MORSE INDEX OF MULTIPLICITY ONE ALLEN-CAHN MINIMAL HYPERSURFACES

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Abstract. In this paper, we characterize the Morse index of nondegenerate minimal hypersurfaces $\Sigma^{n-1} \subset (M^n, g)$ that are limit interfaces of solutions to the Allen-Cahn equation, under the multiplicity one assumption. Our result combined with the existence theory of the first author [9] and of the first author with Gaspar [8], and with the proof of the Multiplicity One Conjecture for $n = 3$ by Chodosh and Mantoulidis [6], establishes, in dimension three and for the Allen-Cahn setting, a Morse theory for the area functional proposed by the last two authors in a series of papers ([18], [21], [23], [29]). This predicts the existence for generic metrics of a sequence $\{\Sigma_k\}$ of minimal hypersurfaces with $\mathrm{index}(\Sigma_k) = k$ for every $k \in \mathbb{N}$ and $\mathrm{area}(\Sigma_k) \sim k^{\frac{1}{n}}$ as $k \to \infty$.

1. Introduction

Minimal surfaces are ubiquitous in Geometry but they are very hard to find. Despite this, Yau conjectured ([44], 1982) that every compact three-dimensional manifold contains infinitely many smooth, closed, immersed, minimal surfaces. The best result until very recently was due to Almgren ([2], 1965) and Pitts ([32], 1981), who developed a deep min-max theory for the area functional and proved that every compact Riemannian manifold (M^n, g), with $3 \leq n \leq 7$, contains at least one smooth, embedded, closed minimal hypersurface. If $n \geq 8$, the minimal hypersurface can be singular in a codimension 7 set by Schoen-Simon ([33], 1981) regularity theory.

Few years ago ([22]), the last two authors introduced the idea of using multiparameter sweepouts of mod two flat cycles into the problem. Since the space of mod 2 cycles is weakly homotopically equivalent to $\mathbb{RP}^\infty$, this allowed the use of topological techniques inspired by Lusternik-Schnirelmann theory. They were able to prove in [22] the existence of infinitely many closed, embedded, minimal hypersurfaces in manifolds of positive Ricci curvature. More recently, with Irie [12], the last two authors settled Yau's Conjecture in the generic case. They proved that for generic metrics on M^n, $3 \leq n \leq 7$, the union of all smooth, embedded, closed minimal hypersurfaces is dense in M. This was an application of the Weyl Law for

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the Volume Spectrum ([16]), which also leads to the existence of an equidistributed sequence of closed minimal hypersurfaces ([25]).

There is yet another approach to the generic case of Yau's Conjecture that has been suggested by the last two authors in a series of papers ([18], [21], [23], [29]). This concerns a program to obtain a Morse-theoretic description of the set of minimal hypersurfaces. The last two authors conjectured that for a generic metric on M^n, $3 \leq n \leq 7$, there exists a sequence $\{\Sigma_k\}$ of two-sided minimal hypersurfaces with $\mathrm{index}(\Sigma_k) = k$ for every $k \in \mathbb{N}$. This is called the Morse Index Conjecture in [24].

The last two authors proposed a program to prove this conjecture based on three main components: the use of min-max constructions over multi-parameter sweepouts to obtain existence results, the characterization of the Morse index of min-max minimal hypersurfaces under the multiplicity one assumption, and a proof of the Multiplicity One Conjecture. The Multiplicity One Conjecture states that for a generic metric any component of a min-max minimal hypersurface is two-sided and has multiplicity one. Precise statements for these conjectures can be found in [24] (see also [21]).

The Morse Index Conjecture follows if one can implement the three parts of the program, no matter if this is done in the Almgren-Pitts setting or in the Allen-Cahn setting that is explained below. The first part of the program has been done both in the Almgren-Pitts setting ([22]) and in the Allen-Cahn context ([8]). In our paper, we will accomplish the second part of the program for Allen-Cahn minimal hypersurfaces. The second part in the Almgren-Pitts setting is finished in [24]. Finally, the third part (a proof of the Multiplicity One Conjecture) has been achieved for $n = 3$ in exciting work of Chodosh and Mantoulidis [6] for the Allen-Cahn context. Hence the program has been completed in dimension three for the Allen-Cahn regularization, which leads to a proof of the Morse Index Conjecture for closed three-manifolds.

The possible presence of integer multiplicities is a key difficulty in the theory. Another difficulty special to the Almgren-Pitts min-max theory comes from the fact that the notions of convergence are very weak. The theory is based on Geometric Measure Theory and there is no Hilbert space structure or Palais-Smale condition to check.

This makes the Allen-Cahn ε-regularization of the theory, the subject of our paper, a very natural tool in the investigation of these conjectures. The area functional is approximated by a one-parameter Palais-Smale family of energies E_ε defined on $W^{1,2}(M)$, and minimal hypersurfaces are obtained as limit interfaces of the level sets of solutions in the singular limit as $\varepsilon \to 0$. The connection of the Allen-Cahn energies with minimal surfaces goes back to Modica ([27], 1987) and Sternberg ([37], 1988), but it was in the PhD thesis of the first author ([9]) that a min-max theory in Riemannian manifolds was first developed.

In [9], the first author constructed a family of mountain-pass critical points u_ε in a general compact Riemannian manifold (M^n, g), and proved that their level sets accumulate, as $\varepsilon \to 0$, at a closed minimal hypersurface Σ that is smooth and embedded outside a set of Hausdorff dimension $n-8$. This gives an independent PDE-based proof of the aforementioned existence theorem of Almgren-Pitts-Schoen-Simon (based on the regularity works of [11], [38], [39], [43]). The reverse direction, namely the construction of solutions to the Allen-Cahn equation that accumulate at a minimal hypersurface that is given a priori, was done by Pacard and Ritoré [31] (see also [30]) in the nondegenerate case.

In [8], with Gaspar, the first author went further and developed a min-max theory for the family of energies E_ε over multiparameter sweepouts of functions in $W^{1,2}(M)$. This can be seen as an ε-regularization of the min-max theory for the area functional. Similarly as in the case of mod two flat cycles, there is an $\mathbb{RP}^\infty$ structure coming from the fact that $E_\varepsilon(u) = E_\varepsilon(-u)$ for any $\varepsilon > 0$ and $u \in W^{1,2}(M)$.

The goal of our paper is to prove, in the Allen-Cahn setting and under the multiplicity one assumption, the Morse index characterization of minimal hypersurfaces conjectured by the last two authors. We do this by showing that the Allen-Cahn solution coincides, for small $\varepsilon > 0$, with the solution constructed by Pacard and Ritoré [31] by perturbation methods. This reduces the Morse Index Conjecture to obtaining a proof of the Multiplicity One Conjecture in the Allen-Cahn setting.

1.1. **Statements.**

The Allen-Cahn equations ([3]) on a compact Riemannian manifold have the form

$$\varepsilon \Delta_g u - \frac{W'(u)}{\varepsilon} = 0, \tag{1}$$

where $\varepsilon > 0$ and W is a double-well potential with unique global minima at ± 1. The typical example is $W(u) = \frac{1}{4}(1-u^2)^2$.

Solutions to this equation are critical points of the energy

$$E_\varepsilon(u) = \int_M \varepsilon \frac{|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}.$$

The Morse index of a solution u is defined to be the index of the second variation quadratic form $E''_\varepsilon(u)(\cdot,\cdot)$ in the Sobolev space $W^{1,2}(M)$. If W is an even function, we have that $E_\varepsilon(u) = E_\varepsilon(-u)$.

