THE STRENGTH OF SOME COMBINATORIAL PRINCIPLES RELATED TO RAMSEY'S THEOREM FOR PAIRS

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Abstract. We study the reverse mathematics and computability-theoretic strength of (stable) Ramsey's Theorem for pairs and the related principles COH and DNR. We show that SRT^2_2 implies DNR over RCA_0 but COH does not, and answer a question of Mileti by showing that every computable stable 2-coloring of pairs has an incomplete Δ^0_2 infinite homogeneous set. We also give some extensions of the latter result, and relate it to potential approaches to showing that SRT^2_2 does not imply RT^2_2.

1. Introduction

In this paper we establish some results on the reverse mathematics and computability-theoretic strength of combinatorial principles related to Ramsey's Theorem for pairs. This topic has attracted a large amount of recent research (see for instance [2, 4, 9, 10]), but certain basic questions still remain open.

For a set X, let $[X]^2 = \{Y \subset X \mid |Y| = 2\}$. A 2-*coloring* of $[\mathbb{N}]^2$ is a function from $[\mathbb{N}]^2$ into $\{0, 1\}$. A set $H \subseteq \mathbb{N}$ is *homogeneous* for a 2-coloring C of $[\mathbb{N}]^2$ if C is constant on $[H]^2$. Ramsey's Theorem for pairs (RT^2_2) is the statement in the language of second-order arithmetic that every 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set. A 2-coloring C of $[\mathbb{N}]^2$ is *stable* if for each $x \in \mathbb{N}$ there exists a $y \in \mathbb{N}$ and a $c < 2$ such that $C(\{x, z\}) = c$ for all $z > y$. Stable Ramsey's Theorem for pairs (SRT^2_2) is RT^2_2 restricted to stable colorings.

It follows from work of Jockusch [5, Theorem 5.7] that if $n > 2$ then Ramsey's Theorem for n-tuples is equivalent to arithmetical comprehension (ACA_0), but Seetapun [11] showed that RT^2_2 does not imply ACA_0. (All implications and nonimplications discussed here are over the standard base

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All of the authors thank the Institute for Mathematical Sciences of the National University of Singapore for its generous support during the period July 18 – August 15, 2005, when part of this research was carried out. The first author was partially supported by NSF grants DMS-0200465 and DMS-0500590. The second author was supported by NSFC Grand International Joint Project 'New Directions in Theory and Applications of Models of Computation', No. 60310213. The fourth author was partially supported by NSF grant DMS-0140120. The fifth author was partially supported by NSF grant DMS-0501167.

theory RCA_0 of reverse mathematics. For background on reverse mathematics and discussions of many of the techniques used below, see Simpson [12].)

A long-standing open question in reverse mathematics is whether RT^2_2 implies Weak König's Lemma (WKL_0), the statement that every computable infinite binary tree has an infinite path. (That WKL_0 does not imply RT^2_2 follows from a result of Jockusch [5, Theorem 3.1] discussed below.) As is well-known, WKL_0 is equivalent to the statement that for each set A, there is a $0,1$-valued function function f that is diagonally noncomputable relative to A (where a total function f is *diagonally noncomputable* if $\forall e\,(f(e) \neq \Phi_e(e))$.) A natural way to weaken this statement is to drop the requirement that f be $0,1$-valued, and allow it to take arbitrary values in ω; the corresponding axiom system has been named DNR. In Section 2 we show that RT^2_2 implies DNR over RCA_0. In other words, whereas we do not know whether RT^2_2 implies WKL_0, we have a partial result toward this implication. In fact, we show that the possibly weaker system SRT^2_2 already implies DNR. It is not known whether SRT^2_2 is strictly weaker than RT^2_2; we will discuss this question further below.

An infinite set X is *cohesive* for a family $R_0, R_1, \ldots$ of sets if for each i, one of $X \cap R_i$ or $X \cap \overline{R}_i$ is finite. COH is the principle stating that every family of sets has a cohesive set. Having seen that $\mathrm{RCA}_0 + \mathrm{SRT}^2_2 \vdash \mathrm{DNR}$, and recalling that RT^2_2 is equivalent over RCA_0 to $\mathrm{SRT}^2_2 + \mathrm{COH}$ (see [2, Lemma 7.11] and [9, Corollary A.1.4]), we proceed to compare COH and DNR. As noted by Cholak, Jockusch, and Slaman [2, Lemma 9.14], even WKL_0 does not imply COH, so certainly DNR does not imply COH. We establish that COH does not imply DNR in Section 3. This result was independently and simultaneously obtained by Hirschfeldt and Shore [4, Corollary 2.21], and as we will see, the main ideas of the proof were already present in [2].

Jockusch [5, Theorem 3.1] constructed a computable 2-coloring of $[\mathbb{N}]^2$ with no Δ^0_2 infinite homogeneous set. On the other hand, computable stable 2-colorings always have Δ^0_2 infinite homogeneous sets. Indeed, the problem of finding an infinite homogeneous set for a computable stable 2-coloring is essentially the same as the problem of finding an infinite subset of either A or $\overline{A}$ for a Δ^0_2 set A. More precisely, we have the following. If A is Δ^0_2 then there is a computable stable 2-coloring C of $[\mathbb{N}]^2$ such that if H is homogeneous for C then $H \subseteq A$ or $H \subseteq \overline{A}$. Conversely, if C is a computable stable 2-coloring of $[\mathbb{N}]^2$ then there is a Δ^0_2 set A such that any infinite set B with $B \subseteq A$ or $B \subseteq \overline{A}$ computes an infinite homogeneous set for C. (See [5, Proposition 2.1] and [2, Lemma 3.5], or [9, Claim 5.1.3].)

Cholak, Jockusch, and Slaman [2, Theorem 3.1] showed that every computable 2-coloring of $[\mathbb{N}]^2$ has a low_2 infinite homogeneous set, and suggested the possibility of separating SRT^2_2 and RT^2_2 by showing that every computable *stable* 2-coloring of $[\mathbb{N}]^2$ has a *low* infinite homogeneous set. Such a result, if relativizable, would allow us to build an ω-model of SRT^2_2 consisting entirely of low sets, which would therefore not be a model of RT^2_2.

(An ω-model of second-order arithmetic is one whose first-order part is standard, and such a model is identified with its second-order part.) However, Downey, Hirschfeldt, Lempp, and Solomon [3] constructed a computable stable 2-coloring of $[\mathbb{N}]^2$ with no low infinite homogeneous set.

Mileti [9, Theorem 5.3.7] showed that for each $X <_{\mathrm{T}} 0'$ there is a computable stable 2-coloring of $[\mathbb{N}]^2$ with no X-computable infinite homogeneous set. (He also showed that this is true for any low_2 set X.)

