The Dimension Spectrum Conjecture for Lines

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Kolmogorov Complexity

Definition

Fix a universal Turing machine U. Let u be a finite binary string. The *Kolmogorov complexity of u* is

$$K(u) = \min\{|\pi| \mid \pi \in \{0,1\}^*, \text{ and } U(\pi) = u\}.$$

Definition

Let $n, r \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = K(u),$$

where $u = x{\upharpoonright}r$ is the first nr bits in the binary representation of x.

Effective Dimension

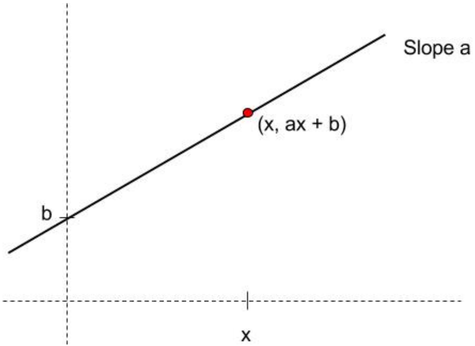
Definition (J. Lutz, Mayordomo)

Let $x \in \mathbb{R}^n$. The *effective dimension of x* is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

- $0 \leq \dim(x) \leq n$.
- x is ML-random $\implies \dim(x) = n$
- x is computable $\implies \dim(x) = 0$.
- quantitative, fine-grained measure of the algorithmic randomness of x.

What are the (effective) dimensions of points on a line $L_{a,b}$ with slope a and intercept b?



The *dimension spectrum* of a line is

$$\mathrm{sp}(L_{a,b}) = \{\dim(x, ax + b) \mid x \in [0, 1]\}.$$

Why Lines?

- Algorithmic randomness perspective:
  - Lines in $\mathbb{R}^2$ are the simplest non-trivial sets.
  - Cannot claim to understand effective dimension without understanding the dimension spectrum of planar lines.
- Deep connections with (classical) geometric measure theory:
  - Proof of the DSC in higher dimensions would solve the Kakeya conjecture.
  - The principle obstruction for the DSC is present in many of the important unsolved problems in geometric measure theory
    - Kakeya conjecture, Furstenberg set conjecture, dimension of sum-product sets, Kauffman's projection bounds,...

Geometric Measure Theory (Detour)

- *Hausdorff dimension* gives quantitative notion of the size of *sets*.
  - Fine grained notion, allowing us to distinguish Lebesgue measure 0 sets.

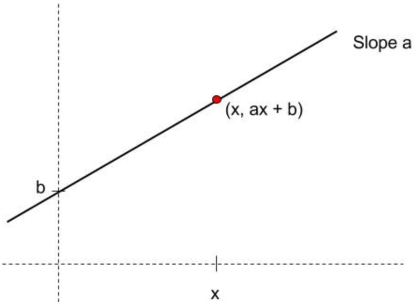
Theorem (J. Lutz and N. Lutz)

Let $E \subseteq \mathbb{R}^n$. Then

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x).$$

- We can attack problems in classical geometric measure theory using algorithmic techniques.
- Any non-trivial lower bounds on the **effective** dimension of points on a line in $\mathbb{R}^3$ would improve the best-known bounds of the notorious Kakeya conjecture.

Dimension of Points on a Line



The *dimension spectrum* of a line L with slope a and intercept b is the set

$$\mathrm{sp}(L) = \{\dim(x, ax + b) \mid x \in \mathbb{R}\}.$$

Conjecture

For every a, b, $\mathrm{sp}(L_{a,b})$ contains an interval of length 1.

Previous Results

Theorem (Turetsky '11)

The set of points in $\mathbb{R}^n$ of (effective) dimension 1 is connected.

As a consequence, for every line L, $1 \in \operatorname{sp} L$.

Previous Results

Theorem (N. Lutz and Stull)

Let $(a, b) \in \mathbb{R}^2$. *Then for every* $x \in \mathbb{R}$,

$$\dim(x, ax + b) \geq \dim^{a,b}(x) + \min\{dim(a, b), \dim^{a,b}(x)\}.$$

Corollary (N. Lutz and Stull)

If $\dim(a, b) < 1$, *then*

$$\mathrm{sp}(L_{a,b}) \supseteq [2\dim(a, b), 1 + \dim(a, b)].$$

If $\dim(a, b) \geq 1$, *then*

$$2 \in \mathrm{sp}(L_{a,b}).$$

This theorem gives improved bounds on Furstenberg sets for certain values of α and β.

Framework

Fix a line $L_{a,b}$. Assume that $\dim(a, b) = d < 1$. Let $x \in [0, 1]$ be random relative to (a, b).

