

# Prime Models of Theories of Computable Linear Orderings

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## Abstract

We answer a long-standing question of Rosenstein by exhibiting a complete theory of linear orderings with both a computable model and a prime model, but no computable prime model. The proof uses the relativized version of the concept of limitwise monotonic function.

A linear ordering is *computable* if both its domain and its order relation are computable; it is *computably presentable* if it is isomorphic to a computable linear ordering. (There are natural generalizations of these notions to other kinds of structures; see for instance [10] for details.) There is a large body of research on computable linear orderings ([4] gives an extensive overview). Much of this work has been focused on the relationship between classical and effective order types, but it is also interesting to take an approach inspired by classical model theory and study the relationship between effective order types and theories of linear orderings, asking, for instance, what kinds of computable linear orderings exist within the models of a given theory of linear orderings.

Taking this approach, Rosenstein ended his book *Linear Orderings* [12] by asking whether a complete theory of linear orderings with a computable model and a prime model must have a computable prime model. This question was repeated in the “Problem Sessions” section of [11] (Problem 7.20), and it has recently been included in [3] (Question 3.18). In this paper, we give a negative answer to Rosenstein’s question.

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We assume familiarity with basic notions and results from computability theory and model theory (standard references are [13] and [5], respectively). For structures  $\mathcal{A}$  and  $\mathcal{B}$  in the same language, we will write  $\mathcal{A} \approx_k \mathcal{B}$  to mean that player  $\exists$  has a winning strategy in the Ehrenfeucht-Fraïssé game  $\text{EF}_k[\mathcal{A}, \mathcal{B}]$  of length  $k$ . Recall that if  $\mathcal{A} \approx_k \mathcal{B}$  for all  $k \in \omega$  then  $\mathcal{A} \equiv \mathcal{B}$ . See Section 3.3 of [5] for details.

We will use the following relativized version of a notion due originally to Khisamiev [6].

**1 Definition.** Let  $\mathbf{a}$  be a Turing degree. A function  $f$  is  *$\mathbf{a}$ -limitwise monotonic* if there exists an  $\mathbf{a}$ -computable binary function  $g$  such that, for all  $n, s \in \omega$ ,  $g(n, s) \leq g(n, s+1)$  and  $\lim_t g(n, t)$  exists and is equal to  $f(n)$ . If  $\mathbf{a} = \mathbf{0}$  then  $f$  is *limitwise monotonic*.

Limitwise monotonic functions were first used by Khisamiev [6] in studying the question of which abelian  $p$ -groups are computably presentable. Later, in [7], he used them to obtain the analog of the main result of this paper for abelian groups, namely that there exists a complete theory of abelian groups with both a computable model and a prime model, but no computable prime model. (See [8] for a survey of these and related results.)

Khousainov, Nies, and Shore [9], who introduced the term “limitwise monotonic”, used limitwise monotonic functions in the construction of an uncountably categorical but not countably categorical theory all of whose countable models are computably presentable except for the prime model. The relativized version given above has been used by Coles, Downey, and Khousainov [2] in the construction of a computable linear ordering with a  $\Pi_2^0$  initial segment that is not computably presentable.

It was shown in [9] that there exists a  $\Delta_2^0$  set that is not the range of a limitwise monotonic function. As noted in [2], that proof relativizes, so for any degree  $\mathbf{a}$  there exists a set that is  $\Delta_2^0$  relative to  $\mathbf{a}$  but is not the range of an  $\mathbf{a}$ -limitwise monotonic function. In particular, there exists a  $\Delta_3^0$  set that is not the range of a  $\mathbf{0}'$ -limitwise monotonic function. We will only need the weaker fact that there exists a  $\Sigma_3^0$  set that is not the range of a  $\mathbf{0}'$ -limitwise monotonic function.

We will also use the following notions.

**2 Definition.** Let  $\mathfrak{K}$  be the set of all linear orderings with endpoints such that every element other than the left endpoint has an immediate predecessor and every element other than the right endpoint has an immediate successor. ( $\mathfrak{K}$  consists of all nonempty finite linear orderings and all linear orderings of type  $\omega + (\omega^* + \omega) \cdot \alpha + \omega^*$ , where  $\alpha$  is any linear ordering.)

