A LOOK AT THE FUNCTORS TOR AND EXT

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Abstract. In this paper I will motivate and define the functors Tor and Ext. I will then discuss a computation of Ext and Tor for finitely generated abelian groups and show their use in the Universal Coefficient Theorem in algebraic topology.

Contents

1. Introduction

In this paper I will be discussing the functors Ext and Tor. In order to accomplish this I will introduce some key concepts including exact sequences, the tensor product on modules, Hom, chain complexes, and chain homotopies. I will then show how to compute the Ext and Tor groups for finitely generated abelian groups following a paper of J. Michael Boardman, and discuss their import to the Universal Coefficient Theorems in Algebraic Topology. The primary resources referenced include Abstract Algebra by Dummit and Foote [3], Algebraic Topology by Hatcher [5], and An Introduction to the Cohomology of Groups by Peter J. Webb [7].

For this paper I will be assuming a first course in algebra, through the definition of a module over a ring.

2. Exact Sequence

Definition 2.1. An *exact sequence* is a sequence of algebraic structures X, Y, Z and homomorphisms φ, ψ between them

$$\cdots \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow \cdots$$

such that $\operatorname{Im}(\varphi) = \ker(\psi)$

For the purposes of this paper, X, Y, Z will be either abelian groups or R-modules.

Definition 2.2. A *short exact sequence* is an exact sequence of the form

$$0 \xrightarrow{0} X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \xrightarrow{0} 0.$$

Because $\ker(\varphi) = \mathrm{Im}(0) = 0$ this means φ is injective. Similarly, because $\mathrm{Im}(\psi) = \ker(0) = Z$ ψ is surjective. Furthermore, any long exact sequence

$$\cdots \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow \cdots$$

can be, at any point, broken up into a short exact sequence as follows:

$$0 \longrightarrow \mathrm{Im}(\varphi) \xrightarrow{\varphi} Y \xrightarrow{\psi} Y \Big/ \ker(\psi) \longrightarrow 0\ .$$

3. Hom

Definition 3.1. Let X, Y left R-modules, for some ring R. Then we define $\mathrm{Hom}_R(X, Y)$ to be the collection of all R-module homomorphism from X into Y:

$$\{\varphi : X \to Y \mid \varphi(x + x') = \varphi(x) + \varphi(x'),\ \varphi(rx) = r\varphi(x) \text{ for } x, x' \in X,\ r \in R\}.$$

Although this construction is, on the face of it, a set we consider it group with operation defined by $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for $\varphi, \psi \in \mathrm{Hom}_R(X, Y),\ x \in X$. Since Y and R-module, and in particular an abelian group, we get that $\varphi(x) + \psi(x) = \psi(x) + \varphi(x)$ implying that $\mathrm{Hom}_R(X, Y)$ is abelian. The above construction has two properties that we will find useful. The first such property is that Hom commutes with finite direct sums.

Theorem 3.2. *For X, Y, Z R-modules, $Hom(X \oplus Y, Z) \cong Hom(X, Z) \oplus Hom(Y, Z)$ and $Hom(X, Y \oplus Z) \cong Hom(X, Y) \oplus Hom(X, Z)$*

Proof. For the first, consider the map $\varphi : X \to X \oplus Y$ defined by $x \mapsto (x, 0)$. Since this is the restriction of a homomorphism, it is a homomorphism. Similarly, define $\psi : Y \to X \oplus Y$ with $y \mapsto (0, y)$ Then let $f \in \mathrm{Hom}(X \oplus Y, Z)$, so $f : X \oplus Y \to Z$ a homomorphism. Thus $f \circ \varphi : X \to Z$ and $f \circ \psi : Y \to Z$ are homomorphisms. Because a homomorphism into a direct sum is uniquely determined by maps into either variable, this gives a map homomorprhism into $\mathrm{Hom}(X, Z) \oplus \mathrm{Hom}(Y, Z)$.

For the other direction, let $f_1 \in \mathrm{Hom}(X, Z), f_2 \in \mathrm{Hom}(Y, Z)$. Then $f(x, y) = f_1(x) + f_2(y)$ is a homomorphism. Composing with $f \in \mathrm{Hom}(X \oplus Y, Z)$ we get $(f \circ \varphi, f \circ \psi)$, and upon applying the map in the other direction we get back f. Similarly, composing in the other direction yields identity, and so the functions are inverses.