In light of these results, Mileti [9, Question 5.3.8] asked whether there is an infinite Δ^0_2 set A such that every infinite Δ^0_2 subset of A or $\overline{A}$ is complete (i.e., has degree $\mathbf{0}'$); in other words, whether there is a computable stable 2-coloring of $[\mathbb{N}]^2$ such that any Δ^0_2 infinite homogeneous set is complete. Hirschfeldt gave a negative answer to this question; this previously unpublished result appears as Corollary 4.10 below. In Theorem 4.5, we modify the proof of this result to show that, in fact, if $C_0, C_1, \ldots >_{\mathrm{T}} 0$ are uniformly Δ^0_2, then for every Δ^0_2 set A there is a Δ^0_2 subset X of either A or $\overline{A}$ such that $\forall i\,(C_i \nleq_{\mathrm{T}} X)$. In proving that RT^2_2 does not imply ACA_0, Seetapun [11] showed that if $C_0, C_1, \ldots >_{\mathrm{T}} 0$ then every 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set that does not compute any of the C_i. Our result can be seen as a Δ^0_2 analogue of this theorem. The restriction to stable colorings is of course necessary in this case, since as mentioned above, there are 2-colorings of pairs with no Δ^0_2 infinite homogeneous set.

There is still a large gap between the negative answer to Mileti's question and the result of Downey, Hirschfeldt, Lempp, and Solomon [3] mentioned above. In particular, we would like to know the answer to the following question.

Question 1.1. Let A be Δ^0_2. Must there be an infinite subset of either A or $\overline{A}$ that is both Δ^0_2 and low_2?

A relativizable positive answer to this question would lead to a separation between SRT^2_2 and RT^2_2, since it would allow us to build an ω-model of RCA_0 + SRT^2_2 that is not a model of RT^2_2, as we now explain. We begin with the ω-model $\mathcal{M}_0$ consisting of the computable sets. Let C_0 be a stable 2-coloring of $[\mathbb{N}]^2$ in $\mathcal{M}_0$. Assuming a positive answer to Question 1.1, we have an infinite homogeneous set H_0 for C_0 that is both Δ^0_2 and low_2. Note that H_0' is low over $0'$ and c.e. over $0'$.

Now let $\mathcal{M}_1$ be the ω-model consisting of the H_0-computable sets, and let C_1 be a stable 2-coloring of $[\mathbb{N}]^2$ in $\mathcal{M}_1$. Again assuming a (relativizable) positive answer to Question 1.1, we have an infinite homogeneous set H_1 for C_1 such that $H_0 \oplus H_1$ is both Δ^0_2 in H_0 and low_2. As before, $(H_0 \oplus H_1)'$ is low over $0'$. It may no longer be c.e. over $0'$, but it is 2-CEA over $0'$ (that is, it is c.e. in and above a set that is itself c.e. in and above $0'$).

Now let $\mathcal{M}_2$ be the ω-model consisting of the $H_0 \oplus H_1$-computable sets, and continue in this way, making sure that for every i and every stable 2-coloring C of $[\mathbb{N}]^2$ in $\mathcal{M}_i$, we have $C_j = C$ for some j. Let $\mathcal{M} = \bigcup_i \mathcal{M}_i$. By construction, $\mathcal{M}$ is an ω-model of RCA_0 + SRT^2_2, and for every set X in

$\mathcal{M}$, we have that X' is low over $0'$ and m-CEA over $0'$ for some m. By the extension of Arslanov's Completeness Criterion given by Jockusch, Lerman, Soare, and Solovay [6], no such X can have PA degree over $0'$ (that is, X cannot be the degree of a nonstandard model of arithmetic with an extra predicate for $0'$). However, Jockusch and Stephan [8, Theorem 2.1] showed that a degree contains a p-cohesive set (that is, a set that is cohesive for the collection of primitive recursive sets) if and only if its jump is PA over $0'$. Thus $\mathcal{M}$ is not a model of COH, and hence not a model of RT^2_2.

Note that to achieve the separation described above, it would be enough to show (in a relativizable way) that every Δ^0_2 set A has a subset of either it or its complement that is both Δ^0_2 and low_n for some n (which may depend on A). However, we do not even know whether every Δ^0_2 set has a subset of either it or its complement that is both Δ^0_2 and nonhigh.

The ultimate refutation of this approach to separating SRT^2_2 and RT^2_2 would be to build a computable *stable* 2-coloring of $[\mathbb{N}]^2$ for which the jump of every infinite homogeneous set has PA degree over $0'$. (Without the condition of stability, such a coloring was built by Cholak, Jockusch, and Slaman [2, Theorem 12.5].) Indeed, such a construction (if relativizable) would show that every ω-model of RCA_0 + SRT^2_2 is a model of RT^2_2, as we now explain. Suppose that such stable colorings exist, and let $\mathcal{M}$ be an ω-model of RCA_0 + SRT^2_2. Relativizing the result of Jockusch and Stephan [8, Theorem 2.1] on p-cohesive sets mentioned above, we can show that $\mathcal{M}$ is a model of COH. But as mentioned above, SRT^2_2 + COH is equivalent to RT^2_2 over RCA_0, so $\mathcal{M}$ is a model of RT^2_2.

2. SRT^2_2 IMPLIES DNR

The proof that SRT^2_2 implies DNR over RCA_0 is naturally given in two parts: first we show that each ω-model of SRT^2_2 is a model of DNR, and then that we can in fact carry out the proof of this implication in RCA_0, that is, using only Σ^0_1-induction.

2.1. **The argument for ω-models.** A set A is *effectively bi-immune* if there is a computable function f such that for each e, if $W_e \subseteq A$ or $W_e \subseteq \overline{A}$, then $|W_e| < f(e)$.

Lemma 2.1. *There is an effectively bi-immune set $A \leqslant_T 0'$. In fact, we can choose the function f witnessing the bi-immunity of A to be defined by $f(e) = 3e + 2$.*

Proof. We build A in stages, via a $0'$-computable construction. At each stage we decide the value of $A(n)$ for at most three n's. At stage e, we check whether W_e has at least $3e+2$ many elements. If so, then there are at least two elements $n_0, n_1 \in W_e$ at which we have not yet decided the value of A. Let $A(n_0) = 0$ and $A(n_1) = 1$. In any case, if $A(e)$ is still undefined then let $A(e) = 0$. $\square$

We also need the following lemma, which follows immediately from the equivalence mentioned above between finding homogeneous sets for computable stable colorings and finding subsets of Δ^0_2 sets or their complements. A *Turing ideal* is a subset of 2^ω closed under Turing reduction and join. A subset of 2^ω is a Turing ideal if and only if it is an ω-model of RCA_0.

Lemma 2.2. *A Turing ideal $\mathcal{I}$ is an ω-model of SRT^2_2 if and only if for each set A, if $A \leqslant_T C'$ for some $C \in \mathcal{I}$, then there is an infinite $B \in \mathcal{I}$ such that either $B \subseteq A$ or $B \subseteq \overline{A}$.*

We can now prove the implication between SRT^2_2 and DNR for ω-models.

