell) = K_7(\ell, K_{00}, \delta, \epsilon, \eta)$ with $K_7(\ell) \subset K_6(\ell)$, $\hat{\nu}(K_7(\ell)/\Gamma) > 1 - c_7(\delta)$ such that for $\hat{q}_1 \in K_7(\ell)$,*

$$|\mathfrak{B}_0[\hat{q}_1] \cap Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell)\hat{q}_1| \geq (1 - c_7'(\delta))|\mathfrak{B}_0[\hat{q}_1]|.$$

Proof of Claim. Given $\delta > 0$, there exists $c_7''(\delta) > 0$ with $c_7''(\delta) \to 0$ as $\delta \to 0$ and a compact set $K_7' \subset \hat{\Omega}$ with $\hat{\nu}(K_7') > 1 - c_7''(\delta)$, such that for $\hat{q}_1 \in K_7'$, $|\mathfrak{B}_0[\hat{q}_1]| \geq c_6'(\delta)^{1/2}|\mathcal{U}_1^+\hat{q}_1|$. Then, for $\hat{q}_1 \in K_7' \cap K_6$,

$$|\mathfrak{B}_0[\hat{q}_1] \cap (Q_6)^c\hat{q}_1)| \leq |(Q_6)^c\hat{q}_1| \leq c_6'(\delta)|\mathcal{U}_1^+\hat{q}_1| \leq c_6'(\delta)^{1/2}|\mathfrak{B}_0[\hat{q}_1]|.$$

Thus, the claim holds with $c_7(\delta) = c_6(\delta) + c_7''(\delta)$ and $c_7'(\delta) = c_6'(\delta)^{1/2}$. □

Standing Assumption. We assume that $\hat{q}_1 \in K_7(\ell)$.

Let

$$Q_7(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell) = \{u \in Q_6(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell) \ : \ u\hat{q}_1 \in \mathfrak{B}_0[\hat{q}_1]\}.$$

Claim 10.10. *There exists a subset $Q_7^* = Q_7^*(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_7$ with $|Q_7^*\hat{q}_1| \geq (1 - c_7^*(\delta))|\mathfrak{B}_0[\hat{q}_1]|$ such that for $u \in Q_7^*$ and any $t > \ell_7(\delta)$ we have*

$$|\mathfrak{B}_t[u\hat{q}_1] \cap Q_7(\hat{q}_1, \ell)\hat{q}_1| \geq (1 - c_7^*(\delta))|\mathfrak{B}_t[u\hat{q}_1]|, \tag{10.19}$$

where $c_7^*(\delta) \to 0$ as $\delta \to 0$.

Proof. This follows immediately from Lemma 2.13. □

Choice of parameters #3: Choice of δ. We can choose $\delta > 0$ so that

$$c_7^*(\delta) < \theta_1/2, \tag{10.20}$$

where θ_1 is as in Proposition 6.1.

Claim 10.11. *There exist sets $Q_9 = Q_9(\hat{q}_1\Gamma, \hat{q}_1'\Gamma, \ell, K_{00}, \delta, \epsilon, \eta) \subset Q_7^*$ with $|Q_9(\hat{q}_1\Gamma, \ell)\hat{q}_1| \geq (\theta_1/4)|\mathfrak{B}_0[\hat{q}_1]|$ and $\ell_9 = \ell_9(K_{00}, \delta, \epsilon, \eta)$, such that for $\ell > \ell_9$ and $u \in Q_9$,*

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \bigcup_{ij \in \tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(\hat{T}^{\tau(u)}u\hat{q}_1)\right) < 4\eta. \tag{10.21}$$

Proof of claim. Suppose $u \in Q_7^*$. Then, by (10.14) we may write

$$\mathbf{v}(u) = \mathbf{v}'(u) + \mathbf{v}''(u),$$

where $\mathbf{v}'(u) \in \mathbf{E}(\hat{T}^{\tau(u)} u\hat{q}_1)$ and $\|\mathbf{v}''(u)\| \le C(\delta,\epsilon)e^{-\alpha'\ell}$. Then, by Proposition 6.1 applied with $L = L_0(\delta,\eta)$ and $\mathbf{v} = \mathbf{v}'(u)$, we get that for at least θ_1-fraction of $y \in \mathcal{F}_{\mathbf{v}'}[\hat{T}^{\tau(u)} u\hat{q}_1, L]$,

$$d\left(\frac{R(\hat{T}^{\tau(u)}u\hat{q}_1, y)\mathbf{v}'(u)}{\|R(\hat{T}^{\tau(u)}u\hat{q}_1, y)\mathbf{v}'(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 2\eta.$$

Then, for at least θ_1-fraction of $y \in \mathcal{F}_{\mathbf{v}}[\hat{T}^{\tau(u)} u\hat{q}_1, L]$, using Proposition 2.14 (d),

$$d\left(\frac{R(\hat{T}^{\tau(u)}u\hat{q}_1, y)\mathbf{v}(u)}{\|R(\hat{T}^{\tau(u)}u\hat{q}_1, y)\mathbf{v}(u)\|}, \bigcup_{ij\in\tilde{\Lambda}} \mathbf{E}_{[ij],bdd}(y)\right) < 3\eta + C(\epsilon,\delta)e^{3\kappa L}e^{-\alpha'\ell}. \tag{10.22}$$

where κ is as in Proposition 2.14 (d).

We may choose $\ell_9 = \ell_9(K_{00},\epsilon,\delta,\eta)$ so that for $\ell > \ell_9$ the right-hand side of (10.22) is at most 4η. For $y \in \mathcal{F}_{\mathbf{v}}[\hat{T}^{\tau(u)}u\hat{q}_1]$, write $y = T^s u_1 \hat{q}_1$, where $u_1 \in \mathcal{U}_1^+$. Then, (see §5.3),

$$y \in \mathcal{F}_{\mathbf{v}}[\hat{T}^{\tau(u)}u\hat{q}_1, L] \quad \text{if and only if} \quad u_1\hat{q}_1 \in \mathfrak{B}_{\tau(u)-L}[u\hat{q}_1].$$

Therefore, in view of (10.22), (10.19) and (10.20), for at least $(\theta_1/2)$-fraction of $u_1\hat{q}_1 \in \mathfrak{B}_{\tau(u)-L}[u\hat{q}_1]$, $T^{\tau(u_1)}u_1\hat{q}_1 \in K$ and (10.21) holds (with u replaced by u_1).

The collection of "balls" $\{\mathfrak{B}_{\tau(u)-L}[u\hat{q}_1]\}_{u\in Q_7^*(\hat{q}_1\Gamma,\ell)}$ are a cover of $Q_7^*(\hat{q}_1\Gamma,\ell)\hat{q}_1$. These balls satisfy the condition of Lemma 2.12 (b); hence we may choose a pairwise disjoint subcollection which still covers $Q_7^*(\hat{q}_1\Gamma,\ell)\hat{q}_1$. Then, by summing over the disjoint subcollection, we see that the claim holds on a set of measure at least $(\theta_1/2)|Q_7^*\hat{q}_1| \ge (\theta_1/2)(1-c_7^*(\delta))|\mathfrak{B}_0[\hat{q}_1]| \ge (\theta_1/4)|\mathfrak{B}_0[\hat{q}_1]|$. ☐

Choice of parameters #4: Choosing ℓ, $\hat{q}_1$, $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$. Choose $\ell > \ell_9(K_{00},\epsilon,\delta,\eta)$. Now choose $\hat{q}_1 \in K_7(\ell)$, and let $\hat{q}$, $\hat{q}'$, $\hat{q}_1'$ be as in Choice of Parameters #2.

Choice of parameters #5: Choosing u, $\hat{q}_2$, $\hat{q}_2'$, ij, $\hat{q}_{3,ij}$, $\hat{q}_{3,ij}'$ (depending on $\hat{q}_1$, $\hat{q}_1'$, ℓ). Choose $u \in Q_9(\hat{q}_1,\ell) \cap Q_4(\hat{q}_1',\ell)$ so that (10.15) holds. We have $T^{\tau(u)}u\hat{q}_1 \in K$ and $T^{\tau'(u)}u\hat{q}_1' \in K$. By (10.18),

$$|\tilde{\tau}_{(\epsilon)}(\hat{q}_1,u,\ell) - \tilde{\tau}_{(\epsilon)}(\hat{q}_1',u,\ell)| \le C_0(\delta),$$

therefore,

$$T^{\tau(u)}u\hat{q}_1' \in T^{[-C,C]}K,$$

where $C = C(\delta)$.

By the definition of K we can find $C_4(\delta)$ and $s \in [0, C_4(\delta)]$ such that

$$\hat{q}_2 \equiv T^sT^{\tau(u)}u\hat{q}_1 \in K_0, \qquad \hat{q}_2' \equiv T^sT^{\tau(u)}u\hat{q}_1' \in K_0.$$

Since $u \in Q_9(\hat{q}_1,\ell)$ there exists $ij \in \tilde{\Lambda}$ be such that

$$d\left(\frac{\mathbf{v}(u)}{\|\mathbf{v}(u)\|}, \mathbf{E}_{[ij],bdd}(T^{\tau(u)}u\hat{q}_1)\right) \le 4\eta \tag{10.23}$$

Note that $\pi_\Omega(u\hat{q}_1') \in W_1^-[\pi_\Omega(u\hat{q}_1)]$ and $\tilde{\tau}_{ij}(x,t) = \tilde{\tau}_{ij}(\pi_\Omega(x), t)$. Then, by Lemma 5.6,

$$|\lambda_{ij}(u\hat{q}_1, \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau(u))| \le C_4'(\delta).$$

Then, by (10.18) and (5.4),

$$|\lambda_{ij}(u\hat{q}_1, \tau(u)) - \lambda_{ij}(u\hat{q}_1', \tau'(u))| \le C_4''(\delta).$$

Hence, by Proposition 2.14 (e) (cf. Lemma 5.2), (10.8), (10.10) and Lemma 5.6 (applied to the points $\pi_\Omega(\hat{q}_1)$ and $\pi_\Omega(\hat{q}_1') \in W_1^-[\pi_\Omega(\hat{q}_1)]$) we have

$$|t_{ij} - t_{ij}'| \le C_5(\delta). \tag{10.24}$$

Therefore, by (10.9) and (10.11), we have

$$\hat{T}^{t_{ij}}\hat{q}_1 \in K, \quad \text{and} \quad \hat{T}^{t_{ij}}\hat{q}_1' \in \hat{T}^{[-C_5(\delta), C_5(\delta)]}K.$$

By the definition of K, we can find $s'' \in [0, C_5''(\delta)]$ such that

$$\hat{q}_{3,ij} \equiv \hat{T}^{s''+t_{ij}}\hat{q}_1 \in K_0, \quad \text{and} \quad \hat{q}_{3,ij}' \equiv \hat{T}^{s''+t_{ij}}\hat{q}_1' \in K_0.$$

Let $\tau = s + \tau(u)$, $\tau' = s'' + t_{ij}$. Then we have

$$\hat{q}_2 = \hat{T}^{\tau}u\hat{q}_1, \quad \hat{q}_2' = \hat{T}^{\tau}u\hat{q}_1', \quad \hat{q}_{3,ij} = \hat{T}^{\tau'}\hat{q}_1, \quad \hat{q}_{3,ij}' = \hat{T}^{\tau'}\hat{q}_1'.$$

Note that for any $\epsilon' > 0$, if ℓ is sufficiently large,

$$|\tau - \tau'| \le \epsilon'\ell. \tag{10.25}$$

10.3. **Completing the proofs.** Since $\pi_\Omega(\hat{q}_{3,ij}') \in W^-[\pi_\Omega(\hat{q}_{3,ij})]$ and $\pi_\Omega(\hat{q}_2') \in W^-[\pi_\Omega(\hat{q}_2)]$, in view of Lemma 2.5,

$$\mathbf{E}_{[ij],bdd}(\hat{q}_2') = P^-(\hat{q}_2, \hat{q}_2')\mathbf{E}_{[ij],bdd}(\hat{q}_2),$$

and

$$\mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij}') = P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')\mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij}).$$

Note that since q and q' have the same combinatorial future,

$$R(\hat{q}_{3,ij}, \hat{q}_2) = R(\hat{q}_{3,ij}', \hat{q}_2'). \tag{10.26}$$

Let $B : \mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij}) \to \mathbf{E}_{[ij],bdd}(\hat{q}_2)$ denote the restriction of $R(\hat{q}_{3,ij}, \hat{q}_2)$ to $\mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij})$. Let $B' : \mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij}') \to \mathbf{E}_{[ij],bdd}(\hat{q}_2')$ denote the restriction of $R(\hat{q}_{3,ij}, \hat{q}_2)$ to $\mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij}')$. By Proposition 6.2, there exists $C = C(\delta)$ such that

$$\max(\|B\|, \|B^{-1}\|) \le C(\delta) \quad \text{and} \quad \max(\|B'\|, \|(B')^{-1}\| \le C(\delta). \tag{10.27}$$

Claim 10.12. *For all $\mathbf{v} \in \mathbf{E}_{[ij],bdd}(\hat{q}_{3,ij})$, for ℓ sufficiently large,*

$$\|R(\hat{q}_{3,ij}', \hat{q}_2')P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')\mathbf{v} - P^-(\hat{q}_2, \hat{q}_2')R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v}\| \le C_2(\delta)e^{-(\alpha\alpha_3^{-1}/2)\ell}. \tag{10.28}$$

Proof of claim. Let $\mathbf{V}_k(x) = \mathbf{E}_{[ij],bdd}(x) \cap \mathcal{V}_k(x)$. Then, for any ij there exists a subset Δ_{ij} of the Lyapunov exponents such that

$$\mathbf{E}_{[ij],bdd}(x) = \bigoplus_{k \in \Delta_{ij}} \mathbf{V}_k(x).$$

A key point in the proof is that $R(x,y)\mathbf{V}_k(x) = \mathbf{V}_k(y)$ (by Lemma 6.6 and the fact that $\mathbf{E}_{[ij],bdd}(x) \subset \mathbf{E}(x)$). Also, $P^-(x,y)\mathbf{V}_k(x) = \mathbf{V}_k(y)$. In view of Proposition 2.14(a), it is enough to prove (10.28) on $\mathbf{V}_k$ working mod $\mathcal{V}_{\le k-1}$, i.e. to show that for all $\mathbf{v} \in \mathbf{V}_k(\hat{q}_{3,ij})$,

$$\|R(\hat{q}'_{3,ij}, \hat{q}'_2)P^-(\hat{q}_{3,ij}, \hat{q}'_{3,ij})\mathbf{v} - P^-(\hat{q}_2, \hat{q}'_2)R(\hat{q}_{3,ij}, \hat{q}_2)\mathbf{v} + \mathcal{V}_{\le k-1}(\hat{q}'_2)\| \le \le C_2(\delta)e^{-(\alpha\alpha_3^{-1}/2)\ell}. \tag{10.29}$$

By Lemma 2.2 and (10.17), there exists $C = C(\delta)$ such that

$$\|P^-(\hat{q}_2, \hat{q}'_2) - I\| \le C(\delta)e^{-\alpha\alpha_3^{-1}\ell} \tag{10.30}$$

and

$$\|P^-(\hat{q}_{3,ij}, \hat{q}'_{3,ij}) - I\| \le C(\delta)e^{-\alpha\alpha_3^{-1}\ell}. \tag{10.31}$$

Choose $\epsilon' = \alpha\alpha_3^{-1}/4$. Write $\mathbf{w} = (P^-(\hat{q}_{3,ij}, \hat{q}'_{3,ij}) - I)\mathbf{v}$. Then, by (10.3),

$$\|(T^{-\tau'}_{\hat{q}'_{3,ij}})_*\mathbf{w}\| \le e^{(-\lambda_k+\epsilon')\tau'}\|\mathbf{w}\|.$$

Hence,

$$\|(T^{-\tau'}_{\hat{q}'_{3,ij}})_*\mathbf{w} + \mathcal{V}_{\le k-1}(u\hat{q}'_1)\| \le e^{(-\lambda_k+\epsilon')\tau'}\|\mathbf{w}\|,$$

and then, since by (10.3) and Proposition 2.14(a), the norm of $(T^{\tau}_{u\hat{q}'_1})_*$, viewed as a linear map from $\mathfrak{g}/\mathcal{V}_{\le k-1}(u\hat{q}'_1)$ to $\mathfrak{g}/\mathcal{V}_{\le k-1}(\hat{q}'_2)$ is at most $e^{(\lambda_k+\epsilon')\tau}$,

$$\|R(\hat{q}'_{3,ij}, \hat{q}'_2)\mathbf{w}+\mathcal{V}_{\le k-1}(\hat{q}'_2)\| = \|(T^{\tau}_{u\hat{q}'_1})_*((T^{-\tau'}_{\hat{q}'_{3,ij}})\mathbf{w}+\mathcal{V}_{\le k-1}(\hat{q}'_1))\| \le e^{(\lambda_k+\epsilon')\tau}e^{(-\lambda_k+\epsilon')\tau'}\|\mathbf{w}\|.$$

The above equation, together with (10.25), (10.26), (10.27), (10.30) and (10.31) implies (10.29). $\square$

For the next claim, we need a metric on the leafwise measures. There exists a function $\rho : G \to \mathbb{R}^+$ which is integrable with respect to Haar measure on any unipotent subgroup of G. Then, by [EiL2, Theorem 6.30], ρ is integrable with respect to any leafwise measure. Let $\mathcal{M}_\rho$ denote the space of positive Radon measures ω on G for which $\int_G \rho\, d\omega \le 1$ equipped with the weakest topology for which for any continuous compactly supported ϕ the function $\omega \to \int_G \phi\, d\omega$ is continuous. Then, $\mathcal{M}_\rho$ is compact and metrizable, by some metric d' (see e.g. [Kal, Theorem 4.2]). Then, if ω_1 and ω_2 are leafwise measures, we can define $d(\omega_1, \omega_2) = d'(c_1\omega_1, c_2\omega_2)$, where $c_i^{-1} = \int_G \rho\, d\omega_i$.

