Real Projective Space: An Abstract Manifold

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March 10, 2017

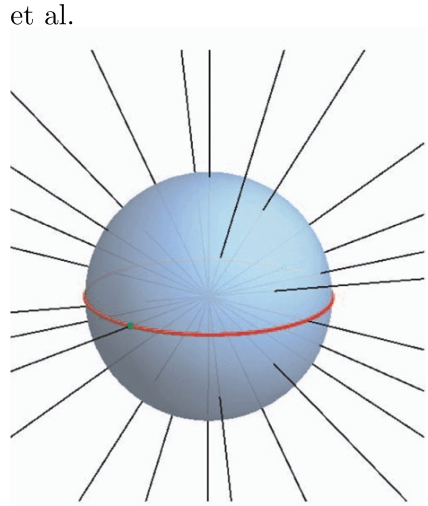
In this talk, we seek to generalize the concept of manifold and discuss *abstract*, or *topological*, manifolds. We'll examine the example of real projective space, and show that it's a compact abstract manifold by realizing it as a quotient space. Ultimately, we'll show that it's not diffeomorphic to the n-sphere by examining a structure called the fundamental group.

1 Why Discuss Abstract Manifolds?

It seems that manifolds, which are spaces that look locally like $\mathbb{R}^n$, would always be easiest to understand in terms of their embeddings in larger Euclidean space. Viewing manifolds as subsets of some larger space $\mathbb{R}^N$ presents them as a natural generalization of curves and surfaces in $\mathbb{R}^3$, and allows helpful ways to visualize them.[1] For example, the Möbius strip is easier to conceptualize as a twisted sheet that we can see in $\mathbb{R}^3$ than it is as a quotient space of a square. And in fact, the Whitney Embedding Thm.[2] guarantees that any k-dimensional manifold can be embedded into $\mathbb{R}^{2k+1}$.

However, the interesting properties of manifolds are not all reliant on the ambient space, and ideally, we'd like the properties of the manifold we discuss to *not* depend on the choice of ambient space.[3] When we examine the manifold alone, not as a submanifold of a larger Euclidean space, we are considering an *abstract manifold*.

Figure 1: $\mathbb{R}P^2$ as a Set of Lines, adapted from Fig. 3 of Demiralp et al.



2 The Real Projective Plane

This talk will focus on one kind of abstract manifold, namely real projective space $\mathbb{R}P^n$. To see why this space has some interesting properties as an abstract manifold, we start by examining the real projective plane, $\mathbb{R}P^2$.

Imagine 3D space. Consider the set of all lines through the origin, extending in either direction to infinity. What does this set look like? It may appear like a sea urchin, but as we'll show this construction actually has the structure of a compact surface.

The visualization on the right shows lines through the origin, cut off at unit length by the unit sphere. But is there a way to visualize the actual surface? The real projective plane does not embed into

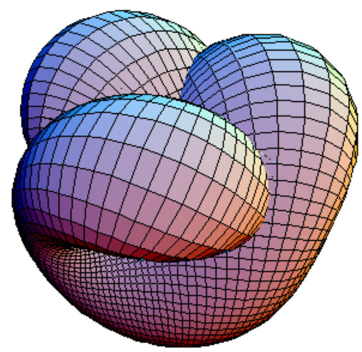
[1] G&P xii
[2] G&P 48
[3] Yuan

$\mathbb{R}^3$, but it does immerse, through, for example, Boy's Surface (shown below). We won't discuss this immersion at length, but include it to show just how confusing an abstract manifold can be when we rely on visualizing it in a larger space! And for $n > 2$, prospects only get bleaker.

3 Projective Space as a Quotient Space

A better way to think of real projective space is as a quotient space of S^n. Now, we arrive at a *quotient space* by making an identification between different points on the manifold. Essentially, we define an equivalence relation, and consider the points that are identified to be "glued" together. So if X is our topological space and X^* is a partition of X into disjoint subsets consisting of points we wish to identify, we define our topology in terms of the *quotient map*, p, that takes X to X^*. Then a subset $U \subset X^*$ is open if and only if its preimage $p^{-1}(U)$ is open in X.[4]

Figure 2: Boy's Surface, from Wikimedia Commons



3.1 A Concrete Example

One of the most basic examples of a quotient space is the identification of the endpoints of an interval to form a circle. To use the notation above, $X = [0, 2\pi]$, $X^* = (0, 2\pi) \cup \{p\}$, and the equivalence relation is simply $0 \sim 2\pi$. Since X^* consists of an interval identified with the point p at both 0 and 2π, the space is a loop and is thus homeomorphic to S^1. We can explicitly write the projection map as $p(x) = e^{ix}$, if we like. This map properly identifies $p(0) = p(1) = (1, 0)$.

