THE BANACH-TARSKI PARADOX

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Abstract. This paper is an exposition of the Banach-Tarski paradox. We will first simplify the theorem by duplicating almost every point in the ball, and then extend our proof to the whole ball.

Contents

1. Introduction

The Banach-Tarski paradox is a theorem which states that the solid unit ball can be partitioned into a finite number of pieces, which can then be reassembled into two copies of the same ball. This result at first appears to be impossible due to an intuition that says volume should be preserved for rigid motions, hence the name "paradox." In fact, the idea of conservation of volume only applies to Lebesgue measurable sets. However, this theorem decomposes the ball using non-Lebesgue measurable sets, that is, sets for which the idea of volume does not apply. The existence of non-Lebesgue measurable sets follows from the Axiom of Choice, which is one reason the axiom is sometimes thought to be controversial.

The proof presented in this paper is based primarily on the one presented by Hendrickson. We will first create a decomposition of the free group on two generators which will allow us to duplicate the group using the group operation. We will then show the existence of a free group of rotations in 3-dimensional space. This will allow us to duplicate almost every point in the ball, and is the main idea of the theorem. Finally, we will show that our work can be applied to the entire solid ball with just a few changes.

Date: January 9, 2015.

2. A Decomposition of the Free Group

In this section we will demonstrate how a free group can be duplicated using the group operation applied to certain partitions. When we allow the group to represent rigid transformations, this will allow us to duplicate the sphere.

Definition 2.1. A *free group* is a group such that any two words on a specified set of generators are different unless their equality follows from the group axioms.

We will be using the free group on two generators, denoted F_2, and denote the generators of F_2 as a and b. Let the group operation be denoted by concatenation. Thus F_2 consists of all expressions that can be built from a, a^{-1}, b, and b^{-1}, as well as the empty expression, e.

Let $S(a)$ denote the set of all expressions in F_2 that begin with a, after having been fully simplified using the group axioms. For example, ab is in $S(a)$, but $aa^{-1}b$ is not, because $aa^{-1}b = b$. Now define $S(a^{-1})$, $S(b)$, and $S(b^{-1})$ similarly. Since every expression in F_2 must be either empty or start with some symbol, we can decompose F_2 into a union of disjoint sets like so:

$$F_2 = \{e\} \cup S(a) \cup S(a^{-1}) \cup S(b) \cup S(b^{-1}).$$

We will now "shift" two of these subsets to obtain two copies of F_2. Let $aS(a^{-1})$ denote the set of all expressions in $S(a^{-1})$ with a concatenated on the left. Define $bS(b^{-1})$ likewise. Thus $aS(a^{-1})$ is the set of all expressions that do not begin with a, and $bS(b^{-1})$ is the set of all expressions that do not begin with b, so we have the following reconstructions of F_2:

$$F_2 = S(a) \cup aS(a^{-1}),$$
$$F_2 = S(b) \cup bS(b^{-1}).$$

Thus after partitioning the group into five parts, we can reassemble four of them into two copies of the entire group. Note that e is included in $aS(a^{-1})$ and $bS(b^{-1})$, so we did not need $\{e\}$ in our reconstructions. Thus we still have $\{e\}$ as a leftover piece. As we do not want to have any extra pieces when we duplicate the ball, we will need to be careful when we apply this deconstruction in Section 3.

3. A Free Group of Rotations

We will now find a free group of rotations about the origin in $\mathbb{R}^3$ with two generators. Choose $\theta = \arccos\frac{1}{3}$ and let A be a rotation by θ about the x axis, and let B be a rotation by θ about the z axis. Thus the following rotations of interest can be represented as these matrices, since $\sin(\arccos\frac{1}{3}) = \sqrt{1-\frac{1}{3^2}} = \frac{2\sqrt{2}}{3}$ and $\cos(\arccos\frac{1}{3}) = \frac{1}{3}$:

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix},$$
$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix},$$
$$B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$
$$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Let $\mathcal{G}$ denote the group generated by A and B. We chose $\theta = \arccos\frac{1}{3}$ because this is an irrational multiple of π, so the orbit of these rotations should have no repetitions. We will now prove this fact.

Lemma 3.1. *If $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ is an expression in $\mathcal{G}$ with length n in reduced form, then $\rho(0,1,0)$ is of the following form, where a, b, and c are integers:*

$$\rho(0,1,0) = \frac{1}{3^n}\left(a\sqrt{2}, b, c\sqrt{2}\right).$$

Proof. We will use a proof by induction. When $n = 0$, $\rho(0,1,0) = (0,1,0)$, so clearly the lemma is true in this case. Let $n > 0$ and suppose the lemma holds for expressions of length $n-1$. Since ρ is of length n, it can be written as $\rho = A\rho'$ or $A^{-1}\rho'$ or $B\rho'$ or $B^{-1}\rho'$ where ρ' is an expression of length $n-1$.