Claim 10.13. *There exists $c_{10}(\delta,\ell)$, with $c_{10}(\delta,\ell) \to 0$ as $\ell \to \infty$ such that*

$$d(f_{ij}(\hat{q}_2), f_{ij}(\hat{q}_2')) \le c_{10}(\delta,\ell). \tag{10.32}$$

In (10.32) we consider $f_{ij}(x)$ to be a measure on G (defined up to scaling) which happens to be supported on the subgroup $\mathcal{E}_{ij}(x)$.

Proof of claim. By Lemma 9.4,

$$f_{ij}(\hat{q}_2) \propto B_* f_{ij}(\hat{q}_{3,ij}), \qquad f_{ij}(\hat{q}_2') \propto B'_* f_{ij}(\hat{q}_{3,ij}'). \tag{10.33}$$

Since $\hat{q}_{3,ij} \in K_0$ and $\hat{q}_{3,ij}' \in K_0$,

$$d(f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_{3,ij}')) \to 0 \text{ as } \ell \to \infty.$$

Then, also by (10.31),

$$d(P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_{3,ij}')) \to 0 \text{ as } \ell \to \infty.$$

Then, applying B' to both sides and using (10.27) and (10.33), we get

$$d(B'P^-(\hat{q}_{3,ij}, \hat{q}_{3,ij}')_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \text{ as } \ell \to \infty.$$

Using (10.28), we get

$$d(P^-(\hat{q}_2, \hat{q}_2')B_* f_{ij}(\hat{q}_{3,ij}), f_{ij}(\hat{q}_2')) \to 0 \text{ as } \ell \to \infty.$$

Then, by (10.33) and (10.30), (10.32) follows. □

Taking the limit as $\eta \to 0$. For fixed δ and ϵ, we now take a sequence of $\eta_k \to 0$ (this forces $\ell_k \to \infty$) and pass to limits (mod Γ) along a subsequence. Let $\tilde{q}_2 \in K_0$ be such that $\tilde{q}_2\Gamma$ is the limit of the $\hat{q}_2\Gamma$, and $\tilde{q}_2' \in K_0$ be such that $\tilde{q}_2'\Gamma$ is the limit of the $\hat{q}_2'\Gamma$. We may also assume that along the subsequence $ij \in \tilde{\Lambda}$ is fixed, where ij is as in (10.23). We get (after possibly replacing $\tilde{q}_2'$ by $\tilde{q}_2'\gamma$ for some $\gamma \in \Gamma$),

$$\frac{1}{C(\delta)}\epsilon \le d(\tilde{q}_2, \tilde{q}_2') \le C(\delta)\epsilon,$$

and in view of (10.23),

$$\tilde{q}_2' \in \mathcal{E}_{ij}[\tilde{q}_2].$$

Now, by (10.32), we have

$$f_{ij}(\tilde{q}_2) \propto f_{ij}(\tilde{q}_2').$$

We have $\tilde{q}_2 \in K_0 \subset K_{00} \cap K_*$, and $\tilde{q}_2' \in K_0 \subset K_*$. This concludes the proof of Proposition 10.2. □

Proof of Proposition 10.1. Take a sequence $\epsilon_m \to 0$. We now apply Proposition 10.2 with $\epsilon = \epsilon_m$. We may assume that $ij \in \tilde{\Lambda}$ is constant along the subsequence. Let $U_2^+(x) = \mathcal{E}_{ij}(x)$.

We get, for each m, a Γ-invariant set $E_m \subset K_*$ with $\hat{\nu}(E_m/\Gamma) > \delta_0$ and with the property that for every $\hat{x} \in E_m$ there exists $\hat{y} \in \mathcal{E}_{ij}[x] \cap K_*$ such that (10.1) and (10.2) hold for $\epsilon = \epsilon_m$. Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m \subset K_*,$$

(so F consists of the points which are in infinitely many E_m). Suppose $\hat{x} \in F$. Then there exists a sequence $\hat{y}_m \to \hat{x}$ such that $\hat{y}_m \in \mathcal{E}_{ij}[x]$, $\hat{y}_m \neq \hat{x}$, and so that $f_{ij}(y_m) \propto f_{ij}(\hat{x})$. We may write $\hat{x} = (x, g)$, $\hat{y}_m = (y_m, \gamma_m g)$. Since $\hat{y}_m \in \mathcal{E}_{ij}[x]$, $y_m = x$ and $\gamma_m \in \mathcal{E}_{ij}(x)$. By (10.1), γ_m tends to the identity of G as $m \to \infty$.

By (10.2)

$$f_{ij}(\hat{x}) \propto (r_{\gamma_m})_* f_{ij}(\hat{x}), \tag{10.34}$$

where $(r_g)_*$ denotes the action on measures induced by right multiplication by g. For $\hat{x} \in F$ let $U^+_{new}(\hat{x})$ denote the maximal connected subgroup of $\mathcal{E}_{ij}(x)$ such that for $n \in U^+_{new}(\hat{x})$,

$$(r_n)_* f_{ij}(\hat{x}) \propto f_{ij}(\hat{x}). \tag{10.35}$$

The set of $n \in \mathcal{E}_{ij}(x)$ satisfying (10.35) is closed, and by (10.34) is not discrete. Therefore, for $\hat{x} \in F$, $U^+_{new}(\hat{x})$ is non-trivial. Let $U^+_2(x)$ denote $\mathcal{E}_{ij}(x)$. By construction, the subgroup $U^+_{new}(\hat{x})$ is constant as $\hat{x}$ varies over $\mathcal{E}_{ij}[\hat{x}] = \{x\} \times U^+_2(x) g$, where we wrote $\hat{x} = (x, g)$.

Suppose $\hat{x} \in F$ and $u \in \mathcal{U}^+_1$. Then, since $f_{ij}(u\hat{x}) = f_{ij}(\hat{x})$, we have that (10.35) holds for $n \in U^+_{new}(u\hat{x})$. Therefore, by the maximality of $U^+_{new}(\hat{x})$, for $\hat{x} \in F$, $u \in \mathcal{U}^+_1$ such that $u\hat{x} \in F$,

$$U^+_{new}(u\hat{x}) = U^+_{new}(\hat{x}). \tag{10.36}$$

Suppose $\hat{x} \in F$, $t < 0$ and $\hat{T}^t\hat{x} \in F$. Then, since the $\mathcal{E}_{ij}[\hat{x}]$ are $\hat{T}^t$-equivariant (see Lemma 9.3) we have that (10.35) holds for $n \in \hat{T}^{-t} U^+_{new}(\hat{T}^t\hat{x})$. Therefore, by the maximality of $U^+_{new}(\hat{x})$, for $\hat{x} \in F$, $t < 0$ with $\hat{T}^t\hat{x} \in F$ we have

$$\hat{T}^{-t} U^+_{new}(\hat{T}^t\hat{x}) = U^+_{new}(\hat{x}), \tag{10.37}$$

and (10.35) and (10.36) still hold.

From (10.35), we get that for $\hat{x} \in F$ and $n \in U^+_{new}(\hat{x})$,

$$(n)_* f_{ij}(\hat{x}) = e^{\beta_{\hat{x}}(n)} f_{ij}(\hat{x}), \tag{10.38}$$

where $\beta_{\hat{x}} : U^+_{new}(\hat{x}) \to \mathbb{R}$ is a homomorphism. Since $\nu(F) > \delta_0 > 0$ and $\hat{T}^t$ is ergodic, for almost all $\hat{x} \in \hat{\Omega}$ there exist arbitrarily large $t > 0$ so that $\hat{T}^{-t}\hat{x} \in F$. Then, we define $U^+_{new}(\hat{x})$ to be $\hat{T}^t U^+_{new}(\hat{T}^{-t}\hat{x})$. (This is consistent in view of (10.37)). Then, (10.38) holds for a.e. $\hat{x} \in \hat{\Omega}$. It follows from (10.38) that for a.e. $\hat{x} \in \hat{\Omega}$, $n \in U^+_{new}(\hat{x})$ and $t > 0$,

$$\beta_{\hat{T}^{-t}\hat{x}}(\hat{T}^{-t} n \hat{T}^t) = \beta_{\hat{x}}(n). \tag{10.39}$$

We can write

$$\beta_{\hat{x}}(n) = L_{\hat{x}}(\log n),$$

where $L_{\hat{x}} : \mathrm{Lie}(U^+)(\hat{x}) \to \mathbb{R}$ is a Lie algebra homomorphism (which is in particular a linear map). Let $K \subset \hat{\Omega}$ be a Γ-invariant set with K/Γ of positive measure for which there exists a constant C with $\|L_{\hat{x}}\| \le C$ for all $\hat{x} \in K$. Now for almost all $\hat{x} \in \hat{\Omega}$ and $n \in U^+_{new}(\hat{x})$ there exists a sequence $t_j \to \infty$ so that $T^{-t_j}\hat{x} \in K$ and $T^{-t_j} n T^{t_j} \to e$, where e is the identity element of U^+_{new}. Then, (10.39) applied to the sequence t_j implies that $\beta_{\hat{x}}(n) = 0$ almost everywhere (cf. [BQ1, Proposition 7.4(b)]). Therefore, for almost all $\hat{x} \in \hat{\Omega}$, the conditional measure of $\hat{\nu}$ along the orbit $U^+_{new}[x]$ is the push-forward of the Haar measure on $U^+_{new}(\hat{x})$.

This completes the proof of Proposition 10.1. □

Proof of Theorem 1.13(a). This argument follows closely [BQ1, §8]. Let $\mathcal{P}(G/\Gamma)$ denote the space of probability measures on G/Γ. For $\alpha \in \mathcal{P}(G/\Gamma)$, let S_α denote the connected component of the identity of the stabilizer of α with respect to the action of G by left-multiplication on G/Γ. Let

$$\mathcal{F} = \{\alpha \in \mathcal{P}(G/\Gamma) \;\; : \;\; S_\alpha \neq \{1\} \text{ and } \alpha \text{ is supported on one } S_\alpha \text{ orbit.}\}$$

The set $\mathcal{F}$ is endowed with the weak-$*$ topology. The group G naturally acts on $\mathcal{F}$. By Ratner's theorems [Ra], $\mathcal{F}$ contains all of the measures invariant and ergodic under a connected non-trivial unipotent subgroup.

Let ν be an ergodic μ-stationary measure on G/Γ. We construct a $\hat{T}^t$-invariant measure $\hat{\nu}$ on $\hat{\Omega}_0$ as in §1, so that (1.9) holds.

By Proposition 10.1 for almost all $\hat{x} = (x, g\Gamma) \in \hat{\Omega}$, there exists a subgroup $U_2^+(x) \subset N^+(x)$ such that the conditional measures $\hat{\nu}|_{U_2^+[\hat{x}]}$ of $\hat{\nu}$ on the $U_2^+(x)$ orbits on the G/Γ-fiber at x are right invariant under a non-trivial unipotent subgroup $U^+_{new}(\hat{x})$ of $U_2^+(x)$. Without loss of generality we may assume that $U^+_{new}(\hat{x})$ is the stabilizer in $U_2^+(x)$ of $\hat{\nu}|_{U_2^+[\hat{x}]}$ (otherwise we replace $U^+_{new}(\hat{x})$ by the stabilizer).

For $(x, g\Gamma) \in \hat{\Omega}$, let

$$\Delta(x, g\Gamma) = \{g' \in G/\Gamma \;\; : \;\; U^+_{new}(x, g'\Gamma) = U^+_{new}(x, g\Gamma)\}.$$

Let $\hat{\nu}_x$ denote the conditional measure of $\hat{\nu}$ on $\{x\} \times G/\Gamma$. We now disintegrate $\hat{\nu}$ under the map $(x, g\Gamma) \to (x, U^+_{new}(x, g\Gamma))$, or equivalently for $\tilde{\mu}$-almost all $x \in \Omega$ we disintegrate $\hat{\nu}_x$ under the map $g\Gamma \to U^+_{new}(x, g\Gamma)$. We get, for almost all $(x, g\Gamma) \in \hat{\Omega}$, probability measures $\tilde{\nu}_{(x,g\Gamma)}$ on G/Γ supported on $\Delta(x, g\Gamma)$ so that for $\tilde{\mu}$-a.e. $x \in \Omega$,

$$\hat{\nu}_x = \int_{G/\Gamma} \tilde{\nu}_{(x,g\Gamma)} \, d\hat{\nu}_x(g\Gamma).$$

By [EiL3, Corollary 3.4] (cf. [BQ1, Proposition 4.3]), for $\hat{\nu}$-a.e. $(x, g\Gamma) \in \hat{\Omega}$, the measure $\tilde{\nu}_{(x,g\Gamma)}$ is (left) $U^+_{new}(x, g\Gamma)$-invariant.

We can do the simultaneous $U^+_{new}(x, g\Gamma)$-ergodic decomposition of all the measures $\tilde{\nu}_{(x,g\Gamma)}$ for almost all $(x, g\Gamma) \in \hat{\Omega}$ to get

$$\tilde{\nu}_{(x,g\Gamma)} = \int_{G/\Gamma} \zeta(x, g'\Gamma)\, d\tilde{\nu}_{(x,g\Gamma)}(g'\Gamma), \tag{10.40}$$

where $\zeta : \hat{\Omega} \to \mathcal{F}$ is a $\hat{\nu}$-measurable map such that for almost all $(x, g\Gamma) \in \hat{\Omega}$, ζ is constant along the fiber $\Delta(x, g\Gamma)$. (In fact, for any $\alpha \in \mathcal{F}$, $\zeta(x, g\Gamma) = \alpha$ if and only if $g\Gamma$ is α-generic for the action of $U^+_{new}(x, g\Gamma)$ on $\Delta(x, g\Gamma)$). Integrating (10.40) over $g\Gamma \in G/\Gamma$ we obtain for almost all $x \in \Omega$,

$$\hat{\nu}_x = \int_{G/\Gamma} \zeta(x, g\Gamma)\, d\hat{\nu}_x(g\Gamma). \tag{10.41}$$

The uniqueness of the ergodic decomposition and the $\hat{T}$ and $\mathcal{U}^+_1$-equivariance of the subgroups $U^+_2(x)$ and $U^+_{new}(\hat{x})$ shows that

$$\zeta(x, g\Gamma) = (T^t_x)_* \zeta(\hat{T}^t(x, g\Gamma)) \tag{10.42}$$

and for $u \in \mathcal{U}^+_1$,

$$\zeta(ux, g\Gamma) = \zeta(x, g\Gamma). \tag{10.43}$$

Let $\hat{\zeta} : \hat{\Omega} \to \Omega \times \mathcal{F}$ be defined by $\hat{\zeta}(x, g\Gamma) = (x, \zeta(x, g\Gamma))$. Then, the push-forward $\hat{\eta} = (\hat{\zeta})_*(\hat{\nu})$ is a $\hat{T}^t$-invariant probability measure on $\Omega \times \mathcal{F}$. Since $\hat{\nu}$ is ergodic, so is $\hat{\eta}$.