Theorem 2.3. *Each ω-model of SRT^2_2 is a model of DNR.*

Proof. Let $\mathcal{I}$ be a Turing ideal that is an ω-model of SRT^2_2. We show that $\mathcal{I}$ contains a diagonally noncomputable function. The proof clearly relativizes to get a function that is diagonally noncomputable relative to X for any $X \in \mathcal{I}$.

Let A be as in Lemma 2.1. By Lemma 2.2, there is an infinite $B \in \mathcal{I}$ such that B is a subset of A or $\overline{A}$. By the choice of A, for all e, if $W_e \subseteq B$ then $|W_e| < 3e + 2$.

Let g be such that $W_{g(e)}$ is the set consisting of the first $3e + 2$ many elements of B (in the usual ordering of ω). For any e, if $W_e = W_{g(e)}$ then $W_e \subseteq B$, and so $|W_e| < 3e+2$. But $|W_{g(e)}| = 3e+2$, so this is a contradiction. Thus $\forall e\, (W_e \neq W_{g(e)})$.

Now let f be a computable function such that $W_{f(e)} = W_{\Phi_e(e)}$ if $\Phi_e(e)\downarrow$, and $W_{f(e)} = \emptyset$ otherwise. Then $h = g \circ f$ is diagonally noncomputable, since it is total and for each e, if $\Phi_e(e)\downarrow$ then $W_{h(e)} \neq W_{f(e)} = W_{\Phi_e(e)}$. But h is also computable in B, and hence belongs to $\mathcal{I}$. □

2.2. **The proof-theoretic argument.** We now simply need to analyze the above proof to ensure that Σ^0_1-induction suffices to carry it out. The formal analog of Lemma 2.2 is the statement that SRT^2_2 is equivalent to the following principle, called D^2_2: For every $0,1$-valued function $d(x,s)$, if $\lim_s d(x,s)$ exists for all x, then there is an infinite set B and a $j < 2$ such that $\lim_s d(x,s) = j$ for all $x \in B$. The equivalence of SRT^2_2 and D^2_2 over RCA_0 is claimed in [2, Lemma 7.10]. However, the argument indicated there for the $\mathrm{D}^2_2 \to \mathrm{SRT}^2_2$ direction appears to require Π^0_1-bounding, which is not provable in RCA_0. It is unknown whether $\mathrm{D}^2_2 \to \mathrm{SRT}^2_2$ is provable in RCA_0. Fortunately, we need only the other direction, since we are starting with the assumption that SRT^2_2 holds. This direction is proved as in [2, Lemma 7.10], and we reproduce the proof here for the reader's convenience. Work in $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$. Let a function $d(x,s)$ be given that satisfies the hypothesis of D^2_2. Give the pair $\{x,s\}$ with $x < s$ the color $d(x,s)$. The infinite homogeneous set produced by SRT^2_2 for this stable coloring satisfies the conclusion of D^2_2.

Theorem 2.4. *$RCA_0 \vdash SRT^2_2 \to DNR$.*

Proof. Given the existence of a set A as in Lemma 2.1 (or more precisely, of a function $d(x, s)$ such that $A(x) = \lim_s d(x, s)$), the definition of the diagonally noncomputable function h given in the proof of Theorem 2.3 can clearly be carried out using D^2_2 and Σ^0_1-induction.

So the only part of the proof of Theorem 2.3 we need to consider more carefully is the construction of A and the satisfaction of all bi-immunity requirements. More precisely, fix a model $\mathcal{M}$ of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$. Within that model, we have an enumeration of the $\mathcal{M}$-c.e. sets $W_0, W_1, \dots$ (where the indices range over all elements of the first-order part of $\mathcal{M}$). We need to show the existence of a function $d(x, s)$ in $\mathcal{M}$ such that $\lim_s d(x, s)$ exists for all x, and for every W_e, if there is a $j < 2$ such that $\forall x \in W_e\, (\lim_s d(x, s) = j)$, then $|W_e| < 3e+2$. (We will actually be able to use $2e+2$ instead of $3e+2$.)

We can build d in much the same way as we built A, but we need to be more careful because we no longer have access to an oracle for $0'$. So we need a computable construction to replace the $0'$-computable construction in the proof of Lemma 2.2. Let R_e be the eth bi-immunity requirement.

In this construction, R_e may control up to two numbers n^0_e and n^1_e at any point in the construction. At stage $t = \langle e, s\rangle$, if $|W_{e,s}| \geqslant 2e + 2$, then for each $i < 2$ such that n^i_e is undefined, define n^i_e to be different from each $n^j_{e'}$ for $e' \leqslant e$, and undefine all $n^j_{e'}$ for $e' > e$. In any case, for each n, if $n = n^j_k$ for some j and k, then let $d(n, t) = j$, and otherwise let $d(n, t) = 0$.

It is now easy to check (in RCA_0) that $\lim_t d(n, t)$ exists for all n, since for each n, either n is never controlled by a requirement, in which case $d(n, t) = 0$ for all t, or there is a stage t at which n is controlled by R_e for some e. In the latter case, since control of a number can only pass to stronger requirements, there are at most e many $u \geqslant t$ such that $d(n, u+1) \neq d(n, u)$.

The last thing we need to check is that each R_e is satisfied. It follows by induction that for each e, there are at most $2e$ many numbers that are ever controlled by any $R_{e'}$ with $e' < e$, and thus there is a stage v_e by which all such numbers have been controlled by such requirements. (This is an instance of Π^0_1-induction, which holds in RCA_0 (see Simpson [12, Lemma 3.10]), using a formula saying that for all finite sequences of size $2e + 1$ of distinct elements and for all t, it is not the case that each element of the sequence has been controlled by some $R_{e'}$ with $e' < e$ by stage t.) So if $|W_e| \geqslant 2e + 2$, then picking a stage $t = \langle e, s\rangle \geqslant v_e$ such that $|W_{e,s}| \geqslant 2e + 2$, the n^i_e must be defined at stage t, and will never be undefined at a later stage, so $\lim_u d(n^i_e, u) = i$. Thus R_e is satisfied. □

3. COH does not imply DNR

In this section we show that COH does not imply DNR over RCA_0. We first recall a connection between diagonally noncomputable functions and special Π^0_1 classes.

Definition 3.1. For $n \geqslant 1$ and $A \in 2^\omega$, a Π^0_n subclass of 2^ω is *A-special* if it has no A-computable members. A class is *special* if it is $\emptyset$-special.

