# Embeddings of Computable Structures

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# Outline

# Order Types...

## Definition

An *order type* is the isomorphism type of a linear order, i.e., an algebraic structure with a irreflexive, antisymmetric, transitive order.

## Notation

$\omega$: *the order type of the non-negative integers*

$\omega^*$: *the order type of the negative integers*

$\zeta$: *the order type of the integers*

$\eta$: *the order type of the rational numbers*

## Definition

An order type is *well-ordered* if it contains no (infinite) descending sequence, i.e., if the order type $\omega^*$ does not embed.
An order type is *scattered* if it contains no (infinite) dense subset, i.e., if the order type $\eta$ does not embed.

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# Computable Order Types...

## Definition

An order type $\tau$ is *computable* if there is a *computable presentation* of $\tau$, i.e., a *computable* binary relation $<$ on $\omega = \{0, 1, 2, \dots\}$ such that $\tau \cong (\omega : <)$.

## Example

The order type $\omega + \omega^*$ is computable as witnessed by the presentation with

$$0 < 2 < 4 < \cdots < 2n < \dots \qquad \cdots < 2n+1 < \cdots < 5 < 3 < 1.$$

## Example

The order type $\eta$ is computable as witnessed by the presentation with

$$\cdots < 3 < \cdots < 1 < \cdots < 4 < \cdots < 0 < \cdots < 5 < \cdots < 2 < \cdots < 6 < \dots.$$

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# The Harrison Ordering...

## Definition

Denote the order type of the least noncomputable ordinal by $\omega_1^{CK}$.

## Theorem (Harrison)

*The order type* $\omega_1^{CK} \cdot (1 + \eta)$ *is computable.*

## Remark

The traditional proof demonstrating that the order type $\omega_1^{CK} \cdot (1 + \eta)$ is computable appeals to the Barwise-Kreisel Compactness Theorem.

In doing so, it constructs a computable presentation having no computable subset of order type $\omega^*$ or $\eta$.

# Properties of Presentations and Order Types...

**Definition**

A presentation of a computable order type is *computably well-ordered* if there is no computable (infinite) descending sequence.

**Definition**

A computable order type is *intrinsically computably well-ordered* if every computable presentation is computably well-ordered.

**Definition**

A presentation of a computable order type is *computably scattered* if there is no computable (infinite) dense subset.

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# Revisiting $\omega + \omega^*$...

**Proposition**

*The order type $\omega + \omega^*$ is not intrinsically computably well-ordered.*

**Proof.**

The presentation from earlier has a computable descending sequence (the odd numbers). □

**Theorem (Denisov; Tennenbaum)**

*There is a computable presentation of the order type $\omega + \omega^*$ that is computably well-ordered.*

**Corollary**

*The order type $\omega + \omega^*$ is not intrinsically computably non-well-ordered.*

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# Revisiting $\omega_1^{CK} \cdot (1 + \eta)$...

**Theorem (Harrison)**

*There is a computable presentation of the order type $\omega_1^{CK} \cdot (1 + \eta)$ that is computably scattered.*

**Corollary**

*The order type $\omega_1^{CK} \cdot (1 + \eta)$ is not intrinsically computably non-scattered.*

**Proposition**

*The order type $\omega_1^{CK} \cdot (1 + \eta)$ is not intrinsically computably scattered.*

**Proof.**

If $\mathcal{L}$ is a computable presentation of the order type $\omega_1^{CK} \cdot (1 + \eta)$, then $\mathcal{L} \cdot (1 + \eta)$ also has order type $\omega_1^{CK} \cdot (1 + \eta)$ and has a computable subset of order type $\eta$. □

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# Outline

# Questions...

**Question**

Is there a computable non-well-ordered order type that is intrinsically computably well-ordered?

**Question**

Is there a computable non-scattered order type that is intrinsically computably scattered?

# Intrinsically Computably Well-Ordered...

## Theorem (Kach and Miller)

*There is a computable non-well-ordered order type that is intrinsically computably well-ordered.*

## Sketch.

The desired order type is the result of starting with the order type

$$\omega^\omega + \cdots + \omega^n + \cdots + \omega^2 + \omega^1 + 1$$

and eliminating, for certain $n$, the copy of $\omega^n$.

The important observation is that any descending sequence separates the order type into two intervals:

- the elements not less than every element of the descending sequence (those part of $\omega^n$ for some finite $n$)
- the elements less than every element of the descending sequence (those part of $\omega^\omega$)

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It therefore suffices to eliminate copies of $\omega^n$ in such a way so that the entire order type is computable, but the order type is not computable if the copy of $\omega^\omega$ is removed.

This is a two step process: characterize when the order type (with the $\omega^\omega$) is computable with limitwise monotonic functions and diagonalize against all computable presentations that appear to be of the right form (without the $\omega^\omega$).

## Theorem (Kach and Miller)

*There is a computable non-scattered order type that is intrinsically computably (hyperarithmetically) scattered.*

## Sketch.

Start with the order type

$$\omega + f(\varepsilon) + \zeta + \qquad + \zeta + f(\varepsilon) + \omega^*$$

and eliminate suborders depending on whether $\sigma \in T$ for an infinite computable tree $T \subseteq \omega^{<\omega}$ with no hyperarithmetic paths. □

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# Intrinsically Hyperarithmetically Well-Ordered...

