

# EVERY 1-GENERIC COMPUTES A PROPERLY 1-GENERIC

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ABSTRACT. A real is called *properly  $n$ -generic* if it is  $n$ -generic but not  $n + 1$ -generic. We show that every 1-generic real computes a properly 1-generic real. On the other hand, if  $m > n \geq 2$  then an  $m$ -generic real cannot compute a properly  $n$ -generic real.

## 1. INTRODUCTION

The notions of measure and category (or in forcing terminology, random (Solovay) and Cohen forcing) have made their way into computability theory via the notions of restricted randomness and genericity. Restricted genericity for Cohen reals was introduced by Jockusch [Joc80], who studied  *$n$ -genericity*, that is, genericity where the forcing relation is restricted to  $n$ -quantifier arithmetic (as Jockusch and Posner [JP78] observed, a real is  $n$ -generic iff for all  $\Sigma_n^0$  sets of strings  $S$ , there is some initial segment  $\sigma$  of  $A$  such that  $\sigma \in S$  or  $\sigma \not\subseteq \tau$  for all  $\tau \in S$ .) Restricted genericity gives rise to a proper hierarchy (every  $n + 1$ -generic real is also  $n$ -generic but not vice versa). Thus, we can define a real to be *properly  $n$ -generic* iff it is  $n$ -generic and not  $n + 1$ -generic. [A related notion, first discussed by Kurtz [Kur81], is that of *weak  $n$ -genericity*. Here a real  $A$  is weakly  $n$ -generic iff  $A$  meets all dense  $\Sigma_n^0$  sets of strings. Kurtz [Kur81] showed that weak genericity refines the genericity hierarchy, with  $n$ -generic  $\supseteq$  weakly  $n + 1$ -generic  $\supseteq$   $n + 1$ -generic.]

The study of reals random at various levels of the arithmetical hierarchy was introduced by Martin-Löf [ML66]. A real  $A$  is called  *$n$ -random* iff for all  $\Sigma_n^0$ -tests  $\{U_n : n \in \mathbb{N}\}$ , we have  $A \notin \bigcap_n U_n$ . Here, a  $\Sigma_n^0$ -test is a (uniform) collection of  $\Sigma_n^0$ -classes  $\{U_n : n \in \mathbb{N}\}$ , such that  $\mu(U_n) \leq 2^{-n}$ , where  $\mu$  is Lebesgue measure. (We refer the reader to Downey, Hirschfeldt, Nies and Terwijn [DHNT] for a general introduction to results relating genericity, randomness and relative computability, as well as to the forthcoming books Nies [Nie] and Downey and Hirschfeldt [DH].)

Both  $n$ -genericity and  $n$ -randomness can be relativized to a given real  $Z$  by replacing  $\Sigma_n^0$  objects by ones that are  $\Sigma_n^0$  relative to  $Z$ . For instance, a real  $A$  is  $n$ -random over  $Z$  iff  $A \notin \bigcap_n U_n$  for all tests  $\{U_n : n \in \mathbb{N}\}$  that are  $\Sigma_n^0$  relative to  $Z$ . It is easy to see that a real is  $n$ -generic iff it is 1-generic over  $\emptyset^{(n-1)}$ . Kurtz [Kur81] showed that this is also true for randomness; that is, a real is  $n$ -random iff it is 1-random over  $\emptyset^{(n-1)}$ .

There are striking similarities between the ways these two notions interact with Turing reducibility. For example, relatively 1-generic reals form minimal pairs, as

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do relatively 2-random reals. Another nice example is van Lambalgen’s Theorem ([VL87]) which says that  $A \oplus B$  is  $n$ -random iff  $A$  is  $n$ -random and  $B$  is  $n$ -random over  $A$ ; Yu [Yu] proved the analogous statement for genericity.

There are interesting distinctions as well. For example, there are complete  $\Delta_2^0$  1-random reals, whereas all 1-generic reals are generalised low.

