# AN OVERVIEW OF LIE THEORY AND PETER WEYL THEOREM

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Abstract. An overview of the basics of Lie Theory including the Closed Subgroup Theorem, Baker-Campbell-Hausdorff formulae, and Lie's Theorems. This discussion culminates with the Peter Weyl Theorem and a generalization of Fourier Analysis to $C(G)$ (where $G$ is a compact Lie Group). My hope is to present this in a manner that is as accessible as possible, and to gradually progress to the more abstract content.

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## 1. Matrix Groups (Introduction)

**Question**: How can we factor an invertible matrix in terms of upper triangular matrices?

*Definition* 1.0.1. Let $F$ be a field. $GL_n(F) \subset M_{n,n}(F)$ denotes the subset of invertible $n \times n$ matrices and $B_n$ denotes the subset of upper triangular matrices.

*Definition* 1.0.2. In $M_{n,n}(\mathbb{R})$ we define the orthogonal group, $O_n$, to be the subset of matrices such that $AA^* = AA^T = Id$. Likewise, in $M_{n,n}(\mathbb{C})$ we define the unitary group, $U_n$, to be the subset of matrices such that $AA^* = A\overline{A^T} = Id$.

**Theorem 1.0.3** (Graham-Schmidt)**.** *If $M \in GL_n(\mathbb{C})$, then there is a factorization $M = UB$ where $U \in U_n, B \in B_n$.*

*Proof.* Enumerate the columns, $M = \begin{bmatrix} | & | & | & \dots & | \\ v_1 & v_2 & v_3 & \dots & v_n \\ | & | & | & \dots & | \end{bmatrix}$, and apply the Graham-Schmidt process to yield an orthonormal basis, $\{u_i\}_{i=1}^n$, satisfying the equations:

$$\begin{aligned} v_1 &= \lambda_{11}u_1 \\ v_2 &= \lambda_{12}u_1 + \lambda_{22}u_2 \\ &\vdots \\ v_n &= \lambda_{1n}u_1 + \lambda_{2n}u_2 + \cdots + \lambda_{nn}u_n \end{aligned}$$

This gives a unitary matrix, $U$, with columns, $u_i$, and a (upper triangular) change of basis matrix, $B$, with entries given by $\begin{cases} B_{ij} = 0 & i > j \\ B_{ij} = \lambda_{ij} & else \end{cases}$ □

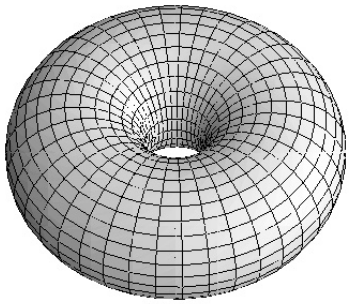
**Remark.** It follows that if $M \in GL_n(\mathbb{C})$, then there is a unique factorization $M = UB$ where $U \in U_n, B \in B_n$ and $U$ has positive, real diagonal entries.

*Definition* 1.0.4. Let $T_n \subset U_n$ denote the maximal torus, in other words, the space of diagonal matrices in $U_n$.

*Example.* In the case $n = 2$, observe that $T_n$ consists of matrices of the form

$$T_n \equiv \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \ \alpha, \beta \in \mathbb{C} : |\alpha| = |\beta| = 1 \right\}$$

and so $T_n \cong S \times S$ where $S$ is the unit circle. This resembles the familiar topological notion of a torus in 3-dimensions.



**Proposition 1.0.5.** *There is a homeomorphism mapping $GL_n(\mathbb{C})/B_n \mapsto U_n/T_n$*

*Proof.* We claim that $B_n \cap U_n = T_n$; the proof follows by induction on $n$. For any matrix, $M \in B_n \cap U_n$, we can decompose into an $(n-1) \times (n-1)$ submatrix, $M'$, with entries, $a_{ij}$, a bottom

$n$th row, $v_n = (u_1, u_2, \ldots, u_{n-1}, v)$, and a rightmost column, $b$ (for sake of avoiding overlap, $b$ only has $n-1$ entries)

$$M = \begin{bmatrix} a_{11} & \ldots & b_1 \\ a_{12} & \ldots & | \\ \vdots & \ddots & b_i \\ \vdots & \vdots & | \\ \text{———} & v_n & \text{———} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & 0 \\ 0 & \ldots & | \\ \vdots & \ddots & 0 \\ \vdots & \vdots & | \\ \text{———} & v_n & \text{———} \end{bmatrix}, \quad MM^* = Id$$

By induction, $M'$ is a diagonal matrix and since $MM^* = Id$, we conclude that $\langle b, v_i \rangle = b_i b_{i+1} == b_1 b_{n-1} = 0$ (where $v_i$ is the $i$th row of $M$) for all $i \leq n-2$ and $\langle u, a_i \rangle = 0$ for $i \leq n-1$. It follows that $b = 0$ and since $M'$ is invertible by hypothesis, we conclude that $u = 0$ as desired.