We can consider three types of sets in the quotient space. The first consists of regions of the circle away from (1,0), which map back to open intervals in the interior of $[0, 2\pi]$; e.g. the circular arc with $\pi/6 < \theta < \pi/3$ maps back to $(\pi/6, \pi/3)$. The second corresponds to regions including (1,0), which map back to unions of open sets at the boundaries of the interval; e.g. the circular arc with $-\pi/3 < \theta < \pi/3$ maps back to $[0, \pi/3) \cup (5\pi/3, 2\pi]$. A somewhat confusing third case is the region $0 \leq \theta < \pi$, which we'd like *not* to be open, since it has a closed boundary on one side. Checking the preimage under p, and recalling that $p^{-1}((1, 0)) = \{0, 2\pi\}$ we find that the preimage $[0, \pi) \cup \{2\pi\}$ is indeed not open.

Hopefully that example makes the formal components of a quotient space a bit more concrete. Now in fact, the example above *does* correspond to projective space—we have $S^1 \cong \mathbb{R}P^1$! Next we'll see why this is the case, and why that's useful for describing projective space.

3.2 Working up from Dimension Zero to Understand the Quotient Identification

In general, I claim that real projective n-space is homeomorphic to an n-sphere with antipodes identified. Symbolically, we can write this statement as $\mathbb{R}P^n \cong S^n/(x \sim -x)$. Why is this true? Let's examine a few smaller-dimensional cases, before inducting to argue it for all n.

[4] Munkres 138-39

3.2.1 $n = 0$

The trivial case here is $n = 0$. Now $\mathbb{R}P^0$, the set of lines through the origin in $\mathbb{R}$, consists of just one element—$\mathbb{R}$ itself. Note that this *doesn't* mean that $\mathbb{R}P^0 \cong \mathbb{R}$. Rather, it means $\mathbb{R}P^0 \cong \{\mathbb{R}\}$. As a singleton set, it's then homeomorphic to just a point: write $\mathbb{R}P^0 \cong \{p\}$. On the other hand, what is a zero-dimensional circle with antipodes (vacuously) identified? Again, it's just a point.

3.2.2 $n = 1$

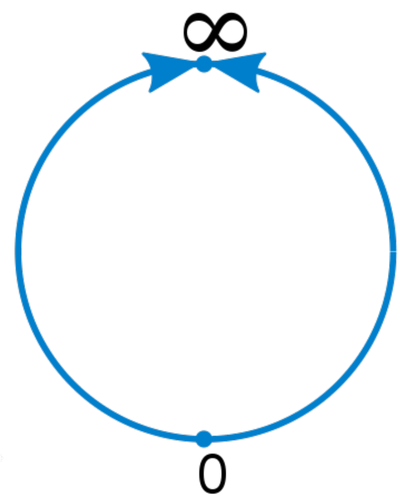
Let's see this in a more exciting example than $n = 0$. The real projective line, we claimed above, is homeomorphic to S^1. Consider the set of lines through the origin in $\mathbb{R}^2$. We can parameterise these lines by their slopes, in terms of the angle they make with the x-axis. Since lines extend in both directions, it only makes sense to look at the region in the upper half of the plane—so restrict $0 \leq \theta \leq \pi$. And the line through 0 also goes through π, so we have the semicircle with its endpoints identified.

But once we identify the endpoints of the semicircle, we glue them together, and complete the loop. We arrive at a space homeomorphic to S^1. This justifies our claim above.

What we wanted though, according to our general statement, was to show it was homeomorphic to $S^1/(x \sim -x)$, not just to S^1. How is this the same? It turns out that our argument above about the semicircle with identified endpoints is exactly the same as if we'd started with a circle and identified antipodes—when antipodes are identified, we may as well take only half of the circle, and then we need to identify the endpoints of the semicircle because they're opposite points. Surprisingly, we get back the circle.

We'll discuss later how this special relationship $S^1 \cong S^1/(x \sim -x)$ also has ramifications for the fundamental group.

Figure 3: $\mathbb{R}P^1$ as a Quotient Space, from Wikimedia Commons



3.2.3 $n = 2$

Now let's see if we can get any more intuition about $\mathbb{R}P^2$ and why its immersion looks so strange. Our claim above means that the projective plane is homeomorphic to the sphere with antipodes identified, and this makes sense, because lines through the origin always intersect the sphere twice, at opposite points. To uniquely determine a line, we can then pick just one of each pair of antipodes such that it lies on the line at unit distance from the origin.