The second statement follows analogously [3]. $\square$

We can consider Hom as a functor: $\mathrm{Hom}_R(-, M)$ for M an R-module. For any map $f : X \to Y$ we get an induced map $f_* : \mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M)$ which is defined to be pre-composition by f. That is, for $\varphi \in \mathrm{Hom}(Y, M), f_*(\varphi) = \varphi \circ f$. Notice that because the directions of arrows are reversed we see that, in fact, Hom is a contravariant functor.

Now, since we have introduced exact sequences we want to consider the interaction between Hom and such sequence. In particular, it would be nice if given a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we got a short exact sequence

$$0 \longleftarrow \mathrm{Hom}_R(X, M) \longleftarrow \mathrm{Hom}_R(Y, M) \longleftarrow \mathrm{Hom}(Z, M) \longleftarrow 0.$$

Unfortunately, this is not the case. However, we do have the following:

Theorem 3.3. *For a short exact sequence*

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

after applying the Hom$(-, M)$ functor we get an exact sequence

$$Hom_R(X, M) \xleftarrow{\varphi_*} Hom_R(Y, M) \xleftarrow{\psi_*} Hom(Z, M) \longleftarrow 0$$

Proof. Suppose $0 \to X \to Y \to Z \to 0$ exact as above. We see that φ_*, ψ_* are compositions of homomorphism, and so must be homomorphisms as well.

To show that $\ker(\psi_*) = \mathrm{Im}(0) = 0$, let $f \in \ker(\psi_*)$. This means that $f \circ \psi = 0$, or that $f(\psi(y)) = 0$ for any $y \in Y$, so f is the zero map, because ψ is surjective. To check that $\ker(\varphi_*) = \mathrm{Im}(\psi_*)$, let $f \in \mathrm{Im}(\psi_*)$. This means that there exists some $f' \in \mathrm{Hom}(Z, M)$ such that $f = f' \circ \psi_*$. Then $\varphi_*(f) = f \circ \varphi = (f' \circ \psi) \circ \varphi = f' \circ 0 = 0$ by the exactness of the original sequence. Thus $\mathrm{Im}(\psi_*) \subseteq \ker(\varphi_*)$. In the other direction let $f \in \ker(\varphi_*)$, so $f \circ \varphi = 0$, given that this is trivial for $f = 0$ let f nonzero. This similarly follows from the exactness of the original sequence. Thus, we have the exactness as desired. □

Since this functor preserves exactness on the left side, we say that Hom is *left exact.*

4. Tensor Product

Next, we turn to the tensor product. The tensor product on R-modules X and Y allows us to create a new module which makes it, essentially, possible to multiply elements of the first two. For a general R we get that the resulting structure is an abelian group. However, for commutative R we get that the resulting structure is itself an R-module.

Definition 4.1. Let R be a commutative ring with unit. Let X be a right R-module and Y a left R-module. Their tensor product, $X \otimes_R Y$, is defined as the free module generated by $X \times Y$ modulo the relation generated by $\{(x + x', y) - ((x, y) + (x', y)), (x, y + y') - ((x, y) + (x, y')), (rx, y) - r(x, y), (x, ry) - r(x, y)\}$ for $x \in X, y \in Y$.

Whenever the ring R is clear from context we will write $X \otimes Y$. Elements in $X \otimes_R Y$ will be sums of generating elements $x \otimes y$. As a result, it suffices to consider generating elements. The tensor product has some useful properties: it is associative, commutative, commutes with direct sum, and tensoring with R yields an identity operation. For all the statements below, let X, Y, Z be R-modules as required, and let R be a commutative ring.

Theorem 4.2. $X \otimes R \cong X$

Proof. Define $\phi : X \otimes R \to X$ by $x \otimes r \mapsto rx$. To show that this is a homomorphism of R-modules we begin with the fact that $\phi(r'(x_1 \otimes r + x_2 \otimes r)) \cong \phi((x_1 + x_2) \otimes r'r) \cong r'r(x_1 + x_2) \cong r'(rx_1 + rx_2) \cong r'(\phi(x_1 \otimes r) + \phi(x_2 \otimes r))$ by properties of the tensor product. In the other direction, consider the map $\psi : x \mapsto x \otimes 1$. This is a homomorphism as $\psi(x_1 + x_2) \cong (x_1 + x_2) \otimes 1 \cong x_1 \otimes 1 + x_2 \otimes 1 \cong \psi(x_1) + \psi(x_2)$. Furthermore, these maps are inverses. In one direction we get that $x \mapsto x \otimes 1 \mapsto x$. In the other we have that $x \otimes r \mapsto rx \mapsto rx \otimes 1$. However, we note that $x \otimes r \cong rx \otimes 1$ by properties of the tensor product previously discussed. □

Theorem 4.3. *The tensor product is associative:* $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$.