By [Ra, Theorem 1.1] the set $\mathcal{G}$ of G-orbits on $\mathcal{F}$ is countable (and therefore countably separated). Let $\hat{\eta}'$ denote the push-forward of $\hat{\eta}$ to $\Omega \times \mathcal{G}$. Since $\hat{\eta}$ is ergodic, so it $\hat{\eta}'$. But the action of $(T^t_x)_*$ on $\mathcal{G}$ is trivial, therefore $\hat{\eta}'$ is supported on the product of Ω and a one-point set. Then, $\hat{\eta}$ is supported on $\Omega \times G\nu_0$, where $\nu_0 \in \mathcal{F}$. Let H denote the stabilizer of ν_0. By the definition of $\mathcal{F}$, ν_0 is supported on a single H-orbit. By the definition of ζ, $g\nu_0$ is $U^+_{new}(x, g\Gamma)$-ergodic. Therefore, the unipotent elements of H act ergodically on ν_0.

We can now write $\zeta(x, g\Gamma) = \theta(x, g\Gamma)\nu_0$, where $\theta : \hat{\Omega} \to G/H$. Then (10.42) and (10.43) hold, with θ in place of ζ.

Let $\sigma_0 : \Omega \to \mathcal{S}^{\mathbb{Z}}$ denote the natural projection. Write $x = (\omega, m)$ where $\omega \in \mathcal{S}^{\mathbb{Z}}$ and $m \in M$, let $\hat{\theta} : \hat{\Omega} \to \mathcal{S}^{\mathbb{Z}} \times M \times G/H$ be defined by $\hat{\theta}(x, g\Gamma) = (\sigma_0(\omega, m), m, \theta(\omega, m, g\Gamma))$, and let $\hat{\lambda}$ denote the pushforward of $\hat{\nu}$ by $\hat{\theta}$. Note that $\hat{\lambda}$ is $\hat{T}$-invariant, projects to $\mu^{\mathbb{Z}}$ and has the $\mathcal{U}^+_1$-invariance property in the sense of §1.4. Therefore, if we disintegrate

$$d\hat{\lambda}(\omega, m, gH) = d\mu^{\mathbb{Z}}(\omega)\, d\lambda_\omega(m, gH),$$

then λ_ω depends only on ω^-. The $\hat{T}$-invariance of $\hat{\lambda}$ translates to the fact that the measure $\lambda \equiv \int_{\mathcal{S}^{\mathbb{Z}}} \lambda_\omega\, d\mu^{\mathbb{Z}}(\omega)$ on $M \times G/H$ is stationary. Finally, integrating (10.41) over $x \in \hat{\Omega}$ we get (1.11) as required. □

For future use, we note that as a consequence of our construction,

$$U^+_{new}(x, g\Gamma) \subset N^+(x) \cap gH^0g^{-1}, \tag{10.44}$$

and for almost all $(x, g\Gamma)$, the $U^+_{new}(x, g\Gamma)$ orbit closure of $g\Gamma$ is given by

$$\overline{U^+_{new}(x, g\Gamma)g\Gamma} = gH^0\Gamma. \tag{10.45}$$

Proof of Theorem 1.13(b). For $\hat{x} = (x, g) \in \Omega$, let $I^+(\hat{x}) \subset N^+(x)$ denote the (right) invariance group of the leafwise measure of $\hat{\nu}$ along $\{x\} \times N^+(x)$. Then, if (1.11) holds, then for almost all $\hat{x}$,

$$I^+(\hat{x}) \supset N^+(x) \cap gH^0g^{-1} \supset U^+_{new}(\hat{x}), \tag{10.46}$$

where for the last inclusion we used (10.44). Then, we can apply the argument labelled "Proof of Theorem 1.14(a)" with $N^+(x)$ in place of $U_2(x)$ and $I^+(\hat{x})$ in place of $U^+_{new}(\hat{x})$ to obtain a Lie subgroup $H' \subset G$ an H'-homogeneous probability measure ν'_0 on G/Γ such that the unipotent elements of H' act ergodically on ν'_0, a finite μ-stationary measure λ' on $M \times G/H'$ such that $\hat{\nu} = \lambda' * \nu'_0$, and also, in view of (10.45), such that for almost all $(x, g\Gamma)$,

$$\overline{I^+(x, g\Gamma)g\Gamma} = g(H')^0\Gamma. \tag{10.47}$$

Then, in view of (10.45), (10.46), and (10.47), we have $H^0 \subset (H')^0$. Then, if we assume that $\dim H^0$ is maximal, we get that $H^0 = (H')^0$, hence the conjugacy class of H^0 is uniquely determined by ν. □

11. Case II

In this section, we will prove Theorem 1.14. Let μ be as in §1 and let M, ϑ be as in §1.3.

11.1. Invariance of the measure.

In §11.1-§11.3 we prove the following, which was proved in a related but different context by Brown and Rodrigues-Hertz in [B-RH, §11.1]:

Proposition 11.1. *Let ν be an ergodic μ-stationary probability measure on $M \times G/\Gamma$, and suppose that Case II holds, (see §1.4). Then ν is $G_{\mathcal{S}}$ invariant with respect to the action of $G_{\mathcal{S}}$ on $M \times G/\Gamma$ given in (1.4).*

Let $\mathcal{F}^+$ denote the σ-algebra whose atoms are $W_1^-[x]$, and let $\hat{\mathcal{F}}^+$ denote the product of $\mathcal{F}^+$ with the Borel σ-algebra $\mathcal{B}(M \times G/\Gamma)$ of $M \times G/\Gamma$ (we will sometimes consider $\mathcal{B}(M \times G/\Gamma)$ also as a σ-algebra of subsets of $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$). Let $\mathcal{Q}$ denote the product of the Borel σ-algebra on $\mathcal{S}^{\mathbb{Z}}$ with the trivial σ-algebra on $M \times G/\Gamma$ and $\mathcal{Q}^+$ denote the product of the σ-algebra $\mathcal{F}^+$ with the trivial σ-algebra on $M \times G/\Gamma$.

In particular, the atoms of $\mathcal{Q}^+$ are of the form $W_1^-[x] \times M \times G/\Gamma$. For a partition α of $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$, let

$$\alpha^+ \equiv \bigvee_{n=0}^{\infty} \hat{T}^{-n}\alpha.$$

The following technical result about existence of finite entropy partitions of $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$ with good properties will be used throught the proof of Proposition 11.1.

Lemma 11.2. *(cf.* [B-RH, Lemma 11.2]*) There exists a finite entropy partition α of $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$ such that each element of α is in $\hat{\mathcal{F}}^+$, and such that for almost all $\hat{x} = (x, m, g\Gamma) \in \mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$, the atom $(\alpha^+ \vee \mathcal{Q}^+)[\hat{x}]$ is contained in $\hat{W}_1^-[\hat{x}]$.*

We defer the proof of this lemma to §11.3.

Lemma 11.3. *Let ν, $\hat{\nu}$ be as in Proposition 11.1, and let α be a finite entropy partition as in Lemma 11.2. Suppose that Case II holds. Then the σ-algebras $(\alpha^+ \vee \mathcal{Q}^+)$ and $\hat{\mathcal{F}}^+$ are equivalent mod $\hat{\nu}$-null sets.*

Recall that two σ-algebra $\mathcal{A}$ and $\mathcal{B}$ are equivalent modulo $\hat{\nu}$-null sets if for every $A \in \mathcal{A}$ there is an $A' \in \mathcal{B}$ so that $\hat{\nu}(A \triangle A') = 0$. We leave the rather straightforward proof of Lemma 11.3 to the reader.

11.2. **An auxiliary construction.** A substantial nuisance in proving Theorem 11.1 is that the entropy of the process defined by $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ may be infinite. To compensate for this fact, we choose the component $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \in \mathcal{S}^{\mathbb{Z}}$ of a point in $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$ in two steps: first we choose for every $j \in \mathbb{Z}$ two candidates for ω_j which we denote by $\omega_j^{(\epsilon)}$ for $\epsilon = 0, 1$, and only then choose randomly which $\omega_j^{(\epsilon)}$ to use at each place independently with equal probability. We now formalize this construction.

Let $\hat{\Omega}'$ denote the space $\mathcal{S}^{\mathbb{Z}} \times M \times G/\Gamma$ and let $\hat{\hat{\Omega}}$ denote the space

$$\hat{\hat{\Omega}} = \mathcal{S}^{\mathbb{Z}} \times \mathcal{S}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}} \times M \times G/\Gamma.$$

Denote an element in $\hat{\hat{\Omega}}$ by $(\omega^{(0)}, \omega^{(1)}, \boldsymbol{\epsilon}, m, g\Gamma)$ where as before we often will write $\omega^{(0)} = (\omega^{(0),-}, \omega^{(0),+})$ etc. By $\omega^{(\boldsymbol{\epsilon})}$ we denote the sequence $(\dots, \omega_{-1}^{(\epsilon_{-1})}, \omega_0^{(\epsilon_0)}, \omega_1^{(\epsilon_1)}, \dots)$; by $\omega^{(\boldsymbol{\epsilon}),-}$ the sequence $(\dots, \omega_{-2}^{(\epsilon_{-2})}, \omega_{-1}^{(\epsilon_{-1})})$. We define a map $\hat{\hat{T}}$ on the space $\hat{\hat{\Omega}}$ similarly to (1.7) by setting

$$\hat{\hat{T}}(\omega^0, \omega^1, \boldsymbol{\epsilon}, m, g\Gamma) = (T\omega^0, T\omega^1, T\boldsymbol{\epsilon}, \omega_0^{\epsilon_0} \cdot m, \vartheta(\omega_0^{\epsilon_0}, m) g\Gamma).$$

Let $\mathcal{Z}$ denotes the σ-algebra of subsets of $\hat{\hat{\Omega}}$ which is a product of the Borel σ-algebra on each $\mathcal{S}^{\mathbb{Z}}$ factor with the trivial σ-algebra on the other factors, and similarly define $\mathcal{E}$ to be the σ-algebra of subsets of $\hat{\hat{\Omega}}$ corresponding to the Borel σ-algebra on $\{0,1\}^{\mathbb{Z}}$ component, i.e. the minimal σ-algebra according to which all $\boldsymbol{\epsilon}_i : i \in \mathbb{Z}$ are measurable functions. Let $\mathcal{E}^+$ denote the minimal σ-algebra according to which all $\boldsymbol{\epsilon}_i : i \geq 0$ are measurable. Finaly, we let $\mathcal{B}(M \times G/\Gamma)$ denote the Borel σ-algebra of $M \times G/\Gamma$ considered also as a σ-algebra of subsets of $\hat{\hat{\Omega}}$.

Let $\mu_{\frac{1}{2},\frac{1}{2}}$ denote the uniform measure on $\{0,1\}$. We define the measure $\hat{\hat{\nu}}$ on $\hat{\hat{\Omega}}$ by

$$d\hat{\hat{\nu}}(\omega^{(0)}, \omega^{(1)}, \boldsymbol{\epsilon}, m, g\Gamma) = d\mu^{\mathbb{Z}}(\omega^{(0)})\, d\mu^{\mathbb{Z}}(\omega^{(1)})\, d\mu^{\mathbb{Z}}_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{\epsilon})\, d\nu_{\omega^{(\boldsymbol{\epsilon})},-}(m, g\Gamma), \tag{11.1}$$

where the measure $\nu_{\omega^{(\boldsymbol{\epsilon})},-}$ is as in (1.9) with $\omega^{(\boldsymbol{\epsilon}),-}$ in place of ω^-. Under the natural map $(\omega^{(0)}, \omega^{(1)}, \boldsymbol{\epsilon}, m, g\Gamma) \mapsto (\omega^{(\boldsymbol{\epsilon})}, m, g\Gamma)$, the measure $\hat{\hat{\nu}}$ is mapped to $\hat{\nu}$, and $\hat{\hat{\nu}}$ is $\hat{\hat{T}}$-invariant.

Proof of Proposition 11.1. Let α be a generating partition for $\hat{T}$ as in Lemma 11.2. Define a partition $\hat{\hat{\alpha}} = \left\{\hat{\hat{A}} : A \in \alpha\right\}$ of $\hat{\hat{\Omega}}$ by setting for every $A \in \alpha$

$$\hat{\hat{A}} = \left\{(\omega^0, \omega^1, \boldsymbol{\epsilon}, m, g\Gamma) \in \hat{\hat{\Omega}} : (\omega^{(\boldsymbol{\epsilon})}, m, g\Gamma) \in A\right\}.$$

Let ε denote the two element partition of $\hat{\hat{\Omega}}$ according to the digit ϵ_0 of a point $(\omega^0, \omega^1, \boldsymbol{\epsilon}, m, g\Gamma) \in \hat{\hat{\Omega}}$. Then the conditions on α given by Lemma 11.2 imply that

$$\bigvee_{n\in\mathbb{Z}} \hat{\hat{T}}^{-n}(\hat{\hat{\alpha}} \vee \varepsilon) \vee \mathcal{Z}$$

is equivalent to the Borel σ-algebra on $\hat{\hat{\Omega}}$ modulo $\hat{\hat{\nu}}$-null sets. It follows form a relativised version of Kolmogorov-Sinai generator theorem [EiLWa, Thm. 2.20] that we may calculate the entropy $h_{\hat{\hat{\nu}}}(\hat{\hat{T}}|\mathcal{Z})$ the following way:

$$h_{\hat{\hat{\nu}}}(\hat{\hat{T}}|\mathcal{Z}) = H_{\hat{\hat{\nu}}}\left(\hat{\hat{T}}(\hat{\hat{\alpha}} \vee \varepsilon)\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\hat{\hat{\alpha}} \vee \varepsilon)\right).$$

The (relative) entropy $h_{\hat{\hat{\nu}}}(\hat{\hat{T}}|\mathcal{Z})$ is at least $\log 2$, since each digit of the sequence $\boldsymbol{\epsilon}$ is chosen i.i.d. with equal probability independently of $\mathcal{Z}$, hence using [EiLWa, Prop. 2.19(1)],

$$h_{\hat{\hat{\nu}}}(\hat{\hat{T}}|\mathcal{Z}) \geq h_{\hat{\hat{\nu}}}(\hat{\hat{T}}, \varepsilon|\mathcal{Z}) = H_{\hat{\hat{\nu}}}\left(\hat{\hat{T}}\varepsilon\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}\varepsilon\right) = \log 2.$$

On the other hand it follows easily from Lemma 11.3 that up to null sets

$$\mathcal{Z} \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\hat{\hat{\alpha}} \vee \varepsilon) = \mathcal{Z} \vee \bigvee_{n=0}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)$$

hence

$$h_{\hat{\nu}}(\hat{\hat{T}}|\mathcal{Z}) = H_{\hat{\nu}}\left(\hat{\hat{T}}(\hat{\hat{\alpha}} \vee \varepsilon)\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\hat{\hat{\alpha}} \vee \varepsilon)\right) \tag{11.2}$$

$$= H_{\hat{\nu}}\left(\hat{\hat{T}}(\hat{\hat{\alpha}} \vee \varepsilon)\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right)$$

$$= H_{\hat{\nu}}\left(\hat{\hat{T}}\varepsilon\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right) +$$

$$H_{\hat{\nu}}\left(\hat{\hat{T}}\hat{\hat{\alpha}}\middle|\mathcal{Z} \vee \bigvee_{n=-1}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right)$$

As $\hat{\hat{T}}\hat{\hat{\alpha}}$ is measurable with respect to the σ-algebra

$$\mathcal{Z} \vee \bigvee_{n=-1}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma),$$

we have that

$$(11.2) = H_{\hat{\nu}}\left(\hat{\hat{T}}\varepsilon\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} (\hat{\hat{T}}^{-n}\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right) \leq \log 2$$

since the cardinality of the partition ε is 2. Since we already know that

$$h_{\hat{\nu}}(\hat{\hat{T}}|\mathcal{Z}) \geq \log 2,$$

the last inequality is in fact an equality.

