

# A SURVEY OF $(\infty, 1)$ -CATEGORIES

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ABSTRACT. In this paper we give a summary of the comparisons between different definitions of so-called  $(\infty, 1)$ -categories, which are considered to be models for  $\infty$ -categories whose  $n$ -morphisms are all invertible for  $n > 1$ . They are also, from the viewpoint of homotopy theory, models for the homotopy theory of homotopy theories. The four different structures, all of which are equivalent, are simplicial categories, Segal categories, complete Segal spaces, and quasi-categories.

## 1. INTRODUCTION

The intent of this paper is to summarize some of the progress that has been made since the IMA workshop on  $n$ -categories on the topic of  $(\infty, 1)$ -categories. Heuristically, a  $(\infty, 1)$ -category is a weak  $\infty$ -category in which the  $n$ -morphisms are all invertible for  $n > 1$ . Practically speaking, there are several ways in which one could encode this information. In fact, at the moment, there are four models for  $(\infty, 1)$ -categories: simplicial categories, Segal categories, complete Segal spaces, and quasi-categories. They have arisen out of different motivations in both category theory and homotopy theory, but work by the author and Joyal-Tierney has shown that they are all equivalent to one another, in that they can be connected by chains of Quillen equivalences of model categories.

From the viewpoint of higher category theory, these comparisons provide a kind of baby version of the comparisons which are being attempted between various definitions of weak  $n$ -category. In [37], Toën actually axiomatizes a theory of  $(\infty, 1)$ -categories and proves that any such theory is equivalent to the theory of complete Segal spaces. In [36], he sketches arguments for proving the equivalences between the four structures used in this paper. Although some of the functors he suggests do not appear to give the desired Quillen equivalences (or at any rate are not used in the known proofs), he gives a good overview of the problem. Another good introduction of the problem can be found in a preprint by Porter [24], and a nice description of the idea behind  $(\infty, 1)$ -categories can be found in [34].

From another point of view, these comparisons are of interest in homotopy theory, as what we are here calling a  $(\infty, 1)$ -category can be considered to be a model for the homotopy theory of homotopy theories, a concept which will be made more precise in the section on simplicial categories. The idea is that a simplicial category is in some way “naturally” a homotopy theory but functors between such are not particularly easy to work with. The goal of finding an equivalent but nicer model led to Rezk’s complete Segal spaces [27] and the task of showing that they were essentially the same as simplicial categories.

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It should be noted that there are other proposed models, including that of  $A_\infty$ -categories. Joyal and Tierney briefly discuss some other approaches to this idea in their epilogue [21].

Furthermore, we should also mention that these structures are of interest in areas beyond homotopy theory and higher category theory. For instance, there are situations in algebraic geometry in which simplicial categories have been shown to provide information that the more commonly used derived category cannot. Given a scheme  $X$ , for example, its derived category  $\mathcal{D}(X)$  does not seem to determine its  $K$ -theory spectrum, whereas its simplicial localization  $\mathcal{L}(X)$  does [38]. Furthermore, the simplicial category  $\mathcal{L}(X)$  forms a stack, which  $\mathcal{D}(X)$  does not [16]. Similar work is also being done using dg categories, which are in many ways analogous to simplicial categories [35]. In particular, the category of dg categories has a model category structure which is defined using the same essential ideas as the model structure on the category of simplicial categories [32].

Also motivated by ideas in algebraic geometry, Lurie uses quasi-categories and their relationship with simplicial categories in his work on higher stacks [22]. The first chapter of his manuscript is also a good introduction to many of the ideas of  $(\infty, 1)$ -categories.

Another application of the model category of simplicial categories can be found in recent work of Douglas on twisted parametrized stable homotopy theory [6]. He uses diagrams of simplicial categories weakly equivalent to the simplicial localization of the category of spectra (i.e., equivalent as homotopy theories to the homotopy theory of spectra) in order to define an appropriate setting in which to do Floer homotopy theory.

In this paper, we will provide some background on each of the four structures and then describe the various Quillen equivalences.

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## 2. BACKGROUND ON MODEL CATEGORIES AND SIMPLICIAL SETS

Since this paper is meant to be an overview, we are not going to go deeply into the details here, and the reader familiar with model categories and simplicial sets may skip to the next section. For the non-expert, we will give the basic ideas behind both model category structures and simplicial sets, as well as other simplicial objects.