Given a sequence of critical points $\{u_{\varepsilon_i}\}$ of E_{ε_i}, with $\varepsilon_i \to 0$, we set $w_{\varepsilon_i} = \Psi \circ u_{\varepsilon_i}$ with $\Psi(t) = \int_0^t \sqrt{W(s)/2}\,ds$. We define V_{ε_i}, the *associated varifold* to u_{ε_i}, by

$$V_{\varepsilon_i}(A) = \frac{1}{\sigma}\int_{-\infty}^{\infty} |\{w_{\varepsilon_i} = t\}|(A)dt,$$

for every Borel set A contained in the $(n-1)$-dimensional Grassmannian $G(M)$. Here $\sigma = \int_{-1}^{1} \sqrt{W(s)/2}ds$ and $|S|$ denotes the $(n-1)$-varifold associated with the $(n-1)$-rectifiable set S. By the coarea formula, one has

$$||V_{\varepsilon_i}||(A) = \frac{1}{\sigma}\int_A |\nabla w_{\varepsilon_i}| = \frac{1}{\sigma}\int_A \sqrt{W(s)/2}\cdot|\nabla u_{\varepsilon_i}|.$$

Our Main Theorem is:

Theorem 1.1. *Let $\{u_{\varepsilon_i}\}$ be a sequence of solutions to the Allen-Cahn equation, with $\varepsilon_i \to 0$, on a closed Riemannian manifold (M^n, g). Suppose that the sequence of associated varifolds $\{V_{\varepsilon_i}\}$ converges to a smooth, two-sided, closed, embedded, non-degenerate minimal hypersurface Γ, with multiplicity one. Then*

$$\mathrm{index}(\Gamma) = \mathrm{index}(u_{\varepsilon_i})$$

for all sufficiently large i.

Moreover, for all sufficiently large i, u_{ε_i} is one of the solutions constructed by Pacard and Ritoré in [31].

A Riemannian metric g is called *bumpy* if every smooth, closed, immersed minimal hypersurface is nondegenerate (i.e. it admits no non-trivial Jacobi fields). White showed in [41, 42] that bumpy metrics are generic in the usual C^∞ Baire sense. By Smith [35] (Theorem 1.1), for a generic metric on M there is a countable set $Y \subset \mathbb{R}$ such that if $\varepsilon \notin Y$, every critical point of E_ε is nondegenerate. Since the intersection of residual sets is residual, we have that for a generic metric on M both White and Smith properties hold true.

For such a metric g and for every $k \in \mathbb{N}$, work of the first author with Gaspar [8] (Theorem 3.3) gives a sequence $\{u_{\varepsilon_i}\}$ of solutions to the Allen-Cahn equation with $\varepsilon_i \to 0$, $|u_{\varepsilon_i}| \le 1$, $E_{\varepsilon_i}(u_{\varepsilon_i}) \le C$ and $\mathrm{index}(u_{\varepsilon_i}) = k$. Theorem A in [9] implies that, if $3 \le n \le 7$, the associated varifolds V_{ε_i} converge to a smooth, closed, embedded minimal hypersurface with integer multiplicities. If the Multiplicity One Conjecture is true in the Allen-Cahn setting, our Theorem 1.1 implies the Morse Index Conjecture.

The Multiplicity One Conjecture for $n = 3$ in the Allen-Cahn setting has been recently announced in remarkable work of Chodosh and Mantoulidis [6]. They prove curvature estimates and strong sheet separation estimates for stable solutions, building on important work of Wang and Wei [40]. Combined with the existence theory of [8] (in particular, the sublinear growth of the area proven in Theorem 3.2 of [8]), this gives a new proof of Yau's Conjecture for generic metrics in dimension three. They prove in [6]:

Theorem 1.2. *(Chodosh-Mantoulidis) Let g be a bumpy metric on a closed three-dimensional manifold M^3. Suppose $\Sigma^2 \subset (M^3, g)$ is a smooth, embedded, closed minimal surface with integer multiplicities that is obtained as the limit of varifolds associated to a sequence of solutions $\{u_{\varepsilon_i}\}$ to the Allen-Cahn equation* (1) *with uniformly bounded index and energy, as $\varepsilon_i \to 0$. Then Σ is two-sided and has multiplicity one.*

By combining our Main Theorem 1.1, Theorem 1.2 of Chodosh-Mantoulidis ([6]), the existence theory of the first author [9] and of the first author with Gaspar [8], we obtain:

Theorem 1.3. *The Morse Index Conjecture is true when $n = 3$, i.e. for a generic metric on M^3 there exists, for every $k \in \mathbb{N}$, a smooth, closed, embedded minimal surface $\Sigma_k \subset M^3$ with*

$$\text{index}(\Sigma_k) = k.$$

1.2. **Remark:** As observed above, Theorem 3.2 of [8] together with Theorem 1.2 also gives:

$$C^{-1}k^{\frac{1}{3}} \leq \text{area}(\Sigma_k) \leq Ck^{\frac{1}{3}}$$

for some $C > 0$. In fact, inspired by the Weyl Law for the Volume Spectrum proven in [16] for the Almgren-Pitts setting, we expect that

$$\lim_{k\to\infty} \frac{\text{area}(\Sigma_k)}{k^{\frac{1}{3}}} = b(3)\text{vol}(M)^{\frac{2}{3}},$$

where $b(3) > 0$ is a universal constant.

1.3. **Remark:** A limiting argument implies that Theorem 1.3 is true for all bumpy metrics. A bumpy metric g can be approximated by a sequence of metrics g_i that satisfy Theorem 1.3. If $\Sigma_{k,i}$ is a minimal surface for g_i with $\text{index}(\Sigma_{k,i}) = k$ and $\text{area}(\Sigma_{k,i}) \leq Ck^{\frac{1}{3}}$, Sharp's Compactness Theorem [34] implies that $\Sigma_{k,i}$, after passing to a subsequence, converges with integer multiplicities to some minimal surface Σ_k of g. Since g is bumpy, Σ_k is nondegenerate and multiplicities have to be equal to one. Therefore the convergence is smooth and it is not difficult to see then that $\text{index}(\Sigma_k) = k$.

Index upper bounds for Allen-Cahn minimal hypersurfaces with no multiplicity assumption have been obtained in Gaspar [7] and Hiesmayr [10]. In the Almgren-Pitts setting, this was done by the last two authors in [21]. Index upper bounds for min-max minimal surfaces played a crucial role in the solution by the last two authors of the Willmore Conjecture [20] (see also [1]). Other studies of the Morse index can be found in [4], [5], [13], [14], [15], [19], [26], [36], [45], [46].

It would be interesting to find an analogue of Theorem 1.3 in dimension two. If $n = 2$, one obtains geodesic networks (not closed geodesics) as limit interfaces. The Allen-Cahn regularization could be used to define the Morse index of these stationary configurations. The one-parameter case has been studied in [17].

1.4. **Organization.** In Section **??** we describe in detail the solutions constructed by Pacard and Ritoré [31]. We follow the presentation in [**?**] and we also estimate the error terms in the expansion of the stability operator of (1) to one order higher than what it was performed in [31, **?**]. Then in Section **??** we use the previous estimates to show that if one solution is close to the Pacard-Ritoré solution with order $\varepsilon^{1+\alpha}$ it is actually close with order

$\varepsilon^{2+\alpha}$. The idea consist in applying the stability operator to the difference of the solutions in order to obtain the extra gain. In Section **??** we show that for the solutions constructed by Pacard and Ritoré, the Morse index of the solution and the minimal surface agree. A similar calculation was made in [**?**, Section 9]. In Section **??** we summarize some of the main results of Wang [**?**] and Wang-Wei [**?**] concerning solutions whose varifold converges to a minimal surface with multiplicity one. Finally, in Section **??** we prove the main theorem by showing that solutions whose varifold converge to a minimal surface with multiplicity one must be $\varepsilon^{2+\alpha}$ close to the solutions constructed by Pacard-Ritoré and thus identical to that solution.

