INTRODUCTION TO CLIFFORD ALGEBRAS AND USES IN REPRESENTATION THEORY

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Abstract. This paper is an introduction to Clifford algebras and a survey of some uses in representation theory. Clifford algebras are a a generalization of the complex numbers that have important uses in mathematical physics. We begin with an introduction to real Clifford algebras and the connection to normed division algebras and braids. We then introduce the tensor construction of the complex Clifford algebra, develop the ideas of root systems and weights of Lie algebras, and construct the spinor module of a Lie algebra from a Clifford algebra.

Contents

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1. Introduction

In this paper, we survey and introduce Clifford algebras across a variety of uses and constructions. Clifford algebras are the result of an attempt to generalize the complex numbers (or more accurately, the quaternions) to higher dimensions. Clifford algebras have a breadth of applications in mathematical physics, appearing almost whenever spinors do. Clifford algebras are also decently versatile in representing other mathematical structures. Both of these uses are examined here.

This paper has two main parts; Sections 2 through 4 focus on Clifford algebras over the real numbers, while Sections 5 through 7 deal with complex Clifford algebras. Real Clifford algebras work well for representing braids and normed division algebras, while we use the complex Clifford algebras to obtain spinor modules.

We begin in Section 2 with an introduction to Clifford algebras over the real numbers and present a particular construction that capitalizes on multilinear algebra to develop an understanding of the structure of Clifford algebras. Section 3 presents connections between Clifford algebras and normed division algebras over $\mathbb{R}$, comparable to generalizations of the complex numbers. Section 4 gives a brief introduction to the Artin braid groups, as well as a discussion of how Clifford algebras can represent these braids and connect them to normed division algebras.

Section 5 introduces the construction of the complex Clifford algebra using tensor methods. Section 6 introduces important concepts in Lie algebras, including the orthogonal Lie algebra, root systems, modules, and the weights and Dynkin indices of these modules. Section 7 concludes the paper with a construction of spinor modules of Lie algebras using Clifford algebras.

2. Clifford Algebras over $\mathbb{R}$ and Multivector Subspaces

2.1. Clifford Algebras over $\mathbb{R}$.

Definition 2.1. Consider a vector space $\mathbb{R}^{p+q}$, for nonnegative integers p and q, equipped with some degenerate quadratic form that we will denote with multiplication. A *real Clifford algebra* is the associative algebra generated by $p+q$ orthonormal basis elements $e_1, ..., e_{p+q}$ such that the following relations hold:

$$e_i{}^2 = 1 \text{ for } 1 \leq i \leq p \tag{2.2}$$

$$e_j{}^2 = -1 \text{ for } p < j \leq p+q \tag{2.3}$$

$$e_i e_j = -e_i e_j \text{ for } i \neq j \tag{2.4}$$

This Clifford algebra therefore has scalar elements and vector elements (those with exactly one basis vector, such as $4e_3$), but we can now also have elements with multiple basis elements, such as $3e_2e_4$, $2.5e_4e_1e_2$, or $3e_2e_1 + 2e_5e_2$. Because of (2.4), which tells us that the basis elements anticommute, we can rewrite some of these elements in the following manner:

Example 2.5.

$$\frac{5}{2}e_4e_1e_2 = -2.5e_1e_4e_2 = \frac{5}{2}e_1e_2e_4$$
$$3e_2e_1 + 2e_5e_2 = -3e_1e_2 - 2e_2e_5$$

Therefore, the element $e_{i_1}...e_{i_k}$ has the same span as the element given by any permutation of those same e_{i_j}, as reordering the basis elements changes the element only by a possible factor of -1. To span our Clifford algebra, we therefore need a basis of every possible unordered combination of the orthonormal basis elements. We can consider each of the $e_1, ..., e_{p+q}$ as independently taking a value 1 or 0, corresponding to being included in the Clifford algebra element or not, respectively. We therefore have 2^{p+q} possible unordered combinations, so our Clifford algebra has dimension 2^{p+q}.

2.2. **Multivector Subspaces of Cl(p,q).** Any element of the Clifford algebra generated from the orthonormal basis vectors of $\mathbb{R}^{p+q}$ with only multiplication of scalars and basis vectors has a definite *grade*. From a real Clifford algebra, take an element v that can be written as the product of a real number and 0 or more of the basis elements generating the Clifford algebra. This element can be expressed as $re_{i_1}...e_{i_k}$ for some real number r and distinct basis elements $e_{i_1}...e_{i_k}$. In this case, the grade of this element is k.