The motivation for a model category structure is a common occurrence in many areas of mathematics. Suppose that we have a category whose morphisms include some, called weak equivalences, which we would like to think of as isomorphisms but do not necessarily have inverses. Two classical examples are the category of topological spaces and (weak) homotopy equivalences between them, and the category of chain complexes of modules over a ring  $R$  and the quasi-isomorphisms, or morphisms which induce isomorphisms on all homology groups. In order to make these maps actually isomorphisms, we could formally invert them. Namely, we could formally add in an inverse to every such map and then add all the necessary composites so that the result would actually be a category. The problem with this approach is that often this process results in a category such that the morphisms between any two given objects form a proper class rather than a set. While it is

common to work in categories with a proper class of objects, it is generally assumed that there is only a set of morphisms between any two objects, even if there is a proper class of morphisms altogether.

Imposing the structure of a model category allows us formally to invert the weak equivalences while keeping the morphisms under control. A model category  $\mathcal{M}$  is a category with three distinguished classes of morphisms, the weak equivalences as already described, plus fibrations and cofibrations, satisfying several axioms. We refer the reader to [13], [15], [17], or the original [25] for these axioms. The importance of these axioms is that they allow us to work with particularly nice objects in the category, called fibrant-cofibrant objects, between which we can define “homotopy classes of maps” even in a situation where the traditional notion of homotopy class (as in topological spaces) no longer makes sense. The axioms of a model category guarantee that every object of  $\mathcal{M}$  has a fibrant-cofibrant replacement, and thus we can define the *homotopy category* of  $\mathcal{M}$ , denoted  $\mathrm{Ho}(\mathcal{M})$ , to be the category whose objects are the same as those of  $\mathcal{M}$  and whose morphisms are homotopy classes of maps between the respective fibrant-cofibrant replacements. In fact, this construction is independent of the choice of such replacements, and the homotopy category, up to equivalence, is independent of the choice of fibrations and cofibrations. There are many examples of categories with two (or more) different model category structures, each leading to the same homotopy category because the weak equivalences are the same even if the fibrations and cofibrations are defined differently.

As an example of a model category, consider the category of topological spaces and the subcategory of weak homotopy equivalences, or maps which induce isomorphisms on all homotopy groups. There is a natural choice of fibrations and cofibrations such that this category has a model category structure. (There is also a model category structure on this category where the weak equivalences are the homotopy equivalences, but the former is considered to be the standard model structure.)

One can then define what it means to have a map between model categories, namely, a functor which preserves essential properties of the model structures. It is convenient to use adjoint pairs of functors to work with model structures, where the left adjoint preserves cofibrations and the right adjoint preserves fibrations. Such an adjoint pair is called a Quillen pair. There is also the notion of Quillen equivalence between model categories, where the adjoint functors preserve the essential homotopical information. In particular, a Quillen equivalence induces an equivalence of homotopy categories, but it is in fact much stronger, in that it preserves higher-order information, as we will discuss further in the next section.

Given this kind of structure and interaction between structures, one can ask the following question, one that can, in fact, be considered a motivation for the research described in this paper. Given a model category  $\mathcal{M}$ , is there a model category  $\mathcal{N}$  which is Quillen equivalent to  $\mathcal{M}$  but which more easily provides information that is difficult to obtain from  $\mathcal{M}$  itself? The properties one might look for in  $\mathcal{N}$  depend very much on the question being asked. Thus, it is hoped that the four model structures given in this paper will be able to provide information about one another.

An important illustration is that of topological spaces and simplicial sets. Heuristically, simplicial sets provide a combinatorial model for topological spaces, and the fact that they are a “model” here means that there is a model category structure on the category of simplicial sets which is Quillen equivalent to the standard model

category structure on the category of topological spaces. Simplicial sets are frequently (but not always) easier to work with because they are just combinatorial objects.

To give a formal definition of a simplicial set, consider the category  $\Delta$  of finite ordered sets  $[n] = \{0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n\}$  for each  $n \geq 0$ , and order-preserving maps between them. (As the notation suggests, one can also consider  $n$  to be a small category.) Let  $\Delta^{op}$  denote the opposite category, where we reverse the direction of all the morphisms. Then a simplicial set is a functor  $X : \Delta^{op} \rightarrow \mathit{Sets}$ . There is a geometric realization functor between the category  $\mathit{SSets}$  of simplicial sets and the category of topological spaces. Specifically, an element of  $X_0$  is assigned to a point, an element of  $X_1$  is assigned to a geometric 1-simplex, and so forth, where identifications are given by the face maps of the simplicial set. Thus, a simplicial set can be regarded as a generalization of a simplicial complex, where the simplices are not required to form “triangles” and a given simplex of degree  $n$  is regarded as a degenerate  $k$ -simplex for each  $k > n$  [14, I.2].