This paper is motivated by a result of Miller and Yu [MY]:

**Theorem 1.1** (Miller and Yu). *Let  $A$  be 1-random over a real  $Z$ , and let  $B$  be 1-random and computable in  $A$ . Then  $B$  is 1-random over  $Z$ .*

[This result follows from van Lambalgen’s theorem in the case that  $Z$  has 1-random degree.] In particular, letting  $Z = \emptyset^{(n-1)}$ , if  $A$  is  $n$ -random and  $B \leq_T A$ , with  $B$  1-random, then  $B$  is  $n$ -random. Asking whether the same property holds for Cohen genericity yields both a similarity and a distinction from the random case. We will show that the analogue of Miller and Yu’s result holds in the generic case, if the bottom real  $B$  is 2-generic:

**Theorem 1.2.** *Let  $A$  be 1-generic over a real  $Z$ , and let  $B$  be 2-generic and computable in  $A$ . Then  $B$  is 1-generic over  $Z$ .*

As a result, it is impossible for, say, a 3-generic real to compute a properly 2-generic real. We mention that Theorem 1.2 may be known, but is not yet found in print, and so we include a proof here.

On the other hand, the analogue of Miller and Yu’s result *always* fails when 2-genericity is reduced to 1-genericity:

**Theorem 1.3.** *Every 1-generic real computes a properly 1-generic real.*

In fact we prove something somewhat stronger.

**Theorem 1.4.** *Every 1-generic real computes a 1-generic real that is not weakly 2-generic.*

We mention some related results: Haught [Hau86] showed that below  $\mathbf{0}'$ , the 1-generic degrees are downward closed; Martin showed that for  $n \geq 2$ , the  $n$ -generic degrees are downward dense (see [Joc80]). More such results are surveyed in [GM03], which gives some applications.

Several questions remain:

**Question 1.5.** *Does a sufficiently generic real compute a weakly 2-generic real that is not 2-generic?*

A degree is properly 1-generic if it contains a 1-generic real but no 2-generic real.

**Question 1.6.** *Does a sufficiently generic real compute a properly 1-generic Turing degree?*

We remark that as all “sufficiently” generic reals share the same “sufficiently” definable properties that are invariant under finite differences, the answers to the questions do not differ between such reals.

**1.1. Notation and terminology.** We work with Cantor space  $2^\omega$ . A *class* is a subset of  $2^\omega$ . For every  $\sigma \in 2^{<\omega}$ , let  $[\sigma]$  denote the clopen interval in  $2^\omega$  defined by  $\sigma$ , i.e.,  $\{X \in 2^\omega : \sigma \subset X\}$ . For any  $W \subseteq 2^{<\omega}$ , we let  $\mathcal{W} = \bigcup_{\sigma \in W} [\sigma]$  be the open class defined by  $W$ . An open class  $\mathcal{O} \subseteq 2^\omega$  is *enumerable* by some Turing

degree  $\mathbf{b}$  (we write that  $\mathcal{O}$  is  $\Sigma_1^0(\mathbf{b})$ ) if it is defined by some  $W$  that is computably enumerable by  $\mathbf{b}$ . We say that  $\mathcal{O}$  is *c.e.* if it is enumerable by  $\mathbf{0}$ . This terminology can be used up the arithmetic hierarchy; thus a  $\Pi_1^0(\mathbf{b})$  class is the complement of an open set enumerable by  $\mathbf{b}$  (equivalently, the set of paths through a tree computable by  $\mathbf{b}$ ); and a  $\Pi_2^0(\mathbf{b})$  class is the intersection of a countable sequence of open sets, uniformly enumerable by  $\mathbf{b}$ .

If  $W \subseteq 2^{<\omega}$ , we say that a string  $\sigma \in 2^{<\omega}$  *decides* (or *forces*)  $W$  if either  $\sigma \in W$  or no extension of  $\sigma$  is in  $W$ . If  $\sigma$  decides  $W$  then either  $[\sigma] \subseteq W$  or  $[\sigma]$  is a subset of the complement of  $W$ . A real  $X \in 2^\omega$  is 1-generic iff for every c.e. class  $\mathcal{W}$ ,  $X$  is not in the boundary of  $\mathcal{W}$  (that is, either  $X \in \mathcal{W}$  or  $X$  is in the complement of the closure of  $\mathcal{W}$ ). Similarly, a real  $X$  is weakly 2-generic if it is a member of all dense classes that are enumerable by  $\mathbf{0}'$ .