Proof. Define $\phi : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ with $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$. We have $\phi((x_1 \otimes y) \otimes z + (x_2 \otimes y) \otimes z) \cong \psi(((x_1 + x_2) \otimes y) \otimes z) \cong (x_1 + x_2) \otimes (y \otimes z) \cong x_1 \otimes (y \otimes z) + x_2 \otimes (y \otimes z) \cong \phi((x_1 \otimes y) \otimes z) + \phi((x_2 \otimes y) \otimes z)$. The map in the other direction follows analogously, and as such the maps are inverses. □

Theorem 4.4. *The tensor product is commutative:* $X \otimes Y \cong Y \otimes X$.

Proof. Define $\phi : X \otimes Y \to Y \otimes X$ by $x \otimes y \mapsto y \otimes x$. We have that $\phi(x_1 \otimes y + x_2 \otimes y) \cong \phi((x_1 + x_2) \otimes y) \cong y \otimes (x_1 + x_2) \cong y \otimes x_1 + y \otimes x_2 \cong \phi(x_1 \otimes y) + \phi(x_2 \otimes y)$. The map in the other direction follows by the symmetry of the definition, yielding an isomorphism. □

Theorem 4.5. *The tensor product commutes with direct sum:* $(X \oplus Y) \otimes_R Z \cong (X \otimes_R Z) \oplus (Y \otimes_R Z)$

Proof. We see that our definition of a tensor product is equivalent to the condition that the following diagram commutes

$$\begin{tikzcd} X \times Y \arrow[rr] \arrow[dr] & & X \otimes Y \arrow[dl] \\ & Z & \end{tikzcd}$$

with the map from $X \otimes Y$ to Z determined uniquely from the map from $X \times Y$ to Z. According to Eisenbud this yields the following universal property:

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

Therefore, the commutativity of the tensor product with direct sum follows from the commutativity of Hom with direct sum, shown above. □

We can also consider the tensor product as a functor $M \otimes_R -$ where M is an R-module for R commutative. For any maps homomorphism $\varphi : X \to Y$ we get map $\Phi : M \otimes_R X \to \otimes_R \to Y$ defined by $\Phi = \mathrm{id} \otimes \varphi$ where we understand $(\mathrm{id} \otimes \varphi)(m \otimes_R x)$ to be $\mathrm{id}(m) \otimes \varphi(x)$. In this case, $M \otimes_R -$ takes R-modules to R-modules, so it is a functor from R-mod to itself. Since our goal to preserve exactness, it would be great if for a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

we got a short exact sequence

$$0 \longrightarrow M \otimes_R X \longrightarrow M \otimes_R Y \longrightarrow M \otimes_R Z \longrightarrow 0$$

Unfortunately, this is not always the case.

However, we do have the following:

Theorem 4.6. *$X \otimes_R -$ is right-exact, meaning for a short exact sequence*

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

we get an exact sequence

$$M \otimes_R X \xrightarrow{\Phi} M \otimes_R Y \xrightarrow{\Psi} M \otimes_R Z \longrightarrow 0.$$

Proof. Define $\Psi = \mathrm{id} \otimes \psi$ and $\Phi = \mathrm{id} \otimes \varphi$. In the case of Ψ we know that if $\Psi(m \otimes y) = 0$ then $\mathrm{id}(m) = 0$ or $\psi(y) = 0$. Since both of these functions are surjective, Ψ must be as well.