2. Notation and results used

Theorem 2.1. *There is $0<\alpha_0<1$ and $C>0$ so that for all $\eta\in C^\infty_c(\Omega_1)$ and $0<\alpha<1$ we have that*

$$\mathcal{P}(\eta):=\Pi(\mathcal{L}_\varepsilon(\eta\circ D_\xi)\circ D_\xi^{-1})-\varepsilon^2 J(\Pi(\eta))$$

satisfies

$$|\mathcal{P}(\eta)|_{C^{0,\alpha}(\Gamma)}=o(\varepsilon^2)|\Pi(\eta)|_{C^{2,\alpha}(\Gamma)}+O(\varepsilon^{\delta^*-\alpha})||\eta^\perp||_{C^{2,\alpha}_\varepsilon(M)}.$$

Furthermore

$$||\eta^\perp||_{C^{2,\alpha}_\varepsilon(M)}\leq C||\mathcal{L}_\varepsilon(\eta\circ D_\xi)\circ D_\xi^{-1}||_{C^{0,\alpha}_\varepsilon(M)}+O(\varepsilon^{1+\delta^*-\alpha})|\Pi(\eta)|_{C^{2,\alpha}(\Gamma)}.$$

Lemma 2.2. *Given $c_0>0,\tau>0$, there is $\varepsilon_1>0, c_1>0$ so that if Ω is a region of M with smooth closed boundary $\partial\Omega$, $\phi\in C^2(\Omega)$ such that*

$$\varepsilon^2\Delta_g\phi-c(x)\phi=0\quad \textit{where}\quad c(x)\geq c_0 \textit{ for all } x\in\Omega,$$

$0<\varepsilon\leq\varepsilon_1$, and the distance function d_∂ to the boundary $\partial\Omega$ is smooth in a 2τ-tubular neighborhood of $\partial\Omega$, then

$$|\phi(x)|\leq|\phi|_{L^\infty(\partial\Omega)}\max\{e^{-c_1 d_\partial(x)/\varepsilon},e^{-c_1\tau/\varepsilon}\}\quad \textit{for all } x\in\Omega.$$

Proof. The proof is standard and consists in using a barriers of the form $e^{-c\beta d_\partial(x)/\varepsilon}$, where $\beta(t)=t$ for $0\leq t<\tau$ and constant for $t>2\tau$. □

3. Index of the Pacard-Ritoré solution

Recall that u_ε denotes the solution constructed by Pacard-Ritoré in [31] whose zero set accumulates on a non-degenerate two-sided minimal hypersurface Γ. In what follows, $d_\Gamma(x)$ denotes the distance of Γ to x and Λ_r denotes all points in M whose distance to Γ is bigger than r. Recall that $\mathcal{L}_\varepsilon$ denotes the linearization operator of the Allen-Cahn equation (1) at u_ε.

The goal is to prove

Theorem 3.1. *For all ε sufficiently small, the Morse index of u_ε coincides with the Morse index of the minimal hypersurface Γ.*

Moreover, there is $a=a(\Gamma)$ so that for all ε sufficiently small $\mathcal{L}_\varepsilon$ has no eigenvalues in $(-a\varepsilon^2,a\varepsilon^2)$.

We first derive several auxiliary results. Similar computations appeared in [**?**, Section 9].

From the fact that $W''(s) \geq -1$ for all s we have that all eigenvalues of $\mathcal{L}_\varepsilon$ are greater or equal than -1.

Lemma 3.2. *Consider $\phi \in C^\infty(M)$ such that $\mathcal{L}_\varepsilon \phi = -\sigma_\varepsilon \phi$, where $|\sigma_\varepsilon| \leq 1/2$, and normalized so that $||\phi||_{L^\infty(M)} = 1$. There is $\eta \in C^\infty_c(\Omega_3)$ such that $||\eta - \phi||_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon^{\mathbb{N}})$ and, for all ε small (independently of ϕ),*

$$\int_M \eta^2 dg \geq 1/2 \int_M \phi^2 dg.$$

Proof. Schauder estimates imply that $||\phi||_{C^{2,\alpha}_\varepsilon(M)}$ is bounded uniformly for all ε.

There is $t_0 >$ so that for all ε sufficiently small we have that $|u_\varepsilon| \geq 2/\sqrt{3}$ on $\Lambda_{\varepsilon t_0}$ and so $W''(u_\varepsilon) - \sigma_\varepsilon \geq 1/2$ on $\Lambda_{\varepsilon t_0}$. Thus we can apply Lemma 2.2 to ϕ on $\Lambda_{\varepsilon t_0}$ and conclude that, for some constant c_1,

$$|\phi(x)| \leq e^{c_1 t_0} \max\{e^{-c_1 d_\Gamma(x)/\varepsilon}, e^{-c_1 \tau/\varepsilon}\} \quad \text{for all } x \in \Lambda_{\varepsilon t_0}.$$

Thus Schauder theory says that $||\phi||_{C^{2,\alpha}_\varepsilon(M\setminus\Omega_5)} = O(\varepsilon^{\mathbb{N}})$. Moreover for all ε uniformly small we have that the maximum of ϕ is attained in Ω_5 and so, from $||\phi||_{C^{2,\alpha}_\varepsilon(M)} = O(1)$, we deduce that for all ε uniformly small (independently of ϕ)

$$\int_{\Omega_5} \phi^2 dg \geq 1/2 \int_M \phi^2 dg \geq O(\varepsilon^n).$$

The result follows from setting $\eta = \chi_3 \phi$. $\square$

The next proposition estimates the lowest eigenvalue of $\mathcal{L}_\varepsilon$.

Proposition 3.3. *There is a constant $d > 0$ such that, for all ε sufficiently small, all eigenvalues of $\mathcal{L}_\varepsilon$ are greater or equal than $-d\varepsilon^2$.*

Proof. Let ϕ be such that $\mathcal{L}_\varepsilon \phi = -\lambda_\varepsilon \phi$, where λ_ε is the first eigenvalue of $\mathcal{L}_\varepsilon$. We assume without loss of generality that $\lambda_\varepsilon \leq 0$ and that $||\phi||_{L^\infty(M)} = 1$.

Considering η given by the previous lemma, it suffices to find $d > 0$ (independent of η) so that for all ε uniformly small (independent of η) we have

$$-\int_M (\mathcal{L}_\varepsilon \eta)\eta dg \geq -d\varepsilon^2 \int_M \eta^2 dg.$$

From $||u_\varepsilon - \omega_\varepsilon \circ D_\xi||_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon^2)$ there is $d_1 > 0$ so that

$$-\int_M (\mathcal{L}_\varepsilon \eta)\eta dg \geq \int_M \varepsilon^2 |\nabla \eta|^2 + W''(\omega_\varepsilon \circ D_\xi)\eta^2 dg - d_1 \varepsilon^2 \int_M \eta^2 dg.$$

The support of η is contained in Ω_3 and so we can see it as being defined in $\Gamma \times \mathbb{R}$. Using Fermi coordinates we have that on Ω_3,

$$D_\xi(x,z) = (x, z - \xi(x)), \quad 2\sqrt{\det g_0} \geq \sqrt{\det g_z} \geq 1/2\sqrt{\det g_0},$$

and, from the minimality of Γ,

$$P(x,z) := \sqrt{\det g_z} - \sqrt{\det g_0} = O(|z|^2)\sqrt{\det g_z}.$$

Setting $\beta(x,t) = \eta \circ D_\xi^{-1}(x,t)$, we have

$$\begin{aligned}
&\int_M \varepsilon^2|\nabla\eta|^2 + W''(\omega_\varepsilon \circ D_\xi)\eta^2 dg \geq \int_{\Gamma\times\mathbb{R}} \varepsilon^2(\partial_z\eta)^2 + W''(\omega_\varepsilon \circ D_\xi)\eta^2 dg_z dz\\
&\geq \int_{\Gamma\times\mathbb{R}} \varepsilon^2(\partial_z\eta)^2 + W''(\omega_\varepsilon \circ D_\xi)\eta^2 dg_0 dz\\
&+ \int_{\Gamma\times\mathbb{R}} (\varepsilon^2(\partial_z\eta)^2 + W''(\omega_\varepsilon \circ D_\xi)\eta^2)O(|z|^2) dg_0 dz\\
&= \int_\Gamma\int_\mathbb{R} \varepsilon^2(\partial_t\beta)^2 + W''(\omega_\varepsilon)\beta^2 dt dg_0\\
&+ \int_\Gamma\int_\mathbb{R} \varepsilon^2(\partial_t\beta)^2 O(|t+\xi|^2) dt dg_0 + \int_\Gamma\int_\mathbb{R} W''(\omega_\varepsilon)(\beta)^2 O(|t+\xi|^2) dt dg_0\\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