Turing functionals are codes of partial computable functions from  $2^\omega$  to  $2^\omega$ . Formally, a Turing functional is a c.e. set  $\Phi \subseteq 2^{<\omega} \times 2^{<\omega}$  that is *consistent*: for  $\sigma' \subseteq \sigma$ , if  $(\sigma', \tau') \in \Phi$  and  $(\sigma, \tau) \in \Phi$ , then  $\tau' \subseteq \tau$ . For  $\sigma \in 2^{\leq\omega}$ , let  $\Phi^\sigma = \bigcup\{\tau : \exists \sigma' \subseteq \sigma [(\sigma', \tau) \in \Phi]\}$ . Consistency is equivalent to having  $\Phi^\sigma$  be well-defined (and hence a string, finite or infinite) for every  $\sigma \in 2^{\leq\omega}$ . If  $X \in 2^\omega$  then  $\Phi^X$  is called total if it is also an element of  $2^\omega$ . We can thus indeed consider  $\Phi$  as a map  $X \mapsto \Phi^X$  defined on  $\text{dom } \Phi$ , which is the collection of all  $X$  such that  $\Phi^X$  is total.

As usual, during a construction, at each stage, all expressions involving dynamic objects are evaluated according to the state of the objects (either constructed or given) at the stage. Usual conventions apply; if during a construction we approximate a  $\Delta_2^0$  set  $A$ , then the value of  $A$  on every  $x$  is carried over from one stage to the next unless we explicitly act to change that value.

If  $\sigma, \tau \in 2^{<\omega}$  then  $\sigma\tau$  denotes the concatenation of  $\sigma$  and  $\tau$ . A digit  $i \in \{0, 1\}$  often stands for the string  $\langle i \rangle$ . If  $\sigma \in 2^{<\omega}$  and  $k < \omega$  then  $\sigma^k$  is the concatenation of  $\sigma$  with itself  $k$  times.

We let  $W_0, W_1, \dots$ , be a uniform enumeration of all c.e. subsets of  $2^{<\omega}$ . The enumeration is arranged so that at every stage  $s > 0$ , there is exactly one string  $\sigma$  and one  $e$  such that  $\sigma$  enters  $W_e$  at stage  $s$ . We also assume that if  $e \geq s$  then  $W_e$  is empty at stage  $s$ .

## 2. A POSITIVE RESULT

We prove Theorem 1.2: Let  $A$  be 1-generic over a real  $Z$ , and let  $B$  be 2-generic and computable in  $A$ . Then  $B$  is 1-generic over  $Z$ .

*Proof of Theorem 1.2.* Let  $A$  be 1-generic over  $Z$  and let  $B \leq_T A$  be 2-generic. Let  $\Phi$  be a Turing functional such that  $\Phi^A = B$ . Let  $W \subseteq 2^{<\omega}$  be c.e. in  $Z$ ; we may assume that  $W$  is closed upwards.

Suppose that  $B \notin W$ . Let  $\tilde{W} = \{\sigma \in 2^{<\omega} : \Phi^\sigma \in W\}$ . Certainly  $\tilde{W}$  is c.e. in  $Z$ . Since  $A \notin \tilde{W}$  and  $A$  is 1-generic over  $Z$ , we know that there is some  $\sigma^* \subset A$  with no extension in  $\tilde{W}$ .

Let  $U = \{\tau \in 2^{<\omega} : \neg \exists \sigma \supseteq \sigma^* [\tau \subseteq \Phi^\sigma]\}$ . The set  $U$  is co-c.e., and  $B \notin U$ , so since  $B$  is 2-generic, there is some  $\tau^* \subset B$  with no extension in  $U$ . Thus if  $\tau \supseteq \tau^*$  then there is a  $\sigma \supseteq \sigma^*$  such that  $\tau \subseteq \Phi^\sigma$ . Since  $\sigma \notin \tilde{W}$ , we have  $\Phi^\sigma \notin W$ . Since  $W$  is closed upwards,  $\tau \notin W$ . Thus  $\tau^*$  has no extension in  $W$ .  $\square$

### 3. COMPUTING PROPERLY 1-GENERIC SETS

In this section we prove theorem 1.4: Every 1-generic real  $X$  computes a 1-generic real that is not weakly 2-generic.

To do this, we construct a Turing functional  $\Gamma$  with the following properties:

- (1) There is a dense  $\Pi_2^0(\mathbf{O}')$  class  $\mathfrak{A}$  that is contained in the domain of  $\Gamma$ , and whose image under  $\Gamma$  consists of 1-generic sets.
- (2) The range of  $\Gamma$  is contained in a nowhere dense  $\Pi_1^0(\mathbf{O}')$  class.