But can we build this inductively? When we identify antipodes on the sphere, we can pick the upper hemisphere to focus on and forget about their mirror images below the xy-plane at $z = 0$. This region, not including the intersection with the plane, has the same structure as the open unit disk $B_1(0) = \{x \in \mathbb{R}^2 \mid ||x - 0|| < 1\}$. We can just flatten it out! As for the boundary of the disk, which corresponds to the sphere's intersection with $z = 0$, its antipodes need to be identified, too. The boundary corresponds exactly to the case $\mathbb{R}P^1$, so we can build $\mathbb{R}P^2$ as $B_1(0) \cup \mathbb{R}P^1$. *How* exactly the circle and disk connect is tricky—we can imagine the disk wrapping *twice* around the circle in order to connect antipodes, and we can start to understand why Boy's surface looks so convoluted.

3.2.4 General n

The process we described above, attaching $B_1(0)$ to $\mathbb{R}P^1$, is called adjoining a k-cell, with $k = 2$ here as the dimension of the open ball, and it results in a topological object called a *CW-complex*. See that this approach also works for $\mathbb{R}P^1 = B_1(0)_1 \sqcup \mathbb{R}P^0$, where $B_1(0)_1$ is the one-dimensional open ball—that is, an open interval—and $\mathbb{R}P^0$ is a point, as we've argued. If we use that point to connect the ends of the interval, we indeed get $S^1 \cong \mathbb{R}P^1$.

Our intuition won't extend much beyond $\mathbb{R}P^2$ because we can't easily visualize a three-dimensional sphere. But it is true that this inductive process continues. If we set $B_1(0)_k$ to be the k-dimensional open ball in $\mathbb{R}^k$, we have

$$\mathbb{R}P^n \cong B_1(0)_n \sqcup \mathbb{R}P^{n-1} \cong B_1(0)_n \sqcup \cdots \sqcup B_1(0)_1 \sqcup B_1(0)_0.$$

From the examples above, and from the inductive construction of $\mathbb{R}P^n$, the quotient identification starts to seem believable. This connection between two different ways of thinking about the manifold will be quite powerful as we try to derive more of its properties.

Now that we have a better understanding of what $\mathbb{R}P^n$ is, without reference to any embedding, we can return to the notion of abstract manifolds. We'll offer a formal definition and show how $\mathbb{R}P^n$ satisfies it, before going on to show how $\mathbb{R}P^n$, as a compact n-dimensional manifold, is *not* diffeomorphic to the compact, n-dimensional manifold S^n.

4 What Are the Properties of an Abstract Manifold?

Once we've forgotten about any surrounding space for our manifold, what properties are we left with? A k-dimensional abstract manifold must of course still be locally diffeomorphic to $\mathbb{R}^k$. It must also satisfy two other topological properties.

4.1 Some Definitions

Before we discuss those properties, it's important to quickly review what a topological space is—a space X, in the form of a set, and a topology τ, which is a set of all open subsets and thus specifies which subsets of our space are open. In Euclidean space, we're fond of the metric topology, in which open sets can all be formed from open balls of certain radii. More generally, a topology only needs to satisfy a few properties. Stated formally,[5]

Definition 4.1. *Topology* A topology τ on a space X satisfies

- $\emptyset \in \tau$
- $X \in \tau$
- Any union of elements in τ is also in τ.
- Finite intersections of elements in τ are in τ.

That is, the empty set and the whole space are always open, as are unions of any set of open sets. Intersections of open sets are only necessarily open if the collection is finite. It's easy to see that our notion of open balls in a metric space satisfies these properties.

[5] Munkres 76

We resort to the general definition of topology here because when we do away with an ambient space, we no longer necessarily have notions of distance, of paths, or of areas and volume.[6] All we can assume for abstract manifolds are a couple basic properties in addition to local Euclidean nature.[7]

Definition 4.2. *Abstract Manifold* An abstract k-dimensional manifold M satisfies

- M is Hausdorff. That is, for any two points $p, q \in M$, there exist disjoint open sets U, V such that $p \in U$ and $q \in V$.
- M is second-countable. That is, the topology for M has a countable basis.
- M is locally Euclidean. For any $x \in M$, there is an open neighborhood of x that is homeomorphic to $\mathbb{R}^k$.