Since for any k, $\hat{\hat{T}}^k\mathcal{Z}^+$ is a sub-σ-algebra of $\mathcal{Z}$, by the monotonicity properties of conditional entropy (cf. [EiLWa, Prop. 1.7])

$$H_{\hat{\nu}}\left(\hat{\hat{T}}\varepsilon\middle|\hat{\hat{T}}\mathcal{Z}^+ \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right) \tag{11.3}$$

$$\geq H_{\hat{\nu}}\left(\hat{\hat{T}}\varepsilon\middle|\mathcal{Z} \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)\right) = \log 2.$$

Using again the fact that the cardinality of the partition ε is 2, $\log 2$ is an upper bound to the first term of the above displayed equation, hence we have equality throughout.

Let f be measurable with respect to the σ-algebra $\mathcal{Z}^+ \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon) \vee \mathcal{B}(M \times G/\Gamma)$. Writing μ^+ for $\prod_{n=0}^{\infty} \mu$, μ^- for $\prod_{n=-\infty}^{-1} \mu$ and similarly for $\mu_{\frac{1}{2},\frac{1}{2}}$, applying (11.1),

$$\begin{aligned}\int f\, d\hat{\hat{\nu}} &= \int d\mu^+(\omega^{(0),+})\, d\mu^+(\omega^{(1),+})\, d\mu^+_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{\epsilon}^+) \\ &\qquad \int d\mu^-(\omega^{(0),-})\, d\mu^-(\omega^{(1),-})\, d\mu^-_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{\epsilon}^-)\, d\nu_{\omega^{(\boldsymbol{\epsilon}),-}}(m, g\Gamma) \\ &\qquad\qquad f(\omega^{(0),+}, \omega^{(1),+}, \boldsymbol{\epsilon}^+, m, g\Gamma) \\ &= \int d\mu^+(\omega^{(0),+})\, d\mu^+(\omega^{(1),+})\, d\mu^+_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{\epsilon}^+) \\ &\qquad \int d\mu^-(\omega'^-)\, d\nu_{\omega'^-}(m, g\Gamma)\, f(\omega^{(0),+}, \omega^{(1),+}, \boldsymbol{\epsilon}^+, m, g\Gamma) \\ &= \int d\mu^+(\omega^{(0),+})\, d\mu^+(\omega^{(1),+})\, d\mu^+_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{\epsilon}^+)\, d\nu(m, g\Gamma)\, f(\omega^{(0),+}, \omega^{(1),+}, \boldsymbol{\epsilon}^+, m, g\Gamma).\end{aligned}$$

In other words, with respect to the measure $\hat{\hat{\nu}}$, the two σ-algebras $\mathcal{B}(M \times G/\Gamma)$ and $\mathcal{Z}^+ \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon)$ are independent, i.e. for every $A \in \mathcal{B}(M \times G/\Gamma)$ and $A' \in \mathcal{Z}^+ \vee \bigvee_{n=0}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon)$ we have that $\hat{\hat{\nu}}(A \cap A') = \hat{\hat{\nu}}(A)\hat{\hat{\nu}}(A')$. By $\hat{\hat{T}}$-invariance of $\hat{\hat{\nu}}$, we similarly have that $\hat{\hat{T}}\mathcal{B}(M \times G/\Gamma)$ and $\hat{\hat{T}}\mathcal{Z}^+ \vee \bigvee_{n=-1}^{\infty} \hat{\hat{T}}^{-n}(\varepsilon)$ are mutually independent.

Let $x = (\omega^0, \omega^1, \boldsymbol{\epsilon}, m, g\Gamma) \in \hat{\hat{\Omega}}$ be distributed according to $\hat{\hat{\nu}}$, and let $(m', g'\Gamma)$ be the $M \times G/\Gamma$-component of $\hat{\hat{T}}^{-1}x$. By definition of $\hat{\hat{T}}$,

$$(m, g\Gamma) = (\omega_{-1}^{\epsilon_{-1}} \cdot m', \vartheta(\omega_{-1}^{\epsilon_{-1}}, m')g'\Gamma),$$

or in the notation of (1.4), $(m, g\Gamma) = \omega_{-1}^{\epsilon_{-1}}.(m', g')$. Both $(m, g\Gamma)$ and $(m', g'\Gamma)$ are distributed according to pushforward with respect to the projection from $\hat{\hat{\Omega}}$ to $M \times G/\Gamma$ — i.e. the stationary measure ν. Moreover by (11.3) we have that ϵ_{-1} considered as a random variable, is independent of $(\omega^0_{-1}, \omega^1_{-1}, m, g\Gamma)$. This implies that for $\mu \times \mu$-a.e. $\omega^0_{-1}, \omega^1_{-1}$,

$$(\omega^0_{-1})_*\nu = (\omega^1_{-1})_*\nu,$$

that is to say there is a fixed measure ν' so that for μ-a.e. ω_{-1} it holds that $(\omega_{-1})_*\nu = \nu'$. Since ν is a μ-stationary measure, it follows that ν is invariant under the support of μ. □

11.3. **Construction of the finite entropy partition.** In this subsection, we will prove Lemma 11.2. (It is proved under a different set of assumptions in [B-RH, §11.2]).

Before starting the proof proper, we will need to deal with some issues related to zero Lyapunov exponents. First, we recall the following well-known lemma, due to Atkinson [At] and Kesten [Ke].

Lemma 11.4. *Let $T : \Omega \to \Omega$ be a transformation preserving a probability measure β. Let $F : \Omega \to \mathbb{R}$ be an L^1 function. Suppose that for β-a.e. $x \in \Omega$,*

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(T^i x) = +\infty.$$

Then $\int_\Omega F\, d\beta > 0$.

Suppose the cocycle $(T_x^n)_*$ has a zero Lyapunov exponent. Let us denote it's index by j_0, so $\lambda_{j_0} = 0$. Then, $\lambda_{j_0-1} > 0$ and $\lambda_{j_0+1} < 0$. We have, for $x \in \Omega$,

$$\operatorname{Lie}(N^-)(x) = \mathcal{V}_{\geq j_0+1}(x).$$

For $x = (x^-, x^+, m) \in \Omega$, let

$$E(x) = E(x^+, m) = \{\mathbf{v} \in \mathfrak{g} \ : \ \limsup_{n\to\infty} \|(T_x^n)_* \mathbf{v}\|_0 < \infty.\} \tag{11.4}$$

Then, (cf. §6) on a set of full measure, E is a subspace, and for a.e. $x = (x^-, x^+, m) \in \Omega$, there exists $C(x) = C(x^+, m)$ such that for all $\mathbf{v} \in E(x)$ and all $n \in \mathbb{N}$,

$$\|(T_x^n)_* \mathbf{v}\|_0 < C(x)\|\mathbf{v}\|_0. \tag{11.5}$$

Furthermore, E is T-equivariant, and $\operatorname{Lie}(N^-)(x) \subset E(x) \subset \mathcal{V}_{\geq j_0}(x)$. Let $V_0(x) = E(x)/\operatorname{Lie}(N^-)(x)$. For consistency, if the cocycle $(T_x^n)_*$ does not have a zero Lyapunov exponent, we set $E(x) = \operatorname{Lie}(N^-)(x)$ and $V_0(x) = \{0\}$.

Lemma 11.5. *There exists a $\mathcal{F}^+$-measurable map $C_0 : \Omega \to \mathbb{R}^+$ bounded a.e. such that for $x \in \Omega$, all $\mathbf{v} \in V_0(x)$ and all $n \in \mathbb{N}$,*

$$C_0(x)^{-1} C_0(T^n x)^{-1} \|\mathbf{v}\|_0 \leq \|(T_x^n)_* \mathbf{v}\|_0 \leq C_0(x) C_0(T^n x)\|\mathbf{v}\|_0, \tag{11.6}$$

where by the norm $\|\cdot\|_0$ we mean the quotient norm induced by the norm $\|\cdot\|_0$ on $\mathfrak{g}$.

Proof. By Lemma 2.3 (and the definition of Ω) for a.e. $x \in \Omega$ there exists a linear map $M_x : V_0(x) \to \mathbb{R}^k$ such that $A(x, n) \equiv M_{T^n x}(T_x^n)_* M_x^{-1}$ has the form (2.14), with the orthogonal matrices fixing a fixed inner product $\langle\cdot,\cdot\rangle'$ on $\mathbb{R}^k$. Furthermore, in view of Lemma 2.5 and Lemma 2.6 the map $x \to M_x$ can be chosen to be $\mathcal{F}^+$-measurable. (Lemma 2.5 and Lemma 2.6 are stated on unstable manifolds, while here we are dealing with stables. Thus, the maps P^+ in the statements of the these lemmas should become P^-. Also, these lemmas are stated in terms of Lyapunov subspaces $\mathcal{V}_i(x)$, but here we are dealing with the quotient $\mathcal{V}_{\geq j_0}(x)/\mathcal{V}_{\geq j_0+1}(x)$, which contains $V_0(x)$. In the translation, the analogues of the maps P^- become the identity map). There exists a compact set $K \subset \Omega$ with $K \in \mathcal{F}^+$ and of positive measure such that the function $C(\cdot)$ of (11.5) is uniformly bounded on K and also $\max(\|M_x\|, \|M_x\|^{-1})$ is uniformly bounded for $x \in K$, where by $\|M_x\|$ we mean the operator norm relative to the norms $\|\cdot\|_0$ on $V_0(x)$ and the norm $\|\cdot\|'$ induced by the inner product $\langle\cdot,\cdot\rangle'$ on $\mathbb{R}^k$.

As in Lemma 2.3, let $e^{\lambda_{0j}(x,n)}$ denote the scaling factors in the conformal blocks of $A(x,n)$, see (2.14). Then, for each j, $\lambda_{0j} : \Omega \times \mathbb{N} \to \mathbb{R}$ is an additive cocycle.

For $x \in K$, let $n_0(x)$ denote the smallest integer n_0 such that $T^{n_0}x \in K$, and let $F : K \to K$ denote the first return map, so $F(x) = T^{n_0(x)}x$. For $x \in K$, let $\lambda'_{0j}(x,1) = \lambda_{0j}(x, n_0(x))$, and let $\lambda'_{0j}(x,\ell) = \sum_{k=0}^{\ell-1} \lambda'_{0j}(F^k x, 1)$. Then, $\lambda'_{0j} : K \times \mathbb{N} \to \mathbb{R}$ is an additive cocycle over the action of F. Furthermore, since $\lambda'_{0j}(x,\ell) = \lambda_{0j}(F^\ell x, n')$ for some $n' = n'(\ell, x) \in \mathbb{N}$, we have, in view of (11.5) and the definition of K, $\lambda'_{0j}(x,\ell) \le C$ for a uniform constant C, all $x \in K$ and all $\ell \in \mathbb{N}$.

We now claim that there exists a function $C_1 : K \to \mathbb{R}$ finite a.e. such that for all $x \in K$, all j and all $\ell \in \mathbb{N}$,

$$-C_1(x) \le \lambda'_{0j}(x,\ell) \le C. \tag{11.7}$$

The upper bound is true by assumption. Suppose that the lower bound fails, i.e. that for x in some set of positive measure, $\liminf_{\ell\to\infty} \lambda'_{0j}(x,\ell) = -\infty$. Then, by the ergodicity of T, the same holds for almost all x, and furthermore, in view of the upper bound in (11.7) and the cocycle condition, for almost all $x \in K$, $\lim_{\ell\to\infty} \lambda'_{0j}(x,\ell) = -\infty$. Then, by Lemma 11.4, we have $\int_K \lambda'_{0j}(\cdot, 1) < 0$. Therefore, $\int_\Omega \lambda_{0j}(\cdot,1) = \int_K \lambda'_{0j}(\cdot,1) < 0$. Then, by e.g. the subadditive ergodic theorem, for a.e. $x \in \Omega$, $\limsup_{n\to\infty} \frac{1}{n}\lambda_{0j}(x,n) < 0$. But, all the Lyapunov exponents of the action of the cocycle $(T^n_x)_*$ restricted to V_0 are 0, which is a contradiction. Therefore, (11.7) holds. In view of the form (11.7), we may assume without loss of generality that C_1 is $\mathcal{F}^+$-measurable.

In view of (11.7), there exists $K' \subset K$ with $K' \in \mathcal{F}^+$ and of positive measure and a constant $C'_1 > 0$ such that for all $x \in K'$ and all $n \in \mathbb{N}$ such that $T^n x \in K$,

$$|\lambda_{0j}(x,n)| \le C'_1. \tag{11.8}$$

It follows from the definitions of E, V_0 and K that there exists $C_3 > 0$ such that for all $x \in K'$ and all n such that $T^n x \in K$, $\|A(x,n)\|' \le C_3$. Now it follows from the form of (2.14) and (11.8) that $\|A(x,n)^{-1}\|' \le C_4$ for some C_4 depending on C_3, C_1 and the dimension. Then, by the definition of K, we have $\|(T^n_x)_*|_{V_0}^{-1}\|_0 \le C_5$ for all $x \in K'$ and all n such that $T^n x \in K$. Now (11.6) follows by considering the smallest $k > 0$ such that $T^k x \in K'$ and the smallest $n' > n$ such that $T^{n'} x \in K$. ☐

The construction of the partition α uses the following:

Lemma 11.6. *(Mañé) Let E be a compact measurable subset of G/Γ, with $\nu(E) > 0$. If $q : E \to (0,1)$ is such that $\log q$ is ν-integrable, then there exists a partition $\mathcal{P}$ of E with finite entropy such that, if $\mathcal{P}(x)$ denotes the atom of $\mathcal{P}$ containing x, then* $\operatorname{diam}\mathcal{P}(x) < q(x)$.

Proof. See [M1] or [M2, Lemma 13.3] □

Proof of Lemma 11.2. We now construct the desired partition α. For any $\delta > 0$ there exists $\mathcal{C}_0 \in \mathcal{F}^+$ of measure at least $1-\delta$, and $T_0 = T_0(\delta)$ such that for $n > T_0$, and any $x \in \mathcal{C}_0$, we have for all $\mathbf{v} \in \mathfrak{g}$,

$$e^{-\kappa n}\|\mathbf{v}\|_0 \leq \|(T_x^n)_*\mathbf{v}\|_0 \leq e^{\kappa n}\|\mathbf{v}\|_0, \tag{11.9}$$

where κ depends only on the Lyapunov spectrum. Then there exists $\mathcal{C}_1 \subset \mathcal{C}_0$ with $\mathcal{C}_1 \in \mathcal{F}^+$ and of positive measure such that for $x \in \mathcal{C}_1$, $T^n x \notin \mathcal{C}_1$ for $1 \leq n \leq T_0$.

Without loss of generality, we may assume that Γ is torsion free. Choose $r > 0$ sufficiently small so that the set $B_r \equiv \{\exp(\mathbf{v}) \ : \ \mathbf{v} \in \mathfrak{g}, \|\mathbf{v}\|_0 \leq r\}$ satisfies $B_r\gamma \cap B_r = \emptyset$ for $\gamma \in \Gamma$ not the identity.

Let $\mathcal{C}_2 = \mathcal{C}_1 \times B_r \subset \mathcal{S}^{\mathbb{Z}} \times G/\Gamma$. For $\hat{x} \in \mathcal{C}_2$, let $n(\hat{x}) \in \mathbb{N}$ be the first return time to $\mathcal{C}_2$. Recall that $\hat{\nu}$ is a $\hat{T}$-invariant measure on $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$.