In fact, we can perform this kind of construction in categories other than sets. A simplicial object in a category  $\mathcal{C}$  is just a functor  $X : \Delta^{op} \rightarrow \mathcal{C}$ . The primary example we will consider in this paper is that of simplicial spaces, or functors  $\Delta^{op} \rightarrow \mathit{SSets}$ . To emphasize the fact that they are simplicial objects in the category of simplicial sets, they are often also called bisimplicial sets.

Given these main ideas, we can now turn to the four different models for  $(\infty, 1)$ -categories, or homotopy theories.

### 3. SIMPLICIAL CATEGORIES

The first of the four categories we consider is that of small simplicial categories. By a simplicial category, we mean what is often called a simplicially enriched category, or a category with a simplicial set of morphisms between any two objects. Given two objects  $a$  and  $b$  in a simplicial category  $\mathcal{C}$ , this simplicial set is denoted  $\mathrm{Map}_{\mathcal{C}}(a, b)$ . This terminology is potentially confusing because the term “simplicial category” can also be used to describe a simplicial object in the category of all small categories. We recover our sense of the term if the face and degeneracy maps are all required to be the identity map on objects.

Simplicial categories have been studied for a variety of reasons, but here we will focus on their importance in homotopy theory, and, in particular, on how a simplicial category can be considered to be a homotopy theory.

We should note here that although for set-theoretic reasons we restrict ourselves to small simplicial categories, or those with only a set of objects, in practice many of the simplicial categories one cares about are large. The standard approach to this problem is to assume that one is working in a larger universe for set theory in which the given category is indeed “small.”

Given a model category  $\mathcal{M}$ , we can consider its homotopy category  $\mathrm{Ho}(\mathcal{M})$ . For many applications, it is sufficient to work in the homotopy category, but it is important to remember that in passing from the original model category to the homotopy category we have lost a good deal of information. Of course, part of the goal was formally to invert the weak equivalences, but in addition the model category possessed higher-homotopical information that the homotopy category has lost. For example, the model category contains the tools needed to take homotopy limits and homotopy colimits.

In a series of papers, Dwyer and Kan develop the theory of simplicial localizations, in which, given a model category  $\mathcal{M}$ , one can obtain a simplicial category which still holds this higher homotopical information. In fact, they construct two different such simplicial categories from  $\mathcal{M}$ , the standard simplicial localization  $LM$  [11] and the hammock localization  $L^H\mathcal{M}$  [10], but the two are equivalent to one another [8, 2.2]. Furthermore, taking the component category  $\pi_0 LM$ , which has the objects of  $LM$  and the morphisms the components of the simplicial hom-sets of  $LM$ , is equivalent to the homotopy category  $\text{Ho}(\mathcal{M})$  [11].

Furthermore, there is a natural notion of “equivalence” of simplicial categories, which is often called a *Dwyer-Kan equivalence* or simply *DK-equivalence*. It is a generalization of the definition of equivalence of categories to the simplicial setting. In particular, a DK-equivalence is a simplicial functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  between two simplicial categories satisfying the following two conditions:

- (1) For any objects  $a, b$  of  $\mathcal{C}$ , the map of simplicial sets

$$\text{Map}_{\mathcal{C}}(a, b) \rightarrow \text{Map}_{\mathcal{D}}(fa, fb)$$

is a weak equivalence.

- (2) The induced functor on component categories  $\pi_0 f : \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$  is an equivalence of categories.

A Quillen equivalence between model categories then induces a DK-equivalence between their simplicial localizations.

More generally, if one is not concerned with set-theoretic issues, we can take the simplicial localization of any category with weak equivalences. Since, in a fairly natural sense, a “homotopy theory” is really some category with “weak equivalences” that we would like to invert, a homotopy theory gives rise to a simplicial category. In fact, the converse is also true: given any simplicial category, it is, up to DK-equivalence, the simplicial localization of some category with weak equivalences [9, 2.5]. Thus, the study of simplicial categories is really the study of homotopy theories.