To see that this suffices, assume that  $X$  is weakly 2-generic (if  $X$  is not weakly 2-generic then we are of course done). The real  $X$  is an element of any dense  $\Sigma_1^0(\mathbf{O}')$  class, hence of any countable intersection of such classes; so  $X \in \mathfrak{A}$ . Then  $\Gamma^X$  (which is computable by  $X$ ) is 1-generic by (1), and is not weakly 2-generic because by (2), it misses a dense open set enumerable by  $\mathbf{O}'$ .

#### 3.1. Discussion.

3.1.1. *Getting property 1.* To make the image of  $\mathfrak{A}$  under  $\Gamma$  consist of 1-generic sets, for each  $e < \omega$ , we must construct a dense, open set  $\mathcal{S}_e$  such that for all  $\sigma \in \mathcal{S}_e$ ,  $\Gamma^\sigma$  decides  $W_e$ ; and further we must make the sequence  $\mathcal{S}_0, \mathcal{S}_1, \dots$  uniformly enumerable by  $\mathbf{O}'$ , so that we can define  $\mathfrak{A} = \bigcap_e \mathcal{S}_e$ . We need to ensure that  $\mathfrak{A}$  is dense; by Baire's theorem, it is sufficient to ensure that each  $\mathcal{S}_e$  is dense.

Consider  $W_0$ . A simple plan for meeting it would be setting  $\mathcal{S}_0 = 2^\omega$  and acting as follows: if  $W_0$  is empty, do nothing; if there is some  $\tau \in W_0$ , let  $\Gamma^\diamond = \tau$ . Now move to  $W_1$ . Of course, this plan is not effective, so we must use the priority method for our construction. Again, a naïve approach would be as follows: While  $W_0$  is empty, do nothing, and let weaker requirements ( $W_1, W_2, \dots$ ) act if they want to. If some string  $\tau$  enters  $W_0$  then injure the weaker requirements and set  $\Gamma^\diamond = \tau$ . The problem here is that we cannot cancel the axioms that our work for  $W_1, W_2, \dots$  had us enumerating into  $\Gamma$ , so if we want to keep  $\Gamma$  consistent, we cannot make the definition we like. The solution is to break up the playing ground into pieces, let weaker requirements work on some of the pieces, and make sure that there is sufficient room for the stronger requirement to act if necessary.

Here is the strategy for  $W_0$ . In the beginning, we mark the interval  $2^\omega = [\langle \rangle]$  to work on  $W_0$ . We break the interval up into infinitely many disjoint subintervals whose union is dense in  $2^\omega$ , say  $[1], [01], [001], [0001], \dots$ . For the time being, each such subinterval believes it has met the  $W_0$ -requirement by forcing its image under  $\Gamma$  into the complement of  $W_0$ , simply because  $W_0$  is still empty. So we can be generous and let each subinterval work for the next requirement  $W_1$ .

At a later stage, some string  $\tau$  enters  $W_0$ . Only finitely many subintervals have been spoiled for  $W_0$ ; so we can define  $\Gamma$  to be  $\tau$  everywhere else. On the spoiled intervals, we need to work again for  $W_0$ ; since definitions of  $\Gamma$  have been made on possibly small subsubintervals, we need to break the spoiled region into small intervals on which we individually work on  $W_0$ .

We let  $\mathcal{S}_0$  be the collection of intervals  $[\sigma]$  that are “good” for  $W_0$ , which are those intervals on which we ensure that  $\Gamma^\sigma$  meets  $W_0$ , and those at which we had a correct belief that  $[\Gamma^\sigma] \cap W_0 = \emptyset$ . This set will in fact be d.c.e., and so certainly  $\Sigma_2^0$ ; and reals in  $\mathcal{S}_0$  will satisfy the  $W_0$  requirement. We need to ensure that  $\mathcal{S}_0$  is dense; this will hold because we break our intervals up into finer and finer subintervals (each time our hopes for an easy win are dashed).

The strategy for weaker  $W_e$  is similar, except that of course we need to take into consideration injury by stronger requirements.