By the classical Kac formula,

$$\int_{\mathcal{C}_2} n \, d\hat{\nu} = \hat{\nu}(\mathcal{C}_2) = 1. \tag{11.10}$$

Let $C_0 : \mathcal{S}^{\mathbb{Z}} \to \mathbb{R}^+$ be as in Lemma 11.5. In view of (11.10) we can choose a $\hat{\mathcal{F}}^+$-measurable function $q : \mathcal{C}_2 \to \mathbb{R}^+$ with the following properties:

(i) $q(\hat{x}) \leq \frac{r}{2}e^{-\kappa n(\hat{x})}$.
(ii) $\int_{\mathcal{C}_2} |\log q| d(\mu^{\mathbb{N}} \times \nu) < \infty$.
(iii) The essential infimum of $q(\hat{x})C_0(\hat{x})$ is 0.

By (ii) and Lemma 11.6 there exists a finite entropy $\mathcal{F}^+$-measurable partition α_0 of $\mathcal{C}_2$ such that if $\hat{x} = (x, g\Gamma)$ and $\hat{y} = (y, \exp(\mathbf{v})g\Gamma) \in \alpha_0[\hat{x}]$ then $\mathbf{v} \leq q(\hat{x})$. Let $K_n = \mathcal{C}_2 \cap \{\hat{x} \ : \ n(\hat{x}) = n\}$, and let α denote the partition of a conull subset of $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$ whose atoms are of the form $T^j(F \cap K_n)$ where $0 \leq j \leq n-1$ and F is an atom of α_0. In view of (i) and (11.9), every atom of $\alpha \vee \mathcal{Q}$ is contained in a set of the form $W_1^-[x] \times B_r g\Gamma$ for some $(x, g\Gamma) \in \mathcal{S}^{\mathbb{Z}} \times G/\Gamma$.

We now claim that α satisfies the conditions of Lemma 11.2. Suppose $\hat{x} = (x, g\Gamma)$ and $\hat{y} = (y, g'\Gamma)$ where $x, y \in \mathcal{S}^{\mathbb{Z}}$ are in the same atom of $\alpha^- \vee \mathcal{Q}$. Then, by the definition of $\mathcal{Q}$, $y \in W_1^-[x]$, so that $y^+ = x^+$. Since $\hat{y} \in \alpha[\hat{x}]$, we may write $g'\Gamma = \exp(\mathbf{v})g\Gamma$ where $\|\mathbf{v}\|_0 < r$. The condition $\hat{y} \in \alpha^-[\hat{x}]$ implies that for all $n > 0$, $\hat{T}^n(x, (\exp \mathbf{v})g\Gamma) \subset \alpha[\hat{T}^n(x, g\Gamma)]$, which implies that there exist $\gamma_n \in \Gamma$ such that $(\exp((T_x^n)_*\mathbf{v}))T_x^n g \in B_r T_x^n g\gamma_n$. But, in view of (i) and (11.9), it follows (by induction on n) that we may take $\gamma_n = e$ for all n. Thus, $\|(T_x^n)_*\mathbf{v}\|_0 \leq r$ for all n, and so $\mathbf{v} \in E(x)$, where $E(x)$ is as in (11.4). If there is no zero Lyapunov exponent, we have $E(x) \subset \operatorname{Lie}(N^-)(x)$, and thus $\mathbf{v} \in \operatorname{Lie}(N^-)(x)$.

Suppose there is a zero exponent. Let $\mathbf{w} = \mathbf{v} + \operatorname{Lie}(N^-)(x) \in V_0(x)$. By (iii) and the ergodicity of $\hat{T}$, for a.e. x there exists a sequence $n_k \to \infty$ such that

$C_0(T^{n_k}x)\|(T_x^{n_k})_*\mathbf{w}\|_0 \to 0$. Nowever, in view of Lemma 11.5, we have, for all $n \in \mathbb{N}$, $C_0(T^n x)\|(T_x^n)_*\mathbf{w}\|_0 > C_0(x)^{-1}\|\mathbf{w}\|_0$. Thus, $\mathbf{w} = 0$ and $\mathbf{v} \in \mathrm{Lie}(N^-)(x)$.

Now the lemma follows in view of (1.10). ☐

11.4. **Reduction to the compactly supported case.** Suppose that M and ϑ are trivial in the sense of §1.3. Then the fact that $\hat{\nu} = \mu^{\mathbb{Z}} \times \nu$ and the $\hat{T}$-invariance of $\hat{\nu}$ implies that ν is $G_{\mathcal{S}}$-invariant. This completes the proof of part (a) of Theorem 1.14. We now begin the proof of part (b) of Theorem 1.14.

Thus, in the rest of §11 we will assume that M and ϑ are trivial, and ν is $G_{\mathcal{S}}$-invariant. This implies that without changing ν, we have the freedom to change μ as long as the support of the new measure is contained in the support of the old measure. The next Lemma states that we can exploit this freedom to reduce to a compactly supported setup.

Lemma 11.7. *Let Z be a (possibly trivial) subgroup of G. Suppose μ is a measure on $\mathcal{S}$ with finite first moment satisfying uniform expansion mod Z. Then, there exists a measure μ' supported on a compact subset of $\mathcal{S}$ which also satisfies uniform expansion mod Z.*

Proof. Note that $\mu^{(N)}$ has finite first moment if μ does. Therefore, after replacing μ by $\mu^{(N)}$, we may assume that for all $\mathbf{v} \in V$,

$$\mu(\sigma_{\mathbf{v}}) > C > 0, \qquad \text{where } \sigma_{\mathbf{v}}(g) = \log \frac{\|g\mathbf{v}\|}{\|\mathbf{v}\|}.$$

For $R \in \mathbb{R}^+$, let χ_R denote the characteristic function of the set $\{g \in G \ : \ \|g\| \leq R\}$. Let $\mu_R = \chi_R \mu$. Then, since $\sigma_{\mathbf{v}}(g) \leq \log\|g\|$, for each $\mathbf{v} \in V$,

$$|\mu(\sigma_{\mathbf{v}}) - \mu_R(\sigma_{\mathbf{v}})| \leq \int_{\|g\| \geq R} \log\|g\|\, d\mu(g)$$

and since μ has finite first moment, the right-hand side of the above equation tends to 0 as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \mu_R(\sigma_{\mathbf{v}}) = \mu(\sigma_{\mathbf{v}}),$$

and the convergence is uniform in $\mathbf{v}$. Thus, there exists $R > 0$ such that

$$\mu_R(\sigma_{\mathbf{v}}) > C/2 > 0$$

for all $\mathbf{v} \in V$. Thus after replacing μ by $\frac{1}{\mu_R(G)}\mu_R$, we may assume that μ is compactly supported. ☐

The fiber entropy. Let ξ be a finite measurable partition of G/Γ. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} H_{\nu_{x^-}}\Big(\bigvee_{i=0}^{n-1} (T_x^i)^{-1}\xi\Big) \equiv \lim_{n\to\infty} -\frac{1}{n} \sum_{A \in \bigvee_{i=0}^{n-1}(T_x^i)^{-1}\xi} \nu_{x^-}(A)\log\nu_{x^-}(A)$$

exists and is constant for $\mu^{\mathbb{Z}}$-a.e. x. We denote its value by $h_{\hat{\nu}}^{G/\Gamma}(\hat{T},\xi)$. Then, we define the fiber entropy $h_{\hat{\nu}}^{G/\Gamma}(\hat{T})$ to be the supremum over all finite measurable partitions ξ of $h_{\hat{\nu}}^{G/\Gamma}(\hat{T},\xi)$.

In view of Lemma 11.7, it is enough to prove Theorem 1.14(b) in the case where the measure μ is compactly supported. Hence we may assume the following hold:

Standing assumptions. μ is a measure on G whose support $\mathcal{S}$ is compact. Also μ satisfies uniform expansion mod Z. The measure ν is a $G_{\mathcal{S}}$-invariant probability measure on G/Γ, with $\hat{\nu} = \mu^{\mathbb{Z}} \times \nu$ an *ergodic* $\hat{T}$-invariant measure on $\mathcal{S}^{\mathbb{Z}} \times G/\Gamma$.

Furthermore, we have

$$h_{\mu^{\mathbb{Z}}\times\nu}^{G/\Gamma}(\hat{T}) = 0,$$

(i.e. the fiber entropy is 0). This is a consequence of the assumption that we are in Case II.

11.5. **Dimensions of invariant measures.** Let $Z \subset G$, $V \subset \mathfrak{g}$ and $K_{\mathcal{S}} \subset \mathrm{Aut}(Z)$ be as in Definition 1.6. Choose a right-invariant and $K_{\mathcal{S}}$-invariant metric $d_Z(\cdot,\cdot)$ on Z, and choose a norm $\|\cdot\|$ on V. Fix $\epsilon > 0$. For $g \in G$ and $r > 0$, let

$$B_{/Z}(r,\epsilon) = \{\exp(\mathbf{v})z \in G \ : \ \mathbf{v} \in V,\, z \in Z,\, d_Z(z,e) < \epsilon \text{ and } \|\mathbf{v}\| \le r\}.$$

We define, for $g\Gamma \in G/\Gamma$, the "mod Z lower local dimension"

$$\dim_{/Z}(\nu, g\Gamma) = \lim_{\epsilon\to 0}\left(\liminf_{r\to 0}\frac{\log \nu(B_{/Z}(r,\epsilon)g\Gamma)}{\log r}\right).$$

(The outer limit exists since the quantity in parenthesis is increasing as a function of ϵ). By the ergodicity of $\hat{T}$, for ν-a.e. $g \in G$, $\dim_{/Z}(\nu, g)$ is independent of g. We denote the common value by $\dim_{/Z}(\nu)$.

Proposition 11.8. *Under the assumptions of Theorem 1.14,* $\dim_{/Z}(\nu) = 0$.

Remark. If there are no zero Lyapunov exponents, this follows from [LX] (which is based on [BPS]). We will give a proof of the trivial special case we need below (allowing for zero exponents).

Let

$$B_0(\epsilon) = \{\exp(\mathbf{v})z \ : \ \mathbf{v} \in V,\, z \in Z,\, \|\mathbf{v}\| \le \epsilon \text{ and } d(z,e) \le \epsilon\}.$$

For $x \in \mathcal{S}^{\mathbb{Z}}$ and $n \in \mathbb{N}$ let $B^n(x,\epsilon)$ denote the "Bowen ball" centered at the identity 1 of G, i.e.

$$B^n(x,\epsilon) = \{h \in G \ : \ \text{ for all } 0 \le m \le n,\, (x_m \dots x_0)h(x_m \dots x_0)^{-1} \in B_0(\epsilon)\}.$$

Lemma 11.9. *For any unit* $\mathbf{v} \in V$ *there exists a positive measure set* $K(\mathbf{v}) \subset \mathcal{S}^{\mathbb{Z}}$*, such that for all* $x \in K(\mathbf{v})$ *there exists* $\eta(\mathbf{v}) > 0$ *and* $N(\mathbf{v}) \in \mathbb{N}$ *so that for all* $n > N(\mathbf{v})$ *and for all unit* $\mathbf{w} \in V$ *with* $\|\mathbf{v}-\mathbf{w}\|_0 < \eta(\mathbf{v})$*,*

$$|\{t \ : \ \exp t\mathbf{w} \in B^n(x,\epsilon)\}| \le e^{-\alpha n}, \tag{11.11}$$

where $\alpha > 0$ depends only on the Lyapunov spectrum.

Proof. Let j_1 be as in Lemma 3.8, so that $\lambda_{j_1} > 0$ and $\lambda_{j_1+1} \le 0$. Fix $\epsilon > 0$ smaller than one quarter the difference between any two Lyapunov exponents. For $\delta > 0$, let $K(\delta) \subset \mathcal{S}^{\mathbb{Z}}$ be as in Lemma 2.16. By the multiplicative ergodic theorem, there exists a subset $K_\delta \subset K(\delta)$ with $\mu^{\mathbb{Z}}(K_\delta) > 1 - 2\delta$ and $N(\delta) \in \mathbb{N}$, such that for any $x \in K_\delta$, any $n > N(\delta)$ and any $\mathbf{w}_i \in \mathcal{V}_i(x)$,

$$e^{(\lambda_i-\epsilon)n}\|\mathbf{w}_i\| \le \|(T_x^n)_*\mathbf{w}_i\| \le e^{(\lambda_i+\epsilon)n}\|\mathbf{w}_i\|. \tag{11.12}$$

(Here we are using the dynamical norm of Proposition 2.14). For any $x \in K_\delta$ and $\mathbf{w} \in \mathfrak{g}$ we may write $\mathbf{w} = \sum_{i=1}^m \mathbf{w}_i$, where $\mathbf{w}_i \in \mathcal{V}_i(x)$. By assumption, μ is uniformly expanding on V. Then, by Lemma 3.8(ii)$'$ for all $\mathbf{v} \in V$ there exists a positive measure subset $K(\mathbf{v}) \subset K_\delta$ such that for $x \in K(\mathbf{v})$, $\mathbf{v} \notin \mathcal{V}_{\ge j_1+1}(x)$. Therefore, we can choose $\eta > 0$ small enough such that for all unit $\mathbf{w}$ with $\|\mathbf{v}-\mathbf{w}\|_0 < \eta$, we have $\|\mathbf{w}_i\| \ge C(\mathbf{v})$ for some $1 \le i \le j_1$. Then, by (11.12) and Proposition 2.14(a), for all unit $\mathbf{w} \in V$ with $\|\mathbf{w}-\mathbf{v}\|_0 > \delta$, and for all $n > N(\delta)$, $\|(T_x^n)_*\mathbf{w}\| > C(\mathbf{v})e^{(\lambda_{j_1}-\epsilon)n}$. Note that $\lambda_{j_1} > 0$. Also in view of Lemma 2.16, we have $\|(T_x^n)_*\mathbf{w}\|_0 > C_1(\mathbf{v})e^{(\lambda_{j_1}-2\epsilon)n}$. This implies (11.11). □

We recall the following:

Lemma 11.10. *For $\epsilon > 0$, $\epsilon'' > 0$, $n \in \mathbb{N}$ and $x \in \mathcal{S}^{\mathbb{Z}}$, let $N(n, x, \epsilon, \epsilon'')$ denote the smallest number of Bowen balls $B^n(x, \epsilon)g\Gamma \subset G/\Gamma$ needed to cover a set of ν-measure at least $1 - \epsilon''$. Then, for $\mu^{\mathbb{Z}}$-a.e. $x \in \mathcal{S}^{\mathbb{Z}}$ and any $0 < \epsilon'' < 1$,*

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log N(n, x, \epsilon, \epsilon'') = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N(n, x, \epsilon, \epsilon'') = h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T}).$$

Proof. The analogous formula for the case of a single measure preserving trasformation is due to Katok [Ka, Theorem I.I]. The precise statement we need is given as [Zhu, Theorem 3.1]. □

Corollary 11.11. *Let $N(n, x, \epsilon, \epsilon'')$ be as in Lemma 11.10. Then for any $\epsilon > 0$, any $0 < \epsilon'' < 1$ and $\mu^{\mathbb{Z}}$-a.e. $x \in \mathcal{S}^{\mathbb{Z}}$,*

$$\lim_{n\to\infty}\frac{1}{n}\log N(n, x, \epsilon, \epsilon'') = 0.$$

Proof. In our setting the fiber entropy $h^{G/\Gamma}_{\mu^{\mathbb{Z}}\times\nu}(\hat{T})$ is zero. Now the statement follows immediately from the fact that for fixed n, x, ϵ'', $N(n, x, \epsilon, \epsilon'')$ is decreasing as a function of ϵ. □

Proof of Proposition 11.8. Let $\epsilon > 0$ be arbitrary. By Lemma 11.9 and the compactness of the unit sphere of V, there exist $x^1, \dots, x^M \in \mathcal{S}^{\mathbb{Z}}$ and $\alpha > 0$ such that for any $g_1, \dots, g_M \in G$, such that for n sufficiently large,

$$\bigcap_{m=1}^{M} B^n(x^m, \epsilon) g_m \subset B_{/Z}(e^{-\alpha n}, \epsilon) g' \qquad \text{for some } g' \in G. \tag{11.13}$$