By the inductive hypothesis, $\rho'(0,1,0) = \frac{1}{3^{n-1}}\left(a\sqrt{2}, b, c\sqrt{2}\right)$ for some integers a, b, and c. By simple calculation using the aforementioned matrices, we find the following:

$$\begin{aligned}
A\rho'(0,1,0) &= \frac{1}{3^n}(3a\sqrt{2}, b-4c, (2b+c)\sqrt{2}),\\
A^{-1}\rho'(0,1,0) &= \frac{1}{3^n}(3a\sqrt{2}, b+4c, (-2b+c)\sqrt{2}),\\
B\rho'(0,1,0) &= \frac{1}{3^n}((a-2b)\sqrt{2}, 4a+b, 3c\sqrt{2}),\\
B^{-1}\rho'(0,1,0) &= \frac{1}{3^n}((a+2b)\sqrt{2}, -4a+b, 3c\sqrt{2}).
\end{aligned}$$

Since a, b, and c are integers, these expressions are all of the desired form. Thus by the principle of mathematical induction we have that $\rho(0,1,0)$ is of the required form for all ρ. □

Theorem 3.2. *$\mathcal{G}$ is a free group, i.e., there is no non-trivial identity in $\mathcal{G}$.*

Proof. Suppose to the contrary that there exists some non-trivial identity ρ in $\mathcal{G}$. Thus $\rho(0,1,0) = (0,1,0)$. By Lemma 3.1, $\rho(0,1,0)$ must be of the form $\frac{1}{3^n}\left(a\sqrt{2}, b, c\sqrt{2}\right)$, so we must have that $a = c = 0$ and $b = 3^n$ where $n > 0$.

Therefore $a \equiv b \equiv c \equiv 0 \mod 3$. It can be shown by induction and listing all possible results of applying A and B modulo 3 that this is not possible. The details are lengthy but not very interesting or elegant, so they will be omitted here. $\square$

4. A Decomposition of (Most of) the Ball

Let $L = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ denote the unit ball. We will now partition the ball missing its center, denoted $L' = L \setminus \{(0, 0, 0)\}$, into the *orbits* of $\mathcal{G}$.

Definition 4.1. Two points, a and b, belong to the same *orbit* if and only if there exists some ρ in $\mathcal{G}$ such that $\rho(a) = b$.

Now we can use the Axiom of Choice to pick one point of L' from each orbit. Call the set of all these points M. Thus we can now reach any point in L' by applying a rotation from $\mathcal{G}$ to the correct element of M.

However, we want every point in L' to be reached by a *unique* rotation in $\mathcal{G}$. We will be partitioning L' according to which kinds of rotations reach each point, so any point that can be reached by different rotations could end up in multiple partitions. Unfortunately, any points along an axis of some rotation can be reached in non-unique ways. For example, if m is in M and ρ is a rotation such that $\rho(m) = (1, 0, 0)$, then $A\rho$ is a different rotation, but $(A\rho)(m) = (1, 0, 0)$, since $(1, 0, 0)$ lies on the x-axis. Thus $(1, 0, 0)$ could potentially end up in multiple partitions if we do not exclude it here.

Let D be the set of all points in L' that are fixed points of rotations in $\mathcal{G}$. Since $\mathcal{G}$ is countable and each rotation in $\mathcal{G}$ has one axis of rotation, the points in D lie on countably many lines. Thus D is a null set (i.e., it has a Lebesgue measure of zero), and so *almost every* point on L' can be reached in a unique way. We will proceed with a partition of $L' \setminus D$ and deal with these fixed points later.

In our decomposition of the free group, we had an extra piece, $\{e\}$, so we need to be a little clever to apply this decomposition to $L' \setminus D$. First we define a set X to make our notation a little easier:

$$X = \bigcup_{i=1}^{\infty} A^{-i} M.$$

So X is the set of all elements of M after rotations consisting entirely of A^{-1} repeated any number of times. Next we will choose a decomposition of four pieces, analogously to the procedure for F_2. Let:

$$\begin{aligned}
P_1 &= S(A)M \cup M \cup X, \\
P_2 &= S(A^{-1})M \setminus X, \\
P_3 &= S(B)M, \\
P_4 &= S(B^{-1})M.
\end{aligned}$$

$$\text{We see that: } L' \setminus D = P_1 \cup P_2 \cup P_3 \cup P_4$$

By rotating two of these pieces, we can get exactly two duplicates of every piece, and so we have a decomposition of $L' \setminus D$ with the desired property. In particular, since:

$$\begin{aligned} AP_2 &= P_2 \cup P_3 \cup P_4 \text{ and} \\ BP_4 &= P_1 \cup P_2 \cup P_4, \text{ we find that} \\ L' \setminus D &= P_1 \cup AP_2, \\ L' \setminus D &= P_3 \cup BP_4. \end{aligned}$$

Thus we have a decomposition which allows us to duplicate all of the ball excepting the center and any points lying on an axis of rotation. These issues will be addressed in the next section.