Definition 2.6. The *grade* of a Clifford algebra element v is the number of unique basis elements needed to generate v.

For example, the element $\frac{5}{2}e_1e_2e_4$ has grade 3. We can divide our Clifford algebra into linear subspaces made up of elements of each grade. The grade subspace of smallest dimension is M_0, the subspace of all scalars (elements with 0 basis vectors). We can also have subspaces like M_1, which consists of all linear combinations of the $p + q$ basis elements. Elements of M_1 are called vectors, elements of M_2 are bivectors, elements of M_3 are trivectors, and so on. We can then write our Clifford algebra $Cl(p, q)$ as the direct sum of all of these subspaces:

$$Cl(p, q) = M_0 \oplus M_1 \oplus \cdots \oplus M_{p+q} \tag{2.7}$$

It is also possible to have elements of the Clifford algebra that are of mixed grade. These are sums of elements from subspaces of different grade, as defined above. The most important of these is the subspace consisting of sums of scalars and vectors, called *paravectors*. Following this pattern, we get biparavectors (sums of vectors and bivectors), trivectors (sums of bivectors and trivectors), and similarly formed higher-grade elements.

More generally, any element of $Cl(p, q)$ can be written as a sum of components of each grade. For x $\in Cl(p, q)$,

$$x = \langle M_0\rangle + \langle M_1\rangle + \cdots + \langle M_{p+q}\rangle, \tag{2.8}$$

where $\langle M_i\rangle$ is the part of x of grade i. All the elements of the Clifford algebra are called *multivectors*, regardless of grade.

2.3. **Anti-Involutions on $Cl(n, 0)$.** There are two basic anti-involutions on $Cl(p, q)$, and their behavior depends on the grade of element it acts on. The first of these anti-involutions is *reversion*

$$x \mapsto x^{\dagger}. \tag{2.9}$$

This reverses the order of the factors in a multivector, acting as

$$e_{i_1} \dots e_{i_k} \mapsto e_{i_k} \dots e_{i_1}.$$

For elements of grade 0 mod 4 or 1 mod 4, this is the identity automorphism, while for elements of other grades, $x^\dagger = -x$. To see this, one can apply (2.4) to each element until the original ordering of factors is reached.

Another anti-involution on $Cl(p, q)$ is Clifford conjugation

$$x \mapsto \bar{x}, \tag{2.10}$$

where the order of factors is reversed, and each vector factor is multiplied by -1. This is the identity automorphism on elements of grade 0 mod 4 or 3 mod 4. Again, this is most easily seen by application of (2.4).

By composition of these two anti-involutions, we get the *grade involution*

$$x \mapsto \bar{x}^\dagger = x^*. \tag{2.11}$$

This is the identity on elements of grade 0 mod 4 or 2 mod 4; that is to say, even elements. Elements of odd grade are eigenvectors of the grade involution with eigenvalue -1. We can now divide the Clifford algebra $Cl(p, q)$ into even and odd elements. The even components of the Clifford algebra make up the *even*, or *second Clifford algebra*.

Definition 2.12. The second Clifford algebra is the algebra of eigenvectors of the grade involution with eigenvalue one, and is written as

$$Cl^+(p, q) := M_0 \oplus M_2 \oplus \dots. \tag{2.13}$$

As we will see later, the second Clifford algebra plays an important role in representing the circular Artin braid group; as we will see much later, the second Clifford algebra also plays an import role in recovering spinor modules.

3. Clifford Algebras and Normed Division Algebras

3.1. Normed Division Algebras.

Definition 3.1. A *normed division algebra* (NDA) over a field is an algebra in which the following hold:

(1) For any element b and nonzero element a of the NDA, $ax = b$ has a unique solution x in the NDA
(2) $|ab| = |a||b|$ for all a and b in the NDA

It is a well-known (and rather interesting) fact that there exist only four NDAs over the real numbers. A proof of this fact can be found in [10]. The first of these is $\mathbb{R}$ itself. The next NDA is the complex numbers $\mathbb{C}$. In moving from $\mathbb{R}$ to $\mathbb{C}$, our algebra is no longer ordered.