Then, by Corollary 11.11, for every $\epsilon' > 0$ and for all sufficiently large n, for each $1 \le m \le M$, there exists compact $Q_m^{(n)} \subset G/\Gamma$ of measure at least $1 - \epsilon''/M$ such that $Q_m^{(n)}$ can be covered by $e^{\epsilon' n}$ Bowen balls of the form $B^n(x^m, \epsilon) g' \Gamma$. Then, $Q^{(n)} = \bigcap_{m=1}^{M} Q_m^{(n)}$ satisfies $\nu(Q^{(n)}) > 1 - \epsilon''$, and also $Q^{(n)}$ can be covered by at most $e^{M\epsilon' n}$ sets of the form

$$\bigcap_{m=1}^{M} B^n(x^m, \epsilon) g_m \Gamma. \tag{11.14}$$

Recall that we are assuming that Γ is torsion free. We may assume that $\epsilon > 0$ is small enough so that for any $g \in G$ with $B_0(\epsilon) g\Gamma \cap Q^{(n)} \neq \emptyset$, we have $B_0(\epsilon) g\gamma \cap B_0(\epsilon) = \emptyset$ for all $\gamma \neq e$. Then, by (11.13) and (11.14), there exists a finite set $\Delta \subset G/\Gamma$ of cardinality at most $e^{M\epsilon' n}$ such that

$$Q^{(n)} \subset \bigcup_{g'\Gamma \in \Delta} B_{/Z}(e^{-\alpha n}, \epsilon) g' \Gamma.$$

Let

$$\Delta' = \{ g'\Gamma \in \Delta \ : \ \nu(B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma) \le \epsilon |\Delta|^{-1} \}.$$

Then,

$$\nu\left(\bigcup_{g'\Gamma \in \Delta'} B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma \right) \le \sum_{g'\Gamma \in \Delta'} \nu(B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma) \le |\Delta| \left(\epsilon |\Delta|^{-1} \right) = \epsilon.$$

Let $\hat{Q}^{(n)} = \bigcup_{g'\Gamma \in \Delta \smallsetminus \Delta'} B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma$. Then, $\nu(\hat{Q}^{(n)}) \ge (1 - 2\epsilon)$, and each $g\Gamma \in \hat{Q}^{(n)}$ is contained in a set of the form $B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma$ with $\nu(B_{/Z}(e^{-\alpha n}, \epsilon) g'\Gamma) > \epsilon |\Delta|^{-1}$. Therefore, for each $g\Gamma \in \hat{Q}^{(n)}$,

$$\nu(B_{/Z}(3e^{-\alpha n}, 3\epsilon) g\Gamma) \ge \epsilon |\Delta|^{-1} \ge \epsilon \, e^{-M\epsilon' n}.$$

Let Q_∞ denote the set of $g\Gamma \in G/\Gamma$ such that $g\Gamma \in \hat{Q}^{(n)}$ for infinitely many n. Then, $\nu(Q_\infty) \ge 1 - 2\epsilon$ and for each $g\Gamma \in Q_\infty$ there exists a sequence $r_k = 3e^{-\alpha n_k}$ with $r_k \to 0$ such that

$$\nu(B_{/Z}(r_k, 3\epsilon) g\Gamma) \ge \epsilon \, r_k^{(M/\alpha)\epsilon'},$$

i.e.

$$\frac{\log \nu(B_{/Z}(r_k, 3\epsilon) g\Gamma)}{\log r_k} \le (M/\alpha)\epsilon' + \frac{|\log \epsilon|}{|\log r_k|}.$$

Since ϵ and ϵ' are arbitrary and $|\log r_k| \to \infty$ as $r_k \to 0$, this implies $\dim_{/Z}(\nu, g\Gamma) = 0$. $\square$

Recall that μ is a probability measure on G, supported on a compact set $\mathcal{S}$. Let V, Z and $K_{\mathcal{S}}$ be as in Definition 1.6. Choose a right-invariant and $K_{\mathcal{S}}$-invariant metric $d_Z(\cdot,\cdot)$ on Z. Choose N, C and a norm $\|\cdot\|$ on V so that (1.1) holds. Let r be small enough so that for any z with $d(z,e) < r$ the exponential map $V \to G$ sending $\mathbf{v}$ to $\exp(\mathbf{v})z$ restricted to the set $\{\mathbf{v} \in V \;:\; \|\mathbf{v}\| \le r\}$ is a diffeomorphism onto its image.

Suppose $0 < \epsilon < r$. Let $d_\epsilon : G \times G \to \mathbb{R}$ be defined by

$$d_\epsilon(g,g') = \begin{cases} \|\mathbf{v}\| & \text{if } g' = \exp(\mathbf{v})zg,\ z \in Z,\ d_Z(z,e) \le \epsilon,\ \mathbf{v} \in V \text{ and } \|\mathbf{v}\| < \epsilon, \\ \epsilon & \text{otherwise.} \end{cases}$$

Let $\delta > 0$ be a small parameter to be chosen later (idependently of ϵ) and let $f_\epsilon : G \times G \to \mathbb{R}$ be defined by

$$f_\epsilon(g_1,g_2) = \sup_{\gamma\in\Gamma} \left(d_\epsilon(g_1,g_2\gamma)^{-\delta} \right).$$

Then, f_ϵ descends to a function $G/\Gamma \times G/\Gamma \to \mathbb{R}$ which we also denote by f_ϵ. For $n \in \mathbb{N}$, let

$$\mathcal{S}^n = \{s_1 \dots s_n \;:\; s_i \in \mathcal{S}\} \subset G.$$

The proof of Theorem 1.14 is based on the following estimate:

Lemma 11.12. *There exists $n \in \mathbb{N}$ sufficiently large (depending only on μ), and $\delta > 0$ sufficiently small (depending only on n and μ), so that the following holds:*

(a) *There exists $c_{max} = c_{max}(\mu,n,\delta) > 0$ such that for any $0 < \epsilon < r$, for all $g_1, g_2 \in G/\Gamma$, and all g in the support of $\mu^{(n)}$,*

$$f_\epsilon(gg_1,gg_2) \le c_{max} f_\epsilon(g_1,g_2). \tag{11.15}$$

(b) *There exists constant $c_0 < 1$ and $c_- < 1$ depending on μ, n and δ such that for any compact subset $K \subset G/\Gamma$, there exists a constant $\epsilon_0 = \epsilon_0(\mu,K,n,\delta) > 0$ and for each $0 < \epsilon < \epsilon_0$ there exists a constant $b = b(K,\epsilon) = b(K,n,\mu,\epsilon,\delta) > 0$ and a function $c : G \times G/\Gamma \times G/\Gamma \to \mathbb{R}^+$ (depending on μ, n, δ, K) such that for g in the support of $\mu^{(n)}$ and $g_1 \in K$,*

$$f_\epsilon(gg_1,gg_2) \le c(g,g_1,g_2) f_\epsilon(g_1,g_2) + b(K,\epsilon). \tag{11.16}$$

In addition, for all $g_1, g_2 \in G/\Gamma$,

$$\int_G c(g,g_1,g_2)\, d\mu^{(n)}(g) \le c_0 < 1, \tag{11.17}$$

and

$$c(g,g_1,g_2) = c_0 \qquad \text{if } g_1 \notin K. \tag{11.18}$$

We stress that c_{max}, and c_0 do not depend on K and ϵ.

Remark. We cannot use the Margulis function from [BQ1, §6.3] since we are not assuming uniform expansion on all exterior powers.

Proof. In the proof, we will be using repeatedly the following obervation. Suppose $g_2 = \exp(\mathbf{v}) z g_1$, where $\mathbf{v} \in V$ and $z \in Z$, and suppose $g \in G_{\mathcal{S}}$. Then,

$$gg_2 = \exp(\mathbf{v}') z' g g_1, \quad \text{where } \mathbf{v}' = \operatorname{Ad}(g)\mathbf{v} \text{ and } z' \in Z \text{ with } d_Z(z', e) = d_Z(z, e). \tag{11.19}$$

Since the support $\mathcal{S}$ of μ is compact, we can choose $R > 10$ such that for all $\mathbf{v} \in V$ and all $g \in \mathcal{S} \cup \mathcal{S}^{-1}$,

$$\frac{1}{R}\|\mathbf{v}\| \le \|\operatorname{Ad}(g)\mathbf{v}\| \le R\|\mathbf{v}\|. \tag{11.20}$$

It follows that for any $n \in \mathbb{N}$, any g in the support of $\mu^{(n)}$, and any $g_1, g_2 \in G$,

$$R^{-n} d_\epsilon(g_1, g_2) \le d_\epsilon(gg_1, gg_2) \le R^n d_\epsilon(g_1, g_2). \tag{11.21}$$

Note that while it is always true that $f_\epsilon(g_1\Gamma, g_2\Gamma) = d_\epsilon(g_1, g_2\gamma)^{-\delta}$ for some $\gamma \in \Gamma$, it is possible that for some $g \in \mathcal{S}^n$ we have $f_\epsilon(gg_1\Gamma, gg_2\Gamma) = d_\epsilon(gg_1, gg_2\gamma')^{-\delta}$ where $\gamma' \neq \gamma$.

We first prove (11.15), where there is no real issue. Suppose $g \in \mathcal{S}^n$, $g_1, g_2 \in G$. Let $\gamma \in \Gamma$ be such that $f_\epsilon(gg_1\Gamma, gg_2\Gamma) = d_\epsilon(gg_1, gg_2\gamma)^{-\delta}$. Then, for any $g \in \mathcal{S}^n$, in view of (11.21) (applied to $g^{-1} \in (\mathcal{S} \cup \mathcal{S}^{-1})^n$),

$$f_\epsilon(gg_1\Gamma, gg_2\Gamma) = d_\epsilon(gg_1, gg_2\gamma)^{-\delta} \le R^{n\delta} d_\epsilon(g_1, g_2\gamma)^{-\delta} \le R^{n\delta} f_\epsilon(g_1\Gamma, g_2\Gamma).$$

This means that (11.15) holds, with $c_{max} = R^{n\delta}$.

Recall that we are assuming that Γ is torsion free. For $\epsilon > 0$, let $B_Z(\epsilon) = \{z \in Z \;:\; d_Z(z, e) < \epsilon\}$. Let $\epsilon_0 = \epsilon_0(K) > 0$ be such that for all $g \in G$ with $g\Gamma \in K$ and all $\gamma \in \Gamma$ with $\gamma \neq e$, $B_Z(\epsilon_0)g \cap B_Z(\epsilon_0)g\gamma = \emptyset$. Suppose $\epsilon < \epsilon_0$. Then there exists $\rho = \rho(K) > 0$ such that for all $g \in G$ with $g\Gamma \in K$ and all $\gamma \in \Gamma$ with $\gamma \neq e$,

$$B_{/Z}(\rho, \epsilon)g \cap B_{/Z}(\rho, \epsilon)g\gamma = \emptyset.$$

Let $\rho' = \rho/R^n$. Then, for $g_1\Gamma \in K$, $g_2\Gamma \in B_{/Z}(\rho', \epsilon)g_1\Gamma$, if we write $f_\epsilon(g_1\Gamma, g_2\Gamma) = d_\epsilon(g_1, g_2\gamma)^{-\delta}$ then for any g in the support of $\mu^{(n)}$,

$$f_\epsilon(gg_1\Gamma, gg_2\Gamma) = d_\epsilon(gg_1, gg_2\gamma)^{-\delta}. \tag{11.22}$$

We now define the function $c(\cdot, \cdot, \cdot)$. Suppose $g_1, g_2 \in G$, $g_1\Gamma \in K$ and $g_2\Gamma \in B_{/Z}(\rho', \epsilon)g_1\Gamma$. Then, there exists unique $\gamma \in \Gamma$ such that $g_2\gamma \in B_{/Z}(\rho', \epsilon)g_1$. Hence, there exists unique $\mathbf{v} \in V$ with $\|\mathbf{v}\| < \rho'$ and z with $d_Z(z, e) < \epsilon$ such that $g_2\gamma = \exp(\mathbf{v}) z g_1$. For g in the support of $\mu^{(n)}$, we set

$$c(g, g_1\Gamma, g_2\Gamma) = \frac{d_\epsilon(gg_1, gg_2\gamma)^{-\delta}}{d_\epsilon(g_1, g_2\gamma)^{-\delta}} = \left(\frac{\|\operatorname{Ad}(g\mathbf{v})\|}{\|\mathbf{v}\|}\right)^{-\delta}. \tag{11.23}$$

If $g_1\Gamma \notin K$ or $g_2\Gamma \notin B_{/Z}(\rho', \epsilon)g_1\Gamma$, we set $c(g, g_1\Gamma, g_2\Gamma) = c_0$. Now (11.17) follows from (11.23) and [EMar, Lemma 4.2]. ([EMar, Lemma 4.2] is stated for semisimple groups, but in fact only uniform expansion, i.e. the conclusion of [EMar, Lemma 4.1] is used in the proof).

Note that there exists $b' = b'(\rho)$ such that if $g_2\Gamma \notin B_{/Z}(\rho', \epsilon)g_1\Gamma$ then $f_\epsilon(g_1\Gamma, g_2\Gamma) \le b'$. Now (11.16) follows from (11.22), (11.23) and (11.15). □

Proposition 11.13. *Suppose for any $R > 0$ and any $g\Gamma \in G/\Gamma$, $\nu(\{zg\Gamma \;:\; d_Z(z, e) < R\}) = 0$. Then, for any $\eta > 0$ there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ and $K'' \subset G/\Gamma$ with $\nu(K'') > 1 - c(\eta)$ where $c(\eta) \to 0$ as $\eta \to 0$ and for any $0 < \epsilon < \epsilon_0$ a constant $C = C(\eta, \epsilon)$ such that for any $g\Gamma \in K''$,*

$$\int_{G/\Gamma} f_\epsilon(g\Gamma, g'\Gamma)\, d\nu(g'\Gamma) < C. \tag{11.24}$$

Proof. Suppose $\eta > 0$ is abitrary, and fix K such that $\nu(K) > 1 - \eta$. We will always assume that $\eta > 0$ is small enough so that

$$(1 - \eta^{1/2}) \log c_0 + \eta^{1/2} \log c_{max} < 0.5 \log c_0 < 0, \tag{11.25}$$

where c_0 and c_{max} are as in Lemma 11.12.

Let $\hat{\Omega} = \mathcal{S}^{\mathbb{Z}} \times G/\Gamma$, $\tilde{\Omega} = \mathcal{S}^{\mathbb{Z}} \times G/\Gamma \times G/\Gamma$. Let $\hat{T} : \hat{\Omega} \to \hat{\Omega}$ denote the skew product map where the fiber is G/Γ, and let $\tilde{T} : \tilde{\Omega} \to \tilde{\Omega}$ denote the skew product where the fiber is $G/\Gamma \times G/\Gamma$. Let $\hat{\nu}$ denote the $\hat{T}$-invariant measure $\mu^{\mathbb{Z}} \times \nu$, and let $\tilde{\nu}$ denote the $\tilde{T}$-invariant measure $\mu^{\mathbb{Z}} \times \nu \times \nu$. We are not assuming that $\tilde{\nu}$ is $\tilde{T}$-ergodic. Let $\mathcal{U}_1^+$ be as in §1. Then, $\mathcal{U}_1^+$ acts on $\hat{\Omega}$ and on $\tilde{\Omega}$ by

$$\sigma \cdot (\omega, g\Gamma) = (\sigma\omega, g\Gamma) \text{ and } \sigma \cdot (\omega, g\Gamma, g'\Gamma) = (\sigma\omega, g\Gamma, g'\Gamma).$$

For each $\hat{\omega} \in \hat{\Omega}$ the orbit $\mathcal{U}_1^+\hat{\omega}$ is isomorphic to a half-infinite Bernoulli shift. We denote the Bernoulli measure on $\mathcal{U}_1^+\hat{\omega}$ by $\mu^{\mathbb{N}}$. We also adapt the same notation for $\tilde{\omega} \in \tilde{\Omega}$.