**3.1.2. Getting property 2.** To ensure that the range of  $\Gamma$  is nowhere dense, we could, whenever we define some axiom  $\Gamma^\sigma = \tau$ , pick some extension  $\tau'$  of  $\tau$  and declare that no value of  $\Gamma$  may ever extend  $\tau'$ . This straightforward approach, however, interferes with the priority mechanism that ensures property (1), in the following way. Suppose that we mark some interval  $[\sigma_0]$  for  $W_1$ , and later define  $\Gamma^{\sigma_1} = \tau$  for some  $\sigma_1 \supset \sigma_0$ , marking  $[\sigma_1]$  for  $W_2$ . We then declare that the range of  $\Gamma$  must be disjoint from  $[\rho]$ , where  $\rho \supset \tau$ . A later  $W_0$  action elsewhere invalidates  $[\sigma_0]$ 's marking, so we mark  $[\sigma_1]$  for  $W_0$ . Then, some string extending  $\rho$  enters  $W_0$ , but  $W_0$  is prohibited from winning by directing  $\Gamma$  through that string on a subinterval of  $[\sigma_1]$ . We will indeed direct  $\Gamma$  to go through some extension  $\tau'$  of  $\tau$  that is incomparable with  $\rho$ , and this presumably will give us another chance of attacking  $W_0$ ; but this process may repeat itself, since following the straightforward approach compels us to first declare some extension  $\rho'$  of  $\tau'$  disjoint from the range of  $\Gamma$ . After infinitely many failed attempts at meeting  $W_0$  we have a real in the range of  $\Gamma$  belonging to the closure of  $\mathcal{W}_0$  but not to  $\mathcal{W}_0$  itself.

This in fact *must* happen, because we made the collection of prohibited intervals a dense c.e. class, ensuring that no element of the range of  $\Gamma$  is even weakly 1-generic (indeed, the recursion theorem and the “slowdown lemma” imply that there is some  $e$  such that  $W_e$  is the set of prohibited intervals, and that every  $\sigma$  is enumerated into  $W_e$  only after it was declared prohibited). The solution is to use the priority mechanism that was introduced for getting property (1). When we define  $\Gamma^\sigma = \tau$  for meeting  $W_e$ , we define one extension to be prohibited with priority  $e$ . This prohibition can be ignored by strings  $\sigma' \subset \sigma$  that are working for stronger  $W_{e'}$ . The whole mechanism does the work for us, so we in fact do not need to use the word “prohibited” during the construction, just to make  $\Gamma^\sigma$  long enough.

**3.2. Construction.** Here is the formal construction.

We build the partial computable functional  $\Gamma : 2^{<\omega} \rightarrow 2^{<\omega}$  by stages. Rather than just being a relation,  $\Gamma$  (as a collection of pairs of strings) will actually be a partial function on strings. We use  $\Gamma(\sigma)$  to denote the value of  $\Gamma$  on  $\sigma$  in this sense; recall that this is distinct from  $\Gamma^\sigma$  which we defined in section 1.1. Thus  $\Gamma(\sigma)$ , if it is ever defined during the construction, is set for ever, whereas  $\Gamma^\sigma$  may keep changing as we keep defining  $\Gamma(\sigma')$  for strings  $\sigma' \subseteq \sigma$ . We let  $\text{dom } \Gamma$  (and its version at stage  $s$ ) also denote the domain of  $\Gamma$  as a function on strings. This should not lead to confusion with the domain of  $\Gamma$  as a function on reals. During the construction, we make sure that if we define  $\Gamma(\sigma)$  at stage  $s$  then no extension of  $\sigma$  is in  $\text{dom } \Gamma[s]$ . This implies that  $\Gamma^\sigma = \Gamma(\sigma)$  from the moment the latter is defined. Of course, to keep  $\Gamma$  consistent, if we want to define  $\Gamma(\sigma) = \tau$  at stage  $s$  then  $\Gamma^\sigma$ , as calculated at the beginning of the stage, must be an initial segment of  $\tau$ .

Further, we will make sure that if  $\sigma$  and  $\sigma'$  in  $\text{dom } \Gamma$  are incompatible, then so are  $\Gamma(\sigma)$  and  $\Gamma(\sigma')$ . Towards getting property 2, we will always have  $\Gamma(\sigma) = \nu_\sigma 1$  for some string  $\nu_\sigma$ , for all  $\sigma \in \text{dom } \Gamma$ .

Towards getting property 1, we will mark certain strings in  $2^{<\omega}$ , and we will define  $\Gamma$  on these marked strings. There will be two possible types of markings for strings: *active* or *satisfied*. We will only mark a string  $\sigma$  as *active for*  $W_e$  at stage

$s$  if there is no  $\tau \in W_e[s]$  which extends  $\Gamma^\sigma[s]$ , and if either  $\sigma$  is satisfied for  $W_{e-1}$  or if a proper substring of  $\sigma$  is active for  $W_{e-1}$ . We will only mark a string  $\sigma$  as *satisfied for  $W_e$*  at stage  $s$  if there is some  $\tau \in W_e$  such that  $\Gamma^\sigma \supset \tau$  and if  $\sigma$  is a proper extension of a string that was active for  $W_e$  at stage  $s-1$ .