The next NDA is the quaternions, $\mathbb{H}$. This algebra is spanned by four elements: the identity element ,1, and three other elements I, J, and K that satisfy the relations

$$I^2 = J^2 = K^2 = -1 \tag{3.2}$$

$$IJK = -1 \tag{3.3}$$

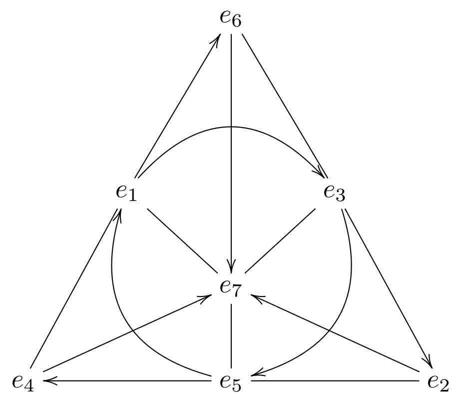

FIGURE 1. The Fano plane

Notice that, under right multiplication on both sides by K, (3.3) implies that

$$IJ = K \tag{3.4}$$

Moving on from the complex numbers to the quaternions loses the commutativity of our algebra.

The final NDA over the real numbers is the octonions, $\mathbb{O}$. The octonions are spanned by one identity element, 1, and 7 other basis elements $e_1, ..., e_7$. These other basis elements follow multiplication relations dictated by the Fano plane, and each squares to -1. For any three elements e_i, e_j, e_k that lie adjacent to each other in that order, with ordering given by the arrows in the Fano plane (Figure 1),

$$e_i e_j = e_k.$$

The octonions are not an associative algebra, and so they have been studied much less than the other NDAs over the real numbers. However, the octonions do contain seven copies of the quaternions, as each of the six straight lines in the Fano plane, together with the identity, generates the quaternion algebra. The seventh copy of the quaternions in the octonions is generated by the identity together with the circle in the middle of the plane, $\{1, e_1, e_3, e_5\}$.

3.2. **Normed Division Algebras and Clifford Algebras.** Clifford algebras were originally born out of an attempt by William Clifford to generalize the quaternions, and so it is no accident that the NDAs over $\mathbb{R}$ are generated in such a similar manner to Clifford algebras over $\mathbb{R}$.

The first NDA over the real numbers that we examined was $\mathbb{R}$, considered as an NDA over itself. Taking the real numbers as the field, the NDA $\mathbb{R}$ is just the span of the identity element, 1. Because it requires no other basis element but the identity in the field, the map

$$\begin{aligned} f : \mathbb{R} &\to Cl(0,0) \\ x &\mapsto x \end{aligned} \tag{3.5}$$

is an isomorphism. Therefore,

$$\mathbb{R} \cong Cl(0,0) \tag{3.6}$$

We can also consider the complex numbers $\mathbb{C}$ as an NDA over the real numbers. The complex numbers can be generated with only the identity, 1, in the real numbers and the imaginary unit i, which squares to -1. In fact, the isomorphism

$$\begin{aligned} f : \mathbb{C} &\to Cl(0,1) \\ a + ib &\mapsto a + e_1 b \end{aligned} \tag{3.7}$$

shows that $\mathbb{C}$ can be identified with $Cl(0,1)$, where $Cl(0,1)$ is generated by 1 and a square root of -1, e_1. Therefore,

$$\mathbb{C} \cong Cl(0,1). \tag{3.8}$$

The quaternions can also be identified with a Clifford algebra, and the isomorphism that accomplishes this is slightly less trivial than the first two. Here, the isomorphism $f : \mathbb{H} \to Cl(0,2)$ is defined by

$$\begin{gathered} f(r) = r \text{ for } r \text{ in } \mathbb{R} \\ f(I) = e_1 \\ f(J) = e_2 \\ f(K) = e_1 e_2. \end{gathered} \tag{3.9}$$

This establishes the relation

$$\mathbb{H} \cong Cl(0,2). \tag{3.10}$$

In crafting this isomorphism, we took advantage of the fact that squaring a product of exactly two basis elements of $Cl(0,n)$ gives -1. In fact, this is also true for a product of exactly two basis elements of $Cl(n,0)$. To see this, use the antisymmetry of basis element multiplication and simplify. We can use this fact to create an isomorphism between $\mathbb{C}$ and $Cl^+(2,0)$.