Now, to show that $\mathrm{Im}(\Phi) = \ker(\Psi)$, let $\sum m_i \otimes y_i \in \mathrm{Im}(\Phi)$. So for some $\sum m_i \otimes x_i, \Phi(\sum m_i \otimes x_i) = \sum m_i \otimes y_i$. This amounts to the fact that $m \otimes y = m \otimes \varphi(x)$ for some $x \in X$. Then $\Phi(\sum m_i \otimes \varphi(x_i)) = \sum m_i \otimes \psi(\varphi(x_i)) = \sum m_i \otimes 0 = 0$ by the exactness of the original sequence. So we have that $\mathrm{Im}(\Phi) \subseteq \ker(\Psi)$. Now let $\sum m_j \otimes y_j \in \ker(\Psi)$, nontrivial. This means that $\sum m_j \otimes \psi(y_j) = 0$ with nonzero m, and so $\psi(y) = 0$. Since the original sequence is exact, this means that $y \in \mathrm{Im}(\varphi)$, and so it must be the case that $m \otimes y \in \mathrm{Im}(\Phi)$. □

5. Homology

One last thing that is necessary before defining the Tor and Ext Functors is the definition of homology. Originating in algebraic topology, the homology group started out as a way to attempt to, essentially, easily count the number of holes of varying dimensions in a topological space X. The approach used for this was then generalized to other contexts within mathematics.

Definition 5.1. A *chain complex*, $\mathcal{X}$, is a sequence of modules X_i and homomorphisms ϕ_i

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\varphi_{n+1}} X_n \xrightarrow{\varphi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\varphi_1} X_0 \longrightarrow 0$$

such that $\phi_n \circ \phi_{n+1} = 0$.

In the context of algebraic topology, these complexes were made up of abelian groups associated to spaces. For the purposes of this paper, they will be R-modules. Then a homology group gives us a way of describing what happens within a chain complex.

Definition 5.2. For a chain complex $\mathcal{X}$ the n$^{\text{th}}$ *homology group* is

$$H_n(\mathcal{X}) := \ker(\phi_n)\big/\mathrm{Im}(\phi_{n+1}).$$

We also want to be able to make comparisons between chain complexes. We say that for chain complexes $\mathcal{X}$ and $\mathcal{Y}$, $f : \mathcal{X} \to \mathcal{Y}$ is a *map of chain complexes* if for each n there exists map $f_n : X_n \to Y_n$ such that the square below commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\varphi_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r] \arrow[d, "f_n"] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "\psi_{n+1}"'] & Y_n \arrow[r] & \cdots
\end{tikzcd}$$

We further say that these is a *chain homotopy* between maps $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{X} \to \mathcal{Y}$ if, for each n, there are maps $s_n : X_n \to Y_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "\varphi_{n+1}"] & X_n \arrow[r, "\varphi_n"] \arrow[dl, "s_n"'] & X_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "\psi_{n+1}"'] & Y_n \arrow[r, "\psi_n"'] & Y_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $\psi_{n+1}s_n + s_{n+1}\varphi_n = f_n - g_n$. We take it for granted that this consitutes an equivalence relation and that such maps induce the same homomorphism on homology [6].

Finally, we introduce one last concept we need for our discussion of Ext and Tor.

Definition 5.3. An R-module P is *projective* if for any modules X, Y, with homomorphisms $\varphi : P \to Y$ and $\psi : X \to Y$ with ψ surjective, there exists $\gamma : P \to X$ the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\gamma"'] \arrow[d, "\varphi"] \\
X \arrow[r, two heads, "\psi"'] & Y
\end{tikzcd}$$

For the sake of intuition, when thinking of projective modules we usually consider free modules, unless told otherwise. Free modules satisfy this universal property, and in this paper we will only be dealing with projective modules that are also free.

Definition 5.4. For an R-module X a *projective resolution*, $\mathcal{P}$, is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that P_i is projective for all i.

The reason we care about projectivity is because if an exact sequence is projective then $\otimes_R$ and Hom_R do, in fact, preserve exactness [3].

6. Ext

The Ext functor is the left-derived functor of Hom. What this amounts to is that Ext depends on the Hom functor. Specifically:

Definition 6.1. For a projective resolution $\mathcal{P}$

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

we define $\mathrm{Hom}_R(\mathcal{P}, Y)$ to be

$$0 \longrightarrow \mathrm{Hom}_R(P_0, Y) \longrightarrow \mathrm{Hom}_R(P_1, Y) \longrightarrow \cdots$$

Definition 6.2. For R-modules X and Y

$$\mathrm{Ext}^n_R(X, Y) := H_n(\mathrm{Hom}(\mathcal{P}, Y))$$

This is well-defined because Ext is independent of the projective resolution we choose.