We now estimate the three terms I_1, I_2 and I_3. To estimate the first term we remark that from the fact the one dimensional Allen-Cahn equation is stable we have the existence of $\gamma > 0$ so that for all $f \in C_0^\infty(\mathbb{R})$

$$\int_\mathbb{R} \varepsilon^2 f'^2 + W''(\omega_\varepsilon) f^2 dt \geq \gamma \int_\mathbb{R} \varepsilon^2((f^\perp)')^2 + (f^\perp)^2 dt.$$

Thus we obtain that

$$I_1 \geq \gamma\varepsilon^2 \int_\Gamma\int_\mathbb{R} (\partial_t\beta^\perp)^2 + \gamma\int_\Gamma\int_\mathbb{R}(\beta^\perp)^2 dt dg_0.$$

With $k = \Pi(\beta) \in C^\infty(\Gamma)$ we have

$$\int_\Gamma\int_\mathbb{R} \beta^2 dt dg_0 = \int_\Gamma\int_\mathbb{R}(\beta^\perp)^2 dt dg_0 + \int_\Gamma k^2 \int_\mathbb{R} (\dot\omega_\varepsilon)^2 dt dg_0 \geq c_\star\varepsilon \int_\Gamma k^2 dg_0. \tag{2}$$

The function $\varepsilon^{-2}t^2|\ddot\omega_\varepsilon|^2$ is uniformly bounded from above with exponential decay and so we obtain from $|\xi|_{L^\infty} = O(\varepsilon^{2(1-\alpha)})$ that for all $x \in \Gamma$

$$\int_\mathbb{R} (t+\xi(x))^2|\ddot\omega_\varepsilon|^2 dt = O(\varepsilon^2)\int_\mathbb{R} \varepsilon^{-2}t^2|\ddot\omega_\varepsilon|^2 dt + O(\varepsilon^3) = O(\varepsilon^3). \tag{3}$$

From the fact that $\partial_t\beta = \partial_t\beta^\perp + k\varepsilon^{-1}\ddot\omega_\varepsilon$, that β has support in Ω_3, (2), and (3), we deduce

$$\begin{aligned}
&\varepsilon^2\int_\Gamma\int_\mathbb{R}(\partial_t\beta)^2|t+\xi|^2 dt dg_0\\
&\quad\leq 2\varepsilon^2\int_\Gamma\int_\mathbb{R}(\partial_t\beta^\perp)^2\chi_1|t+\xi|^2 dt dg_0 + 2\int_\Gamma k^2\int_\mathbb{R}|\ddot\omega_\varepsilon|^2|t+\xi|^2 dt dg_0\\
&\qquad\leq O(\varepsilon^{2+2\delta^*})\int_\Gamma\int_\mathbb{R}(\partial_t\beta^\perp)^2 dt dg_0 + O(\varepsilon^2)\int_\Gamma\int_\mathbb{R}\beta^2 dt dg_0.
\end{aligned}$$

Thus, there is d_2 so that for all ε uniformly small

$$I_1 + I_2 \geq \gamma \int_{\Gamma\times\mathbb{R}} (\beta^\perp)^2 dt dg_0 - 2\varepsilon^2 d_2 \int_{\Gamma\times\mathbb{R}} \beta^2 dt dg_0$$
$$\geq \gamma \int_{\Gamma\times\mathbb{R}} (\beta^\perp)^2 dt dg_0 - \varepsilon^2 d_2 \int_{\Gamma\times\mathbb{R}} \eta^2 dg_z dz.$$

To estimate I_3 we note the existence of d_3 so that for all $x \in \Gamma$

$$\int_{\mathbb{R}} (\dot\omega_\varepsilon)^2 |t+\xi(x)|^2 dt \leq 2 \int_{\mathbb{R}} |\dot\omega_\varepsilon|^2 (t^2 + \xi^2(x)) dt \leq d_3 \varepsilon^3$$

and thus, using (2) and the identity above, we have

$$\int_\Gamma \int_{\mathbb{R}} |W''(\omega_\varepsilon)|\beta^2 |t+\xi|^2 dt dg_0 \leq 4 \int_\Gamma \int_{\mathbb{R}} ((\beta^\perp)^2 \chi_1 + k^2 \dot\omega_\varepsilon^2)|t+\xi|^2 dt dg_0$$
$$\leq O(\varepsilon^{2\delta^*}) \int_{\Gamma\times\mathbb{R}} (\beta^\perp)^2 dt dg_0 + O(\varepsilon^3) \int_\Gamma k^2 dg_0$$
$$\leq O(\varepsilon^{2\delta^*}) \int_{\Gamma\times\mathbb{R}} (\beta^\perp)^2 dt dg_0 + O(\varepsilon^2) \int_{\Gamma\times\mathbb{R}} \beta^2 dt dg_0.$$

Hence, there is a constant d_4 so that

$$I_3 \geq -d_4 \varepsilon^{2\delta^*} \int_{\Gamma\times\mathbb{R}} (\beta^\perp)^2 dt dg_0 - d_4 \varepsilon^2 \int_{\Gamma\times\mathbb{R}} \eta^2 dg_z dz.$$

Putting this all together we find d_5 so that for all ε sufficiently small we have

$$I_1 + I + 2 + I_3 \geq -d_5 \varepsilon^2 \int_{\Gamma\times\mathbb{R}} \eta^2 dg_z dz,$$

which is what we wanted to show. $\square$

For each small ε, consider $\phi_1, \ldots, \phi_j$ orthogonal eigenfunctions of $\mathcal{L}_\varepsilon$ with eigenvalues $\sigma_\varepsilon^1, \ldots, \sigma_\varepsilon^j$ all smaller than one and so that $||\phi_i||_{L^\infty} = 1$ all $i = 1, \ldots, j$. From the previous lemma we see that $\lambda_\varepsilon^i := \sigma_\varepsilon^i/\varepsilon^2$, $i = 1, \ldots, j$ are uniformly bounded and thus subsequentialy converge as $\varepsilon \to 0$ to constants $\lambda_1, \ldots, \lambda_j \leq 1$, $i = 1, \ldots, j$.

Lemma 3.4. *The functions $\Pi(\phi_i) \in C^\infty(\Gamma)$, $i = 1, \ldots, j$, subsequentialy converge as $\varepsilon \to 0$ to nonzero orthogonal eigenfunctions k_i, $i = 1, \ldots, j$, of the Jacobi operator of Γ with eigenvalues λ_i, $i = 1, \ldots, j$, respectively.*

Proof. For simplicity we consider $j = 2$. From Lemma 3.2 we obtain for each ε and $i = 1, 2$, $\beta_i \in C_c^\infty(\Omega_3)$ such that $||\beta_i - \phi_i||_{C^{2,\alpha}_\varepsilon(M)} = O(\varepsilon^{\mathbb{N}})$. Set $\eta_i = \beta_i \circ D_\xi^{-1}$, $i = 1, 2$. Using Theorem 2.1 we deduce that

$$||\eta_i^\perp||_{C^{2,\alpha}_\varepsilon(M)} \leq C||\mathcal{L}_\varepsilon(\beta_i) \circ D_\xi^{-1}||_{C^{0,\alpha}_\varepsilon(M)} + O(\varepsilon^{1+\delta^*-\alpha})|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\mathbb{N}})$$
$$= O(\varepsilon^2)||\eta_i^\perp||_{C^{2,\alpha}_\varepsilon(M)} + O(\varepsilon^{1+\delta^*-\alpha})|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\mathbb{N}}),$$

which means that for all ε uniformly small

$$||\eta_i^\perp||_{C^{2,\alpha}_\varepsilon(M)} \le O(\varepsilon^{1+\delta^*-\alpha})|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\mathbb{N}}),$$

and we also deduce that

$$\mathcal{A}_\varepsilon(\Pi(\eta_i)) := J(\Pi(\eta_i)) - \lambda_\varepsilon^i \Pi(\eta_i), \quad i = 1, 2$$

satisfies

$$\begin{aligned} |\mathcal{A}_\varepsilon(\Pi(\eta_i))|_{C^{0,\alpha}(\Gamma)} &\le o(1)|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\delta^*-\alpha-2})||\eta_i^\perp||_{C^{2,\alpha}_\varepsilon(M)} + O(\varepsilon^{\mathbb{N}}) \\ &\le o(1)|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{2\delta^*-2\alpha-1})|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\mathbb{N}}) \\ &\le o(1)|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} + O(\varepsilon^{\mathbb{N}}), \end{aligned}$$

where the last inequality holds provided we pick α suitably small.