By extending Theorem 1.0.3, any invertible matrix can be uniquely factored into a non-diagonal unitary matrix with positive, real entries and upper triangular matrices. This therefore induces a continuous bijection to $U_n/T_n$ as desired. □

*Definition* 1.0.6. $\Omega_n$ denotes the space of upper triangular matrices with ones on the diagonal and $\overline{\Omega}_n$ is the analogous space of lower triangular matrices. Moreover, for each permutation matrix, $\tau \in Perm(n)$, let $\Omega_\tau \equiv \Omega_n \cap \tau \overline{\Omega}_n \tau^{-1}$.

**Theorem 1.0.7** (Bruhat/Gauss)**.** [C20] *Let $F$ be a field. If $M \in GL_n(F)$ then there is a factorization $M = \Omega\tau B$ where $B \in B_n$ is an upper triangular matrix, $\tau \in Perm(n)$ is a permutation matrix, and $\Omega \in \Omega_n$ is an upper triangular matrix with ones on the diagonal.*

*Proof.* Let $\tau^{-1} \in Perm(n)$ be a permutation matrix that arranges the pivots of $M$ into echelon form. By scaling each of the columns of the swapped matrix, we can produce a matrix with 1's on the diagonals. Likewise, by following the process of Gaussian Elimination, one can subtract multiples of the columns left of the pivot to eliminate all entries below the pivot (i.e. below the diagonal) by multiplying (on the right) by elementary triangular matrices. Call the resultant echelon matrix, $\Omega$. Note that $B$ is unique because of the criteria that $\Omega$ has ones on the diagonals. Moreover, $\Omega$ is uniquely determined by the permutation, $\tau$, since Gaussian Elimination is fixed by the pivot entries of the matrix. □

*Example.* Consider the matrix $M = \begin{pmatrix} 6 & -3 \\ 3 & -2 \end{pmatrix}$. Fix the permutation matrix, $\tau^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that the swapped version of $M$ is $\begin{pmatrix} -3 & 6 \\ -2 & 3 \end{pmatrix}$. We can make the following transformations to reach echelon form.

$$-\frac{1}{3}[1] \mapsto [1] : \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$2[1] + [2] \mapsto [2] : \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

$$-[2] \mapsto [2] : \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Note that here we use the standard Gaussian Row Operations (going in order based on the pivot columns), so that they become column operations in our decomposition when we invert the process.

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$$
$$M = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix} = \Omega\tau B$$

**Remark.** Notice that in our decomposition $\Omega \in \Omega_n$ (trivially), but also that $\tau^{-1}\Omega\tau \in \overline{\Omega}_n$ (or equivalently, $\Omega \in \tau\overline{\Omega}_n\tau^{-1}$). This is because going from right multiplication to left multiplication reverts from column swaps to row swaps.

**Proposition 1.0.8.** [C20] *The Bruhat Decomposition, $M = \Omega\tau B$, is unique if we stipulate that $\Omega \in \Omega_\tau$, and therefore*

$$GL_n(F)/B_n \mapsto \bigsqcup_{\tau \in Perm(n)} \Omega_\tau \tag{1.0.9}$$

*induces an isomorphism.*

*Proof.* The uniqueness follows from the remark above since $\Omega \in \tau\overline{\Omega}_n\tau^{-1}$ means that $\tau$ is fixed. This is simply because we need the cancellation $\tau\overline{\Omega}_n\tau^{-1}\tau = \tau\overline{\Omega}_n$ in order to properly shuffle the rows. Otherwise there will be a permutation to the left and right of $\overline{\Omega}_n$ which fails to meet the desired form of the decomposition (when we transform back to column operations from the row operations as per the remark). □

**Proposition 1.0.10.** [C20] *The subgroup, $\Omega_\tau$, is in bijection with (homeomorphism if $F = \mathbb{C}$) $F^{l_\tau}$ where $l_\tau$ is the number of inversions/crossings in the permutation, $\tau$ (i.e. the number of pairs, $i > j$, such that $\tau(i) < \tau(j)$).*

*Proof.* Consider an arbitrary $\Omega \in \Omega_\tau$; by assumption, $\Omega$ is an upper triangular matrix with ones on the diagonal. Moreover, its conjugate is lower triangular and has $(i,j)$th entries given by

$$\left[\tau\Omega\tau^{-1}\right]_{i,j} = \Omega_{\tau(i),\tau(j)}$$
$$\Omega_{i,j} = 0 \iff i > j \iff \tau^{-1}(i) < \tau^{-1}(j)$$

and thus $\Omega_\tau \cong F^{l_{\tau^{-1}}} = F^{l_\tau}$. □

We conclude this introduction with a compelling application of the Bruhat Decomposition.