Theorem 6.3. *$Ext^n_R(X, Y)$ is independent of projective resolution.*

Proof. Following the proof provided by Dummit and Foote [3], suppose we have two projective resolutions of X, $\mathcal{P}$ and $\mathcal{Q}$:

$$\cdots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} X \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_1 \xrightarrow{\psi_1} Q_0 \xrightarrow{\psi_0} X \longrightarrow 0.$$

We first show that there must be a chain map of chain complexes between these two resolutions. We see the map ψ_0 must be surjective by definition of exactness, and so we can write

$$\begin{tikzcd} & P_0 \arrow[d, "\varphi_0"] \\ Q_0 \arrow[r, two heads, "\psi_0"'] & X \end{tikzcd}$$

and by definition of projective we get that there must exist a map $\gamma_0 : P_0 \to Q_0$ such that $\varphi_0 = \psi_0\gamma_0$. We can now combine our projective resolutions into one diagram as follows:

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r, "\varphi_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\gamma_0"] & X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \cdots \arrow[r] & Q_1 \arrow[r, "\psi_1"'] & Q_0 \arrow[r, "\psi_0"'] & X \arrow[r] & 0. \end{tikzcd}$$

Since $\varphi_0 = \psi_0\gamma_0$, we see that $\varphi_0\varphi_1 = \psi_0\gamma_0\varphi_1$. Since $\varphi_0\varphi_1 = 0$ by definition of exactness, we get that $\mathrm{Im}(\gamma_0\varphi_1) \subseteq \ker(\psi_0)$. This means that we can think of the portion of

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r, "\varphi_1"] & P_0 \arrow[r] \arrow[d, "\gamma_0"] & \cdots \\ \cdots \arrow[r] & Q_1 \arrow[r, "\psi_1"'] & Q_0 \arrow[r] & \cdots \end{tikzcd}$$

as its own diagram

$$\begin{tikzcd} & P_1 \arrow[d, "\gamma_0\varphi_1."] \\ Q_1 \arrow[r, "\psi_1"'] & Q_0 \end{tikzcd}$$

This diagram can be completed a $\gamma_1 : P_1 \to Q_1$ such that $\gamma_1\psi_1 = \varphi_1\gamma_0$, by definition of projectivity. The above process can be repeated in general, and so inductively we get the rest of the maps γ_n:

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r, "\varphi_n"] \arrow[d, "\gamma_n"] & \cdots \arrow[r, "\varphi_2"] & P_1 \arrow[r, "\varphi_1"] \arrow[d, "\gamma_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[d, "\gamma_0"] & X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \cdots \arrow[r] & Q_n \arrow[r, "\varphi_n"'] & \cdots \arrow[r, "\psi_2"'] & Q_1 \arrow[r, "\psi_1"'] & Q_0 \arrow[r, "\psi_0"'] & X \arrow[r] & 0. \end{tikzcd}$$

This gives us a map of chain complex, γ.

We now wish to show that such a map is unique up to chain homotopy. We see that it suffices to show that f is chain homotopic of 0 [3]. Given the above diagram, we see that since P_0 is projective, we may complete the triangle

$$\begin{tikzcd} & P_0 \arrow[d, "\gamma_0"] \\ Q_1 \arrow[r, "\varphi_1"'] & Q_0 \end{tikzcd}$$

with a map $s_0 : P_0 \to Q_1$. In fact, we can do this for a general n:

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r, "\varphi_n"] \arrow[dl, "s_n"'] \arrow[d, "\gamma_n"] & \cdots \arrow[r, "\varphi_2"] & P_1 \arrow[r, "\varphi_1"] \arrow[dl, "s_1"'] \arrow[d, "\gamma_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[dl, "s_0"'] \arrow[d, "\gamma_0"] & X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ \cdots \arrow[r] & Q_n \arrow[r, "\varphi_n"'] & \cdots \arrow[r, "\psi_2"'] & Q_1 \arrow[r, "\psi_1"'] & Q_0 \arrow[r, "\psi_0"'] & X \arrow[r] & 0. \end{tikzcd}$$

This gives us s_n such that $\gamma_n = \psi_{n+1}s_n + s_{n-1}\varphi_n$, a chain homotopy [3].