Theorem 3.2 (Jockusch and Soare [7, Corollary 1.3])**.** *If A computes an element of a special Π^0_2 class, then A computes an element of a special Π^0_1 class.*

Corollary 3.3. *Any diagonally noncomputable function computes an element of a special Π^0_1 class.*

Proof. Consider the special Π^0_2 class

$$\{A \mid \forall x, t\, \exists y\, \exists s > t\, [\langle x, y\rangle \in A \wedge \neg(\Phi_{x,s}(x)\downarrow = y)] \wedge \\ \forall x, a, b\, [(\langle x, a\rangle \in A \wedge \langle x, b\rangle \in A) \to a = b]\}.$$

It is easy to check that any diagonally noncomputable function computes an element of this class. The corollary now follows from Theorem 3.2. $\square$

We now consider the relationship between cohesiveness and special Π^0_1 classes.

Lemma 3.4 (Cholak, Jockusch and Slaman [2, Lemma 9.16])**.** *Let $A \in 2^\omega$, let P be an A-special Π^0_1 class, and let $R_0, R_1, \ldots \leqslant_T A$. Then there is an $\vec{R}$-cohesive set G that does not compute any element of P.*

This lemma is proved using Mathias forcing with A-computable conditions. We will use two results about Mathias forcing, but since we will not work with this notion directly, we refer to [2, Section 9], [1, Section 6], and [4, Section 2] for the relevant definitions. Analyzing the proof of Lemma 3.4, we immediately obtain the following result.

Corollary 3.5 (to the proof of Lemma 3.4)**.** *There is an $m \in \omega$ such that if G is m-A-generic for Mathias forcing with A-computable conditions, then G is cohesive with respect to any collection of sets $\vec{R} \leqslant_T A$.*

It is clear that Lemma 3.4 generalizes to deal with all Π^0_1 classes at once; this is proved directly in [1, Lemma 6.3].

Lemma 3.6 (Binns, Kjos-Hanssen, Lerman, and Solomon [1, Lemma 6.3])**.** *Let P be a Π^0_1 class and let A be a set. Let G be 3-A-generic for Mathias forcing with A-computable conditions. If P is A-special, then P is $(G \oplus A)$-special.*

We are now ready to establish the result in the section heading.

Theorem 3.7. *There is an ω-model of $RCA_0 +$ COH that is not a model of DNR.*

Proof. Let $m \geqslant 3$ be as in Corollary 3.5. Let $A_0 = \emptyset$, and inductively let A_{n+1} be $A_n \oplus G_n$, where G_n is m-A_n-generic for Mathias forcing with A_n-computable conditions. Let $\mathcal{I}$ be the Turing ideal generated by $\{A_n \mid n \in \omega\}$.

Let $\mathcal{M}$ be the ω-model determined by $\mathcal{I}$. If $\vec{R} \in \mathcal{I}$ is a collection of sets then $\vec{R} \leqslant_{\mathrm{T}} A_n$ for some n. By Corollary 3.5, G_n is $\vec{R}$-cohesive. Since $G_n \in \mathcal{I}$, it follows that $\mathcal{M}$ is a model of COH.

On the other hand, if B computes a diagonally noncomputable function, then by Corollary 3.3, there is a special Π^0_1 class P such that B computes an element of P. In other words, P is not B-special. However, if $B \in \mathcal{I}$ then $B \leqslant_{\mathrm{T}} A_n$ for some n. By Lemma 3.6 and induction, P is A_n-special, and hence P is B-special. So if B computes a diagonally noncomputable function, then $B \notin \mathcal{I}$. Thus $\mathcal{M}$ is not a model of DNR. □

So DNR separates SRT^2_2 from COH. That is, SRT^2_2 implies DNR, whereas COH does not.

4. Degrees of homogeneous sets for stable colorings

In this section we give our negative answer to Mileti's question mentioned in the introduction. We will need two auxiliary results. One is an extension of the low basis theorem noted by Linda Lawton (unpublished).

Theorem 4.1 (Lawton)**.** *Let T be an infinite, computable, computably bounded tree, and let $C_0, C_1, \ldots >_T 0$ be uniformly Δ^0_2. Then T has an infinite low path P such that $\forall i\,(C_i \nleqslant_T P)$, and an index of such a P can be $0'$-computed from an index of T.*

This theorem is proved by forcing with Π^0_1 classes, and lowness is achieved just as in the usual proof of the low basis theorem. Steps are interspersed to guarantee cone avoidance, which is possible by the following lemma.

Lemma 4.2. *Let C be a noncomputable set and let Q be a nonempty computably bounded Π^0_1 class. Let Φ be a Turing reduction. Then Q has a nonempty Π^0_1 subclass R such that $\Phi^f \neq C$ for all $f \in R$. Furthermore, there is a fixed procedure that computes an index of R from indices of Q and Φ and an oracle for $C \oplus 0'$.*

Proof. Let U be a computable tree with $Q = [U]$. For each n, let U_n be the set of strings σ in U such that $\Phi^\sigma(n)$ is either undefined or has a value other than $C(n)$. (Here we use the convention that computations with string oracles σ run for at most $|\sigma|$ steps.) Then U_n is a computable tree, and an index of it can be computed from a C-oracle. Note that U_n is infinite for some n, since otherwise C is computable. Furthermore, $\{n \mid U_n \text{ is infinite }\} \leqslant_{\mathrm{T}} C \oplus 0'$, since C can compute an index of U_n as a computable tree, and then $0'$ can determine whether U_n is infinite by asking whether it contains a string of every length. Let $R = [U_n]$ for the least n with U_n infinite. □

Below, we will use the following relativized form of Theorem 4.1, which can be proved in the same way: Let L be a low set. Let T be an infinite, L-computable, L-computably bounded tree, and let $C_0, C_1, \ldots \nleqslant_{\mathrm{T}} L$ be uniformly Δ^0_2. Then T has an infinite low path P such that $\forall i\,(C_i \nleqslant_{\mathrm{T}} P)$, and an index of such a P can be $0'$-computed from indices of L and T.

The other result we will use below is that if $C_0, C_1, \ldots >_{\mathrm{T}} 0$ are uniformly Δ^0_2 and the complement $\overline{A}$ of the Δ^0_2 set A has no infinite Δ^0_2 subset Y such that $\forall i\,(C_i \nleqslant_{\mathrm{T}} Y)$, then A cannot be too sparse.

Definition 4.3. An infinite set Z is *hyperimmune* if for every computable increasing function f, there is an n such that the interval $[f(n), f(n+1))$ contains no element of Z.

If Z is not hyperimmune, then a computable f such that $[f(n), f(n+1)) \cap Z \neq \emptyset$ is said to *witness the non-hyperimmunity of* Z.

Proposition 4.4. *Let A be Δ^0_2. Let $C_0, C_1, \ldots$ be uniformly Δ^0_2 and let L be an infinite Δ^0_2 set such that $C_i \nleq_T L$ for all i. If $A \cap L$ is L-hyperimmune, then there is an infinite Δ^0_2 set $Y \subseteq \overline{A}$ such that $\forall i\,(C_i \nleq_T Y)$.*

Proof. We build Y by finite extensions; that is, we define $\gamma_0 \prec \gamma_1 \prec \cdots$ and let $Y = \bigcup_i \gamma_i$.