The first two conditions prevent pathological behavior, and are necessary if M is to locally behave like Euclidean space. For example, *non*-Hausdorff spaces could have two distinct points be non-separable into two disjoint neighborhoods, and so we could have a sequence converge to *both* points. Intuitively, this doesn't make sense for a locally Euclidean space—in Euclidean space, we can always find open balls small enough around a pair of points such that they don't intersect.

Second-countability, meanwhile, is a condition on the basis for our topology. We give a brief definition below but won't be too concerned with the details.

Definition 4.3. *Basis for a Topology* A basis is a collection $\mathcal{B}$ of subsets of X such that

- For any $x \in X$, there is a $B \in \mathcal{B}$ that contains x.
- For any $B_1, B_2 \in \mathcal{B}$, with some $x \in B_1 \cap B_2$, there is a third basis element B_3 inside the intersection.

Second-countability, then, means that our basis $\mathcal{B}$ is countable. We can imagine this on the line $\mathbb{R}$ by taking the metric topology, and forming a basis containing open balls of rational radius centered at rational points on the number line. Since $\mathbb{Q}$ is countable, the basis is countable. This approach generalizes to $\mathbb{R}^n$, where we take open balls of rational radii centered at points of rational radius. So for a manifold, which behaves locally like $\mathbb{R}^n$, we'd expect the same sort of behavior.

4.2 Demonstrating that $\mathbb{R}P^n$ *Is* an Abstract Manifold

To see that $\mathbb{R}P^n$ is Hausdorff, consider two distinct elements x and y. Thinking of them as two lines in $\mathbb{R}^{n+1}$, we can imagine two cones around each that intersect only at the origin. But we can be a bit more rigorous using our quotient space—we can identify each of x and y with a pair of antipodes on S^n. Call these antipodes $+p$ and $-p$ for x, and $+q$ and $-q$ for y. Then since S^n is a manifold, it is Hausdorff, and we may find[8] four disjoint open neighborhoods—say $+p \in U_+$, $-p \in U_-$, $+q \in V_+$, and $-q \in V_-$.

In fact, by symmetry, we may require that each pair U_+, U_- and V_+, V_- are mirror images. That is, $a \in U_+ \iff -a \in U_-$. This simplifies our next step, because if the sets are symmetric we know that $p(U_+) = p(U_-)$ and $p(V_+) = p(V_-)$. It follows that if $a \in U_+ \cup U_-$ then we also have $-a \in U_+ \cup U_-$, and so by assumption, $-a \notin V_+ \cup V_-$. And that means the quotient map does not identify any points

[6] Yuan
[7] Lee 3
[8] "How can I prove formally..."

between the open U and V sets. That is, $p(U_+) \cap p(V_+) = \emptyset$, and we've found our disjoint open sets around x and y. Hence $\mathbb{R}P^n$ is Hausdorff.

To see that $\mathbb{R}P^n$ is second countable, we can consider the topology of the sphere S^n. This topology is second-countable by assumption that S^n is a manifold. To see this formally, we can remember just for a moment that we actually *do* have an embedding $S^n \hookrightarrow \mathbb{R}^n$, and we can take a countable basis for the topology on S^n simply by intersecting the basis for $\mathbb{R}^n$ with the sphere. Then, any open set in $\mathbb{R}P^n$ is the image under p of an open set in S^n, so if $\mathcal{B}$ is a basis of the sphere, we can take $\{p(B) \mid B \in \mathcal{B}\}$ to be the basis for the projective space.

Finally, we can exhibit a parameterisation to show that $\mathbb{R}P^n$ is locally diffeomorphic to $\mathbb{R}^n$. Following the steps of Lee (6), we start by defining a more convenient quotient space. We consider $\mathbb{R}P^n$, the set of lines, as a subset of $\mathbb{R}^{n+1}$, and we form the quotient map π that takes a point $x \neq 0$ to the line running from the origin to x. This quotient map identifies all points on that line to an element in $\mathbb{R}P^n$.

Now, we specify our open sets. We call $\tilde{U}_i \subset \mathbb{R}^n \setminus \{0\}$ the set where the i^{th} coordinate is *nonzero*, and define $U_i = \pi(U_i)$. We can define a parameterisation φ by examining one case at a time. For the i^{th} coordinate, define $\varphi_i : U_i \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ such that

$$\varphi_i(x_1, ..., x_{n+1}) = \left(\frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i}\right).$$

Then, since the sets U_i cover the space, we've shown a local parameterisation everywhere.