This, by definition of chain homotopy, yields an analogous inverse isomorphism between homology groups of $\mathrm{Hom}(\mathcal{P}, Y)$ and $\mathrm{Hom}(\mathcal{Q}, Y)$, and therefore of their Ext groups. $\square$

One more piece of machinery needed to understand the problem is the Snake Lemma [7]:

Lemma 6.4. *If the diagram*

$$\begin{tikzcd} & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

is commutative and each row is exact, then there is an exact sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \mathit{Coker}\,\alpha \longrightarrow \mathit{Coker}\,\beta \longrightarrow \mathit{Coker}\,\gamma$$

Thus [3], we have that applying $\mathrm{Hom}(-, M)$ to a short exact sequence

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

yields, via the Snake Lemma, a long exact sequence as follow

$$\cdots \longleftarrow \mathrm{Ext}^2(Z, M) \longleftarrow \mathrm{Ext}^1(X, M) \longleftarrow \mathrm{Ext}^1(Y, M) \longleftarrow \mathrm{Ext}^1(Z, M)$$

$$\begin{tikzcd} {} & \mathrm{Hom}(X, M) \arrow[l] \arrow[d, "="] & \mathrm{Hom}(Y, M) \arrow[l] \arrow[d, "="] & \mathrm{Hom}(Z, M) \arrow[l] \arrow[d, "="] & 0 \arrow[l] \\ & \mathrm{Ext}^0(X, M) & \mathrm{Ext}^0(Y, M) & \mathrm{Ext}^0(Z, M) & \end{tikzcd}$$

This gives us a relationship where each Ext^n term measures the failure of exactness of each Ext^{n-1} term.

7. Tor

While Ext is the derived functor of Hom, Tor is the derived functor of $\otimes$.

Definition 7.1. For a projective resolution $\mathcal{P}$

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

Define $\mathcal{P} \otimes Y$ to be

$$\cdots \longrightarrow P_1 \otimes Y \longrightarrow P_0 \otimes Y \longrightarrow X \otimes Y \longrightarrow 0$$

Definition 7.2. For R-modules X and Y

$$\mathrm{Tor}_n^R(X, Y) := H_n(\mathcal{P} \otimes Y)$$

As with Ext, we have independence of projective resolutions

Theorem 7.3. *$Tor_n^R(X, Y)$ is independent of projective resolution.*

Proof. Just as in the case of Ext, given two projective resolutions of Y, $\mathcal{P}$ and $\mathcal{P}'$, we get that the following diagram commutes in all the ways that we require

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "\varphi_n"] \arrow[dl, "s_n"'] \arrow[d, "\gamma_n"] & \cdots \arrow[r, "\varphi_2"] & P_1 \arrow[r, "\varphi_1"] \arrow[dl, "s_1"'] \arrow[d, "\gamma_1"] & P_0 \arrow[r, "\varphi_0"] \arrow[dl, "s_0"'] \arrow[d, "\gamma_0"] & X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\cdots \arrow[r] & Q_n \arrow[r, "\varphi_n"'] & \cdots \arrow[r, "\psi_2"'] & Q_1 \arrow[r, "\psi_1"'] & Q_0 \arrow[r, "\psi_0"'] & X \arrow[r] & 0.
\end{tikzcd}$$

Namely, γ give us a chain map, and the collection of s_n give us a chain homotopy. This yields an isomorphism of homology groups, and thereby Tor groups. $\square$

Similarly, we have that applying $- \otimes M$ to a short exact sequence

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

yields, via the Snake Lemma, a long exact sequence as follow

$$\cdots \longrightarrow \mathrm{Tor}_2(Z, M) \longrightarrow \mathrm{Tor}_1(X, M) \longrightarrow \mathrm{Tor}_1(Y, M) \longrightarrow \mathrm{Tor}_1(Z, M)$$

$$\begin{tikzcd}
{} \arrow[r] & X \otimes M \arrow[r] \arrow[d, "="] & Y \otimes M \arrow[r] \arrow[d, "="] & Z \otimes M \arrow[r] \arrow[d, "="] & 0 \\
& \mathrm{Tor}_0(X, M) & \mathrm{Tor}_0(Y, M) & \mathrm{Tor}_0(Z, M) &
\end{tikzcd}$$

This gives us a relationship where each Tor_n term measure the failure of exactness of each Tor_{n-1} term.

