ASSOCIATED PRIME IDEALS

Let R be a commutative Noetherian ring and M a non-zero finitely generated R-module. Let $I = Ann(M)$. The case $M = R/I$ is of particular interest. We sketch the theory of associated prime ideals of M, following Matsumura (pp 39-40) and, in particular, of the height of ideals. The height of a prime ideal P is the Krull dimension of the localization R_P, that is the maximal length of a chain of prime ideals contained in P. The height of I is the minimum of the heights of P, where P ranges over the prime ideals that contain I (or, equivalently, are minimal among the prime ideals that contain I).

Definition 0.1. The support of M, $Supp(M)$, is the set of prime ideals such that the localization M_P is non-zero.

A prime containing a prime in $Supp(M)$ is also in $Supp(M)$. For example, the support of R/I is $V(I)$.

Definition 0.2. The associated primes of M are those primes P that coincide with the annihilator of some non-zero element $x \in M$. Observe that $I \subset P$ for any such P since $I = Ann(M) \subset Ann(x)$. The set of associated prime ideals of M is denoted $Ass(M)$ or, when necessary for clarity, $Ass_R(M)$.

The definition of localization implies the following obervation.

Lemma 0.3. *M_P is non-zero if and only if there is an element $x \neq 0$ in M such that $Ann(x) \subset P$. Therefore $Ass(M)$ is contained in $Supp(M)$.*

Proposition 0.4. *Let $\mathscr{S}$ be the set of ideals of the form $Ann(x)$ for some $x \neq 0$ in M. An ideal P that is maximal in the set $\mathscr{S}$ is prime. In particular, $Ass(M)$ is non-empty.*

Proof. Let $rs \in P$, where $P = Ann(x)$ and $s \notin P$. Then $r \in Ann(sx) \supset Ann(x)$. By the maximality of P, $Ann(sx) = Ann(x)$ and therefore $r \in P$. □

Corollary 0.5. *The set of zero-divisors for M is the union of the primes in $Ass(M)$.*

Proof. Clearly, any element of an associated prime is a zero-divisor for M. If $rx = 0$, then $(r) \subset Ann(x)$, and $Ann(x) \subset P$ for some $P \in Ass(M)$ by the proposition. □

Proposition 0.6. *If S is a multiplicative subset of R and N is a finitely generated R_S-module, then $Ass_R(N) = Ass_{R_S}(N)$, where we view $Spec(R_S)$ as contained in $spec(R)$. Therefore $Ass_R(M_S) = Ass_{R_S}(M_S)$.*

Proof. Inspection of the definitions. See Matsumura, p. 39. □

Corollary 0.7. *P is in $Ass_R(M)$ if and only if PR_P is in $Ass_{R_P}(M_P)$.*

Proposition 0.8. *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is exact, then*

$$Ass(M) \subset Ass(M') \cup Ass(M'')$$

Proof. If $P = Ann(x)$, then Rx is a copy N of R/P contained in M. Since P is prime $Ann(y) = P$ for any non-zero $y \in N$. If $N \cap M' \neq 0$, this implies $P \in Ann(M')$. If $N \cap M' = 0$, then N is isomorphic to its image in M'' and $P \in Ann(M'')$. $\square$

Proposition 0.9. *There is a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong R/P_i$ *for some prime ideal* P_i.

Proof. Proceed inductively, starting with $M_1 = Rx$ where $Ann(x)$ is prime. If $M_i \neq M$, choose $P_i \in Ass(M/M_{i-1})$ to obtain a copy of R/P_i in M/M_{i-1}. $\square$

Theorem 0.10. *$Ass(M)$ is a finite subset of $Supp(M)$, and the minimal elements of $Ass(M)$ and $Supp(M)$ coincide.*

Proof. The set of prime ideals containing any non-zero proper ideal in a commutative Noetherian ring is finite, by consideration of the topology on $Spec(R)$. But we have a different proof here: the finiteness of $Ass(M)$ follows inductively from the previous two propositions. Let P be a minimal element of $Supp(M)$. Then $M_P \neq 0$ and, by minimality and results above,

$$\emptyset \neq Ass_R(M_P) = Ass_R(M) \cap Spec(R_P) \subset Supp(M) \cap Spec(R_P) = \{P\}.$$

Therefore $P \in Ass(M)$. $\square$