A first approach to applying the techniques of homotopy theory to a category of simplicial categories itself was first given by Dwyer and Kan [11]. In this paper, they define a model category structure on the category of simplicial categories with a fixed set  $\mathcal{O}$  of objects. The idea was then proposed that the homotopy theory of (all) simplicial categories was essentially the “homotopy theory of homotopy theories.” Dwyer and Spalinski mention this concept at the end of their survey paper [13], and the idea was further explored by Rezk [27], whose ideas we will return to in the next section. The author then showed in [2] that the category of all small simplicial categories with the DK-equivalences has a model category structure, thus formalizing the idea.

In order to define the fibrations in this model structure, we need the following notion. If  $\mathcal{C}$  is a simplicial category and  $x$  and  $y$  are objects of  $\mathcal{C}$ , a morphism  $e \in \text{Map}_{\mathcal{C}}(x, y)_0$  is a *homotopy equivalence* if the image of  $e$  in  $\pi_0 \mathcal{C}$  is an isomorphism.

**Theorem 3.1.** [2, 1.1] *There is a model category structure on the category  $\mathcal{S}\mathcal{C}$  of small simplicial categories defined by the following three classes of morphisms:*

- (1) *The weak equivalences are the DK-equivalences.*  
(2) *The fibrations are the maps  $f : \mathcal{C} \rightarrow \mathcal{D}$  satisfying the following two conditions:*

- For any objects  $x$  and  $y$  in  $\mathcal{C}$ , the map

$$\text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{D}}(fx, fy)$$

is a fibration of simplicial sets.

- For any object  $x_1$  in  $\mathcal{C}$ ,  $y$  in  $\mathcal{D}$ , and homotopy equivalence  $e: fx_1 \rightarrow y$  in  $\mathcal{D}$ , there is an object  $x_2$  in  $\mathcal{C}$  and homotopy equivalence  $d: x_1 \rightarrow x_2$  in  $\mathcal{C}$  such that  $fd = e$ .

(3) The cofibrations are the maps which have the left lifting property with respect to the maps which are fibrations and weak equivalences.

The advantage of this model category is that its objects are fairly straightforward. As mentioned above, there is a reasonable argument for saying that simplicial categories really are homotopy theories. The disadvantage here lies in the weak equivalences, in that they are difficult to identify. Thus, it was natural to look for a model with nicer weak equivalences.

#### 4. COMPLETE SEGAL SPACES

Complete Segal spaces are probably the most complicated objects to define of the four models described in this paper, but from the point of view of homotopy theory, they might be the easiest to use because the corresponding model structure gives what Dugger calls a presentation for the homotopy theory [7]. They are defined by Rezk [27] whose purpose was explicitly to find a nice model for the homotopy theory of homotopy theories.

A complete Segal space is first a simplicial space. It should be noted that we require that certain of our objects be fibrant in the Reedy model structure on the category of simplicial spaces [26]. This structure is defined by levelwise weak equivalences and cofibrations, but its importance here is that several of our constructions will be homotopy invariant because the objects involved satisfy this condition.

We begin by defining Segal spaces, for which we need the Segal map assigned to a simplicial space. As one might guess from its name, the Segal map is first defined by Segal in his work with  $\Gamma$ -spaces [29]. Let  $\alpha^i: [1] \rightarrow [k]$  be the map in  $\mathbf{\Delta}$  such that  $\alpha^i(0) = i$  and  $\alpha^i(1) = i + 1$ , defined for each  $0 \leq i \leq k - 1$ . We can then define the dual maps  $\alpha_i: [k] \rightarrow [1]$  in  $\mathbf{\Delta}^{op}$ . For  $k \geq 2$ , the Segal map is defined to be the map

$$\varphi_k: X_k \rightarrow \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_k$$

induced by the maps

$$X(\alpha_i): X_k \rightarrow X_1.$$

**Definition 4.1.** [27, 4.1] A Reedy fibrant simplicial space  $W$  is a *Segal space* if for each  $k \geq 2$  the map  $\varphi_k$  is a weak equivalence of simplicial sets. In other words, the Segal maps

$$\varphi_k: W_k \rightarrow \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_k$$

are weak equivalences for all  $k \geq 2$ .