8. In the Case of Finitely Generated Abelian Groups

Given the nice symmetry seen before for $\otimes$ and Hom we are able to conveniently compute the $\mathrm{Ext}_{\mathbb{Z}}$ and $\mathrm{Tor}^{\mathbb{Z}}$ groups for all finitely generated abelian groups, following the paper of J. Michael Boardman [2]. Recall that a a group G is a finitely generated abelian group if and only if it is isomorphic to

$$G \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z} \big/ n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \big/ n_j\mathbb{Z}$$

for some number of copies of $\mathbb{Z}$ and some finite j.

Let G, H finitely generated abelian groups. Due to the commutativity of both Hom and $\otimes$ with direct sum, it suffices to consider the following cases: $\mathrm{Ext}^i(\mathbb{Z},\mathbb{Z})$, $\mathrm{Ext}^i(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$, $\mathrm{Ext}^i(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$, $\mathrm{Tor}_i(\mathbb{Z},\mathbb{Z})$, $\mathrm{Tor}_i(\mathbb{Z},\mathbb{Z}/n\mathbb{Z})$, and $\mathrm{Tor}_i(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$. Furthermore, we only consider the case where $i = 1$, and as such omit the index from our notation. We will see that for all $i > 1$ the Ext and Tor groups will be trivial.

For $\mathrm{Ext}(\mathbb{Z},\mathbb{Z})$ we note that $\mathbb{Z}$ is itself projective, and so has projective resolution $\mathcal{P}$ given by

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Thus, $\mathrm{Hom}(\mathcal{P},\mathbb{Z})$ is

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \longrightarrow 0$$

of which the homology groups are trivial. Therefore $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0$.

To compute $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$ we begin with the projective resolution for $\mathbb{Z}/n\mathbb{Z}$, which is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}\big/n\mathbb{Z} \longrightarrow 0.$$

Applying the Hom functor we get

$$0 \longleftarrow \mathrm{Ext}(\mathbb{Z}\big/n\mathbb{Z},\mathbb{Z}) \longleftarrow \mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \longleftarrow \mathrm{Hom}(\mathbb{Z}\big/n\mathbb{Z},\mathbb{Z}) \longleftarrow 0$$

This yields that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

For $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$ we, again, begin with the long exact sequence for $\mathbb{Z}/n\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}\big/n\mathbb{Z} \longrightarrow 0$$

Applying the Hom functor we get

$$0 \longleftarrow \mathrm{Ext}(\mathbb{Z}\big/n\mathbb{Z},\mathbb{Z}\big/m\mathbb{Z}) \longleftarrow \mathbb{Z}\big/n\mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z}\big/m\mathbb{Z} \longleftarrow \mathrm{Hom}(\mathbb{Z}\big/n\mathbb{Z},\mathbb{Z}\big/m\mathbb{Z}) \longleftarrow 0$$

Taking homology groups this yields that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$ is $(\mathbb{Z}/m\mathbb{Z})/n(\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ where d is the greatest common divisor of n and m [2].

For $\mathrm{Tor}(\mathbb{Z},\mathbb{Z})$ we recall that $\mathbb{Z}\otimes\mathbb{Z} \cong \mathbb{Z}$, and so as in the case of Ext we have trivial homology groups, so $\mathrm{Tor}(\mathbb{Z},\mathbb{Z}) = 0$

For $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$ we use the same projective resolution of $\mathbb{Z}\big/n$ as in the case of Ext:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}\big/n\mathbb{Z} \longrightarrow 0.$$

Upon tensoring with $\mathbb{Z}$ we get the exact sequence

$$0 \longrightarrow \mathrm{Tor}(\mathbb{Z}\big/n\mathbb{Z},\mathbb{Z}) \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}\big/n\mathbb{Z} \longrightarrow 0.$$

We notice that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$ must be 0, since Tor is trivial whenever either of the inputs are projective.

To show $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ we begin with the same projective resolution as before:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}\Big/n\mathbb{Z} \longrightarrow 0.$$

Upon tensoring with $\mathbb{Z}/m\mathbb{Z}$ we get

$$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}\Big/m \longrightarrow \mathbb{Z} \otimes \mathbb{Z}\Big/m/Z \longrightarrow \mathbb{Z}\Big/n/\mathbb{Z} \otimes \mathbb{Z}\Big/m\mathbb{Z} \longrightarrow 0.$$

This is equivalent to

$$0 \longrightarrow \mathbb{Z}\Big/m\mathbb{Z} \longrightarrow \mathbb{Z}\Big/m\mathbb{Z} \longrightarrow \mathbb{Z}\Big/(n+m)\mathbb{Z} \longrightarrow 0.$$

Taking homology groups this yields that $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ is $\mathbb{Z}/d\mathbb{Z}$ where d is the greatest common divisor of n and m.