By a *block* in a linear ordering  $\mathcal{R}$  we will mean an interval  $[a, b] \in \mathfrak{K}$  of  $\mathcal{R}$  such that, in  $\mathcal{R}$ ,  $a$  has no immediate predecessor and  $b$  has no immediate successor. If a block has order type  $\alpha$  then we call it an  $\alpha$ -block. (This is not the standard use of the word *block*, which is usually reserved for what we call  $n$ -blocks,  $n \in \omega$ .) Note that there exists a formula  $\varphi(x, y)$  in the language of linear orderings such that, for any linear ordering  $\mathcal{R}$  and  $a, b \in \mathcal{R}$ ,  $\mathcal{R} \models \varphi(a, b) \Leftrightarrow [a, b]$  is a block in  $\mathcal{R}$ .

Let  $\alpha_0, \alpha_1, \dots$  be order types. The *shuffle* of  $\{\alpha_0, \alpha_1, \dots\}$  is the (unique) order type of a linear ordering obtained by partitioning the rationals into dense sets  $A_0, A_1, \dots$  and, for each  $i \in \omega$ , replacing each element of  $A_i$  by a linear ordering of type  $\alpha_i$ .

**3 Theorem.** *There exists a complete theory of linear orderings with a computable model and a prime model, but no computable prime model.*

*Proof.* Let  $S \subset \omega - \{0, 1\}$  be a  $\Sigma_3^0$  set that is not the range of a  $\mathbf{0}'$ -limitwise monotonic function. Let  $\mathcal{L}$  and  $\widehat{\mathcal{L}}$  be linear orderings whose order types are the shuffles of  $S$  and  $S \cup \{\omega + \omega^*\}$ , respectively, and let  $T$  be the theory of  $\mathcal{L}$ . We will show that  $\mathcal{L}$  is the prime model of  $T$ ,  $\widehat{\mathcal{L}}$  is also a model of  $T$ ,  $\mathcal{L}$  is not computably presentable, and  $\widehat{\mathcal{L}}$  is computably presentable.

**4 Lemma.** *Let  $\mathcal{L}'$  be a nonempty linear ordering. The following are equivalent.*

1. *Each element of  $\mathcal{L}'$  is in some block,  $\mathcal{L}'$  has no  $n$ -blocks for  $n \notin S$ ,  $\mathcal{L}'$  has neither a leftmost nor a rightmost block, and, for any pair of distinct blocks in  $\mathcal{L}'$  and  $n \in S$ , there is an  $n$ -block between the two given blocks.*
2.  *$\mathcal{L}'$  is a model of  $T$ .*
3.  *$\mathcal{L}$  can be elementarily embedded in  $\mathcal{L}'$ .*

*Proof.* Obviously  $3 \Rightarrow 2$ , and  $2 \Rightarrow 1$  since each of the statements in 1 can be expressed by a sentence in  $T$ . We show that  $1 \Rightarrow 3$ .

Fix  $\mathcal{L}'$  satisfying 1. Clearly, there is an embedding  $f : \mathcal{L} \rightarrow \mathcal{L}'$  sending each  $n$ -block of  $\mathcal{L}$  to an  $n$ -block of  $\mathcal{L}'$ . Let  $k \in \omega$  and let  $\vec{a}$  be a tuple of elements of  $\mathcal{L}$ . For each  $\mathcal{R} \in \mathfrak{K}$  and  $k \in \omega$ ,  $\mathcal{R} \approx_k \mathcal{F}$  for all sufficiently large finite linear orderings  $\mathcal{F}$  (this is Exercise 6.11 in [12]). Using this fact, it is easy to show that  $(\mathcal{L}, \vec{a}) \approx_k (\mathcal{L}', f(\vec{a}))$ . (Basically, the fact that, for each block  $B$  in  $\mathcal{L}'$ , there is a dense set of blocks in  $\mathcal{L}$  each of which is  $\approx_k B$  gives player  $\exists$  an obvious winning strategy in the game  $\text{EF}_k[(\mathcal{L}, \vec{a}), (\mathcal{L}', f(\vec{a}))]$ .) Since  $k$  was arbitrary,  $(\mathcal{L}, \vec{a}) \equiv (\mathcal{L}', f(\vec{a}))$ . Since  $\vec{a}$  was arbitrary,  $f$  is an elementary embedding.  $\square$

**5 Corollary.**  $\mathcal{L}$  is the prime model of  $T$ .

**6 Corollary.**  $\widehat{\mathcal{L}}$  is a model of  $T$ .

The following lemma was proved in [2].