Fix a precision r. Assume that $K_r(a, b) = dr$. We want to prove that

$$K_r(x, ax + b) \geq K_r(x, a, b) = (1 + d)r.$$

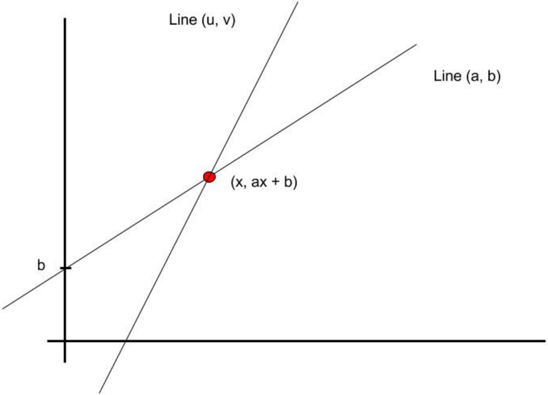
It suffices to show that, given a 2^{-r} approximation of $(x, ax + b)$, we can compute a 2^{-r} approximation of (x, a, b).

- How can we compute (an approximation of) a line only given a point?
- The line $L_{a,b}$ is special - it is of low complexity.
- Want to show that it is essentially the *only* low complexity line intersecting $(x, ax + b)$.

Framework

Want to show that (a, b) is essentially the *only* low complexity line intersecting $(x, ax + b)$.

- If it weren't, then x would not be random relative to (a, b).
- Makes use of the simple geometric fact that any two lines intersect at a unique point.



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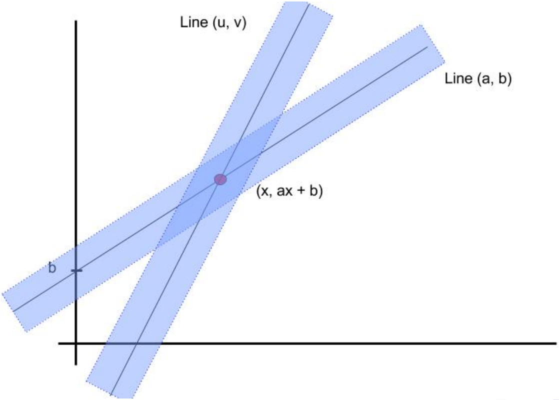
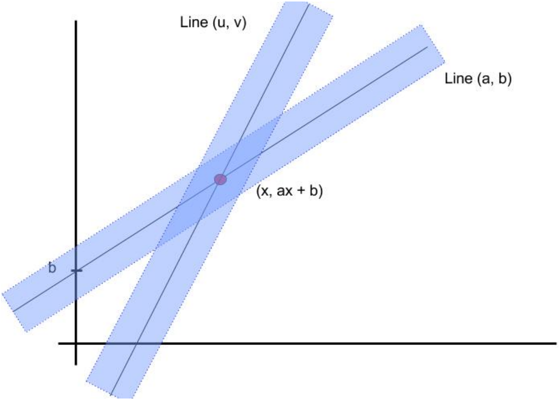


Framework



Suppose that (u, v) intersects $(x, ax + b)$. Let $t = -\log \|(a, b) - (u, v)\|$. Then

$$K^{a,b}_{r-t}(x) \leq K^{a,b}_{r}(u, v).$$

Framework

Suppose that (u, v) intersects $(x, ax + b)$, and $K_r(u, v) \leq dr$. Let $t = -\log \|(a, b) - (u, v)\|$. Then

$$K^{a,b}_{r-t}(x) \leq K^{a,b}_r(u, v).$$

Since x is random relative to (a, b),

$$r - t \leq K^{a,b}_r(u, v).$$

Since (u, v) shares the first t bits with (a, b), and $K_r(u, v) \leq dr$,

$$r - t \leq dr - dt.$$

This cannot happen if $d < 1$, and therefore (a, b) is the unique line such that

- (a, b) intersects $(x, ax + b)$, and
- $K_r(a, b) \leq dr$.

Framework

- The general intersection lemma shows that
$$s(r-t) \leq K_r^{a,b}(u,v) \leq d(r-t),$$
where $s = \dim^{a,b}(x)$.
- This proof makes essential use of the assumption that s was greater than d.
- The obstacle when s is smaller than d seems very deep.
  - Heart of the difficulty of the Kakeya conjecture, Furstenberg set conjecture,...

Dimension Spectrum Conjecture

Goal: Given a line (a, b), for every $s \in (0, 1]$, construct a point x such that

- $\dim^{a,b}(x) = s$.
- $\dim(x, ax + b) = s + \min\{\dim(a, b), 1\}$.

For simplicity, let $\dim(a, b) = d < 1$ and let r be a precision such that $K_r(a, b) = dr$. Want to construct a point x (finite binary string) such that

- $K_r^{a,b}(x) = sr$.
- $K_r(x, ax + b) = (s + d)r$.