Therefore, the Tor and Ext groups of G, H will be direct sum of the above [2].

9. Application: Universal Coefficient Theorem

One of the principal applications of Ext and Tor functors, and how I came to this REU topic, is the Universal Coeffcient Theorems in Algebraic Topology. The goal of the Universal Coefficient Theorem, for both homology and cohomology, is to give a way to express the homology (resp. cohomology) with coefficients in a group G in terms of homology (resp. cohomology) with coefficients in $\mathbb{Z}$.

The Universal Coefficient Theorem for Homology, as presented by Hatcher[5], states:

Theorem 9.1. *If C is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C; G) \longrightarrow Tor(H_{n-1}(C); G) \longrightarrow 0.$$

for all n and all G, and these sequence split, though not naturally.

If you are like me, the inclusion of the Tor term in this definition is a source of confusion. To justify it's existence there, Hatcher gives us that $C_n(X; G) \cong C_n(X) \otimes G$[5].

So, for a chain complex

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \longrightarrow \cdots$$

when we want to get a corresponding chain complex

$$\cdots \longrightarrow C_n(X; G) \longrightarrow C_{n-1}(X; G) \longrightarrow \cdots$$

we really get

$$\cdots \longrightarrow C_n(X) \otimes G \longrightarrow C_{n-1}(X) \otimes G \longrightarrow \cdots$$

However, from previous sections we know that tensoring does not necessarily preserve exactness, so it does not suffice to just tensor with G and hope for the best.

Hatcher considers, at each C_n we have the short exact sequence of chain complexes given by

$$\begin{tikzcd}
0 & 0 \\
\mathrm{Im}\,\partial_n \arrow[u] \arrow[r] & \mathrm{Im}\,\partial_{n-1} \arrow[u] \\
C_n \arrow[u] \arrow[r] & C_{n-1} \arrow[u] \\
\ker\partial_n \arrow[u] \arrow[r] & \ker\partial_{n-1} \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

After tensoring this with G, and with the use of the snake lemma, this yields the long exact sequence of homology groups

$$\cdots \longrightarrow \ker\partial_n \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Im}\,\partial_n \otimes G \longrightarrow \cdots$$

Hatcher then, again, breaks this up into short exact sequences

$$0 \longrightarrow \mathrm{Coker}(i_n\otimes) \longrightarrow H_n(C;G) \longrightarrow \ker(i_{n-1}\otimes) \longrightarrow 0$$

where i_n is the inclusion map. Hatcher identifies $\mathrm{Coker}(i_n\otimes)$ with $H_n(C)\otimes G$, and shows that the term $\ker(i_{n-1}\otimes)$ must be the measure of the failure of exactness of the sequence

$$\mathrm{Im}\partial_{n+1}\otimes G \longrightarrow \ker\partial_n \longrightarrow H_n(C)\otimes G \longrightarrow 0.$$

and so it must be the Tor term. Thus he gets the short exact sequence present in the proof.

The Universal Coefficient Theorem for *Cohomoogy*, as presented by Hatcher[5], states:

Theorem 9.2. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n;G)$ are determined by split exact sequences*

$$0 \longrightarrow \mathit{Ext}(H_{n-1}(C);G) \longrightarrow H^n(C;G) \longrightarrow \mathit{Hom}(H_n(C);G) \longrightarrow 0.$$

The justification for the existence of the Ext term in the sequence follows similarly to Tor in the previous one with the additional notes that instead of tensoring as before, since we want to switch the directions of the arrows, we must actually consider $\mathrm{Hom}(C_n,G)$ and that there is a map between $H^n(C;G))$ and $\mathrm{Hom}(H_n(C),G)$ that allows us to make the appropriate considerations[5].

Acknowledgments

I am grateful to Colin Aitken, whose help in learning this material was invaluable. I am grateful to Professor Peter May for giving me the opportunity me participate in this REU and for his patience in providing feedback and guidance. I am grateful to Skye Binegar for his comments on the second draft of this paper. Finally, I am grateful to the all the participants of the University of Chicago REU of 2018 for helping me grow as a student of mathematics.

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