We have $||\beta_i||_{L^\infty(M)} = 1 + o(1)$, $i = 1, 2$ and thus $|\Pi(\eta_i)|_{L^\infty(\Gamma)} = O(1)$, $i = 1, 2$ as well. Thus we conclude from the inequalities above and standard Schauder theory that, for $i = 1, 2$, $|\Pi(\eta_i)|_{C^{2,\alpha}(\Gamma)} = O(1)$ and, after passing to a subsequence, converges in $C^{2,\alpha}$ to an eigenfunction $k_i \in C^\infty(\Gamma)$ of the Jacobi operator with eigenvalue λ_i. Furthermore, we also obtain from the same inequalities that $||\eta_i^\perp||_{C^{2,\alpha}_\varepsilon(M)} = o(\varepsilon)$, $i = 1, 2$, provided we pick α suitably small.

From the decomposition $\eta_i = \eta_i^\perp + \Pi(\eta_i)\dot{\omega}_\varepsilon$, $||\eta_i||_{L^\infty(M)} = 1 + o(1)$, and $||\eta_i^\perp||_{L^\infty(M)} = o(\varepsilon)$, we have that $|\Pi(\eta_i)|_{L^\infty(\Gamma)} = 1 + o(1)$, and thus k_i is not zero for $i = 1, 2$. We are left to show that k_1 and k_2 are orthogonal.

We use Fermi coordinates (x, z). From $|\det D_\xi| = 1 + o(\varepsilon)$, and

$$\sqrt{\det g_z} - \sqrt{\det g_0} = o(1)\sqrt{\det g_0} \quad \text{on } \Omega_3,$$

we have

$$\begin{aligned} \int_M \Pi(\eta_1)\Pi(\eta_2)(\dot{\omega}_\varepsilon)^2|\det D_\xi|dg &= o(\varepsilon) + \int_{\Gamma\times\mathbb{R}} \Pi(\eta_1)\Pi(\eta_2)(\dot{\omega}_\varepsilon)^2 dt dg_0 \\ &= o(\varepsilon) + c_\star\varepsilon \int_\Gamma \Pi(\eta_1)\Pi(\eta_2)dg_0 \end{aligned}$$

$$\int_M \Pi(\eta_1)\eta_2^\perp\dot{\omega}_\varepsilon|\det D_\xi|dg = o(\varepsilon), \quad \int_M \Pi(\eta_2)\eta_1^\perp\dot{\omega}_\varepsilon|\det D_\xi|dg = o(\varepsilon),$$

$$\int_M \eta_1^\perp\eta_2^\perp|\det D_\xi|dg = o(\varepsilon)$$

and so from the orthogonality of ϕ_1, ϕ_2 we deduce

$$\begin{aligned} O(\varepsilon^{\mathbb{N}}) = \int_M \beta_1\beta_2 dg &= \int_M \eta_1^\perp\eta_2^\perp|\det D_\xi|dg + \int_M \eta_1^\perp\Pi(\eta_2)\dot{\omega}_\varepsilon|\det D_\xi|dg \\ &+ \int_M \eta_2^\perp\Pi(\eta_1)\dot{\omega}_\varepsilon|\det D_\xi|dg + \int_M \Pi(\eta_1)\Pi(\eta_2)(\dot{\omega}_\varepsilon)^2|\det D_\xi|dg \\ &= o(\varepsilon) + c_\star\varepsilon\int_\Gamma \Pi(\eta_1)\Pi(\eta_2)dg_0. \end{aligned}$$

Making $\varepsilon \to 0$, it follows at once that k_1, k_2 are orthogonal.

□

Proof of Theorem 3.1. Suppose that $\{\phi_1, \ldots, \phi_j\} \subset C^\infty(M)$ is an orthonormal set of eigenfunctions for $\mathcal{L}_\varepsilon$ with negative eigenvalues. From Proposition 3.3 and Lemma 3.4 we obtain that the Jacobi operator of Γ has at least j non-negative eigenvalues (counted with multiplicity). The non-degeneracy assumption of Γ implies that its Morse index is at least j. Thus the Morse index of Γ is greater or equal than the Morse index of u_ε for all ε sufficiently small. The opposite inequality was proven by [7, ?].

The last statement regarding the eigenvalues of $\mathcal{L}_\varepsilon$ follows from the same type of arguments because the Jacobi operator of Γ has no eigenvalues near zero. □

4. Multiplicity one solutions - after Wang and Wei

Given u_ε a solution to the Allen-Cahn equation (1) consider the energy ratios

$$E_\varepsilon(p,r) = \frac{r^{1-n}}{\omega_{n-1}} \int_{B_r(p)} \varepsilon|\nabla u|^2 + \frac{W'(u)}{\varepsilon} dg,$$

where ω_{n-1} is the volume of the unit $n-1$-sphere. This quantity satisfies a monotonicity type formula proven [8] that we will use repeatedly in what follows without further mentioning it.

On the set of points where $|\nabla u_\varepsilon|(x) \neq 0$, Tonegawa [?] introduced the generalized second fundamental form $|B(u_\varepsilon)|(x)$ as

$$|B(u_\varepsilon)|^2(x) = \frac{|\nabla^2 u_\varepsilon|^2(x) - |\nabla^2 u_\varepsilon(\nu,\cdot)|^2(x)}{|\nabla u_\varepsilon|^2(x)} \quad \text{where } \nu(x) = \frac{\nabla u_\varepsilon(x)}{\nabla u_\varepsilon(x)}.$$

Note that $|B(u_\varepsilon)|(x)$ bounds the second fundamental form of the hypersurface $\{u_\varepsilon = u_\varepsilon(x)\}$ at x.

The following regularity theorem follows directly from the work of Wang [?] and Wang-Wei [40].

Theorem 4.1. *Given $0 < b < 1$, there are $\bar r, \bar\varepsilon, \bar\rho, \bar\delta, \delta$ all small and $\bar D, \bar R$ large so that if for some $\varepsilon < \bar\varepsilon$, $\varepsilon\bar R < r < \bar r$, a solution to the Allen-Cahn equation u_ε satisfies, for some $p \in \{|u| \le 1-b\}$,*

$$E_\varepsilon(p,r) \le c_\star(1+\bar\rho)$$

then

$$|\nabla u_\varepsilon| \neq 0 \quad \textit{in} \quad B_{\bar\delta r}(p) \cap \{|u| \le 1-b\}$$

and

$$(\bar\delta r - \operatorname{dist}(x,p))|B(u_\varepsilon)(x)| \le \bar D \quad \textit{for all } x \in B_{\bar\delta r}(p) \cap \{|u| \le 1-b\}.$$

Moreover, $\{u=0\} \cap B_{\delta r}(p)$ is contained in the image under the exponential map of a graph of a function with uniformly bounded Lipschitz norm defined over some hyperplane of T_pM.

Proof. The proof follows at once from Theorem 9.1 in [**?**] and Theorem 2.4 in [**?**] via a classical scaling and point picking argument. We simply sketch the proof and leave the details to the reader.