For a string σ and a set X, we write $\sigma \sqsubset X$ to mean that $\{n < |\sigma| \mid \sigma(n) = 1\} \subseteq X$.

Begin with γ_0 defined as the empty sequence. At stage $s = \langle e, i\rangle$, given the finite binary sequence $\gamma_s \sqsubset \overline{A} \cap L$, we $0'$-computably search for either

(1) an m and extensions $\gamma_s\sigma_0$ and $\gamma_s\sigma_1$ such that $\Phi_e^{\gamma_s\sigma_0}(m)\downarrow \neq \Phi_e^{\gamma_s\sigma_1}(m)\downarrow$ and $\gamma_s\sigma_k \sqsubset \overline{A} \cap L$ for $k = 0, 1$; or
(2) an m such that for all extensions $\gamma_s 0^m \sigma \sqsubset L$, either $\Phi_e^{\gamma_s 0^m \sigma}(m)\uparrow$ or $\Phi_e^{\gamma_s 0^m \sigma}(m)\downarrow \neq C_i(m)$.

We claim one of these must be found. Suppose not. Then for every m we can find an extension $\gamma_s 0^m \sigma_0 \sqsubset L$ such that $\Phi_e^{\gamma_s 0^m \sigma_0}(m)\downarrow = C_i(m)$. Since $C_i \nleq_{\mathrm{T}} L$, there must be infinitely many m for which there is also an extension $\gamma_s 0^m \sigma_1 \sqsubset L$ such that $\Phi_e^{\gamma_s 0^m \sigma_1}(m)\downarrow \neq C_i(m)$. So we can L-computably enumerate an infinite set M such that for each $m \in M$, there are $\gamma_s 0^m \sigma_k \sqsubset L$ for $k = 0, 1$ such that $\Phi_e^{\gamma_s 0^m \sigma_0}(m)\downarrow \neq \Phi_e^{\gamma_s 0^m \sigma_1}(m)\downarrow$. Let $m \in M$. Since we are assuming that case 1 above does not hold, there must be a k such that $\gamma_s 0^m \sigma_k \not\sqsubset \overline{A} \cap L$. So letting l_m be the maximum of $|\gamma_s 0^m \sigma_k|$ for $k = 0, 1$, we are guaranteed the existence of an element of $A \cap L$ in the interval $[m, l_m)$. Now we can find $m_0, m_1, \ldots \in M$ such that $m_{j+1} > l_{m_j}$, and define $f(j) = m_j$. Then f is a witness to the non-L-hyperimmunity of $A \cap L$, contrary to hypothesis.

So one of the two cases above must eventually hold. If case 1 holds, let k be such that $\Phi_e^{\gamma_s\sigma_k}(m) \neq C_i(m)$ and define $\gamma'_s = \gamma_s\sigma_k$. If case 2 holds, define $\gamma'_s = \gamma_s 0^m$. In either case, let $\gamma_{s+1} \sqsubset \overline{A} \cap L$ be an extension of γ'_s such that $\gamma_{s+1}(j) = 1$ for some $j > |\gamma_s|$. Such a string must exist since $\gamma'_s \sqsubset \overline{A} \cap L$ and $\overline{A} \cap L$ is infinite (as otherwise $A \cap L$ would be cofinite within L, and hence not L-hyperimmune). This definition ensures that $\Phi_e^Y \neq C_i$. $\square$

We are now ready to prove the main result of this section.

Theorem 4.5. *Let A be Δ^0_2 and let $C_0, C_1, \ldots >_T 0$ be uniformly Δ^0_2. Then either A or $\overline{A}$ has an infinite Δ^0_2 subset X such that $C_i \nleq_T X$ for all i.*

Proof. Assume that $\overline{A}$ has no infinite Δ^0_2 subset Y such that $C_i \nleq_{\mathrm{T}} Y$ for all i. We use Proposition 4.4 to build an infinite Δ^0_2 set X such that $C_i \nleq_{\mathrm{T}} X$ for all i, via a $0'$-computable construction satisfying the following requirements:

$$R_{e,i} : \Phi_e^X \text{ total } \Rightarrow \exists n\, (\Phi_e^X(n) \neq C_i(n)).$$

We first discuss how to satisfy the single requirement $R_{0,0}$. By Proposition 4.4 (with $L = \omega$), A is not hyperimmune. Suppose we have a computable function f witnessing the non-hyperimmunity of A. Let the computable, computably bounded tree $\widehat{T}$ consist of the nodes $(m_0, \ldots, m_{k-1})$ with $f(j) \leqslant m_j < f(j+1)$ for all $j < k$. Such a node represents a guess that $m_j \in A$ for each $j < k$. Note that the choice of f ensures that $\widehat{T}$ has at least one path along which all such guesses are correct.

Now prune $\widehat{T}$ as follows. For each node $\sigma = (m_0, \ldots, m_{k-1})$, if there are nonempty $F_0, F_1 \subseteq \operatorname{rng}(\sigma)$ and an n such that $\Phi_0^{F_0}(n)\downarrow \neq \Phi_0^{F_1}(n)\downarrow$ with uses bounded by the largest element of $F_0 \cup F_1$, then prune $\widehat{T}$ to ensure that σ is not extendible to an infinite path. Note that we can do this pruning in such a way as to end up with a computable tree T.

Now $0'$ can determine whether T is finite. If so, then we can find a leaf σ of T such that $\operatorname{rng}(\sigma) \subset A$. There are nonempty $F_0, F_1 \subseteq \operatorname{rng}(\sigma)$ and an n such that $\Phi_0^{F_0}(n)\downarrow \neq \Phi_0^{F_1}(n)\downarrow$ with uses bounded by the largest element z of $F_0 \cup F_1$, so if we let k be such that $\Phi_0^{F_k}(n) \neq C_0(n)$ and define X so that $X \upharpoonright z+1 = F_k \upharpoonright z+1$, then we ensure that $\Phi_0^X(n) \neq C_0(n)$.