A nice property of Segal spaces is the fact that they can be regarded as analogous to simplicial categories, in that we can define their “objects” and “morphisms” in a meaningful way. Given a Segal space  $W$ , its set of objects, denoted  $\text{ob}(W)$ , is the set of 0-simplices of the space  $W_0$ , namely, the set  $W_{0,0}$ . Given any two objects  $x, y$

in  $\text{ob}(W)$ , the mapping space  $\text{map}_W(x, y)$  is the fiber of the map  $(d_1, d_0): W_1 \rightarrow W_0 \times W_0$  over  $(x, y)$ . Given a 0-simplex  $x$  of  $W_0$ , we denote by  $\text{id}_x$  the image of the degeneracy map  $s_0: W_0 \rightarrow W_1$ . We say that two 0-simplices of  $\text{map}_W(x, y)$ , say  $f$  and  $g$ , are homotopic, denoted  $f \sim g$ , if they lie in the same component of the simplicial set  $\text{map}_W(x, y)$ .

Given  $f \in \text{map}_W(x, y)_0$  and  $g \in \text{map}_W(y, z)_0$ , there is a composite  $g \circ f \in \text{map}_W(x, z)_0$ , and this notion of composition is associative up to homotopy. The homotopy category  $\text{Ho}(W)$  of  $W$ , then, has as objects the set  $\text{ob}(W)$  and as morphisms between any two objects  $x$  and  $y$ , the set  $\text{map}_{\text{Ho}(W)}(x, y) = \pi_0 \text{map}_W(x, y)$ .

Finally, a map  $g$  in  $\text{map}_W(x, y)_0$  is a homotopy equivalence if there exist maps  $f, h \in \text{map}_W(y, x)_0$  such that  $g \circ f \sim \text{id}_y$  and  $h \circ g \sim \text{id}_x$ . Any map in the same component as a homotopy equivalence is itself a homotopy equivalence [27, 5.8]. Therefore we can define the space  $W_{\text{hoequiv}}$  to be the subspace of  $W_1$  given by the components whose zero-simplices are homotopy equivalences.

We then note that the degeneracy map  $s_0: W_0 \rightarrow W_1$  factors through  $W_{\text{hoequiv}}$  since for any object  $x$  the map  $s_0(x) = \text{id}_x$  is a homotopy equivalence. Therefore, we have the following definition:

**Definition 4.2.** [27, §6] A *complete Segal space* is a Segal space  $W$  for which the map  $s_0: W_0 \rightarrow W_{\text{hoequiv}}$  is a weak equivalence of simplicial sets.

We can now consider some particular kinds of maps between Segal spaces. Note that, as the name suggests, these maps are very similar in spirit to the weak equivalences in  $\mathcal{SC}$ .

**Definition 4.3.** A map  $f: U \rightarrow V$  of Segal spaces is a *DK-equivalence* if

- (1) for any pair of objects  $x, y \in U_0$ , the induced map

$$\text{map}_U(x, y) \rightarrow \text{map}_V(fx, fy)$$

is a weak equivalence of simplicial sets, and

- (2) the induced map  $\text{Ho}(f): \text{Ho}(U) \rightarrow \text{Ho}(V)$  is an equivalence of categories.

We are now able to describe the important features of the complete Segal space model category structure.

**Theorem 4.4.** [27, 7.2, 7.7] *There is a model structure  $\mathcal{CSS}$  on the category of simplicial spaces such that*

- (1) *The weak equivalences between Segal spaces are the DK-equivalences.*
- (2) *The cofibrations are the monomorphisms.*
- (3) *The fibrant objects are the complete Segal spaces.*

What makes the model category  $\mathcal{CSS}$  so nice to work with is the fact that the weak equivalences between the fibrant objects, the complete Segal spaces, are easy to identify.

**Proposition 4.5.** [27, 7.6] *A map  $f: U \rightarrow V$  between complete Segal spaces is a DK-equivalence if and only if it is a levelwise weak equivalence.*

To avoid further technical detail, we have not defined what a general weak equivalence is in  $\mathcal{CSS}$ , but the interested reader can find it in Rezk's paper [27, §7]. The important point is that, when working with the complete Segal spaces, the weak equivalences are especially convenient.

## 5. SEGAL CATEGORIES

We now turn to our third model, that of Segal categories. These are natural generalizations of simplicial categories, in that they can be regarded as simplicial categories with composition only given up to homotopy. They first appear in the literature in a paper of Dwyer, Kan, and Smith [12], where they are called special  $\Delta^{op}$ -diagrams of simplicial sets. In particular, Segal categories are again a kind of simplicial space.

We begin with the definition of a Segal precategory.

**Definition 5.1.** A *Segal precategory* is a simplicial space  $X$  such that  $X_0$  is a discrete simplicial set.

As with the Segal spaces in the previous section, we can use the Segal maps to define Segal categories.