We won't go into great detail about this map, but give the formula in order for some concreteness. That it is continuous with a continuous inverse can be checked without much difficulty, and it at least makes sense in the case $n = 1$, where φ_1 sends $(x_1, x_2) \mapsto \left(\frac{x_2}{x_1}\right)$, which is the slope of the line from 0 to (x_1, x_2) and can be identified with an angle on S^1.

4.3 $\mathbb{R}P^n$ Is Compact

In the past few sections, we've seen the power of the quotient space in describing properties of $\mathbb{R}P^n$ as an abstract manifold. We now argue that $\mathbb{R}P^n$ also inherits compactness from S^n.

The sphere S^n is a compact manifold because it is closed and bounded in $\mathbb{R}^n$. This follows from the Heine-Borel Thm., and is invariant of the ambient space.

For $\mathbb{R}P^n$, compactness follows from surjectivity of our quotient map, p. Explicitly, let $\{U_\alpha\}_{\alpha \in A}$ be a possibly infinite cover of $\mathbb{R}P^n$. We can take the preimages $\{p^{-1}(U_\alpha) \mid \alpha \in A\}$ to form an open cover of S^n. Then, by compactness of S^n, we can find a finite subcover $\{p^{-1}(U_i)\}$. Passing these back through p, we get a finite subcover $\{U_i\}$ of $\mathbb{R}P^n$.

5 The Fundamental Group

In the last part of this talk, we'll introduce the idea of the fundamental group, and combine it with our conception of real projective space as a quotient space to find that real projective n-space is *not* diffeomorphic with the n-sphere.

The fundamental group of a topological space concerns the different forms of loops that can exist in that space, and how they can be combined. We can think of these loops as maps from S^1 into the space, and observe the paths that they trace out on the surface, which start and end at the same point.

We'll quickly give the definition of homotopy,[9] which is a fairly intuitive concept when you can visualize it.

Definition 5.1. *Homotopy* We say that two functions f_1 and f_2 from $X \to Y$ are homotopic if we have a larger function that continuously deforms one into the other. We write this map as F and parameterise it by $t \in [0, 1]$, which we could think of as time. Formally, $F(x, t) : X \times [0, 1] \to Y$ is such that $F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$.

We consider two loops to be the same if there is a homotopy between them—that is, if we can continuously deform one into the other. So an element of the group is a homotopy class of maps S^1 into the space X. These elements can be combined by adjoining two loops at the same point, and tracing out one then the other. There isn't time in this talk to do justice to the fascinating ways that the group structure of these loops can be seen in covering spaces of our original space, but it's something worth researching if you've never seen it before!

Instead, the critical idea we are going to use is that two spaces can only be diffeomorphic if they have the same fundamental group. If two spaces are diffeomorphic, any kind of path in one must exist in the other.

5.1 A Few Examples

The simplest fundamental group that a space can have is just the trivial group, consisting of one element. When a space has trivial fundamental group, all loops in the space can be deformed into each other, and hence into a point. We call such a space *simply connected*. Examples of contractible spaces are $\mathbb{R}^n$, which we shrink by the homotopy $F(x, t) = (1 - t) \cdot x$, and S^n for $n > 1$. Imagine on the 2-sphere, we can slide any loop around the sphere until it contracts to a point. In higher dimensions, this is still the case. We show this by finding a point p not in the image of our map $S^1 \to S^n$, taking the stereographic projection from $S^n \setminus \{p\} \to \mathbb{R}^{n-1}$, and using that $\mathbb{R}^{n-1}$ is simply connected. We do run into some subtleties in guaranteeing that such a point p exists, but this can be resolved with another theorem, by Sard.[10]

However, for S^1, the fundamental group is much more interesting. Since we can map a loop into S^1 by mapping it to a point, or by wrapping it around the circle once, or by wrapping it around twice, and so on, we find that the fundamental group of the circle is in fact $\mathbb{Z}$. As shown in the figure below, we identify each kind of loop in S^1 with the number of times it wraps around the circle, and arrive at the group structure of the integers.