Dimension Spectrum Conjecture

Let $\dim(a, b) = d < 1$ and let r be a precision such that $K_r(a, b) = dr$. Want to construct a point x (finite binary string) such that

- $K_r^{a,b}(x) = sr$.

Two immediate ideas:

- Take random, relative to (a, b), string and every change every $\frac{1}{s}$th bit to 0.
  - Constructions of Furstenberg sets from geometric measure theory seem to rule this out.
- Take random, relative to (a, b), string and set all bits after index sr to 0.
  - Runs into the main obstacle.

Dimension Spectrum Conjecture

We use the structure of the problem to remove the main obstacle:

- Take random, relative to (a, b), string of length sr and concatenate the first $r - sr$ bits of a.

Thus, our string x satisfies

$$K_r^{a,b}(x) = sr.$$

Moreover, for all $r' \leq sr$,

$$K_{r'}^{a,b}(x) = r'.$$

Key point: For precisions less than sr, we are essentially in the high complexity case we know how to solve.

Dimension Spectrum Conjecture

Suppose that (u, v) intersects $(x, ax + b)$, and $K_r(u, v) \leq dr$. Let $t = -\log \|(a, b) - (u, v)\|$, and suppose that $t \geq r - sr$.

$$K^{a,b}_{r-t}(x) \leq K^{a,b}_r(u, v).$$

Since x is random relative to (a, b) at precisions less than sr,

$$r - t \leq K^{a,b}_r(u, v) \leq dr - dt.$$

Therefore (a, b) is the unique line such that

- (u, v) intersects $(x, ax + b)$,
- $K_r(u, v) \leq dr$, and
- $t = -\log \|(a, b) - (u, v)\| \geq r - sr$

We would be done if we could restrict our search to lines such that

$$t = -\log \|(a, b) - (u, v)\| \geq r - sr$$

Dimension Spectrum Conjecture

Given a 2^{-r} approximation of $(x, ax + b)$:

1. We have access to the first r bits of x.
2. Thus, we know the first $r - sr$ bits of a.
3. Combining these with our approximation of $(x, ax + b)$ we can compute the first $r - sr$ bits of b.
4. Hence, we know the first $r - sr$ bits of (a, b), and can restrict our search for lines (u, v) such that
   - (u, v) intersects $(x, ax + b)$
   - $K_r(u, v) \leq dr$, and
   - $t = -\log \|(a, b) - (u, v)\| \geq r - sr$
5. (a, b) is the only such line, and so $K_r(x, ax + b) \geq K_r(x, a, b) = s + d$.

Dimension Spectrum Conjecture

Full proof of low dimensional lines ($\dim(a, b) \leq 1$)

- Choose very sparse set of precisions r such that $K_r(a, b) = dr$, and modify the bits of x.
  - At these precisions, the previous argument works.
  - For other precisions, need a slightly different approach.

High dimensional lines ($\dim(a, b) > 1$)

- This argument doesn't immediately work.
  - It will only prove that $K_r(x, ax + b) \geq (s + 1)r$, but we need this to be an **equality**.
- In this case, we use a non-constructive argument.
  - Consider strings $x_0, \ldots, x_{r-sr}$. The point x_m encodes first m bits of a.
  - We can upper bound the point corresponding to x_0, and we have a good lower bound for x_{r-sr}.
  - Using a discrete version of MVT, we show that *some* point has dimension $(s + 1)$.

The End

Thank you!

The Kakeya Conjecture

Question

Let $E \subseteq \mathbb{R}^n$ be a set containing a line in every direction (a Kakeya set). How big must E be?.

- Besicovitch: Can have measure 0.
- Davies: In $\mathbb{R}^2$, Kakeya sets must have Hausdorff dimension 2.
- For $n > 2$ this is still an open question.

Conjecture (Kakeya Conjecture)

Every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

Furstenberg Sets

Definition

Let $\alpha, \beta \in (0,1]$. A set of Furstenberg type with parameters α and β is a subset $F \subseteq \mathbb{R}^2$ such there is a set $J \subseteq S^1$ (set of directions) satisfying the following.

- $\dim_H(J) \geq \beta$.
- For every $e \in J$, there is a line l_e in the direction of e such that $\dim_H(F \cap l_e) \geq \alpha$.

Open question: For α, β, how big must a set of Furstenberg type with parameters α and β be?

Theorem (Molter and Rela)

For all $\alpha, \beta \in (0,1]$ and every set $E \in F_{\alpha,\beta}$,

$$\dim_H(E) \geq \alpha + \max\{\tfrac{\beta}{2}, \alpha + \beta - 2\}.$$