**7 Lemma** (Coles, Downey, and Khoussainov). *For any computably presentable linear ordering  $\mathcal{R}$  obtained by replacing each element of the countable dense linear ordering by a finite block, the set of all  $n \in \omega$  such that  $\mathcal{R}$  contains an  $n$ -block is the range of a  $\mathbf{0}'$ -limitwise monotonic function.*

**8 Corollary.**  $\mathcal{L}$  is not computably presentable.

In [1], Ash, Jockusch, and Knight used a worker argument to show that if  $A \subseteq \omega - \{0\}$  is  $\Sigma_3^0$  then the shuffle of  $A \cup \{\omega\}$  is computably presentable. It is not hard to adapt their proof to show that  $\widehat{\mathcal{L}}$  is computably presentable; we give a different proof avoiding the use of workers.

**9 Lemma.**  $\widehat{\mathcal{L}}$  is computably presentable.

*Proof.* We will build a computable linear ordering  $L = (|L|, \prec) \cong \widehat{\mathcal{L}}$  in stages, using a recursive procedure. At each stage we will have a finite collection of pairs of elements which we currently want to be the endpoints of blocks of given types. Between each pair of adjacent blocks, we will start a new version of the construction, thus guaranteeing that any type of block that exists in  $L$  exists densely in  $L$ . By doing this, we ensure that all we have to require of our basic construction is that it build an  $\omega + \omega^*$ -block and that it place each element of  $L$  in some block of type  $n \in S$  or  $\omega + \omega^*$ .

Let  $\Psi$  be a computable relation such that  $S(n) \Leftrightarrow \exists t \forall u \exists v (\Psi(n, t, u, v))$  for all  $n \in \omega$ . The idea of the construction is as follows.

We build an  $\omega + \omega^*$ -block. For each  $n, t \in \omega$ , we begin to build an  $n$ -block  $B$ . For each  $u \in \omega$ , we start to make  $B$  into an  $\omega + \omega^*$ -block, but preserving the original  $n$ -block within it, until we find a  $v$  such that  $\Psi(n, t, u, v)$ . If such a  $v$  is found then we declare the part that has been added to the original  $n$ -block in  $B$  to be a new  $\omega + \omega^*$ -block, thus making  $B$  again an  $n$ -block.

There are two possible fates for  $B$ . If  $\exists u \forall v (\neg \Psi(n, t, u, v))$  then  $B$  becomes an  $\omega + \omega^*$ -block. Otherwise,  $B$  remains an  $n$ -block in the limit. Thus, if  $\exists t \forall u \exists v (\Psi(n, t, u, v))$  then we build at least one  $n$ -block, while otherwise we build no  $n$ -blocks.

We now proceed with the construction. When we mention new numbers we mean numbers larger than any that have previously appeared in the construction.

*stage 0.* Choose two new numbers  $x$  and  $y$ , add them to  $|L|$ , declare that  $x \prec y$ , and declare that we want  $[x, y]$  to be an  $\omega + \omega^*$ -block.

*stage  $s + 1$ .* For each pair of numbers  $x$  and  $y$  for which we currently want  $[x, y]$  to be an  $\omega + \omega^*$ -block, if  $[x, y]$  currently has  $n$  many elements  $x = a_0 \prec \cdots \prec a_{n-1} = y$  then pick a new number  $z$  and add it to  $L$  between  $a_{\lfloor n/2 \rfloor}$  and  $a_{\lfloor n/2 \rfloor + 1}$ .

For each  $1 < n \leq s$ , choose new numbers  $x_{n,s}$  and  $y_{n,s}$ , add them to  $|L|$ , declare that  $x_{n,s} \prec y_{n,s} \prec z$  for all  $z$  already in  $L$ , declare that we want  $[x_{n,s}, y_{n,s}]$  to be an  $n$ -block, and add  $n - 2$  many new numbers to  $[x_{n,s}, y_{n,s}]$ .

Let  $n, t \leq s$  be such that we currently want  $[x_{n,t}, y_{n,t}]$  to be an  $n$ -block. If, for some  $u \leq s$ ,  $\neg \Psi(n, t, u, v)$  for all  $v \leq s$ , then, for the least such  $u$ , choose a new number  $z_{n,t,u}$ , place it in  $L$  immediately to the right of  $y_{n,t}$ , declare that we want  $[x_{n,t}, z_{n,t,u}]$  to be an  $\omega + \omega^*$ -block, declare that we no longer want  $[x_{n,t}, y_{n,t}]$  to be an  $n$ -block, and add  $n - 1$  many new numbers to  $[y_{n,t}, z_{n,t,u}]$ . (Note that this last action ensures that when numbers are added to  $[x_{n,t}, z_{n,t,u}]$  to attempt to make it into an  $\omega + \omega^*$ -block, they will be added to the right of  $y_{n,t}$ .)