Let $\hat{K} = \mathcal{S}^{\mathbb{Z}} \times K$. Then $\hat{\nu}(\hat{K}) \ge 1 - \eta$. Let $\psi : \hat{\Omega} \to \mathbb{R}$ denote the characteristic function of $\hat{K}^c$, and for $\hat{\omega} \in \hat{\Omega}$ let

$$\psi^*(\hat{\omega}) = \sup_{j > 1} \frac{1}{j} \sum_{i=1}^{j} \psi(\hat{T}^{-in}(\hat{\omega})).$$

Then, by the maximal ergodic theorem, for any $\lambda \in \mathbb{R}$,

$$\lambda \hat{\nu}(\{\hat{\omega} \in \hat{\Omega} \;:\; \psi^*(\hat{\omega}) > \lambda\}) \le \int_{\psi^* > \lambda} \psi \, d\hat{\nu} \le \int_{\hat{\Omega}} \psi \le \eta. \tag{11.26}$$

Choose $\lambda = \eta^{1/2}$, and let $\Upsilon = \{\hat{\omega} \in \hat{\Omega} \;:\; \psi^*(\hat{\omega}) \le \eta^{1/2}\}$. Then, in view of (11.26),

$$\hat{\nu}(\Upsilon) \ge 1 - \eta^{1/2}.$$

Write $x = (\dots, x_{-1}, x_0, x_1, \dots)$. Set $c : \tilde{\Omega} \to \mathbb{R}$ by

$$c(x, g_1\Gamma, g_2\Gamma) = c(x_0, g_1\Gamma, g_2\Gamma),$$

and set

$$\bar{c}(x, g_1\Gamma, g_2\Gamma) = \begin{cases} c(x_0, g_1\Gamma, g_2\Gamma) & \text{if } g_1\Gamma \in K \\ c_{max} & \text{if } g_1\Gamma \in K^c. \end{cases} \tag{11.27}$$

Let $\Delta_0(\tilde{\omega}) = \bar{\Delta}(\tilde{\omega}) = 1$ and for any $j \in \mathbb{N}$ and $\tilde{\omega} \in \tilde{\Omega}$, let

$$\Delta_j(\tilde{\omega}) = \prod_{i=1}^{j} c(\tilde{T}^{in}(\tilde{\omega})), \quad \bar{\Delta}_j(\tilde{\omega}) = \prod_{i=1}^{j} \bar{c}(\tilde{T}^{in}(\tilde{\omega})), \tag{11.28}$$

Fix $k \in \mathbb{N}$. By repeatedly applying Lemma 11.12 we get, for all $u \in \mathcal{U}_1^+$,

$$f_\epsilon(\tilde{T}^{nk}(u\tilde{x})) \le \bar{\Delta}_k(u\tilde{x}) f_\epsilon(\tilde{x}) + b(K, \epsilon) \sum_{j=1}^{k} \bar{\Delta}_{k-j}(\tilde{T}^{jn}(u\tilde{x})). \tag{11.29}$$

Let χ_Υ denote the characteristic function of $\Upsilon \subset \hat{\Omega}$. We may consider χ_Υ to be a function from $\tilde{\Omega} \to \mathbb{R}$ (by ignoring the third component). Let $\tilde{f}_\epsilon : \tilde{\Omega} \to \mathbb{R}$ be defined by

$$\tilde{f}_\epsilon(x, g\Gamma, g'\Gamma) = \chi_\Upsilon(x, g\Gamma) f_\epsilon(g\Gamma, g'\Gamma). \tag{11.30}$$

Let

$$\xi(j, k, u, \hat{x}) = \sum_{i=1}^{k-j} \psi(\hat{T}^{-in}(\hat{T}^{kn}(u\hat{x}))),$$

i.e. $\xi(j, k, u, \hat{x})$ is the number of $j \le s < k$ such that $\hat{T}^{sn}(u\hat{x}) \in K^c$. By (11.27), (11.28), and (11.18), for any $0 \le j \le k-1$,

$$\bar{\Delta}_{k-j}(\tilde{T}^{jn}(u\tilde{x})) \le (c_{max}/c_0)^{\xi(j,k,u,\hat{x})} \Delta_{k-j}(\tilde{T}^{jn}(u\tilde{x})), \tag{11.31}$$

Suppose $T^{kn}(ux) \in \Upsilon$. Then, by (11.30) and the definition of Υ, for all j, $\xi(j, k, u, \hat{x}) < \eta^{1/2}(k-j)$, and thus, by (11.29) and (11.31), for all $u \in \mathcal{U}_1^+$,

$$\begin{aligned} \tilde{f}_\epsilon(\tilde{T}^{nk}(u\tilde{x})) \le (c_{max}/c_0)^{\eta^{1/2}k} \Delta_k(u\tilde{x}) f_\epsilon(\tilde{x}) + \\ + b(K, \epsilon) \sum_{j=1}^{k} (c_{max}/c_0)^{\eta^{1/2}(k-j)} \Delta_{k-j}(T^{jn}(u\tilde{x})). \end{aligned} \tag{11.32}$$

By (11.17), and (11.28), for any $0 \le j < k$,

$$\int_{\mathcal{U}_1^+ \tilde{x}} \Delta_{k-j}(\tilde{T}^{jn}(u\tilde{x}))\, d\mu^{\mathbb{N}}(ux) \le c_0^{(k-j)}.$$

Substituting into (11.32), we get for any $\tilde{x} \in \tilde{\Omega}$, using (11.25),

$$\int_{\mathcal{U}_1^+\tilde{x}} \tilde{f}_\epsilon(\tilde{T}^{kn}(\tilde{y}))\, d\mu^{\mathbb{N}}(\tilde{y}) \leq f_\epsilon(\tilde{x}) c_{max}^{\eta^{1/2}k} c_0^{(1-\eta^{1/2})k} + b(K,\epsilon) \sum_{j=1}^{k} c_{max}^{\eta^{1/2}(k-j)} c_0^{(1-\eta^{1/2})(k-j)} \leq$$

$$\leq c_0^{k/2} f_\epsilon(\tilde{x}) + b(K,\epsilon) \sum_{j=1}^{k} c_0^{(k-j)/2} \leq c_0^{k/2} f_\epsilon(\tilde{x}) + b(K,\epsilon)/(1 - c_0^{1/2}).$$

Therefore, for all $\tilde{x} \in \tilde{\Omega}$ with $f_\epsilon(\tilde{x}) < \infty$,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_{\mathcal{U}_1^+\tilde{x}} \tilde{f}_\epsilon(\tilde{T}^{kn}(\tilde{y}))\, d\mu^{\mathbb{N}}(\tilde{y}) \leq b(K,\epsilon)/(1 - c_0^{1/2}). \tag{11.33}$$

By our assumption, $\tilde{\nu}(\{\tilde{x} \in \tilde{\Omega} \;:\; f_\epsilon(\tilde{x}) = \infty\}) = 0$. By the ergodic theorem (applied to the non-necessarily ergodic transformation $\tilde{T}$), there exists a function $\phi : \tilde{\Omega} \to \mathbb{R} \cup \{\infty\}$ such that for almost all $\tilde{x} \in \tilde{\Omega}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \tilde{f}_\epsilon(\tilde{T}^{kn}(\tilde{x})) = \phi(\tilde{x}) \tag{11.34}$$

and

$$\int_{\tilde{\Omega}} \phi\, d\tilde{\nu} = \int_{\tilde{\Omega}} \tilde{f}_\epsilon\, d\tilde{\nu}. \tag{11.35}$$

Then, by (11.33), (11.34) and Fatou's lemma, for almost all $\tilde{x} \in \tilde{\Omega}$,

$$\int_{\mathcal{U}_1^+\tilde{x}} \phi(\tilde{y})\, d\mu^{\mathbb{N}}(\tilde{y}) \leq b(K,\epsilon)/(1 - c_0^{1/2}).$$

Integrating over $\tilde{x}$ and using (11.35), we get

$$\int_{\tilde{\Omega}} \tilde{f}_\epsilon(\tilde{\omega})\, d\tilde{\nu}(\tilde{\omega}) \leq b(K,\epsilon)/(1 - c_0^{1/2}) < \infty. \tag{11.36}$$

Let

$$K' = \{g\Gamma \in G/\Gamma \;:\; \mu^{\mathbb{Z}}(\{x \in \mathcal{S}^{\mathbb{Z}} \;:\; (x, g\Gamma) \in \Upsilon\}) > 1/2.\}.$$

Then, by Fubini's theorem, $\nu(K') \geq (1 - 2\eta^{1/2})$, and thus $\nu(K') \to 1$ as $\eta \to 0$. Then, in view of (11.30) and (11.36),

$$\int_{K'} \left(\int_{G/\Gamma} f_\epsilon(g\Gamma, g'\Gamma)\, d\nu(g'\Gamma) \right) d\nu(g\Gamma) \leq 2b(K,\epsilon)/(1 - c_0^{1/2}).$$

Therefore, there exists $K'' \subset K'$ with $\nu(K'') \geq \nu(K') - \eta$ such that for $g\Gamma \in K''$,

$$\int_{G/\Gamma} f_\epsilon(g\Gamma, g'\Gamma)\, d\nu(g'\Gamma) \leq 2\eta^{-1} b(K,\epsilon)/(1 - c_0^{1/2}).$$

This completes the proof of Proposition 11.13. $\square$

Proof of Theorem 1.14. For $r > 0$, let

$$B_Z(r) = \{z \in Z \ : \ d_Z(z, e) < r\}.$$

We first show that for some $R > 0$ and some $g\Gamma \in G/\Gamma$, $\nu(B_Z(R)g\Gamma) > 0$. Suppose not. Choose $\eta > 0$ and suppose $0 < \epsilon < \epsilon_0(\eta)$, where ϵ_0 is as in Proposition 11.13. Let K'', C be as in Proposition 11.13. Then it follows from (11.24) that for all $r > 0$ and all $g\Gamma \in K''$,

$$\nu(B_{/Z}(r, \epsilon)g\Gamma) \leq C(\eta, \epsilon) r^{\delta},$$

hence

$$\frac{\log \nu(B_{/Z}(r, \epsilon)g\Gamma)}{\log r} \geq \delta - \frac{|\log C(\eta, \epsilon)|}{|\log r|}.$$

This implies $\dim_{/Z}(\nu, g\Gamma) \geq \delta$, contradicting Proposition 11.8.

Thus, there exists $R > 0$ and $g\Gamma \in G/\Gamma$ such that

$$\phi_R(g\Gamma) \equiv \nu(B_Z(R)g\Gamma) > 0.$$

Then by the ergodicity of $\hat{T}$, there exists $\epsilon_1 > 0$ and a set $\Psi \subset G/\Gamma$ of full ν-measure such that $\phi_R(g\Gamma) = \epsilon_1$ for all $g\Gamma \in \Psi$. We now pick $g_1\Gamma \in \Psi$, and then inductively pick $g_k\Gamma \in \Psi$ such that

$$g_k\Gamma \notin \bigcup_{i=1}^{k-1} B_Z(2R)g_i\Gamma.$$

Then, since the sets $B_Z(R)g_i\Gamma$, $1 \leq i \leq k$ are pairwise disjoint, for any k,

$$1 = \nu(G/\Gamma) \geq \nu(\bigcup_{i=1}^{k} B_Z(R)g_i\Gamma) = \sum_{i=1}^{k} \nu(B_Z(R)g_i\Gamma) = k\epsilon_1.$$

Thus, $k \leq \epsilon_1^{-1}$ and so the process must stop after finitely many steps. This shows that for some $k \leq \epsilon_1^{-1}$,

$$\Psi \subset \bigcup_{i=1}^{k} B_Z(2R)g_i\Gamma,$$

and thus ν is supported on finitely many compact pieces of Z-orbits. □

12. Proof of Theorem 1.2.

Suppose $\mathcal{G}$ is a Zariski-connected algebraic group generated by unipotents over $\mathbb{C}$. Then, $\mathcal{G}$ has no non-trivial characters, and thus the radical of $\mathcal{G}$ is equal to its unipotent radical $R_u(\mathcal{G})$.

Let $\pi_{ss} : \mathcal{G} \to \mathcal{G}/R_u(\mathcal{G})$ denote the quotient map. Thus, if $\mathcal{G}$ is a Zariski-connected algebraic group generated by unipotents over $\mathbb{C}$, then the quotient $\pi_{ss}(\mathcal{G})$ is semisimple.

We recall the following well known result of Furstenberg ([F, Theorem 8.6]):

Proposition 12.1. *Suppose $\mathcal{G}$ is a semisimple algebraic group with no compact factors, and V is a $\mathcal{G}$-vector space with no fixed $\mathcal{G}$-vectors. Let μ be any probability measure on $\mathcal{G}$ such that the group generated by the support of μ is Zariski dense in $\mathcal{G}$. Then there exists $\lambda > 0$ such that for any $\mathbf{v} \in V \smallsetminus \{0\}$ and $\mu^{\mathbb{N}}$-a.e. $\omega \in \mathcal{G}^{\mathbb{N}}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{\|\omega_n \dots \omega_0 \mathbf{v}\|}{\|\mathbf{v}\|} \geq \lambda > 0.$$

Lemma 12.2. *Suppose $\mathcal{G}$ is an algebraic group generated by unipotents over $\mathbb{C}$ and V is a $\mathcal{G}$-vector space. Let μ be any probability measure on $\mathcal{G}$ such that the group generated by the support of μ is Zariski dense in $\mathcal{G}$. Then μ satisfies a type of weak bounceback condition. More precisely, let*

$$\mathbf{F}_V = \{\mathbf{v} \in V \;\;:\;\; \text{for a.e. } x \in \mathcal{S}^{\mathbb{Z}},\ \limsup_{n\to\infty} \frac{1}{n} \log \|(T_x^n) \cdot \mathbf{v}\| \leq 0\}$$

and for $x \in \mathcal{S}^{\mathbb{Z}}$, let

$$N_V^-(x) = \{\mathbf{v} \in V \;\;:\;\; \limsup_{n\to\infty} \frac{1}{n} \log \|(T_x^n) \cdot \mathbf{v}\| < 0\}.$$

Then, for a.e. $x \in \Omega$, $\mathbf{F}_V \cap N_V^-(x) = \{0\}$.

Proof. It is easy to see that $\mathbf{F}_V$ is $\mathcal{G}$-invariant. Let $\rho_F : \mathcal{G} \to GL(\mathbf{F}_V)$ denote the restriction homomorphism. Then $\rho_F(\mathcal{G})$ is also generated by unipotents over $\mathbb{C}$.

In view of Proposition 12.1, $\pi_{ss}(\rho_F(\mathcal{G}))$ is compact. Therefore, $\rho_F(\mathcal{G})$ is compact by unipotent, and thus $\mathbf{F}_V \cap N_V^-(x) = \{0\}$ for a.e. $x \in \mathcal{S}^{\mathbb{Z}}$. □

Lemma 12.3. *Suppose $\mathcal{G}$ is an algebraic group generated by unipotents over $\mathbb{C}$ and W is a $\mathcal{G}$-vector space. Let μ be any probability measure on $\mathcal{G}$ such that the group generated by the support of μ is Zariski dense in $\mathcal{G}$. Suppose θ is an ergodic μ-stationary probability measure on $W \smallsetminus \{0\}$. Then, θ is $\mathcal{G}$-invariant.*

Furthermore, if θ is not supported on a proper $\mathcal{G}$-invariant subspace of W, the $\mathcal{G}$-action on W factors through a compact subgroup M' of $GL(W)$, and θ is supported on a single orbit of M'.

Proof. Without loss of generality, we may assume that θ is not supported on a proper $\mathcal{G}$-invariant subspace of W (or else we replace W by that subspace). Since W is a $\mathcal{G}$-vector space, there is a linear representation $\rho : \mathcal{G} \to GL(W)$. Since the radical of $\mathcal{G}$ is the unipotent radical, we have $\rho(\mathcal{G}) \in SL(W)$. Then, by (the proof of) [F, Theorem 1.2], $\rho(\mathcal{G})$ is compact. Since the Furstenberg-Poisson boundary of a compact group is trivial (e.g. by Louiville's theorem), θ is $\rho(\mathcal{G})$-invariant. Then, by ergodicity, θ is supported on a single $\mathcal{G}$-orbit. □

Proof of Theorem 1.2. Let ν be an ergodic μ-stationary measure on G'/Γ'. If Case II holds, then by Theorem 1.14 (with ϑ trivial) we get that ν is $G_{\mathcal{S}}$-invariant as required. Therefore, we may assume that Case I holds. The measure μ satisfies the bounceback condition by Lemma 12.2. Thus, Theorem 1.13 applies (with ϑ trivial). Let the subgroup $H \subset G'$ and the measures λ on G'/H and ν_0 on G'/Γ' be as in Theorem 1.13. We choose H so that $\dim(H)$ is maximal.