In the beginning, the entire space  $2^\omega = [\langle \rangle]$  is marked *active* for  $W_0$ . (When we mark a string  $\sigma$  for  $W_e$  we also say that the clopen set  $[\sigma]$  is marked for  $W_e$ .)

At stage  $s+1$ :

Assume that at the end of stage  $s$ , all the properties desired of  $\Gamma$  as described above hold. Assume that if a string  $\sigma$  is active for  $W_e$ , then strings of the form  $\sigma 0^k 1$  are either active for  $W_{e+1}$  or have no extension that is active for any  $W_{e'}$ . Finally, for all  $\sigma \in 2^{<\omega}$ , assume that  $\Gamma^\sigma = \Gamma^{\sigma'}$  for some  $\sigma' \subseteq \sigma$  which is active for some  $W_{e'}$  at stage  $s$ .

**Step 1.** A string  $\tau$  is enumerated into some  $W_{e'}$ . Suppose that there is some  $\sigma$  that is active for  $W_e$ , such that  $\Gamma^\sigma \subseteq \tau$  (there will be at most one such  $\sigma$ ). Do the following:

- Remove all markings of strings  $\sigma' \supseteq \sigma$ . Note that such markings will pertain to  $W_{e'}$  with  $e' \geq e$ .
- Let  $m$  be larger than any number mentioned so far in the construction.
- Define  $\Gamma(\sigma 0^s) = \tau 1^m$ . Mark  $[\sigma 0^s]$  as satisfied for  $W_e$  and as active for  $W_{e+1}$ .
- For every  $\sigma' \in 2^m$  extending  $\sigma 0^k 1$  for some  $k < s$ , mark  $[\sigma']$  as active for  $W_e$ ; find some  $\tau_{\sigma'} \supset \Gamma^{\sigma'}[s]$  of length  $m$  that is incompatible with  $\tau 1^m$ , and define  $\Gamma(\sigma') = \tau_{\sigma'} \sigma' 1$ . Note that the length of  $\Gamma^{\sigma'}[s]$  is much smaller than  $m$  because it was defined prior to the current stage. Note also that, by choice of  $m$ , none of these  $\sigma'$  have ever been marked at an earlier stage, or have any extensions which have been marked at an earlier stage. Finally, note that any extension of  $\sigma$  that was active (for any  $W_{e'}$ ) at stage  $s$  has an extension  $\sigma'$  that is active for  $W_e$  at stage  $s+1$ .

Note that all the properties described at the beginning of the stage still hold.

**Step 2.** Inductively for  $e \leq s$ , for every  $[\sigma]$  that is currently active for  $W_e$  (either it was already active at the beginning of the stage, or was marked as active during the stage so far), for  $k \leq s$ , if  $[\sigma 0^k 1]$  is not marked (and hence by assumption and induction nor is any extension of it), then mark it as active for  $W_{e+1}$  and define  $\Gamma(\sigma 0^k 1) = \Gamma^\sigma 0^k 1 1$ .

This completes the construction. An illustration of a typical turn of events is given in Figures 1-3.

**3.3. Verification.** In the construction, for any string  $\sigma$ , there is indeed at most one stage  $s$  where we define  $\Gamma(\sigma)$ . When we do this it is because  $\sigma$  was marked active for some  $W_e$  at stage  $s$ . (Recall that whenever we mark  $\sigma$  satisfied for some  $W_e$ , then we also mark  $\sigma$  active for  $W_{e+1}$ .) Conversely, whenever a string  $\sigma$  is marked active or satisfied at a stage  $s$ , we define  $\Gamma(\sigma)$  at that stage.