First one shows that $|\nabla u_\varepsilon| \neq 0$ in $B_{\bar{\delta}r}(p) \cap \{|u| \leq 1-b\}$ by arguing by contradiction. If that did not happen, one would produce a sequence of solutions $\{u_i\}_{i\in\mathbb{N}}$ to the Allen-Cahn equation (with $\varepsilon = 1$) on increasingly larger balls $(B_{R_i}(0), g_i)$ (where g_i approaches the Euclidean metric) such that $|\nabla u_i|(0)$ tends to zero, $|u_i(0)| \leq 1-b$, and u_i converges in C^2 to a solution u to the Allen-Cahn equation (with $\varepsilon = 1$) on $\mathbb{R}^n$ having $E_1(0, r) \leq c_\star$ for all $r > 0$. From Theorem 9.1 in [**?**] one has that u must be the standing wave solution, which contradicts the fact that $|\nabla u(0)|$ should be zero.

After that, one argues again by contradiction to obtain the uniform bounds for $|B(u_\varepsilon)|$. Like before, after a standard point picking and scaling argument, we obtain a sequence of solutions u_{ε_i} on increasingly large balls $(B_{R_i}(0), g_i)$ (where g_i approaches the Euclidean metric) with $R_i/\varepsilon_i \to \infty$ and such that $|u_{\varepsilon_i}(0)| \leq 1-b$, $|B(u_{\varepsilon_i})|(0) = 1$, $|B(u_{\varepsilon_i})|(x) \leq 2$ for all $x \in B_{S_i}(0)$, where S_i also tends to infinity, and such that $E_{\varepsilon_i}(0, R_i)) \leq c_\star(1 + 1/i)$.

Note that ε_i must tend to zero because otherwise we would further rescale u_{ε_i} to obtain solutions $\{v_i\}_{i\in\mathbb{N}}$ to the Allen-Cahn equation with $|B(v_i)|(0) = \varepsilon_i^{-1}$ that would converge strongly to a smooth solution u to the Allen-Cahn equation on $\mathbb{R}^n$ (with $\varepsilon = 1$) and $E_1(0, r) \leq c_\star$ for all r. Such solution must be the standing wave and this is a contradiction.

From Theorem 1 of [**?**] we have that the associated varifold to u_{ε_i} converges to a stationary integral varifold V with area ratios less or equal than one and so V must be a hyperplane passing through the origin. Moreover $\{u_{\varepsilon_i} \leq 1-b\} \cap B_r(0)$ converges in the Hausdorff sense to $V \cap B_r(0)$ for all $r > 0$ and so, due to the uniform bounds on $|B(u_{\varepsilon_i})|$, $\{u_i = t\} \cap B_r(0)$ can be written as a union of N graphs over the hyperplane V for all $|t| \leq 1-b$ and i sufficiently large. From the monotonicity formula proven by [8] we have $\limsup_{i\to\infty} E_{\varepsilon_i}(0, r) \leq c_\star$ for all r and we obtain that for all i sufficiently large, and all $r > 0$, that $N = 1$, i.e. $\{u_i = 0\} \cap B_r$ is a single graph defined over V (this fact was proven in [**?**, Section 5] and follows from Proposition 5.5). Theorem 2.4 in [40] (applied to balls of increasingly larger radius) implies that $|B(u_{\varepsilon_i})|$ converges to zero on compact sets, which give us a contradiction.

The last statement in the theorem follows from the same type of arguments. □

To state the next theorem we need to introduce some notation. First notice that for every D there is $0 < \delta < 1/2$ so that for all r small if $\Gamma \subset M$ is a hypersurface with second fundamental form bounded by Dr^{-1} in $B_{2r}(p)$ and $p \in \Gamma$, Fermi coordinates for Γ are defined in $B^{n-1}(r) \times (-2\delta r, 2\delta r)$, where $B^{n-1}(r)$ is the geodesic ball of radius r in Γ centered at p. We omit the dependence of the point p in this notation as it will be clear from context that we are always referring to a fix point in the hypersurface. We set $\mathcal{D}(s)$

to be the cylinder described in Fermi coordinates by $B^{n-1}(p) \times (-s, s)$, whenever it is well defined.

Given Fermi coordinates, recall the definition of χ_i, Ω_i, $i = 1, \dots, 5$, D_ξ in ?????. We will also use

$$\tilde{\omega}_\varepsilon(t) = \chi_4 \omega_\varepsilon(t) + (1 - \chi_4)\mathrm{sign}(t).$$

Note that with $\dot{\tilde{\omega}}_\varepsilon := \varepsilon \frac{d}{dt}\tilde{\omega}_\varepsilon$, we have

$$\dot{\tilde{\omega}}_\varepsilon - \chi_4 \dot{\omega}_\varepsilon = O_{C^{2,\alpha}_\varepsilon(\mathbb{R})}(\varepsilon^{\mathbb{N}})$$

for all $0 < \alpha < 1$ and similar estimates hold for $\ddot{\tilde{\omega}}_\varepsilon$ and $\dddot{\tilde{\omega}}_\varepsilon$.

Given Ω a subset of Γ (or M) and ϕ a function defined on Ω, we denote by $|\phi|_{C^{k,\alpha}_\lambda}(\Omega)$ (or $||\phi||_{C^{k,\alpha}_\lambda}(\Omega)$) the $C^{k,\alpha}$ norm on Ω computed with respect to the metric $\lambda^{-2}g$. (PUT IN SOME OTHER SECTION)

The next theorem appears in [**?**]. We state a slightly different version that is more suitable for our purposes.

Theorem 4.2. *Given K, there are $\bar{r}, \bar{\varepsilon}$ small and, for each $0 < \theta < 1$, $\bar{K} = \bar{K}(\theta)$ large so that if for some $\varepsilon < \bar{\varepsilon}$, and $r \leq \bar{r}$, u_ε is a solution to the Allen-Cahn equation* (1) *such that*

- *$\{u_\varepsilon = 0\} \cap B_{2r}(p)$ is contained in the graph of a Lipschitz function f_ε defined over a hyperplane in T_pM with $|f_\varepsilon| < r/2$ and Lipschitz constant bounded by K;*
- *the second fundamental form of $\{u_\varepsilon = 0\}$ is bounded by Kr^{-1} in $B_{2r}(p)$;*

then considering Fermi coordinates (x, z) for $\{u_\varepsilon = 0\}$ in $B^{n-1}(r) \times (-\delta r, \delta r)$ we have that:

a) *the existence of $\xi \in C^{2,\theta}(B^{n-1}(3/4r))$ with*
$$|\xi|_{C^{2,\theta}_{\varepsilon r}(B^{n-1}(3/4r))} \leq \bar{K}\varepsilon^3;$$
b) *$\phi := u_\varepsilon - \tilde{\omega}_\varepsilon \circ D_\xi$ has $||\phi||_{C^{2,\theta}_{r\varepsilon}(\mathcal{D}(3/4r))} \leq \bar{K}\varepsilon^2$;*
c) *For all $x \in B^{n-1}(r)$*
$$\int_{-\delta r}^{\delta r} \phi(x, z)\dot{\tilde{\omega}}_\varepsilon \circ D_\xi(x, z)dz = 0;$$
d) *the mean curvature of $\{u_\varepsilon = 0\}$ is bounded in $C^{0,\theta}_r$ by $\bar{K}\varepsilon^{1-\theta}r^{-1}$ in $\mathcal{D}(3/4r)$;*
e) *the mean curvature of $\{u_\varepsilon = 0\}$ and $\Gamma(\xi) := \{(x, \xi(x)) : x \in B^{n-1}(r)\}$ is bounded in $C^{0,\theta}_r$ by $\bar{K}\varepsilon^{2-2\theta}1r^{-1}$ in $\mathcal{D}(3/4r)$.*

Proof. The hypothesis we require are exactly those required by Wang and Wei in [**?**, Section 7]. All statements but the last one were proven in the Euclidean case by Wang and Wei in [**?**] from Section 8 to Section 16 but the proofs extend straightforwardly to the Riemannian setting. We do a minor change that instead of multiplying the approximate solution ω_ε by a cut-off functions supported in a tubular neighborhood of radius proportional to

$\varepsilon|\log\varepsilon|$ (see [**?**, Section 9.1]), we multiply it by a cut-off function supported in a tubular neighborhood of radius proportional to ε^{δ^*}. We do this so that it is consistent with the cut-off functions used by Pacard in [**?**] and in this case the approximate solution is $\tilde{\omega}_\varepsilon$.