Let ρ_H be as in §1, and let $L \subset G'$ denote the stabilizer of ρ_H in G'. Then, L is an H-envelope. The measure λ on G'/H projects to a μ-stationary measure $\tilde{\lambda}$ on G'/L. We may identify G'/L with the orbit of ρ_H in $\bigwedge^{\dim H}(\mathfrak{g}')$. Thus, we may think of $\tilde{\lambda}$ as a μ-stationary measure on $\bigwedge^{\dim H}(\mathfrak{g}')$. By ergodicity, we may assume that $\tilde{\lambda}$ is supported on a single $\overline{G}^Z_{\mathcal{S}}$-orbit. After possibly replacing H and L by a conjugate and making the corresponding change to ν_0, we may assume that $\tilde{\lambda}$ is supported on $\overline{G}^Z_{\mathcal{S}} L \subset G'/L$, and thus λ is supported on $\overline{G}^Z_{\mathcal{S}} L/H \subset G'/H$. By Lemma 12.3, $\tilde{\lambda}$ is $\overline{G}^Z_{\mathcal{S}}$-invariant.

If $L = H^0$ then ν is $\overline{G}^Z_{\mathcal{S}}$-invariant, which implies the statement of Theorem 1.2. Thus, we may assume that H^0 is a proper normal subgroup of L.

The group $\pi_{ss}(\overline{G}^Z_{\mathcal{S}})$ is semisimple. Let M'' denote the product of the compact factors of $\pi_{ss}(\overline{G}^Z_{\mathcal{S}})$. Let $G^+ \subset \overline{G}^Z_{\mathcal{S}}$ denote the subgroup of $\overline{G}^Z_{\mathcal{S}}$ generated by unipotent elements. Then G^+ is a normal subgroup of $\overline{G}^Z_{\mathcal{S}}$, and the quotient $M' \equiv \overline{G}^Z_{\mathcal{S}}/G^+$ is compact. Then, M' is a quotient of M''. Let $\pi_{M'} : \overline{G}^Z_{\mathcal{S}} \to M'$ denote the natural map.

By assumption, there exists a compact set $\mathcal{K}$ such that $\overline{G}^Z_{\mathcal{S}} \cdot \rho_H \subset \mathcal{K} \cdot \rho_H$. This implies that ρ_H is fixed by any unipotent element of $\overline{G}^Z_{\mathcal{S}}$. Since G^+ is generated by the unipotent elements of $\overline{G}^Z_{\mathcal{S}}$, this implies that $G^+ \cdot \rho_H = \rho_H$, i.e. $G^+ \subset L$.

Let W denote the smallest $\overline{G}^Z_{\mathcal{S}}$-invariant subpace of $\bigwedge^{\dim H}(\mathfrak{g}')$ containing the $\overline{G}^Z_{\mathcal{S}}$ orbit of ρ_H. Then, since G^+ is normal and stabilizes ρ_H, G^+ fixes every point of W. Thus, $M' = \overline{G}^Z_{\mathcal{S}}/G^+$ acts on W. Let M_0 be the stabilizer of $\rho_H \in M'$. Then, $\tilde{\lambda}$ is supported on a single orbit $M' \cdot \rho_H$ of M' and thus we can think of $\tilde{\lambda}$ as a measure on $M \equiv M'/M_0$. Let $\pi_M : \overline{G}^Z_{\mathcal{S}} L \to M$ denote the natural map, i.e. $\pi_M(g') = m' M_0 \in M$ where $m' \in M'$ is such that $g' \cdot \rho_H = m' \cdot \rho_H$. (Note that neither the domain nor the codomain of π_M is a group). Then, for all $g_1, g_2 \in \overline{G}^Z_{\mathcal{S}} L$ with $\pi_M(g_1) = \pi_M(g_2)$, we have $g_2 \in g_1 L$.

Let $G_0^+ = \overline{G}^Z_{\mathcal{S}} \cap L = \{g \in \overline{G}^Z_{\mathcal{S}} \;:\; \pi_{M'}(g) \in M_0\}$ denote the stabilizer of ρ_H.

Choose a bounded measurable map $s : M \to \pi_{ss}^{-1}(M'') \subset \overline{G}^Z_{\mathcal{S}}$ with $\pi_M \circ s : M \to M$ the identity map. We say that $s : M \to \pi_{ss}^{-1}(M'')$ is a section.

We now define a cocycle $\bar{\vartheta} : \overline{G}^Z_{\mathcal{S}} \times M \to G_0^+ \subset L$. Note that for $g \in \overline{G}^Z_{\mathcal{S}}$ and $m \in M$,

$$\pi_M(gs(m)) = \pi_M(s(g \cdot m)).$$

We set, for $g \in \overline{G}^Z_{\mathcal{S}}$ and $m \in M$,

$$gs(m) = s(g \cdot m)\bar{\vartheta}(g, m),$$

so that $\bar{\vartheta}(g,m) \in L$. Also, since g, $s(m)$ and $s(g\cdot m)$ all belong to $\overline{G}_{\mathcal{S}}^{Z}$, we have $\bar{\vartheta}(g,m) \in \overline{G}_{\mathcal{S}}^{Z}$. Thus, $\bar{\vartheta}(g,m) \in \overline{G}_{\mathcal{S}}^{Z} \cap L = G_0^{+}$.

Also, since $s(m) \in \pi_{ss}^{-1}(M'')$ and $s(gm) \in \pi_{ss}^{-1}(M'')$, and $\pi_{ss}^{-1}(M'')$ is normal in $\overline{G}_{\mathcal{S}}^{Z}$, we have

$$\bar{\vartheta}(g,m) \in g\pi_{ss}^{-1}(M''). \tag{12.1}$$

We now claim that $\bar{\vartheta} : G \times M \to L$ is a cocycle. Indeed,

$$g_1 g_2 s(m) = g_1(g_2 s(m)) = g_1 s(g_2 m)\bar{\vartheta}(g_2,m) = s(g_1 g_2 m)\bar{\vartheta}(g_1, g_2 m)\bar{\vartheta}(g_2,m),$$

so that

$$\bar{\vartheta}(g_1 g_2, m) = \bar{\vartheta}(g_1, g_2 m)\bar{\vartheta}(g_2, m),$$

as required. Thus, $\overline{G}_{\mathcal{S}}^{Z}$ acts on $M \times L$ by

$$g' \cdot (m,g) = (g'\cdot m, \bar{\vartheta}(m,g')g). \tag{12.2}$$

Now since L is an H-envelope, $L \cap \Gamma'$ is discrete in L, and H^0 is a normal Lie sugroup of L with $H^0 \cap \Gamma'$ discrete in H^0. Let $G = L/H^0$, let $\pi_{H^0} : L \to G$ denote the quotient map, and let $\Gamma = \pi_{H^0}(\Gamma' \cap L) \subset G$. Then, Γ is a discrete subgroup of G. Let $\vartheta = \bar{\vartheta} \circ \pi_{H^0}$, so $\vartheta : \overline{G}_{\mathcal{S}} \to G$ is a cocycle. Then, $\overline{G}_{\mathcal{S}}^{Z}$ acts on $M \times G/\Gamma$ by

$$g' \cdot (m, g\Gamma) = (g' \cdot m, \vartheta(g',m)g\Gamma). \tag{12.3}$$

We now claim that the action (12.3) satisfies the weak bounceback condition (3.13). Let

$$\{0\} \subset V_1 \subset V_2 \cdots \subset V_k = \mathfrak{g}$$

denote a maximal invariant flag for the action of the subgroup $G_0^{+}/H \subset L/H$ on $\mathfrak{g} = \mathrm{Lie}(L/H)$. Suppose $x \in \Omega$ and we have $\mathbf{v} \in \mathbf{F}_{\geq j_1+1}(x) \cap \mathrm{Lie}(N^-)(x)$ with $\mathbf{v} \neq 0$. (Recall that if we write $x = (\omega, m)$ where $\omega \in \mathcal{S}^{\mathbb{Z}}$ and $m \in M$, then $\mathbf{F}_{\geq j_1+1}(x)$ depends on x only via the coordinate m). Let i be maximal so that $\mathbf{v} \in V_i$ (so in particular $\mathbf{v} \notin V_{i-1}$). Let $V = V_i/V_{i-1}$, and let $\mathbf{w} \equiv \mathbf{v} + V_{i-1} \in V$. Note that G_0^{+}/H acts on V_i/V_{i-1} by some semisimple quotient G_i. If G_i is compact, then $\mathbf{v} \notin \mathrm{Lie}(N^-)(x)$, which is a contradiction. Therefore, we may assume that G_i is not compact. Note that G_i is also a semisimple quotient of G_0^{+} and $\overline{G}_{\mathcal{S}}^{Z}$, and let $\pi_i : \overline{G}_{\mathcal{S}}^{Z} \to G_i$ denote the quotient map. Then, in view of (12.1), for any $g \in \overline{G}_{\mathcal{S}}^{Z}$ and any $m \in M$,

$$\pi_i(\vartheta(g,m)) = \pi_i(g).$$

Thus on V we have an i.i.d. random walk with measure $\pi_i(\mu)$, and thus by Proposition 12.1 and the fact that $\mathbf{v} \in \mathbf{F}_{\geq j_1+1}(x)$ we have $\mathbf{w} = \{0\}$. This implies $\mathbf{v} \in V_{i-1}$ which is a contradiction. Thus the claim is proved.

Suppose $g' \in \overline{G}_{\mathcal{S}}^{Z} L \subset G'$. We may write

$$g' = s(\pi_M(g'))g, \qquad \text{for some } g \in G'. \tag{12.4}$$

Since $\pi_M(g') = \pi_M(s(\pi_M(g')))$, we have $g \in L$. Let $\bar{f} : \overline{G}^Z_{\mathcal{S}} L \to M \times L$ be defined by $\bar{f}(g') = (\pi_M(g'), g)$, where g as in (12.4). We now claim that for $g' \in \overline{G}^Z_{\mathcal{S}}$ and $g \in \overline{G}^Z_{\mathcal{S}} L$,

$$\bar{f}(g'g) = g' \cdot \bar{f}(g),$$

where the action on the left is that of (12.2). Indeed, $\pi_M(g'g) = g' \cdot \pi_M(g)$, and

$$s(\pi_M(g'g))^{-1} g'g = s(g' \cdot \pi_M(g)))^{-1} g' s(\pi_M(g)) s(\pi_M(g))^{-1} g = \bar{\vartheta}(g', \pi_M(g)) s(\pi_M(g))^{-1} g$$

as required.

Note that $\bar{f}$ descends to a function $f : \overline{G}^Z_{\mathcal{S}} L/H \to M \times G/\Gamma$. We have, for $g' \in \overline{G}^Z_{\mathcal{S}}$ and $g \in \overline{G}^Z_{\mathcal{S}}/H$,

$$f(g'g) = g' \cdot f(g), \tag{12.5}$$

where the action on the right hand side is that of (12.3).

Let $\theta = f_*(\lambda)$. Then, in view of (12.5), θ is μ-stationary, where the action on $(M, G/\Gamma)$ is given by (12.3). If Case I holds for θ, then by Theorem 1.13 there exists $H' \subset G$ with $\dim H' > 0$ and we have $\theta = \lambda' * \nu_0'$ where ν_0' is an H'-homogeneous probability measure on G/Γ, with the unipotent elements of H' acting ergodically on ν_0', and λ' is a stationary measure on $M \times G/H'$. Then, since $\lambda = f_*^{-1}(\theta)$ we have

$$\lambda = f_*^{-1}(\theta) = f_*^{-1}(\lambda' * \nu_0') = f_*^{-1}(\lambda') * \nu_0' = \lambda'' * \nu_0'$$

where λ'' is a μ-stationary probability measure on $G'/H'H$. Thus, we may write $\nu = \lambda'' * (\nu_0' * \nu_0)$ where $\nu_0' * \nu_0$ is a $H'H$-homogeneous probability measure with the unipotent elements of $H'H$ acting ergodically on $\nu_0' * \nu_0$. Since $\dim(H'H) > \dim(H)$, this contradicts the maximality of $\dim(H)$.

Thus, we may assume that case II holds for θ. Let $\hat{\theta}$ be as in (1.9) (with θ in place of ν). Then, by Theorem 1.14, $\hat{\theta}$ is a product of the Bernoulli measure $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ and the measure θ on $M \times G/\Gamma$.

Let $\phi : \mathcal{S}^{\mathbb{Z}} \times G/\Gamma \to S^{\mathbb{Z}} \times \overline{G}^Z_{\mathcal{S}} L/H$ be given by $\phi(\omega, m, g\Gamma) = (\omega, f^{-1}(m, g\Gamma))$. Then, $\phi_*(\hat{\theta})$ is a $\hat{T}$-invariant measure on $S^{\mathbb{Z}} \times G'/H$ which is a product of the Bernoulli measure $\mu^{\mathbb{Z}}$ on $\mathcal{S}^{\mathbb{Z}}$ and the measure $\lambda = f_*^{-1}(\theta)$ on G'/H. Then, the $\hat{T}$-invariance of $\mu^{\mathbb{Z}} \times \lambda$ translates into the $G_{\mathcal{S}}$-invariance of λ. Since $\nu = \lambda * \nu_0$ this implies that ν is $G_{\mathcal{S}}$-invariant. □

Proof of Theorem 1.3. By assumption, $\overline{G}^Z_{\mathcal{S}}$ is semisimple with no compact factors. Then μ satisfies uniform expansion mod Z where Z is the centralizer of $\overline{G}^Z_{\mathcal{S}}$ in G' (see e.g. [EMar, Lemma 4.1]). Let ν be an ergodic μ-stationary measure on G'/Γ'. Therefore, we can apply Theorem 1.7. If (b) of Theorem 1.7 holds, then by ergodicity, ν is finitely supported (and thus homegeneous). Therefore we may assume that (a) of Theorem 1.7 holds.

Let the subgroup $H \subset G'$ and the measures λ on G'/H and ν_0 on G'/Γ' be as in Theorem 1.7. We choose H so that $\dim(H^0)$ is maximal.

Let ρ_H be as in §1, and let $L \subset G'$ denote the stabilizer of ρ_H in G'. Then, L is an H-envelope. The measure λ on G'/H projects to a μ-stationary measure $\tilde{\lambda}$ on G'/L. We may identify G'/L with the orbit of ρ_H in $\bigwedge^{\dim H}(\mathfrak{g}')$. Let W denote the linear span of this orbit. Thus, we may think of $\tilde{\lambda}$ as a μ-stationary measure on the vector space W. By Lemma 12.3, the action of $\overline{G}_{\mathcal{S}}^Z$ on W factors through a compact quotient of $\overline{G}_{\mathcal{S}}^Z$. By assumption, $\overline{G}_{\mathcal{S}}^Z$ has no compact quotients, and thus $\overline{G}_{\mathcal{S}}^Z \subset L$.

If $L = H^0$ then ν is homogenous, and we are done. Thus we may assume that H^0 is a proper subgroup of L. Let $G = L/H^0$ and let $\Gamma = H/H^0$.

We now apply Theorem 1.7 to the μ-stationary measure λ on $G/\Gamma \cong L/H$. Note that we still have uniform expansion mod Z' where Z' is the centralizer of $\overline{G}_{\mathcal{S}}^Z$ in L. Thus, as above, if (b) of Theorem 1.7 holds, then λ is finitely supported, which implies that ν is homogeneous. Thus, we may assume that (a) of Theorem 1.7 holds. But then, $\lambda = \lambda' * \nu_0'$, where ν_0' is invariant by a subgroup $H' \subset G$ with $\dim H' > 0$. But then $\nu = \lambda' * (\nu_0' * \nu_0)$ and $\nu_0' * \nu_0$ is $(H')^0 H$-invariant, contradicting the maximality of H. □

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Index of Notation

Department of Mathematics, University of Chicago, Chicago, Illinois 60637, USA,
Email address: eskin@math.uchicago.edu

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem Israel 9190401,
Email address: elon@math.huji.ac.il