**Definition 5.2.** A *Segal category*  $X$  is a Segal precategory  $X : \Delta^{op} \rightarrow \mathbb{S}Sets$  such that for each  $k \geq 2$  the Segal map

$$\varphi_k : X_k \rightarrow \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_k$$

is a weak equivalence of simplicial sets.

A model category structure  $SeCat_c$  for Segal categories is given by Hirschowitz and Simpson [16]. In fact, they generalize the definition to that of a Segal  $n$ -category and give a model structure for Segal  $n$ -categories for any  $n \geq 1$ . The idea behind this generalization is used for both the Simpson and Tamsamani definitions of weak  $n$ -category [30], [33].

The author gives a new proof of this model structure, just for the case of Segal categories, from which it is easier to characterize the fibrant objects [3]. It should be noted that, as in the case of  $CSS$ , this model structure is actually defined on the larger category of Segal precategories. However, the fibrant-cofibrant objects are Segal categories [1].

To define the weak equivalences in  $SeCat_c$ , we first note that there is a functor  $L_c$  assigning to every Segal precategory a Segal category [3, §5]. Then, if we are working with a Segal category  $X$ , we can define its “objects,” “mapping spaces,” and “homotopy category” just as we did for a Segal space.

**Theorem 5.3.** *There is a cofibrantly generated model category structure  $SeCat_c$  on the category of Segal precategories with the following weak equivalences, fibrations, and cofibrations.*

- *Weak equivalences are the maps  $f : X \rightarrow Y$  such that the induced map  $map_{L_c X}(x, y) \rightarrow map_{L_c Y}(fx, fy)$  is a weak equivalence of simplicial sets for any  $x, y \in X_0$  and the map  $Ho(L_c X) \rightarrow Ho(L_c Y)$  is an equivalence of categories.*
- *Cofibrations are the monomorphisms. (In particular, every Segal precategory is cofibrant.)*
- *Fibrations are the maps with the right lifting property with respect to the maps which are both cofibrations and weak equivalences.*

It should not be too surprising that the weak equivalences in this case are again called DK-equivalences, since the same idea underlies the definition in each of the three categories we have considered.

Furthermore, there is also second model structure  $\mathcal{SeCat}_f$  on this same category, with the same weak equivalences but different fibrations and cofibrations. Thus each leads to the same homotopy theory, and in fact they are Quillen equivalent, but the slight difference between the two is key in comparing Segal categories with the other models. We will not define the fibrations and cofibrations in  $\mathcal{SeCat}_f$  here, as they are technical and unenlightening in themselves, but we refer the interested reader to [3, §7] for the details. As the subscript suggests, the initial motivation was to find a model structure with the same weak equivalences but in which the fibrations, rather than the cofibrations, were given by levelwise fibrations of simplicial sets. As it turns out, such a description does not work, but one is not far off thinking of the fibrations as being levelwise.

## 6. QUASI-CATEGORIES

Quasi-categories are perhaps the most mysterious, as far as why they are equivalent to the others. While each of the other models consists of simplicial spaces, or objects easily related to simplicial spaces, quasi-categories are simplicial sets, and thus simpler than any of the others. Thus, they may also provide a good model to use when we actually want to compute something for a given homotopy theory. They were first defined by Boardman and Vogt [4] and are sometimes called weak Kan complexes.

Recall that in the category of simplicial sets we have several particularly important objects. For each  $n \geq 0$ , there is the  $n$ -simplex  $\Delta[n]$  and its boundary  $\dot{\Delta}[n]$ . If we remove the  $k$ th face of  $\dot{\Delta}[n]$ , we get the simplicial set denoted  $V[n, k]$ . Given any simplicial set  $X$ , a horn in  $X$  is a map  $V[n, k] \rightarrow X$ . A Kan complex is a simplicial set such that every horn factors through the inclusion map  $V[n, k] \rightarrow \Delta[n]$ . A quasi-category is then a simplicial set  $X$  such that every horn  $V[n, k] \rightarrow X$  factors through  $\Delta[n]$  for each  $0 < k < n$ . Such horns are called the inner horns. Further details on quasi-categories can be found in Joyal's papers [18] and [20].

Like the cases for Segal categories and complete Segal spaces, the model structure  $\mathcal{QCat}$  for quasi-categories is defined on a larger category, in this case the category of simplicial sets. To define the weak equivalences, we need some definitions.