On the other hand, if T is infinite then by Theorem 4.1, $0'$ can find a low path P of T such that $C_i \nleqslant_{\mathrm{T}} P$ for all i. There must be an n such that either $\Phi_0^Y(n)\uparrow$ for every $Y \subseteq \operatorname{rng}(P)$ or there is a $Y \subseteq \operatorname{rng}(P)$ such that $\Phi_0^Y(n)\downarrow \neq C_0(n)$, since otherwise we could P-compute $C_0(n)$ for each n by searching for a finite $F \subset \operatorname{rng}(P)$ such that $\Phi_0^F(n)\downarrow$. But by the construction of T, this means that there is an n such that for every infinite $Y \subseteq \operatorname{rng}(P)$, either $\Phi_0^Y(n)\uparrow$ or $\Phi_0^Y(n)\downarrow \neq C_0(n)$. So if we now promise to make $X \subseteq \operatorname{rng}(P)$, we ensure that $\Phi_0^X \neq C_0$. Notice that we can make such a promise because $C_i \nleqslant_{\mathrm{T}} P$ for all i, and hence $C_i \nleqslant_{\mathrm{T}} \operatorname{rng}(P)$ for all i (since P is an increasing sequence), which implies that $A \cap \operatorname{rng}(P)$ is infinite.

Let us now consider how to satisfy another requirement, say $R_{0,1}$. The action taken to satisfy $R_{0,0}$ results in either a finite initial segment of X being determined, or a promise being made to keep X within a given infinite low set that does not compute any of the C_i. We can handle both cases at once by assuming that we have a number r_1 and an infinite low set L_1 containing the finite set F_1 of numbers less than r_1 currently in X, such that $C_i \nleqslant_{\mathrm{T}} L_1$ for all i. We want $X \upharpoonright r_1 = F_1$ and $X \subseteq L_1$.

Suppose that we have an L_1-computable function g witnessing the non-L_1-hyperimmunity of $A \cap L_1$. We can then proceed much as we did for $R_{0,0}$, but taking r_1 and L_1 into account, in the following way. We can assume that $g(0) \geqslant r_1$. Define $\widehat{T}$ to consist of the nodes $(m_0, \ldots, m_{k-1})$ with $g(j) \leqslant m_j < g(j+1)$ and $m_j \in L_1$ for all $j < k$. For each node $\sigma = (m_0, \ldots, m_{k-1})$, if there are nonempty $G_0, G_1 \subseteq \operatorname{rng}(\sigma)$ and an n such that $\Phi_1^{F_1 \cup G_0}(n)\downarrow \neq \Phi_1^{F_1 \cup G_1}(n)\downarrow$ with uses bounded by the largest element of

$G_0 \cup G_1$, then prune $\widehat{T}$ to ensure that σ is not extendible to an infinite path, thus obtaining a new L_1-computable tree T.

If T is finite then find a leaf σ of T such that $\mathrm{rng}(\sigma) \subset A \cap L_1$. Then there is a nonempty $G \subseteq \mathrm{rng}(\sigma)$ and an n such that $\Phi_1^{F_1 \cup G}(n) \downarrow \neq C_0(n)$ with use bounded by the largest element z of G, so if we define X such that $X \upharpoonright z+1 = (F_1 \cup G) \upharpoonright z+1$ then we ensure that $\Phi_1^X(n) \neq C_0(n)$.

If T is infinite then $0'$ can find a low path P of T. If we now promise that all future elements of X will be in $\mathrm{rng}(P)$, we ensure that $\Phi_1^X \neq C_0$ as before. Notice that we can make such a promise because $\mathrm{rng}(P) \subseteq L_1$ and, as before, $A \cap \mathrm{rng}(P)$ is infinite.

Thus we can satisfy $R_{0,1}$, and the action we take results in a number r_2 and an infinite low set L_2 that does not compute any of the C_i (and contains the finite set F_2 of numbers less than r_2 currently in X) such that we want $X \upharpoonright r_2 = F_2$ and $X \subseteq L_2$. In other words, we are in the same situation we were in after satisfying $R_{0,0}$, and we could now proceed to satisfy another requirement as we did $R_{0,1}$.

However, there is a crucial problem with proceeding in this way for all the $R_{e,i}$ at once, which is that we know no $0'$-computable way to determine the witnesses to non-hyperimmunity required by the construction. The best we can do is guess at them. That is, we have a $0''$-partial computable function w such that if l is a lowness index for an infinite set L (that is, $\Phi_l^{0'} = L'$) then $\Phi_{w(l)}^L$ witnesses the non-L-hyperimmunity of $A \cap L$.

We are now ready to describe our construction. We give our requirements a priority ordering by saying that $R_{e,i}$ is stronger than $R_{e',i'}$ if $\langle e,i\rangle < \langle e',i'\rangle$. All numbers added to X at a stage s of our construction will be greater than s, thus ensuring that $X \leqslant_{\mathrm{T}} \emptyset'$. Let X_s be the set of numbers added to X by the beginning of stage s.

Throughout the construction, we run a $0'$-approximation to w. Associated with each $R_{e,i}$ are a number $r_{\langle e,i\rangle}$ and a low set $L_{\langle e,i\rangle}$ with lowness index $l_{\langle e,i\rangle}$ (all of which might change during the construction). If the approximation to $w(l_{\langle e,i\rangle})$ changes, then for all $\langle e',i'\rangle \geqslant \langle e,i\rangle$ the strategy for $R_{e',i'}$ is immediately canceled, $R_{e',i'}$ is declared to be unsatisfied, and $r_{\langle e',i'\rangle}$, $L_{\langle e',i'\rangle}$, and $l_{\langle e',i'\rangle}$ are reset to the current values of $r_{\langle e,i\rangle}$, $L_{\langle e,i\rangle}$, and $l_{\langle e,i\rangle}$, respectively. It is important to note that the approximation to w continues to run during the action of a strategy at a fixed stage. That is, we may find a change in the approximation to some $w(l_{\langle e,i\rangle})$ with $\langle e,i\rangle \leqslant \langle e',i'\rangle$ in the middle of a stage s at which we are trying to satisfy $R_{e',i'}$. If this happens then we immediately end the stage and cancel strategies as described above.

Initially, all requirements are unsatisfied. At the beginning of stage 0, for every e,i, let $r_{\langle e,i\rangle} = 0$ and $L_{\langle e,i\rangle} = \omega$, and let $l_{\langle e,i\rangle}$ be a fixed lowness index for ω.

At stage s, let $R_{e,i}$ be the strongest unsatisfied requirement and proceed as follows.

We have a number $r_{\langle e,i\rangle}$ and a low set $L_{\langle e,i\rangle}$ with lowness index $l_{\langle e,i\rangle}$, such that $L_{\langle e,i\rangle}$ contains $X_s \restriction r_{\langle e,i\rangle}$, and $C_j \nleqslant_{\mathrm{T}} L_{\langle e,i\rangle}$ for all j. As before, we want to ensure that $X \restriction r_{\langle e,i\rangle} = X_s \restriction r_{\langle e,i\rangle}$ and $X \subseteq L_{\langle e,i\rangle}$. Let v be the current approximation to $w(l_i)$ and let $g = \Phi_v^{L_{\langle e,i\rangle}}$. By shifting the values of g if necessary, we can assume that $g(0) \geqslant \max(r_{\langle e,i\rangle}, s)$. Define $\widehat{T}$ to consist of the nodes $(m_0, \dots, m_{k-1})$ with $g(j) \leqslant m_j < g(j+1)$ and $m_j \in L_{\langle e,i\rangle}$ for all $j < k$. Note that g may not be total, in which case $\widehat{T}$ is finite.