Let  $n, t, u \leq s$  be such that  $z_{n,t,u}$  is defined and we currently want  $[x_{n,t}, z_{n,t,u}]$  to be an  $\omega + \omega^*$ -block. If  $\Psi(n, t, u, s)$  then let  $w_{n,t,u}$  be the current immediate successor of  $y_{n,t}$ , declare that we want  $[x_{n,t}, y_{n,t}]$  to be an  $n$ -block, and declare that we want  $[w_{n,t,u}, z_{n,t,u}]$  to be an  $\omega + \omega^*$ -block.

For each  $x \prec y \prec w \prec z$  such that we currently want  $[x, y]$  and  $[w, z]$  to be blocks and there are currently no elements in  $(y, w)$ , begin a new version of the construction, following the steps described above but placing all elements in  $(y, w)$ . Similarly, for the  $\prec$ -least element  $a$  currently in  $L$ , begin a new version of the construction to the left of  $a$ , and for the  $\prec$ -greatest element  $b$  currently in  $L$ , begin a new version of the construction to the right of  $b$ .

This completes the construction. Since every element we add to  $L$  is a new number,  $L$  is computable. Furthermore, each element of  $L$  is in some block, each block we build is either finite or of type  $\omega + \omega^*$ , and we build at least one  $\omega + \omega^*$ -block. Since we repeat the construction between each pair of blocks, all we have to show to verify that  $L \cong \widehat{\mathcal{L}}$  is that we build an  $n$ -block if and only if  $n \in S$ .

Let  $n, t \in \omega$ ,  $n > 1$ . If  $\forall v (\neg \Psi(n, t, u, v))$  holds for some  $u \in \omega$  then, for the least such  $u$ , we eventually define  $z_{n,t,u}$ . (This happens at the least stage  $s > n, t, u$  such that  $\forall u' < u \exists v < s (\Psi(n, t, u', v))$ .) We declare that we want  $[x_{n,t}, z_{n,t,u}]$  to be an  $\omega + \omega^*$ -block, and we never change our mind, so, in this case,  $[x_{n,t}, y_{n,t}]$  is contained in an  $\omega + \omega^*$ -block.

Otherwise, for each  $z_{n,t,u}$  we define, we eventually define  $w_{n,t,u}$  and  $[w_{n,t,u}, z_{n,t,u}]$

becomes an  $\omega + \omega^*$ -block. In this case,  $[x_{n,t}, y_{n,t}]$  is an  $n$ -block.

If  $n \notin S$ ,  $n > 1$ , then, by the definition of  $\Psi$ ,  $\forall t \exists u \forall v (\neg \Psi(n, t, u, v))$ , so  $[x_{n,t}, y_{n,t}]$  is contained in an  $\omega + \omega^*$ -block for each  $t \in \omega$ , and hence we build no  $n$ -blocks. On the other hand, if  $n \in S$  then, for some  $t \in \omega$ ,  $\forall u \exists v (\Psi(n, t, u, v))$ , which means that  $[x_{n,t}, y_{n,t}]$  is an  $n$ -block. Thus we build an  $n$ -block if and only if  $n \in S$ .  $\square$

The theorem follows from Corollaries 5, 6, and 8 and Lemma 9.  $\blacksquare$

We conclude by remarking that sets that are not the range of a limitwise monotonic function can be used to build other complete theories having both a computable and a prime model, but no computable prime model. Khisamiev's construction of such a theory of abelian groups has already been mentioned; the following is another example. Let  $A \subset \omega - \{0\}$  be a  $\Sigma_2^0$  set that is not the range of a limitwise monotonic function. Let  $\mathcal{E}$  be the equivalence structure consisting of one equivalence class of size  $n$  for each  $n \in A$  and let  $T$  be the theory of  $\mathcal{E}$ . It is not hard to check that  $\mathcal{E}$  is the prime model of  $T$  and is not computably presentable, while the structure consisting of one equivalence class of size  $n$  for each  $n \in A$  and  $\aleph_0$  many equivalence classes of size  $\aleph_0$  is also a model of  $T$  and is computably presentable.

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