First, consider the nerve functor  $N : \mathcal{Cat} \rightarrow \mathcal{SSets}$ , where  $\mathcal{Cat}$  denotes the category of small categories. This functor has a left adjoint  $\tau_1 : \mathcal{SSets} \rightarrow \mathcal{Cat}$ . Given a simplicial set  $X$ , the category  $\tau_1(X)$  is called its fundamental category. We can then define a functor  $\tau_0 : \mathcal{SSets} \rightarrow \mathcal{Sets}$ , where  $\tau_0(X)$  is the set of isomorphism classes of objects of the category  $\tau_1(X)$ . Now, if  $X^Y$  denotes the simplicial set of maps  $Y \rightarrow X$ , for any pair  $(X, Y)$  of simplicial sets we define  $\tau_0(Y, X) = \tau_0(X^Y)$ . A weak categorical equivalence is a map  $A \rightarrow B$  of simplicial sets such that the induced map  $\tau_0(B, X) \rightarrow \tau_0(A, X)$  is an isomorphism of sets for any quasi-category  $X$ .

**Theorem 6.1.** [20] *There is a model category structure  $\mathcal{QCat}$  on the category of simplicial sets in which the weak equivalences are the weak categorical equivalences and the cofibrations are the monomorphisms. The fibrant objects of  $\mathcal{QCat}$  are the quasi-categories.*

## 7. QUILLEN EQUIVALENCES

The origins of the comparisons between these various structures seem to be in various places. The question of simplicial categories and Segal categories is a

fairly natural one, since a Segal category is essentially a simplicial category up to homotopy. It is addressed partially by Dwyer, Kan, and Smith [12], but they do not give a Quillen equivalence, partly because their work predates both model structures by several years. Schwänzl and Vogt also address this question, using topological rather than simplicial categories [28].

Rezk defines complete Segal spaces with the comparison with simplicial categories in mind [27]. While his functor from simplicial categories to complete Segal spaces naturally factors through Segal categories, he does not mention this fact as such. Initially, there did not seem to be a need to bring in the Segal categories from this point of view, but further investigation led to skepticism that his functor had the necessary adjoint to give a Quillen equivalence.

Toën mentions all four models and conjectures the relationships between them in [36]. As mentioned in the introduction, some but not all of these functors are the ones used in the known proofs. Toën further axiomatizes the notion of “theory of  $(\infty, 1)$ -categories” in [37]. He gives six axioms for a model structure to satisfy in order to be such a theory, and he shows that any such model category structure is Quillen equivalent to Rezk’s complete Segal space model structure. As far as the author knows, these axioms have not been verified for the other three models given in this paper, but it seems likely that they should hold.

Some of the comparisons are also mentioned in work by Simpson, who sketches an argument for comparing the Segal categories and complete Segal spaces [31].

The author showed in [3] that simplicial categories are equivalent to Segal categories, which are in turn equivalent to the complete Segal spaces. However, the adjoint pairs go in opposite directions and therefore cannot be composed into a single Quillen equivalence. It is still unknown, as far as we know, whether there is a direct Quillen equivalence. Further work by Joyal and Tierney has shown that there are Quillen equivalences between quasi-categories and each of the other three models.

We now look in more detail at these Quillen equivalences. Let us begin with the Segal categories and complete Segal spaces. Since the underlying category of  $\mathcal{C}\mathcal{S}\mathcal{S}$  is the category of simplicial spaces and the underlying category of  $\mathcal{S}e\mathcal{C}at_c$  is the category of simplicial spaces with a discrete space in degree zero, there is an inclusion functor  $I: \mathcal{S}e\mathcal{C}at_c \rightarrow \mathcal{C}\mathcal{S}\mathcal{S}$ . This functor has a right adjoint  $R: \mathcal{C}\mathcal{S}\mathcal{S} \rightarrow \mathcal{S}e\mathcal{C}at_c$  which acts as a discretization functor. In particular, if it is applied to a complete Segal space  $W$ , the result is a Segal category which is DK-equivalent to it.

**Theorem 7.1.** [3, 6.3] *The adjoint pair*

$$I: \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{C}\mathcal{S}\mathcal{S} : R$$

*is a Quillen equivalence.*

Then, as we mentioned in the section on Segal categories, the two model structures  $\mathcal{S}e\mathcal{C}at_c$  and  $\mathcal{S}e\mathcal{C}at_f$  are Quillen equivalent.

**Theorem 7.2.** [3, 7.5] *The identity functor induces a Quillen equivalence*

$$id: \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}e\mathcal{C}at_c : id.$$

Turning to simplicial categories, the nerve functor  $N: \mathcal{S}\mathcal{C} \rightarrow \mathcal{S}e\mathcal{C}at_f$  has a left adjoint  $F$ , which can be considered a rigidification functor.