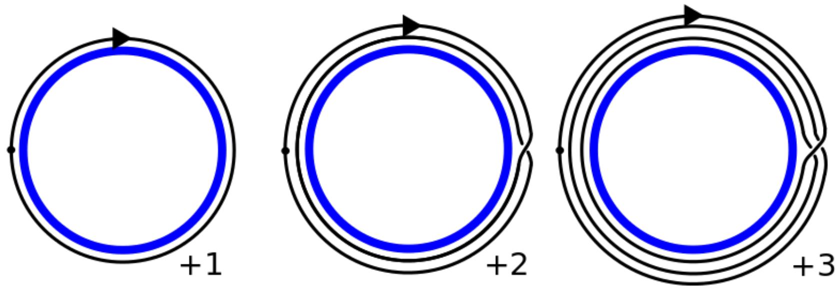


Figure 4: Fundamental Group of S^1, from Wikimedia Commons

[9] G&P 33

[10] see Math 132 Problem Set 4, Problem §1.7.6 from G&P

5.2 Showing $\mathbb{R}P^n \ncong S^n$ for $n > 1$

From our quotient relationship $\mathbb{R}P^n \cong S^n/(x \sim -x)$, we might already expect that $\mathbb{R}P^n \ncong S^n$ for $n > 1$, but by using the fundamental group we can see more clearly what the differences are between each space. As explained above, we actually have $\mathbb{R}P^1 \cong S^1$, so these spaces share the fundamental group of $\mathbb{Z}$.

Meanwhile, the fundamental group of S^n for $n > 1$ is trivial. But for $\mathbb{R}P^n$ for $n > 1$, we actually have fundamental group $\mathbb{Z}/2$ instead. Looking at the case $n = 2$, we consider the space as the unit disk with antipodes identified, which we explained earlier as one way to view the space. Then, in addition to having loops in the interior of the space, which contract to points, we also have "loops" that run between antipodes, which are identified. Since we can't move the boundary of the disk, or shift the antipodes to the interior, there is no way to deform this kind of loop down to a point. So we have a non-trivial class of loops in the space, and hence non-identity element in the group. It turns out this is the only other element, and so our two-element set is isomorphic to the cyclic group of order 2, or $\mathbb{Z}/2$.

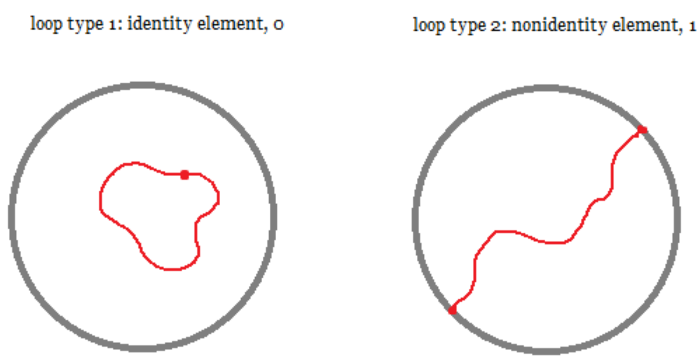


Figure 5: Two Types of Loops in $\mathbb{R}P^2$

A similar argument shows the difference between loops in the interior of the solid ball $B_1(0)_3$ within and loops running between antipodes on the surface of S^2. Since these form $\mathbb{R}P^3$ according to our construction in part 3.2.4, that shows $\mathbb{R}P^3$ also has fundamental group $\mathbb{Z}/2$.

Inductively, we can go on to show $\mathbb{R}P^n$ has fundamental group $\mathbb{Z}/2$. If we were to give a more rigorous proof of the fundamental group of $\mathbb{R}P^n$, then covering spaces would come in handy.[11] However, the really interesting part of this example is seeing the two different kinds of loops in the quotient-identified sphere. This clever argument makes it a bit more intuitive why higher-dimensional real projective space differs from the n-sphere.

6 Summary

In this talk, we presented real projective space as a kind of abstract manifold. Even in two dimensions, the space becomes very tricky to visualize becaues it has no embedding in $\mathbb{R}^3$. But after trying a couple different ways of thinking about the space, we found that a quotient relationship with the n-sphere allowed us to prove several statements about the properties of the manifold—for example, that it was

[11] Munkres 373

compact. Finally, we elucidated the *differences* between $\mathbb{R}P^n$ and S^n by examining their fundamental groups. We saw that the quotient relationship introduced a kind of loop running between antipodes, showing that the space is quite different from the simply connected sphere.

If this was all new to you, hopefully you were able to follow parts of the arguments in spite of the myriad definitions. And if some or all of this was familiar to you, hopefully this talk provided a good recap. Either way, I hope I convinced you of how cool real projective space is, and how thinking about it as an abstract manifold can sometimes make more sense than trying to visualize it in its ambient space.

Thank you!

7 References

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8 Acknowledgements

Professor George Melvin gave excellent guidance and advice on this project.