First, notice that in the case of $Cl(2,0)$, its second Clifford algebra $Cl^+(2,0)$ is composed of scalars, bivectors, and sums of scalars and bivectors. Consider the map

$$\begin{aligned} f : \mathbb{C} &\to Cl^+(2,0) \\ a + ib &\mapsto a + be_1 e_2 \end{aligned} \tag{3.11}$$

This is an isomorphism identifying $\mathbb{C}$ with $Cl^+(2,0)$. However, (3.8) implies that

$$Cl^+(2,0) \cong Cl(0,1) \tag{3.12}$$

This is one instance of an isomorphism that will prove useful in the following section. In general, it is true that

$$Cl^+(p,q) \cong Cl(q,p-1) \tag{3.13}$$

This is established by the following isomorphism:

$$f : Cl(q, p-1) \to Cl^+(p, q) \tag{3.14}$$
$$e_i \mapsto e_i e_{p+q}$$

By repeated use of (2.4), one can see that this acts as the identity on even elements of $Cl(q, p-1)$, and these elements will not have the factor e_{p+q} in their simplest forms, when the vector factors are ordered by index. The odd elements of $Cl(q, p-1)$ will be unchanged, except each will have an extra factor of e_{p+q}.

4. Circular Artin Braid Groups and Clifford Algebras

4.1. **Artin Braid Groups.** The Artin braid group for n strands, written B_n, can be considered as the group of actions on a set of n strands, all held down next to each other at one end. B_n is generated by $n-1$ elementary braids $\sigma_1 \dots \sigma_{n-1}$, where σ_i corresponds to crossing the i^{th} strand under the $(i+1)^{th}$ strand.

Formally, the group is generated by the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \tag{4.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \dots, n-2. \tag{4.2}$$

The *circular* Artin braid group for n strands, written B_n^c, can be thought of as modifying the Artin braid group by bringing the first and n^{th} strands together at the fixed end, forming a circle. B_n^c can be generated by the same relations, with the additional requirement that

$$\sigma_n \sigma_1 \sigma_n = \sigma_1 \sigma_n \sigma_1. \tag{4.3}$$

4.2. **Clifford Braiding Theorem.** In what they call the Clifford Braiding Theorem, Kauffman and Lomonaco [8] showed that Clifford algebras of the form $Cl(n, 0)$ contain partial representations of circular Artin braid groups. This is accomplished by representing the elementary braids in terms of scalars and bivectors in $Cl(n, 0)$. For the partial representation of B_n^c, we consider the Clifford algebra $Cl(n, 0)$. Then, we represent

$$\sigma_i = \frac{1}{\sqrt{2}}(1 + e_{i+1} e_i) \text{ for } 1 \le i < n \tag{4.4}$$

$$\sigma_n = \frac{1}{\sqrt{2}}(1 + e_1 e_n). \tag{4.5}$$

One can check that these generators obey the same relations as those given above that define the Artin braid group, and so this representation is suitable for many braids with few factors. Notice that these generators only use scalars and bivectors of $Cl(n, 0)$, and so we actually only use elements of $Cl^+(n, 0)$.

As an example, consider the braid group $B_2^c = B_2$. We claim that $Cl^+(2, 0)$ contains a partial representation of this group. The lone generator here is

$$\sigma_1 = \frac{1}{\sqrt{2}}(1 + e_2 e_1),$$

where σ_2 is not needed to generate the representation because $\sigma_1^{-1} = \sigma_2$.

However,

$$\sigma_1^8 = 1. \tag{4.6}$$

Because the generator is cyclic, there are braids in B_2^c we cannot represent in $Cl^+(2,0)$, such as the braid σ_1^8, which would not be the identity. This problem persists with every braid group, and so the Clifford algebras do not provide a complete representation of all possible braids in the circular Artin braid group.

4.3. **Normed Division Algebras and Circular Artin Braid Groups.** As we have already seen, there exist isomorphisms between Clifford algebras $Cl^+(n,0)$ and $Cl(0,n-1)$, as well as isomorphisms between certain Clifford algebras and NDAs over $\mathbb{R}$.

Therefore, we should be able to extend our representations of circular braid groups in Clifford algebras to representations of circular braid groups in NDAs. In our last example, we found that we could represent B_2^c by $Cl^+(2,0)$. Since $Cl^+(2,0) \cong Cl(0,1) \cong \mathbb{C}$,we can write

$$\begin{aligned} \sigma_1 &= \frac{1}{\sqrt{2}}(1+i) \\ \sigma_1^{-1} &= \frac{1}{\sqrt{2}}(1-i), \end{aligned} \tag{4.7}$$

where we find the inverse by complex conjugation. Using a similar procedure, one can write a representation of B_3^c in terms of the quaternions.

Note that the octonions are not associative, while the Clifford algebra is defined as an associative algebra, so a Clifford algebra (or NDA) over $\mathbb{R}$ representation of an Artin braid group is not possible.