Their estimates are in stretched coordinates and so ξ corresponds to εh in their notation. For the last statement, they only showed that the mean curvature of $\Gamma(\xi)$ decays like $\varepsilon^{1-\theta}$ but it follows from what they did that in fact decays faster. We indicate how to proceed for the sake of completeness. We assume $r=1$ without loss of generality.

In what follows we use Fermi coordinates (x,z) for $\{u_\varepsilon=0\}$ in $B^{n-1}(r)\times(-\delta r,\delta r)$.

From identity (9.4) in [**?**] we have that in Fermi coordinates

$$\begin{aligned}&\varepsilon^2(\Delta_{g_z}\phi+\partial^2_{zz}\phi)-\varepsilon^2H(x,z)\partial_z\phi-W''(\tilde{\omega}_\varepsilon\circ D_\xi)\phi\\&\quad=-\varepsilon(H(x,z)+\Delta_{g_z}\xi(x))\dot{\tilde{\omega}}_\varepsilon\circ D_\xi+\mathcal{R}(\phi)-|\nabla\xi|^2\ddot{\tilde{\omega}}_\varepsilon\circ D_\xi+O_{C^{0,\theta}_\varepsilon}(\varepsilon^3),\end{aligned}$$

where $\mathcal{R}(\phi)$ is quadratic in ϕ and so, from b) we have $\mathcal{R}(\phi)=O_{C^{0,\theta}_\varepsilon}(\varepsilon^3)$. The term $\varepsilon^2H(x,z)\partial_z\phi$ and $|\nabla\xi|^2\ddot{\tilde{\omega}}_\varepsilon\circ D_\xi$ have even higher order and so the equation simplifies to

$$\begin{aligned}&\varepsilon^2(\Delta_{g_z}\phi+\partial^2_{zz}\phi)-W''(\tilde{\omega}_\varepsilon\circ D_\xi)\phi\\&\qquad\qquad=-\varepsilon(H(x,z)+\Delta_{g_z}\xi(x))\dot{\tilde{\omega}}_\varepsilon\circ D_\xi+O_{C^{0,\theta}_\varepsilon}(\varepsilon^3).\end{aligned}\tag{4}$$

We now integrate this identity in the z component against $\dot{\tilde{\omega}}_\varepsilon\circ D_\xi$ and estimate the $C^{0,\theta}(B^{n-1}(3/4))$ norm of all terms. We start with

$$F(x):=\int_{-\delta}^{\delta}(\varepsilon^2(\Delta_{g_z}\phi+\partial^2_{zz}\phi)-W''(\tilde{\omega}_\varepsilon\circ D_\xi))\dot{\tilde{\omega}}_\varepsilon\circ D_\xi dz.$$

We have

$$\begin{aligned}&\int_{-\delta}^{\delta}\varepsilon^2\Delta_{g_z}\phi\,\dot{\tilde{\omega}}_\varepsilon\circ D_\xi dz\\&\qquad=\int_{-\delta}^{\delta}\varepsilon^2(\Delta_{g_z}-\Delta_{g_0})\phi\,\dot{\tilde{\omega}}_\varepsilon\circ D_\xi dz+\int_{-\delta}^{\delta}\varepsilon^2\Delta_{g_0}\phi\,\dot{\tilde{\omega}}_\varepsilon\circ D_\xi dz\\&\qquad:=I_1+I_2.\end{aligned}$$

The term I_1 has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by

$$\varepsilon^{2-\theta}||\phi||_{C^{2,\theta}_\varepsilon(\mathcal{D}(3/4))}+\varepsilon^{1-\theta}|\xi|_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))}||\phi||_{C^{2,\theta}_\varepsilon(\mathcal{D}(3/4))}$$

and so from a) and b) we have that I_1 has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by $\varepsilon^{4-\theta}$. We also have, after using c) in the second indentity,

$$\begin{aligned} I_2 &= \Delta_{g_0} \int_{-\delta}^{\delta} \varepsilon^2 \phi \, \dot{\tilde{\omega}}_\varepsilon \circ D_\xi dz \\ &\quad - \varepsilon \int_{-\delta}^{\delta} (2\langle \nabla \phi, \nabla \xi \rangle + \phi \Delta_{g_0} \xi) \ddot{\tilde{\omega}}_\varepsilon \circ D_\xi dz + \int_{-\delta}^{\delta} \phi |\nabla \xi|^2 \dddot{\tilde{\omega}}_\varepsilon \circ D_\xi dz \\ &= -\varepsilon \int_{-\delta}^{\delta} (2\langle \nabla \phi, \nabla \xi \rangle + \phi \Delta_{g_0} \xi) \ddot{\tilde{\omega}}_\varepsilon \circ D_\xi dz + \int_{-\delta}^{\delta} \phi |\nabla \xi|^2 \dddot{\tilde{\omega}}_\varepsilon \circ D_\xi dz \\ &\hspace{20em} := C_1 + C_2. \end{aligned}$$

The term C_1, C_2 have $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by

$$\varepsilon^{-\theta} |\xi|_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))} ||\phi||_{C^{2,\theta}_\varepsilon(\mathcal{D}(3/4))}$$

and

$$\varepsilon^{-1-\theta} |\xi|_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))} ||\phi||_{C^{2,\theta}_\varepsilon(\mathcal{D}(3/4))}$$

respectively. Thus from a) and b) we have that I_2 has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by $\varepsilon^{4-\theta}$.

On Ω_4 we have $\partial_z D_\xi = 1$ and so

$$\begin{aligned} &\int_{-\delta}^{\delta} (\varepsilon^2 \partial^2_{zz} \phi - W''(\tilde{\omega}_\varepsilon \circ D_\xi) \phi) \dot{\tilde{\omega}}_\varepsilon \circ D_\xi dz \\ &\qquad = \int_{-\delta}^{\delta} (\varepsilon^2 \partial^2_{tt} (\phi \circ D_\xi^{-1}) - W''(\tilde{\omega}_\varepsilon)(\phi \circ D_\xi^{-1})) \dot{\tilde{\omega}}_\varepsilon dt \\ &\qquad = \int_{-\delta}^{\delta} \phi \circ D_\xi^{-1} (\varepsilon^2 \partial^2_{zz} \dot{\tilde{\omega}}_\varepsilon - W''(\tilde{\omega}_\varepsilon) \dot{\tilde{\omega}}_\varepsilon) dt \\ &\qquad = O_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))}(\varepsilon^{\mathbb{N}}), \end{aligned}$$

where in the last identity we used b) and the fact that

$$||\omega_\varepsilon \circ D_\xi - \tilde{\omega}_\varepsilon \circ D_\xi||_{C^{2,\theta}_\varepsilon(M)} = O(\varepsilon^{\mathbb{N}}).$$

Hence there is $\bar{K} = \bar{K}(\theta)$ so that

$$|F|_{C^{0,\theta}(B^{n-1}(3/4))} \leq \bar{K} \varepsilon^{4-\theta} \tag{5}$$

We now move on to estimate the right side of (4). With $Q(x) = \partial_z H(x, 0)$ we set

$$E_1(x,z) := H(x,z) - H(x,0) - zQ(x), \quad E_2(x,z) := \Delta_{g_z} \xi(x) - \Delta_{g_0} \xi(x)$$

and

$$E_3(x) := H(x,0) + \Delta_{g_0} \xi(x) + Q(x) \xi(x) - H(\Gamma_\xi)(x, \xi(x)).$$