5. Fixed Points and the Center

We will now deal with the issue of fixed points.

Definition 5.1. We say that sets X and Y are *equidecomposable* if X can be partitioned into a finite number of parts, and the parts can be reassembled into Y by rigid motions (rotations and translations) only.

Theorem 5.2. *$L' \setminus D$ and L' are equidecomposable.*

Proof. Since the points of D lie on countably many axes, we can find a line l through the origin that does not intersect D. Again, since the points of D lie on countably many axes, we can find an angle θ such that a rotation by θ about l never maps a point in D onto another point in D. I.e., if we let ρ be the rotation by θ, then $\rho^n(D) \cap \rho^m(D) = \emptyset$ for all distinct integers n and m. Define a set E as follows.

$$E = D \cup \rho(D) \cup \rho^2(D) \cup \rho^3(D) \cup \cdots$$

Clearly, $L' = E \cup (L' \setminus E)$, which is equidecomposable with $\rho(E) \cup (L' \setminus E)$. By the way we defined E, we have that $\rho(E) = E \setminus D$, so $\rho(E) \cup (L' \setminus E) = (E \setminus D) \cup (L' \setminus E) = L' \setminus D$. Thus we have that L' is equidecomposable with $L' \setminus D$. □

Since we now know that $L' \setminus D$ is equidecomposable with L', the fixed points of rotations in $\mathcal{G}$ are not a problem. Thus it follows that a ball without a center can be duplicated. We will now address the center point.

Lemma 5.3. *A circle is equidecomposable with a circle missing a single point.*

Proof. Suppose we are dealing with the unit circle, $S^1 = \{(x, y) : 1 = x^2 + y^2\}$, missing the point at $(1, 0)$. We are using the unit circle simply to make the notation easier, the same argument will work for any circle.

Let A be the set of all points of the form $(\cos n, \sin n)$, where n is a positive integer. Since π is irrational, $(\cos n, \sin n) \neq (\cos m, \sin m)$ for distinct integers n and m. Thus A is countably infinite and does not include $(1, 0)$. Let B be the set of all points in $S^1 \setminus \{1, 0\}$ that are not in A.

Rotate A about the origin by -1 radians. Denote this rotated set A'. This rotation maps $(\cos 1, \sin 1)$ onto $(1, 0)$, our missing point. Since each point of A is 1 radian apart and A is countably infinite, every point that was originally in A is still in A'. Thus $A' = A \cup \{(1, 0)\}$, and so $S^1 \setminus \{(1, 0)\} = A \cup B$ is equidecomposable with $A' \cup B = S^1$. □

Theorem 5.4. *A ball without its center is equidecomposable with the ball.*

Proof. By simply constructing a circle inside the ball that intersects the center of the ball, we have by Lemma 5.3 that the ball is equidecomposable with the ball missing the center. □

6. The Final Proof

Now that all the small details have been dealt with, we can easily prove the full theorem.

Theorem 6.1 (Banach-Tarski Paradox)**.** *The unit ball is equidecomposable with two copies of itself.*

Proof. It has been shown in Section 4 that the ball without a center or any points lying on an axis of rotation is equidecomposable with two copies of itself. By Theorem 5.2, the ball missing its center is equidecomposable with two copies of itself. Finally, by applying Theorem 5.4, we see that the solid ball is equidecomposable with two copies of itself. □

Acknowledgments. It is a pleasure to thank my mentor, Rachel Vishnepolsky, for everything she has done for me throughout the REU program. I'd also like to thank Peter May for giving me the opportunity to research this topic and for his patience which allowed me to gain a deeper understanding of the ideas behind this paper.

References

[1] Anders O.F. Hendrickson The Banach-Tarski Paradox. `http://home.snc.edu/andershendrickson/presentations/HendricksonBanachTarski.pdf`
[2] C. E. Silva. Invitation to Ergodic Theory. American Mathematical Society. 2007.
[3] Terence Tao. The Banach-Tarski Paradox. `https://www.math.ucla.edu/~tao/preprints/Expository/banach-tarski.pdf`
[4] Tom Weston. The Banach-Tarski Paradox. `http://people.math.umass.edu/~weston/oldpapers/banach.pdf`