LINEAR ALGEBRA, WINTER 2018, FINAL

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This final exam was posted online on Thursday, March 14, and is due by 11:00 on Thursday, March 21 in Pallav's mailbox. Collaboration is not allowed, nor is the use of outside materials and textbooks. Treil's book and your class notes may be used to remember definitions, but not to copy calculations or proofs.

Problem 1. Construct a linear map from $\mathbb{R}^3$ to itself whose range is spanned by the vectors $(1,3,2)^T$ and $(3,-1,1)^T$. What is the kernel of your map?

Problem 2. The *rank* of a linear map is the dimension of its range. Let A and B be two linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Show that $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$.

Problem 3. Show that for any $n \times n$ matrix M with non-zero determinant there is a permutation matrix P so that we can write $PM = AB^T$ where both A and B are upper-triangular matrices. (hint: what does this have to do with row and column operations?)

Problem 4. Compute the characteristic polynomial, the eigenvalues, and the algebraic and geometric eigenspaces of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Verify by a direct calculation that the Cayley-Hamilton theorem holds for this matrix.

Problem 5. Let V denote the vector space of complex polynomials of degree at most 3. Let $D : V \to V$ be the linear map that takes a polynomial to its derivative.

1. show that V is isomorphic (as a vector space) to $\mathbb{C}^4$ by choosing an explicit basis.
2. write D as a matrix in the chosen basis.
3. compute the characteristic polynomial of D.
4. find the eigenvalues.

Problem 6. Let M be an $n \times n$ real matrix with $M^T = M^{-1}$ and suppose n is odd. Show that there is a nonzero (real) vector v with $Mv = v$ or $Mv = -v$.

Problem 7. Let $(,)$ be an inner product on $\mathbb{R}^n$. We do *not* assume that $(,)$ is the 'standard' dot product.

1. Show that there is a unique matrix Q for which $(v,w) = w^T Q v$ for all vectors $v, w \in \mathbb{R}^n$.
2. Show that Q is symmetric, and that its eigenvalues are all *positive* real numbers.
3. Now let's specialize to the case that $n = 2$. Let γ be the set of vectors v in $\mathbb{R}^2$ for which $(v,v) = 1$ with respect to the given inner product. Show that γ is an ellipse centered at the origin.
4. Conversely, suppose that γ is any ellipse in $\mathbb{R}^2$ centered at the origin. Show there is a unique inner product $(,)$ for which the set of vectors v with $(v,v) = 1$ is γ.

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