There are uniform constants L_1 and L_2 so that

$$||z^{-2} E_1||_{C^{0,\theta}(\mathcal{D}(3/4))} \leq L_1, ||z^{-1} E_2||_{C^{0,\theta}_\varepsilon(\mathcal{D}(3/4))} \leq L_2 \varepsilon^{-2} |\xi|_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))} \tag{6}$$

and, because $|\xi|_{C^{2,\theta}(B^{n-1}(3/4))}$ can be made uniformly small just depending on $\bar{\varepsilon}$,

$$|E_3|_{C^{0,\theta}(B^{n-1}(3/4))} \leq L_2 \varepsilon^{-2(2+\theta)} |\xi|^2_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))}. \tag{7}$$

We have

$$H(x,z) + \Delta_{g_z}\xi(x) = H(\Gamma_\xi)(x,\xi(x)) + (z-\xi(x))Q(x) + E_1 + E_2 + E_3$$

and so, as a function of $x \in B^{n-1}$ (but omitting x from some terms to make notation cleaner),

$$\int_{-\delta}^{\delta} \varepsilon(H(x,z) + \Delta_{g_z}\xi)(\dot{\omega}_\varepsilon \circ D_\xi)^2 dz = \varepsilon H(\Gamma_\xi) \int_{-\delta}^{\delta} (\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$$
$$+ Q\varepsilon \int_{-\delta}^{\delta} (z-\xi)(\dot{\omega}_\varepsilon \circ D_\xi)^2 dz + \varepsilon \int_{-\delta}^{\delta} (E_1 + E_2 + E_3)(\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$$

The term $\int_{-\delta}^{\delta} (\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$ is a constant bounded from above and below by ε. The term $\int_{-\delta}^{\delta} (z-\xi)(\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$ is identical to $\int_{-\delta}^{\delta} t(\dot{\omega}_\varepsilon)^2 dt$ and thus zero because the integrand is odd. In virtue of (6) the term

$$\varepsilon \int_{-\delta}^{\delta} E_1 (\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$$

has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by $\varepsilon^4 + \varepsilon^{2-\theta}|\xi|^2_{C^{0,\theta}_\varepsilon(B^{n-1}(3/4))}$ and thus bounded uniformly by $\varepsilon^{4-\theta}$ by a). The term

$$\varepsilon \int_{-\delta}^{\delta} E_2 (\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$$

has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by

$$\varepsilon^{-1-\theta}|\xi|_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))}(\varepsilon^2 + \varepsilon^{1-\theta}|\xi|_{C^{0,\theta}_\varepsilon(B^{n-1}(3/4))})$$

and thus bounded uniformly by $\varepsilon^{4-\theta}$ by a). In virtue of (7) the term

$$\varepsilon \int_{-\delta}^{\delta} E_3 (\dot{\omega}_\varepsilon \circ D_\xi)^2 dz$$

has $C^{0,\theta}(B^{n-1}(3/4))$ norm bounded uniformly by $\varepsilon^{-2-2\theta}|\xi|^2_{C^{2,\theta}_\varepsilon(B^{n-1}(3/4))}$ and thus bounded uniformly by $\varepsilon^{4-2\theta}$ by a). Thus we obtain

$$H(\Gamma_\xi) = \int_{-\delta}^{\delta} \varepsilon^{-1}(H(x,z) + \Delta_{g_z}\xi)(\dot{\omega}_\varepsilon \circ D_\xi)^2 dz + O_{C^{0,\theta}(B^{n-1}(3/4))}(\varepsilon^{2-2\theta}).$$

Combining this with (4) and (5) we deduce that for some other constant $\bar{K} = \bar{K}(\theta)$

$$|H(\Gamma_\xi)|_{C^{0,\theta}(B^{n-1}(3/4))} \leq \bar{K}\varepsilon^{2-2\theta}.$$

$\square$

5. Proof of Main Theorem

Let $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$, be sequence of solutions to the Allen-Cahn equation (with $\varepsilon_i \to 0$) on a closed manifold (M^n, g) whose associated varifold converges to a two-sided closed embedded non-degenerate minimal hypersurface Γ with multiplicity one.

We obtain from Theorem 1 in [?] that for all $p \in \Gamma$ and r sufficiently small, $E_{\varepsilon_i}(p, r) < c_\star(1 + \bar{\rho})$, where $\bar{\rho}$ is the constant given by Theorem 4.1 (with $b = 3/4$) and that $\{u_{\varepsilon_i} = 0\}$ converges in Hausdorff distance to Γ.

Thus we obtain from Theorem 4.1 that $\{u_{\varepsilon_i} = 0\}$ has second fundamental form uniformly bounded and that, with respect to Fermi coordinates of Γ, can be written as the graph of a function f_i defined over Γ, where the f_i converges pointwise to zero and its Lipschitz norm is bounded. Theorem 4.2 implies that the mean curvature of $\{u_{\varepsilon_i} = 0\}$ converges to zero in $C^{0,\theta}$ norm and so we obtain from standard elliptic theory that $|f_i|_{C^{2,\theta}(\Gamma)}$ also converges to zero.

From Theorem 4.2 we obtain the existence of ξ_i (which we can assume to be defined in Fermi coordinates over Γ instead of Γ_i) with $C^{2,\theta}(\Gamma)$ norm tending to zero and such that denoting, its graph by $\Gamma(\xi_i)$, we have that its mean curvature $H(\Gamma(\xi_i))$ has $C^{0,\theta}$ norm bounded uniformly by $\varepsilon_i^{2-2\theta}$. From the fact that Γ is a nondegenerate minimal surface we deduce from standard elliptic theory that $|\xi_i|_{C^{2,\theta}(\Gamma)}$ is bounded uniformly by $\varepsilon_i^{2-2\theta}$.

We now show that for all i sufficiently large $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$ is in the Pacard-Ritoré form described in PUT SECTION.

From Theorem 4.2 a) we also have that the Hausdorff distance between $\{u_{\varepsilon_i} = 0\}$ and $\Gamma(\xi_i)$ is bounded by ε_i^3 and thus the Hausdorff distance between

Moreover, Theorem 4.2 also states that the function $\phi_i = u_i^1 - \omega_{\varepsilon_i} \circ D_{\xi_i}$, which is defined in a fixed tubular neighborhood T of Γ, has $||\phi_i||_{C^{2,\theta}_{\varepsilon_i}(T)} = O(\varepsilon^2)$.

Denoting by u_{ε_i} the solution constructed by Pacard-Ritoré [31], we have from the fact that $|\xi_i|_{C^{1,\theta}(\Gamma)} = O(\varepsilon^2)$ that $u_{\varepsilon_i} - \omega_{\varepsilon_i} \circ D_{\xi_i}$ has $C^{1,\theta}_\varepsilon(\Gamma)$-norm of order $O(\varepsilon^2)$. Hence, setting $\beta_i = u_1^i - u_{\varepsilon_i}$, we have that $||\beta_i||_{C^{2,\theta}_{\varepsilon_i}(T)} = O(\varepsilon^2)$. If T' denotes a slightly smaller tubular neighborhood we have from Lemma 2.2 that $||\beta_i||_{L^\infty(M\setminus T')} = O(\varepsilon^{\mathbb{N}})$ and so, applying standard Schauder theory, $||\beta_i||_{C^{2,\theta}_{\varepsilon_i}(M)} = O(\varepsilon^2)$.

Using Theorem **??** we have that $||\beta_i||_{C^{2,\theta}_{\varepsilon_i}(M)} = O(\varepsilon^{2+\alpha})$ for some $\alpha > 0$. Recall that $\mathcal{L}_{\varepsilon_i}$ denotes the linearization of the Allen-Cahn equation at u_{ε_i}. A simple computation shows that $\mathcal{L}_{\varepsilon_i}\beta_i = O(\beta_i)\beta_i$ and so the correspondent quadratic form Q_i has $|Q_i(\beta_i, \beta_i)| \leq O(\varepsilon^{2+\alpha}) \int_M \beta_i^2 dg$. On the other hand we know from Theorem 3.1 that $\mathcal{L}_{\varepsilon_i}$ has no eigenvalues of order $O(\varepsilon^{2+\alpha})$, which means that $\beta_i = 0$ for all i sufficiently large. In particular, the Morse index of u_i^1 equals the Morse index of Γ for all i sufficiently large.

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