5. Clifford Algebras over $\mathbb{C}$

5.1. **Tensor Construction of $Cl(M)$.** In order to consider Clifford algebras over the complex numbers, we construct the Clifford algebra in terms of a module M. In this paper, we set $M = \mathbb{C}^n$ and equip it with some nonsingular symmetric bilinear form $(\cdot,\cdot)$.

Definition 5.1. Let $T(M)$ be the tensor algebra generated by the module M, and let K be the ideal in $T(M)$ generated by all $z \otimes z - (z,z)1$ for all $z \in M$, where 1 is the identity tensor of the same shape and size as $z \otimes z$. Then the quotient $T(M)/K$ is the Clifford algebra $Cl(M)$ with respect to the bilinear form $(\cdot,\cdot)$.

Notation 5.2. From this point on, unless explicitly stated otherwise, we will write $x \otimes y$ as xy for $x, y \in Cl(M)$.

Proposition 5.3. *$xy + yx = 2(x,y)$ for all $x, y \in Cl(M)$.*

Proof. By the above definition of $Cl(M)$,

$$(x+y)(x+y) - (x+y, x+y)1 = 0$$

Expanding, we see that

$$x^2 + xy + yx + y^2 - (x,x)1 - (y,x)1 - (x,y)1 - (y,y)1 = 0$$

However, $x^2 - (x,x)1 = 0$ and $y^2 - (y,y)1 = 0$, so

$$xy + yx = 2(x,y),$$

where we have used the fact that the bilinear form on M is symmetric. $\square$

In fact, in the case of $M = \mathbb{C}^n$, the Clifford algebra $Cl(M)$ may also be described as the associative algebra generated by n basis elements $e_1, \ldots, e_n$ such that

$$\{e_i, e_j\} = 2\delta_{ij}1, \tag{5.4}$$

where $\{a, b\} = ab + ba$ is the usual anticommutator.

5.2. **The Second Clifford Algebra.** From this new description of a Clifford algebra $Cl(M)$, we can recover the second Clifford algebra, $Cl^+(M)$. To do this, we describe the grade involution that we used to extract the second Clifford algebra before in terms of tensors and our module M.

Consider the involution that maps all vectors in M to their negatives

$$t \mapsto t^* \text{ for all } t \in Cl(M).$$

This involution is such that $t^{**} = t$ for any $t \in Cl(M)$, and more particularly $z^* = -z$ for any $z \in M$. Every element of $Cl(M)$ is simultaneously an eigenvector of this involution with eigenvalue 1 or -1. We can divide $Cl(M)$ into these two parts, and note that, as with the grade involution, the second Clifford algebra is the algebra of elements in $Cl(M)$ with eigenvalue 1 under the above involution. As before,

$$Cl^+(M) = M_0 \oplus M_2 \oplus \ldots, \tag{5.5}$$

where now M_k is the set of elements in $Cl(M)$ that require tensoring exactly k elements of M.

6. Lie Algebras

Definition 6.1. A Lie algebra over $\mathbb{C}$ is a non-associative algebra equipped with a Lie bracket $[\cdot, \cdot]$. Objects a, b, and c must satisfy the following:

(1) $[a, b] = -[b, a]$
(2) $[a + b, c] = [a, c] + [b, c]$
(3) (The Jacobi identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

6.1. **Root Systems.** We begin with this import theorem [11]:

Theorem 6.2. *Given a complex Lie algebra g, there exists a subalgebra h, known as the Cartan subalgebra, such that*

$$g = \bigoplus_{\alpha \in h^*} g_\alpha, \tag{6.3}$$

where

$$g_\alpha = \{v \in g \mid [b, v] = \alpha(b)v \ \forall b \in h \tag{6.4}$$

Definition 6.5. If $\alpha \neq 0$ and $g_\alpha \neq \{0\}$, then α is a *root* of the Lie algebra g, and g_α is a *root space.*

Each Lie algebra has a single root system, but many Lie algebras can share the same root system. Therefore, root systems are a helpful way to classify Lie algebras and reveal interesting things about their structure.

Example 6.6. In this section, our example will be $A_1 \cong B_1 \cong C_1 = sl(2, \mathbb{C})$, the smallest simple Lie algebra as classified by Cartan and Killing. This is the Lie algebra with three generating elements E, F, and H, such that

$$[E, F] = H,\ [H, E] = 2E,\ [H, F] = -2F. \tag{6.7}$$

We want to find the subalgebra such that all elements of g are simultaneously eigenvectors of $[H, \cdot]$. From the relations above, one can inspect that the Cartan subalgebra is $h = \langle H \rangle$, the span of H.