## Corollary

*For each computable ordinal $\alpha$, there is a computable non-well-ordered order type that is intrinsically $\mathbf{0}^{(\alpha)}$ well-ordered.*

## Theorem (Harrison)

*If a computable presentation has no hyperarithmetic descending sequence, then it has order type $\omega_1^{CK} \cdot (1 + \eta) + \alpha$ for some computable ordinal $\alpha$.*

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*There is no computable non-well-ordered order type that is intrinsically hyperarithmetically well-ordered.*

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# Directed Graphs...

## Question

Is there a computable directed graph having an infinite path but no computable infinite path?

## Theorem (Kach, Levin, and Solomon)

*It suffices to start with an infinite computable tree $T \subseteq \omega^{<\omega}$ having no computable paths. Form the directed graph $\mathcal{G}_T$ via the mapping*

*Then a computable embedding would yield a computable path through $T$.*

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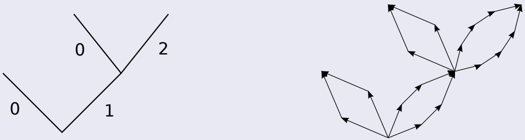


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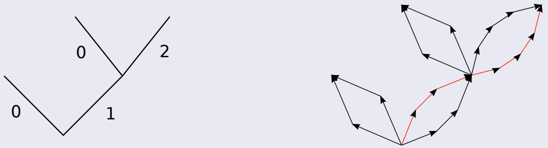


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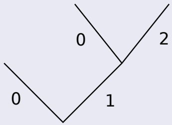


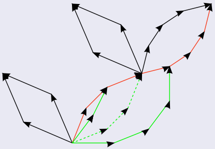

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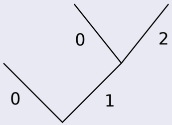


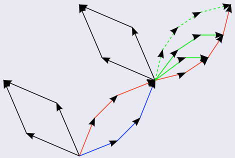

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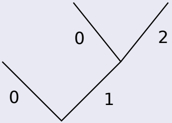


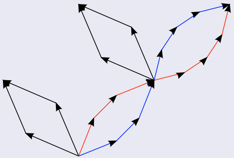

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# More Examples...

## Question

If $\mathcal{C}$ is a class of computable algebraic structures, are there computable structures $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{C}$ so that $\mathcal{S}_1$ classically embeds into $\mathcal{S}_2$ but for which there is no computable embedding between any computable presentations?

## Corollary (Hirshfeldt, Khoussainov, Shore, and Slinko)

*There are such examples within the classes of commutative semigroups, two step nilpotent groups, undirected graphs, lattices, and rings.*

## Corollary (Kach, Levin, and Solomon)

*There are such examples within the class of computable ordered fields.*

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# BAs, ACFs, and Equivalence Structures...

**Question**

Is this phenomena present throughout all natural classes of algebraic structures?

**Theorem (Kach, Levin, and Solomon)**

*The class of computable Boolean algebras, the class of algebraically closed fields, and the class of computable equivalence structures fail to exhibit this phenomena.*

**Remark**

The proofs for Boolean algebras and equivalence structures is fundamentally different than for algebraically closed fields. For the former two, it suffices (and is necessary) to change the presentation of $\mathcal{S}_2$. For the latter, it suffices to change the presentation of either $\mathcal{S}_1$ or $\mathcal{S}_2$.

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# Embedding Properties...

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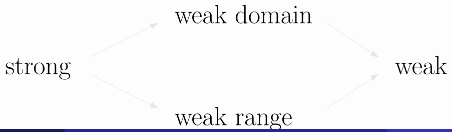
A class $\mathcal{C}$ of algebraic structures has the $***$ *embedding property* if for all computable presentations $\mathcal{S}_1$ and $\mathcal{S}_2$ of structures in $\mathcal{C}$ such that $\mathcal{S}_1$ classically embeds into $\mathcal{S}_2$

- *strong*: it is unnecessary to change the presentations
- *weak domain*: it suffices to change the presentation of $\mathcal{S}_1$
- *weak range*: it suffices to change the presentation of $\mathcal{S}_2$
- *weak*: it suffices to change the presentations of $\mathcal{S}_1$ and $\mathcal{S}_2$

to obtain a computable embedding between computable presentations.

## Theorem (Kach, Levin, and Solomon)

*The pictured implications hold (trivially). No other implications hold.*



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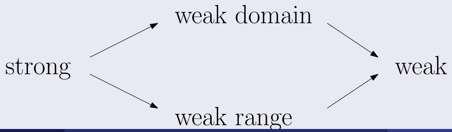

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# Questions on Linear Orders...

**Conjecture**

There is a computable non-scattered linear order that is intrinsically computably well-ordered.

**Question**

Was the choice of $\omega^*$ and $\eta$ important? Is it the case that for every computable (infinite) order type $\tau_1$, there is a computable order type $\tau_2$ such that $\tau_1$ classically embeds into $\tau_2$ but does not computably embed for any computable presentations?

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# Questions on the Embedding Properties...

**Question**

Is there a natural class of algebraic structures that has the weak domain embedding property but not the weak range embedding property?

Is there a natural class of algebraic structures that has the weak embedding property but neither the weak domain embedding property nor the weak range embedding property?

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Which of the embedding properties do other classes of algebraic structures have or not have? In particular, the class of fields? The class of reduced abelian *p*-groups?

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