For each node $\sigma = (m_0, \dots, m_{k-1})$, if there are nonempty $G_0, G_1 \subseteq \sigma$ and an n such that $\Phi_e^{X \restriction r_i \cup G_0}(n)\downarrow \neq \Phi_e^{X \restriction r_i \cup G_1}(n)\downarrow$ with uses bounded by the largest element of $G_0 \cup G_1$, then prune $\widehat{T}$ to ensure that σ is not extendible to an infinite path, thus obtaining a new $L_{\langle e,i\rangle}$-computable tree T.

We want to $0'$-effectively determine whether T is finite. More precisely, the question we ask $0'$ is whether the pruning process described above ever results in all the nodes at some level of $\widehat{T}$ becoming non-extendible. A positive answer means T is finite. If g is total then a negative answer means T is infinite. However, if g is not total, so that $\widehat{T}$ is finite, we may still get a negative answer, because the pruning process may get stuck waiting forever for a level of $\widehat{T}$ to become defined.

If the answer to our question is positive, then look for a leaf σ of T such that $\operatorname{rng}(\sigma) \subset A \cap L_{\langle e,i\rangle}$. If no such leaf exists, then either $L_{\langle e,i\rangle}$ is finite or $v \neq w(l_{\langle e,i\rangle})$, so end the stage and cancel the strategies for $R_{\langle e',i'\rangle}$ with $\langle e',i'\rangle \geqslant \langle e,i\rangle$ as described above. (That is, declare $R_{\langle e',i'\rangle}$ to be unsatisfied, and reset $r_{\langle e',i'\rangle}$, $L_{\langle e',i'\rangle}$, and $l_{\langle e',i'\rangle}$ to the current values of $r_{\langle e,i\rangle}$, $L_{\langle e,i\rangle}$, and $l_{\langle e,i\rangle}$, respectively.) Otherwise, there are a nonempty $G \subseteq \operatorname{rng}(\sigma)$ and an n such that $\Phi_e^{X \restriction r_i \cup G}(n)\downarrow \neq C_i(n)$ with use bounded by the largest element of G. Let $r_{\langle e,i\rangle+1}$ be the largest element of G, let $L_{\langle e,i\rangle+1} = L_{\langle e,i\rangle}$, and let $l_{\langle e,i\rangle+1} = l_{\langle e,i\rangle}$. Put every element of G into X.

If the answer to our question is negative, then use the relativized form of Theorem 4.1 to $0'$-effectively obtain a low path P of T such that $C_j \nleqslant_{\mathrm{T}} P$ for all j, and a lowness index $l_{\langle e,i\rangle+1}$ for $L_{\langle e,i\rangle+1} = X \restriction r_i \cup \operatorname{rng}(P)$. If g is not total, then the construction in the proof of Theorem 4.1 will still produce such an $L_{\langle e,i\rangle+1}$ and $l_{\langle e,i\rangle+1}$, but $L_{\langle e,i\rangle+1}$ may be finite. (Which will of course be a problem for weaker priority requirements, but in this case the strategy for $R_{e,i}$ will eventually be canceled, and hence $L_{\langle e,i\rangle+1}$ will eventually be redefined.) Let $r_{\langle e,i\rangle+1} = r_{\langle e,i\rangle}$. Search for an element of $A \cap L_{\langle e,i\rangle+1}$ greater than $\max\{r_{\langle e',i'\rangle} \mid \langle e',i'\rangle \leqslant \langle e,i\rangle\}$ not already in X and put this number into X. If $L_{\langle e,i\rangle+1}$ is infinite, such a number must be found. Otherwise, such a number may not exist, but this situation can only happen if the approximation to $w(l_{\langle e',i'\rangle})$ at the beginning of stage s is incorrect for some $\langle e',i'\rangle \leqslant \langle e,i\rangle$, in which case the strategy for $R_{e,i}$ will be canceled, and the stage ended as described above.

In either case, if the action of the strategy for $R_{e,i}$ has not been canceled, then declare $R_{e,i}$ to be satisfied, and for $\langle e',i'\rangle > \langle e,i\rangle$, declare $R_{e',i'}$ to be unsatisfied, let $r_{\langle e',i'\rangle+1} = r_{\langle e,i\rangle+1}$, let $L_{\langle e',i'\rangle+1} = L_{\langle e,i\rangle+1}$, and let $l_{\langle e',i'\rangle+1} = l_{\langle e,i\rangle+1}$.

This completes the construction. Since every element entering X at stage s is in A and is greater than s, we have that X is a Δ_2^0 subset of A. Furthermore, at each stage a number is added to X unless the strategy acting at that stage is canceled, so once we show that every requirement is eventually permanently satisfied, we will have shown that X is infinite.

Assume by induction that for all $\langle e',i'\rangle < \langle e,i\rangle$, the requirement $R_{e',i'}$ is eventually permanently satisfied, and that $r_{\langle e,i\rangle}$, $L_{\langle e,i\rangle}$, and $l_{\langle e,i\rangle}$ eventually reach a final value, for which $L_{\langle e,i\rangle}$ is infinite. Let s be the least stage by which this situation obtains and the approximation to $w(l_{\langle e,i\rangle})$ has settled to a final value v. Note that at stage $s-1$, either the strategy for some $R_{\langle e',i'\rangle}$ with $\langle e',i'\rangle < \langle e,i\rangle$ acted, or the approximation to $w(l_{\langle e,i\rangle})$ changed, so at the beginning of stage s, it must be the case that $R_{e,i}$ is the strongest unsatisfied requirement. Thus at that stage the strategy for $R_{e,i}$ acts, and the function $g = \Phi_v^{L_{\langle e,i\rangle}}$ it works with at that stage is in fact a witness to the non-$L_{\langle e,i\rangle}$-hyperimmunity of $A \cap L_{\langle e,i\rangle}$. Thus $R_{e,i}$ will become satisfied at the end of the stage, and $r_{\langle e,i\rangle+1}$, $L_{\langle e,i\rangle+1}$, and $l_{\langle e,i\rangle+1}$ will not be redefined after the end of the stage.

If the tree T built at stage s is finite, then a leaf σ of T is found such that $\mathrm{rng}(\sigma) \subset A \cap L_{\langle e,i\rangle}$, and there are a nonempty $G \subseteq \mathrm{rng}(\sigma)$ and an n such that $\Phi_e^{X \upharpoonright r_{\langle e,i\rangle} \cup G}(n)\downarrow \neq C_i(n)$ with use bounded by the largest element $r_{\langle e,i\rangle+1}$ of G. Since $r_{\langle e,i\rangle+1}$ is never again redefined, $\Phi_e^X(n) \neq C_i(n)$, and thus the requirement $R_{e,i}$ is satisfied. Furthermore, $L_{\langle e,i\rangle+1}$ is defined to be $L_{\langle e,i\rangle}$ at this stage, and hence is infinite.