We can now go about finding the roots of the algebra g. First, note that all elements of g are in the span of E, F, and H. The Lie bracket is linear in either argument, so for every $x \in \langle H \rangle$ and b in the Cartan subalgebra,

$$[b, x] = \alpha(b)x \Rightarrow \alpha(b) = 0.$$

However, for $e \in \langle E \rangle$, we have

$$[H, E] = \alpha(H)E = 2E.$$

Since α is a linear function, the entire span of E represents g_α.

Now, we also note that

$$[H, F] = -\alpha(H)F = -2F.$$

Therefore, the span of F is $g_{-\alpha}$. We have found that

$$sl(2, \mathbb{C}) = g = \bigoplus_{\{0,\alpha,-\alpha\} \in h^*} g_\alpha, \tag{6.8}$$

where

$$g_0 = \langle H \rangle,\ g_\alpha = \langle E \rangle,\ g_{-\alpha} = \langle F \rangle \tag{6.9}$$

6.2. **Weights in Lie Algebra-Modules.**

Definition 6.10. For a Lie algebra g, a g-module is a complex vector space V together with a bilinear form

$$\_ \cdot \_ : g \times V \to V$$

such that, for all a,$b \in g$ and $v \in V$,

$$[a, b] \cdot v = a \cdot (b \cdot v) - b \cdot (a \cdot v) \tag{6.11}$$

Remark 6.12. Finding a g-module is essentially equivalent to finding a representation of g, where the bilinear form first applies ρ to the element a of g, then applies $\rho(a)$ to an element of V. The associativity of the maps in $im(\rho)$ guarantees that (6.11) is satisfied.

We now state the following theorem [11], somewhat resembling the above theorem on root space decomposition.

Theorem 6.13. *For any Lie algebra g and g-module V,*

$$V = \bigoplus_{\lambda \in h^*} V_\lambda \tag{6.14}$$

where

$$V_\lambda = \{v \in V \mid b \cdot v = \lambda(b)v\ \forall b \in h \tag{6.15}$$

Definition 6.16. If $\lambda \neq 0$ and $V_\lambda \neq \{0\}$, then λ is a *weight*, and V_λ is a *weight space.*

Example 6.17. Here, we continue our example from the previous subsection with $A_1 \cong B_1 \cong C_1 = sl(2,\mathbb{C})$ as our Lie algebra g. Because our Lie algebra has not changed, the Cartan subalgebra of g is still $h = \langle H \rangle$.

We choose a representation of $sl(2,\mathbb{C})$, the space of 2 by 2 matrices over $\mathbb{C}$ with zero trace. We let

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \; H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and we let the bilinear form be left multiplication, while the Lie bracket is commutation. Here, we will use $\mathbb{C}^2$ as our g-module.

Our weight spaces are then the eigenspaces of H. By inspection, these are the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, with eigenvalues of 1 and -1, respectively. Since these two vectors span the entire V, this is the complete weight space decomposition of the g-module, with weights $\lambda(H) = 1$ and $\lambda(H) = -1$.

6.3. **The Dynkin Index.** One characterization of an algebra g's g-module is its *Dynkin index*. Informally, this describes the relationship between a Lie algebra's roots and its g-module's weights. In order to give the formula for the Dynkin index, we first need the following definitions.

Definition 6.18. For a Lie algebra g, its *highest root* is a positive root that cannot be written as the sum of two other positive roots.

Definition 6.19. The *Killing form* is a map from $g \times g$ to $\mathbb{C}$. It is given by

$$B(a,b) = tr[ad_a ad_b], \tag{6.20}$$

the trace of the product of the adjoint representations. It is normalized so that, for a highest root γ,

$$B(\gamma,\gamma) = 2. \tag{6.21}$$

Notation 6.22. Let $\Delta(V)$ be the set of weights of V.

Proposition 6.23. *Let g be a complex simple Lie algebra with highest root γ. Let V be a g-module. Then the Dynkin index m_V is given by the following formula:*

$$m_V = \frac{1}{2} \sum_{\lambda \in \Delta(V)} dim(V_\lambda) B(\lambda,\gamma)^2 \tag{6.24}$$

Example 6.25. Again, we return to our example of $g = sl(2,\mathbb{C})$. The highest root of g is the function α such that $[H,E] = \alpha(H)E = 2E$, so we let this be γ. We also saw before that the weights are λ and $-\lambda$ such that $H \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $H \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Note that $\lambda = \frac{1}{2}\gamma$. $dim(V_\lambda)$ is 1, so we can now write

$$m_V = \frac{1}{2}((\frac{1}{2}B(\gamma,\gamma))^2 + (\frac{1}{2}B(\gamma,\gamma))^2)$$

The Killing form is normalized as in Equation 6.20, so this simplifies to

$$m_V = \frac{1}{2}(1+1) = 1$$

So for this representation of $sl(2,\mathbb{C})$, the Dynkin index is 1.