We only take action to mark a string  $\sigma$  as active or satisfied for  $W_e$  at stage  $s+1$  if there is some substring of  $\sigma$  that is active or satisfied for  $W_{e-1}$  at stage  $s+1$ , and there is no extension of  $\sigma$  on which  $\Gamma$  is already defined (and hence no

FIGURE 1. An interval is active for  $W_1$ , some subintervals are active for  $W_2$ , and those subintervals have subintervals active for  $W_3$ :



FIGURE 2. Action was taken for the middle subinterval. Its mark and the marks of its subintervals were removed. It is broken into smaller subintervals; on one we have a positive win for  $W_2$  and so it is satisfied for  $W_2$  and active for  $W_3$ ; on the rest we go back to work on  $W_2$ , so they are active for  $W_2$

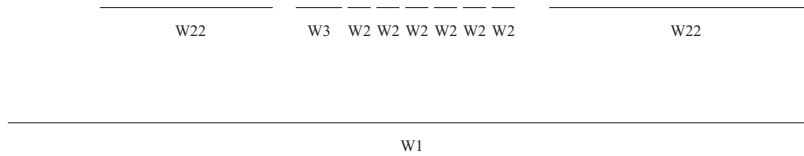
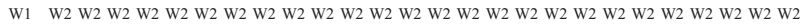


FIGURE 3. Later, action is taken for the original  $W_1$ -interval. The previous  $W_2$  and  $W_3$  markings are cancelled, and the original interval is broken into small subintervals:



extension of  $\sigma$  has ever been marked at an earlier stage). Thus, if some string  $\sigma$  has its markings removed at a stage, it will never again be marked at a later stage.

We define  $S_e$ , the success set for  $W_e$ . At the beginning of a stage  $s$ , the approximation for this success set is  $S_e[s]$ . We let  $S_e^+[s]$  be the collection of those strings that are satisfied for  $W_e$  at stage  $s$ . We let  $S_e^-$  be the collection of strings that are active for  $W_e$  at stage  $s$ . We let  $S_e[s] = S_e^+ \cup S_e^-$ . The idea here is simple: if  $\sigma$  is satisfied for  $W_e$  then  $\Gamma_\sigma$  extends some string in  $W_e$ . Thus a string will be later removed from  $S_e^+$  if action is taken for some  $\sigma' \subset \sigma$  which is active for  $W_{e'}$  for some  $e' < e$ . On the other hand, if  $\sigma$  is active for  $W_e$  at stage  $s$ , then no extension of  $\Gamma^\sigma$  has yet been found in  $W_e$ ; so if it remains active for ever, then it negatively meets

the  $W_e$ -requirement. Otherwise, the mark is removed and  $\sigma$  leaves  $S_e^-$ . Of course the string can leave  $S_e^-$  also if some  $\sigma' \subset \sigma$  acts.

So by the construction, since each string  $\sigma$  can become marked at most once, each  $S_e$  is d.c.e. (uniformly). [In fact  $\bigcup_e S_e$  is d.c.e.] Thus  $\mathfrak{A} = \bigcap_e S_e$  is a  $\Pi_2^0(\mathbf{0}')$  class.

By the construction, for  $\sigma, \sigma' \in \text{dom } \Gamma$  (i.e., for  $\sigma$  and  $\sigma'$  that were ever marked at some stage), there exist  $\nu_\sigma$  and  $\nu_{\sigma'} \in 2^{<\omega}$  such that  $\Gamma(\sigma) = \nu_\sigma 1$ ,  $\Gamma(\sigma') = \nu_{\sigma'} 1$ . If  $\sigma \perp \sigma'$ , then we ensured  $\Gamma(\sigma) \perp \Gamma(\sigma')$ , in fact, we ensured that  $\nu_\sigma \perp \nu_{\sigma'}$ .

**Lemma 3.1.** *Suppose that  $\sigma \in \bigcup_e S_e$  (at the end of time). Then for no  $\sigma'$  do we have  $\Gamma(\sigma') \supseteq \nu_\sigma 0$ .*

*Proof.* Suppose for a contradiction that there is some  $\sigma'$  such that  $\Gamma(\sigma') \supseteq \nu_\sigma 0$ . We cannot have  $\sigma$  and  $\sigma'$  comparable; if  $\sigma' \supset \sigma$  then  $\Gamma(\sigma') \supset \Gamma(\sigma) = \nu_\sigma 1$ ; and if  $\sigma' \subset \sigma$  then  $\Gamma(\sigma') \subseteq \Gamma(\sigma)$ .

Thus  $\sigma' \perp \sigma$ . But then, by our construction,  $\nu_{\sigma'} \perp \nu_\sigma$ , and so we cannot have  $\Gamma(\sigma') = \nu_{\sigma'} 1 \supseteq \nu_\sigma 0$ .  $\square$

We now are ready to verify properties (1) and (2). For simplicity of notation, we let (at every stage)  $S_{-1} = \{\langle \rangle\}$ .