**Theorem 7.3.** [3, 8.6] *The adjoint pair*

$$F: \text{SeCat}_f \rightleftarrows \text{SC} : N$$

*is a Quillen equivalence.*

Turning to the quasi-categories, Joyal and Tierney have shown that there are in fact two different Quillen equivalences between  $\mathcal{QCat}$  and  $\mathcal{CSS}$ . For the first of these equivalences, the map  $i_1^*$ , which associates to a complete Segal space  $W$  the simplicial set  $W_{*0}$ , has a left adjoint  $p_1^*$ .

**Theorem 7.4.** [21] *The adjoint pair of functors*

$$p_1^*: \mathcal{QCat} \rightleftarrows \mathcal{CSS} : i_1^*$$

*is a Quillen equivalence.*

The second Quillen equivalence between these two model categories is given by a total simplicial set functor  $t_! : \mathcal{CSS} \rightarrow \mathcal{QCat}$  and its right adjoint  $t^!$ .

**Theorem 7.5.** [21] *The adjoint pair*

$$t_!: \mathcal{CSS} \rightleftarrows \mathcal{QCat} : t^!$$

*is a Quillen equivalence.*

Even one of these Quillen equivalences would be sufficient to show that all four of our model categories are equivalent to one another, but, interestingly, Joyal and Tierney go on to prove that there are also two different Quillen equivalences directly between  $\mathcal{QCat}$  and  $\text{SeCat}_c$ . The first of these functors is analogous to the pair given in Theorem 7.4; the right adjoint functor  $j^* : \text{SeCat}_c \rightarrow \mathcal{QCat}$  assigns to a Segal precategory  $X$  the simplicial set  $X_{*0}$ . Its left adjoint is denoted  $q^*$ .

**Theorem 7.6.** [21] *The adjoint pair*

$$q^*: \mathcal{QCat} \rightleftarrows \text{SeCat}_c : j^*$$

*is a Quillen equivalence.*

The second Quillen equivalence between these two model categories is given by the map  $d^* : \text{SeCat}_c \rightarrow \mathcal{QCat}$ , which assigns to a Segal precategory its diagonal, and its right adjoint  $d_*$ .

**Theorem 7.7.** [21] *The adjoint pair*

$$d^*: \text{SeCat}_c \rightleftarrows \mathcal{QCat} : d_*$$

*is a Quillen equivalence.*

Finally, Joyal has also related the quasi-categories to the simplicial categories directly. There is the coherent nerve functor  $\tilde{N}: \text{SC} \rightarrow \mathcal{QCat}$ , first defined by Cordier and Porter [5]. Given a simplicial category  $X$  and the simplicial resolution  $C_*[n]$  of the category  $[n] = (0 \rightarrow \dots \rightarrow n)$ , the coherent nerve  $\tilde{N}(X)$  is defined by

$$\tilde{N}(X)_n = \text{Hom}_{\text{SC}}(C_*[n], X).$$

This functor has a left adjoint  $J: \mathcal{QCat} \rightarrow \text{SC}$ .

**Theorem 7.8.** [19] *The adjoint pair*

$$J: \mathcal{QCat} \rightleftarrows \text{SC} : \tilde{N}$$

*is a Quillen equivalence.*

Thus, we have the following diagram of Quillen equivalences of model categories:

$$\begin{array}{ccccccc}
 \mathcal{SC} & \rightleftarrows & \mathcal{SeCat}_f & \rightleftarrows & \mathcal{SeCat}_c & \rightleftarrows & \mathcal{CSS} \\
 & & & & \updownarrow & & \\
 & & & & \mathcal{QCat} & & 
 \end{array}$$

$\mathcal{SC} \rightleftarrows \mathcal{SeCat}_f \rightleftarrows \mathcal{SeCat}_c \rightleftarrows \mathcal{CSS}$   
 $\mathcal{SC} \rightleftarrows \mathcal{QCat}$   
 $\mathcal{SeCat}_f \rightleftarrows \mathcal{QCat}$   
 $\mathcal{SeCat}_c \rightleftarrows \mathcal{QCat}$   
 $\mathcal{CSS} \rightleftarrows \mathcal{QCat}$

The single double-headed arrows indicate that in these cases either direction can be chosen to be a left (or right) adjoint, depending on which Quillen equivalence is used.

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