If T is infinite, then $L_{\langle e,i\rangle+1}$ is defined to contain the range of a path of T, and hence is infinite. Furthermore, $L_{\langle e,i\rangle+1}$ is never redefined, and by the way X is defined, $X \subseteq A \cap L_{\langle e,i\rangle+1}$. There must be an n such that either $\Phi_e^Y(n)\uparrow$ for every $Y \subseteq L_{\langle e,i\rangle+1}$ or there is a $Y \subseteq L_{\langle e,i\rangle+1}$ such that $\Phi_0^Y(n)\downarrow \neq C_i(n)$, since otherwise we could $L_{\langle e,i\rangle+1}$-compute $C_i(n)$ for each n by searching for a finite $F \subset L_{\langle e,i\rangle+1}$ such that $\Phi_e^F(n)\downarrow$. But by the definition of T, this means that there is an n such that for every infinite $Y \subseteq L_{\langle e,i\rangle+1}$, either $\Phi_e^Y(n)\uparrow$ or $\Phi_e^Y(n)\downarrow \neq C_i(n)$. So since $X \subseteq L_{\langle e,i\rangle+1}$, we have $\Phi_e^X \neq C_i$, and hence the requirement $R_{e,i}$ is satisfied. $\square$

Theorem 4.5 gives the negative answer to Mileti's question mentioned above.

Corollary 4.6 (Hirschfeldt)**.** *Every Δ_2^0 set has an incomplete infinite Δ_2^0 subset of either it or its complement. In other words, every computable stable 2-coloring of $[\mathbb{N}]^2$ has an incomplete Δ_2^0 infinite homogeneous set.*

We can improve on this result by using the following unpublished result due to Jockusch.

Proposition 4.7 (Jockusch)**.** *Let Z be hyperimmune. Then there is a 1-generic $G \leqslant_T Z \oplus 0'$ such that $Z \subseteq G$.*

Proof. We build G by finite extensions; that is, we define $\gamma_0 \prec \gamma_1 \prec \cdots$ and let $G = \bigcup_i \gamma_i$. Let $S_0, S_1, \dots$ be an effective listing of all c.e. sets of finite binary sequences.

Begin with γ_0 defined as the empty sequence. At stage i, given the finite binary sequence γ_i, search for an extension $\alpha \in S_i$ of $\gamma_i 1^{f(n)}$. If one is found then let $f(n+1) = |\alpha|$.

If f is total then, since Z is hyperimmune, there is an n such that the interval $[|\gamma_i| + f(n), |\gamma_i| + f(n+1))$ contains no element of Z. So $Z \oplus 0'$-computably search for either such an interval or for an n such that $f(n+1)$ is undefined. In the first case, let α be as above and let $\gamma_{i+1} = \alpha$. In the second case, let $\gamma_{i+1} = \gamma_i 1^{f(n)}$.

It is now easy to check by induction that $Z \subseteq G$ and that G meets or avoids each S_i. $\square$

Corollary 4.8. *Let $X \subset Y$ be such that X is Y-hyperimmune. Then there are $G, H \leqslant_T X \oplus Y'$ such that*

1. *$H \leqslant_T G \oplus Y$,*
2. *G is 1-generic relative to Y,*
3. *$X \subseteq H \subset Y$, and*
4. *$Y \setminus H$ is infinite.*

Proof. Let $h(0) < h(1) < \cdots$ be the elements of Y, and let $Z = h^{-1}(X)$. By Proposition 4.7 relativized to Y, there is a $G \leqslant_{\mathrm{T}} Z \oplus Y'$ such that G is 1-generic relative to Y and $Z \subseteq G$. Let $H = h(G)$. Since $h \leqslant_{\mathrm{T}} Y$ and h is increasing, we have $H \leqslant_{\mathrm{T}} G \oplus Y$, and $X \subseteq H \subset Y$ by the definition of h. Finally, $Y \setminus H = h(\overline{G})$, and hence is infinite. $\square$

Corollary 4.9. *Let A be a Δ^0_2 set such that $\overline{A}$ has no infinite low subset, and let L be low. Then $A \cap L$ is not L-hyperimmune.*

Proof. Suppose that $A \cap L$ is L-hyperimmune. We can apply Corollary 4.8 to $X = A \cap L$ and $Y = L$ to obtain G and H as above. Since $A \cap L$ and L' are both Δ^0_2, so is G. Since G is also 1-generic relative to L, and L is low, $G \oplus L$ is low. But $H \oplus L \leqslant_{\mathrm{T}} G \oplus L$, and hence $H \oplus L$ is low. Thus $L \setminus H$ is an infinite low subset of $\overline{A}$, which is a contradiction. $\square$

Corollary 4.10 (Hirschfeldt)**.** *Let A be a Δ^0_2 set such that $\overline{A}$ has no infinite low subset. Then A has an incomplete infinite Δ^0_2 subset.*

Proof Sketch. The proof is similar to that of Theorem 4.5. Instead of working with the given sets C_i, we build a Δ^0_2 set C while satisfying the requirements

$$R_e : \Phi^X_e \text{ total } \Rightarrow \exists n\, (\Phi^X_e(n) \neq C(n)).$$

At stage s, we work with the least unsatisfied requirement R_i. We have a number r_i and a low set L_i with lowness index l_i, such that L_i contains

$X_s \upharpoonright r_i$. We define $\widehat{T}$ as before. For each node $\sigma = (m_0, \ldots, m_{k-1})$, if there is a nonempty $G \subseteq \sigma$ such that $\Phi_e^{X \upharpoonright r_i \cup G}(s)\downarrow$ with use bounded by the largest element of G, then we prune $\widehat{T}$ to ensure that σ is not extendible to an infinite path, thus obtaining a new L_i-computable tree T.

If T is finite, then we look for a leaf σ of T and a G as above, let r_{i+1} be the largest element of G, define $C(s) \neq \Phi_i^{X \upharpoonright r_i \cup G}(s)$, let $L_{i+1} = L_i$, let $l_{i+1} = l_i$, and put every element of G into X.

If T is infinite, we $0'$-effectively obtain a low path P of T and a lowness index l_{i+1} for $L_{i+1} = X \upharpoonright r_i \cup \operatorname{rng}(P)$. We then let $r_{i+1} = r_i$ and $C(s) = 0$, search for an element of $A \cap L_{i+1}$ greater than $\max\{r_j \mid j \leqslant i\}$ not already in X, and put this number into X.

The further details of the construction are as before, and the verification that it succeeds in satisfying all the requirements is similar. □

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