7. The Spinor Module

7.1. **The Orthogonal Lie Algebra.** Given a module $M = \mathbb{C}^n$ equipped with a symmetric bilinear form as in the previous section, one can define an orthogonal Lie algebra. Taking $M = \mathbb{C}^n$ as in the previous section, we can examine the complex orthogonal Lie algebra.

Definition 7.1. The complex orthogonal Lie algebra on $\mathbb{C}^n$, written $L = so(n, \mathbb{C})$, is the complex vector space of all linear operators α on M that are antisymmetric with respect to the bilinear form on $M = \mathbb{C}^n$.

Any associative algebra can be considered as a Lie algebra if we let the Lie bracket be the commutator.

Theorem 7.2. *The subalgebra M_2 of $Cl(M)$ is isomorphic to the orthogonal Lie algebra, L.*

To prove this statement, we first need the following Lemma.

Lemma 7.3. *For any a,b, and c in M, the following relation holds:*

$$[[a,b],c] = 4(b,c)a - 4(a,c)b \tag{7.4}$$

Proof. First, recall the Jacobi identity from Definition 6.1. Using this and the antisymmetry of the Lie bracket, we obtain

$$[[a,b],c] = [a,[b,c]] + [b,[c,a]]$$

We now use the identity

$$\{x,\{y,z\}\} - \{y,\{x,z\}\} = [[x,y],z]. \tag{7.5}$$

We can reverse the order of the arguments on the right side and multiply by -1 on the left to make the following step. Recall that $\{x,y\}$ is the anticommutator applied to x and y, defined by $\{x,y\} = xy + yx$.

$$[[a,b],c] = \{c,\{b,a\}\} - \{b,\{c,a\}\} + \{a,\{c,b\}\} - \{c,\{a,b\}\}$$

The first and last terms cancel. Recall that $\{x,y\} = 2(x,y)1$, and we have

$$[[a,b].c] = 4(b,c)a - 4(a,c)b$$

□

We are now ready to prove Theorem 7.2.

Proof. First, note that any element of M_2 can be written as the commutator of two elements in M, and so

$$M_2 = [M,M] \tag{7.6}$$

We can also note that, for any $m_2 \in M_2$ and $m, m* \in M$,

$$[m_2,[m,m*]] = -[m,[m*,m_2]] - [m*,[m_2,m]] = [[m_2,m],m*] - [[m_2,m*],m]. \tag{7.7}$$

Using Lemma 7.3, we can state that

$$[M_2, M] \subset M \tag{7.8}$$

We are now ready to make the following series of statements:

$$[M_2,M_2] = [M_2,[M,M]] \subset [[M_2,M],M] \subset [M,M] = M_2 \tag{7.9}$$

Therefore, M_2 is closed under commutation, making it a subalgebra of M.

We now define a map f on M_2 such that $f(m)x = [m, x]$ for $x \in M$. This map is antisymmetric with respect to the bilinear form on M, and so $f : M_2 \to L$ has its image in the orthogonal Lie algebra. This map is injective, so

$$dim(M_2) = dim(im(f)) = \binom{n}{2} = dim(L)$$

The map f is therefore an isomorphism between M_2 and L. $\square$

7.2. **The Spinor Module.** As we have seen, we can use a fundamental module M to generate a Clifford algebra $Cl(M)$. Once we have its orthogonal Lie algebra as above, we can find L-modules with the same rules as the g-modules above. All of these L-modules can be created from the fundamental module M with varying tensor techniques, with the exception of the spinor and semi-spinor modules. Here, we extract the spinor and semi-spinor modules from the Clifford algebra.

We will actually be treating $Cl(M)$ as a module over its embedded orthogonal Lie Algebra L. Let the bilinear map $\_ \cdot \_ : L \times Cl(M) \to Cl(M)$ be left multiplication by L. Here, we must distinguish between the odd and even n case. When $n = 2l+1$ is odd, we write $L = B_l$. The even case gives rise to the semispinor module, so in this paper we will work with the odd case, which gives us the spinor module. For the semispinor module, see Belinfante [3].