**Lemma 3.2.** *Each  $S_e$  is dense, and so  $\mathfrak{A} = \bigcap_e S_e$  is dense too.*

*Proof.* We take some  $\sigma \in S_{e-1}$  and show that  $S_e$  is dense in  $[\sigma]$ . Suppose that  $\sigma$  is put into  $S_{e-1}$  at stage  $s_0$ .

We first note that if some  $[\sigma'] \subseteq [\sigma]$  is ever marked active for  $W_e$  at a stage  $s \geq s_0$ , then there is a subinterval of  $[\sigma']$  that is (permanently) in  $S_e$ . This is because either no action is taken for  $\sigma'$ , in which case the string  $\sigma'$  is in  $S_e$ , or action is taken for  $\sigma'$  at some stage  $s$ , in which case  $\sigma' 0^s$  is in  $S_e$ . The point is that these markings cannot be eliminated by action below  $\sigma'$ , because such action would remove  $\sigma$  from  $S_{e-1}$  (note that there are no marked strings between  $\sigma$  and  $\sigma'$ ).

Let  $\rho \supset \sigma$ . By the above argument, to show that  $[\rho] \cap S_e \neq \emptyset$ , it suffices to show that there exists some  $[\sigma'] \subseteq [\rho]$  such that  $\sigma'$  is compatible with  $\rho$  and  $[\sigma']$  is marked active for  $W_e$  at some point. If  $\sigma \in S_{e-1}^+$ , then at the same stage when  $\sigma$  was marked as satisfied for  $W_{e-1}$ ,  $\sigma$  was also marked as active for  $W_e$ , so we are done. Suppose  $\sigma \in S_{e-1}^-$ . Then there was some stage after which  $\sigma$  was always active for  $W_{e-1}$ . Since  $\rho \supset \sigma$ , there is some  $k \geq 0$  such that  $\rho$  is compatible with  $\sigma 0^k 1$ . Then at the least stage  $s$  such that  $k \leq s$  and  $\sigma$  was active for  $W_{e-1}$  at stage  $s$ ,  $\sigma 0^k 1$  was marked active for  $W_e$  in step 2 of the construction.  $\square$

**Lemma 3.3.** *Suppose that  $X \in S_e$ . Then some initial segment of  $\Gamma^X$  determines  $W_e$ .*

(Note that we do not assume that  $\Gamma^X$  is total.)

*Proof.* Suppose that  $\sigma \in S_e$  and  $\sigma \subset X$ . If  $\sigma \in S_e^+$ , then  $\Gamma(\sigma)$  extends some string in  $W_e$ . If  $\sigma \in S_e^-$ , then no  $\tau \supset \Gamma(\sigma)$  is ever enumerated into  $W_e$ .  $\square$

**Corollary 3.4.**  *$\mathfrak{A} \subseteq \text{dom } \Gamma$  and every  $Y \in \Gamma[\mathfrak{A}]$  is 1-generic.*

*Proof.* The second part follows immediately from Lemma 3.3 (and the first part). For the first part, apply Lemma 3.3 to the sets  $W_{e_n} = 2^{\geq n}$  for  $n \in \omega$ .  $\square$



**Lemma 3.5.**  $\Gamma[2^\omega]$  is contained in a nowhere dense  $\Pi_1^0(\mathbf{0}')$  class.

*Proof.* Let  $T$  be the downward closure of the range of  $\Gamma$ , this time viewed as a function on strings. That is, let

$$T = \{\tau : \exists \sigma [\tau \subseteq \Gamma(\sigma)]\}.$$

$T$  is c.e., and so  $[T]$ , the class of paths through  $T$ , is a  $\Pi_1^0(\mathbf{0}')$  class that contains the image of  $\Gamma$  on  $2^\omega$ .

The class  $[T]$  is closed, so to show that it is nowhere dense, it suffices to show that it does not contain any interval. Suppose for a contradiction that  $[\rho]$  is an interval contained in  $[T]$ , which means that every extension of  $\rho$  is in  $T$ . By the definition of  $T$ , there is some  $\sigma \in \text{dom } \Gamma$  such that  $\Gamma(\sigma) \supseteq \rho$ . By Lemma 3.2, there is some  $\sigma'$  extending  $\sigma$  which is in  $\bigcup_e S_e$ . Then  $\nu_{\sigma'} \supseteq \tau$ , and by Lemma 3.1,  $\nu_{\sigma'}$  is not on  $T$ ; this is a contradiction.  $\square$

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