The spinor module is a vector space N for which there exists an isomorphism $Cl^+(M) \cong lin(N)$, so we can consider N as the representation on which $Cl^+(M)$ acts linearly. In fact, this connection shows that

$$Cl^+(M) \cong lin(N) \cong N \otimes N^* \cong N \otimes N, \tag{7.10}$$

where we use the fact that N is self-dual [3]. So considered as an L-module, $Cl^+(M)$ is isomorphic to $N \otimes N$. As we will see, N itself is also an L-module.

To extract the spinor module, consider the $2l + 1$ basis elements $\sigma_1, \ldots, \sigma_{2l+1}$, which generate $Cl(M)$ for $M = \mathbb{C}^{2l+1}$. Now, we can create $2l$ elements

$$u_i = \frac{1}{2}(\sqrt{-1}\sigma_{2i}\sigma_{2l+1} + \sigma_{2i-1}\sigma_{2l+1}) \text{ for } 1 \le i \le l \tag{7.11}$$

$$v_i = \frac{1}{2}(\sqrt{-1}\sigma_{2i}\sigma_{2l+1} - \sigma_{2i-1}\sigma_{2l+1}) \text{ for } 1 \le i \le l \tag{7.12}$$

that satisfy the following relations:

$$\{u_i, v_j\} = \delta_{ij},\ \{u_i, u_j\} = \{v_i, v_j\} = 0. \tag{7.13}$$

These elements and their commutators generate M_2, which as we showed earlier, is isomorphic to the Lie algebra B_l that we want a module for.

The Cartan subalgebra of B_l is spanned by the l elements of the form $[v_i, u_i]$. The root vectors in B_l, therefore, are each of the invidual u_i and v_j, as well as the commutators $[u_i, u_j]$ and $[v_i, v_j]$ for $i \neq j$. The commutators $[u_i, v_j]$ are also root vectors for $i \neq j$.

Definition 7.14. The *extreme vectors* of a Lie algebra L's L-module are those vectors in the L-module that are mapped to zero under left multiplication by any positive root vector.

Each of the v_i is a positive root, while the u_i are negative roots. To see this, one can use (7.5) and the defining relations in (7.13) to simplify the expressions $[[v_i, u_i], v_i]$ and $[[v_i, u_i], u_i]$. Therefore, the extreme vectors x of B_l all must have the property $v_i x = 0$. When we restrict our module to the second Clifford algebra $Cl^+(M)$, this is also a sufficient condition to show that x is an extreme vector [3]. Let U be the subalgebra generated by the u_i, and let V be the subalgebra generated by the v_i. Using the above condition for being an extreme vector of B_l, we can see that the extreme vectors form a subspace $X = v_1 \ldots v_l U$.

Because $B_l \cong M_2 = UV$, $B_l Ux \subset Ux$ for any extreme vector $x \in X$. Therefore, for each extreme vector $x \in X$, Ux is a B_l-module under left multiplication. In fact, the module $N = Uv_1 \ldots v_l$ is the particular spinor module described above. Since U is generated from l basis elements, the spinor module $N = Uv_1 \ldots v_l$ is a 2^l-dimensional module over L.

Remark 7.15. Each of these modules, as a Lie algebra-module, has an associated Dynkin index. In physics, for a spinor module with Dynkin index n, the *spin* is $\frac{n}{2}$.

Example 7.16. Returning to our example of $L = A_1 \cong B_1 \cong C_1$, we can now find spinor modules for $L = B_1$, which corresponds to $M = \mathbb{C}^3$.

Following the previous construction, we obtain just two basis elements for $Cl^+(M)$:

$$u = \frac{1}{2}(\sqrt{-1}\sigma_2\sigma_3 + \sigma_1\sigma_3) \tag{7.17}$$

$$v = \frac{1}{2}(\sqrt{-1}\sigma_2\sigma_3 - \sigma_1\sigma_3) \tag{7.18}$$

Since the span of u, v, and uv is $M_2 \cong L = B_l$, we can identify the earlier used basis of B_l with u and v by

$$E = v, \ F = u, \ H = [v, u]. \tag{7.19}$$

Viewing the algebra in terms of u and v or in terms of E, F, and H, the Cartan subalgebra $h = H = [v, u]$ agrees in either view. The extreme vectors make up the space $X = vU$, where $U = \langle 1, u \rangle$. So the particular spinor module for B_1 is

$$N = \langle v, uv \rangle = \langle E, FE \rangle \tag{7.20}$$

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