**Theorem 1.0.11.** [S95] *For any prime, $p$, and $n \in \mathbb{N}$ the identity holds*

$$\prod_{k=1}^{n}\left(p^{k-1} + p^{k-2} + \cdots + p + 1\right) = \sum_{\tau \in \mathbb{S}_n} p^{l_\tau} \tag{1.0.12}$$

*Proof.* [S95] Consider the finite group, $G = GL_n(\mathbb{F}_p)$. To count the order of $G$, we first observe that there are $p^n - 1$ choices for the first column (it cannot be zero). For the second column, we require that it not be multiple of the first column, and this leaves $p^n - p$ choices. Continuing this process, the total order of the group is $|G| = \prod_{k=1}^{n}(p^n - p^{k-1})$. To compute the order of the corresponding triangular subgroup, $|B_n|$, we observe that there are $(p-1)$ possible values for each entry on the diagonal (these cannot be zero as otherwise the matrix is not invertible) and $p$ choices for each of $\frac{n(n-1)}{2}$ the entries above the diagonal. Thus, we conclude that $|B_n| = (p-1)^n p^{\frac{n(n-1)}{2}}$

Applying Proposition 1.0.8 gives

$$|G/B_n| = \frac{|G|}{|B_n|} = \frac{\prod_{k=1}^{n}(p^n - p^{k-1})}{p^{\frac{n(n-1)}{2}}(p-1)^n} = \frac{\prod_{k=1}^{n}(p^n - p^{k-1})}{\prod_{k=1}^{n} p^{k-1}(p-1)} = \prod_{k=1}^{n} \frac{p^k - 1}{(p-1)}$$

$$= \prod_{k=1}^{n} \left(p^{k-1} + p^{k-2} + \cdots + p + 1\right) = \sum_{\tau \in Perm(n)} |\Omega_\tau| = \sum_{\tau \in \mathbb{S}_n} p^{l^\tau}$$

as desired. Note that the final equality comes from Proposition 1.0.10. □

*Example.* In the case of $GL_3(F)$ where $F = \mathbb{F}_5$, the left hand side gives

$$\prod_{k=1}^{3} \left(p^{k-1} + p^{k-2} + \cdots + 1\right) = (5^2 + 5 + 1)(5 + 1) = 186$$

and looking at $\mathbb{S}_3 = \{Id, (12), (13), (23), (123), (132)\}$ gives

$$\sum_{\tau \in \mathbb{S}_3} 5^{l_\tau} = 5^0 + 5^1 + 5^3 + 5^1 + 5^2 + 5^2 = 186$$

**Theorem 1.0.13.** *This can be generalized further; for any $q, n \in \mathbb{N}$ the identity holds*

$$\prod_{k=1}^{n} \left(q^{k-1} + q^{k-2} + \cdots + q + 1\right) = \sum_{\tau \in \mathbb{S}_n} q^{l_\tau} \tag{1.0.14}$$

*Proof.* Let $f(x) = \sum_{\tau \in \mathbb{S}_n} x^{l_\tau}$ and $g(x) = \prod_{k=1}^{n} \left(x^{k-1} + x^{k-2} + \cdots + x + 1\right)$. Since $f(p) = g(p)$ for any prime, $p$, this implies that $(f - g)(x) = 0$ for infinitely many values, and thus $f(x) = g(x)$ in general. □

**Corollary 1.0.15.** *For any $q, n \in \mathbb{N}$ the sum $\sum_{\tau \in \mathbb{S}_n} q^{l_\tau}$ is divisible by $(q + 1)$.*

**Remark.** We have unearthed a generating function for the number of permutations in $\mathbb{S}_n$ which have $m$ inversions. Referring back case of $\mathbb{S}_3$, this we observe that

$$\sum_{\tau \in \mathbb{S}_3} x^{l_\tau} = (x^2 + x + 1)(x + 1) = x^3 + 2x^2 + 2x + 1$$

and thus there is 1 permutation with 0 inversions, 1 with 3 inversions, 2 with 2 inversions, and 2 with 1 inversion.

## 2. Introduction to Lie Groups

### 2.1. **Smooth Manifolds.**

*Definition* 2.1.1. Recall that a $k$-manifold, $M$, is a topological space such that there is an open neighborhood around each point $x \in M$ given by $U_x \subset M$ that is homeomorphic via a chart, $\varphi_x : U_x \to V_x$, to an open subset of $V_x \subset \mathbb{R}^k$.

*Definition* 2.1.2. A $k$-manifold, $M$, is smooth if there is an atlas, or covering, $M = \bigcup_{\alpha \in \mathcal{I}} U_\alpha$, with smoothly related charts

$$\forall \alpha, \beta \in \mathcal{I} : \ \varphi_{\alpha,\beta} \equiv \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta), \ \ \varphi_{\alpha,\beta} \in C^\infty(M) \tag{2.1.3}$$

*Example.* In $\mathbb{R}^n$, if $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a smooth function such that $DF$ is surjective at the origin, then $F^{-1}(0)$ is a smooth $k$ manifold by the Implicit Function Theorem. Simple examples such as the unit sphere follow naturally. In this case, $f(x_1, x_2, \ldots, x_n) = 1 - x_1^2 - x_2^2 - \cdots - x_n^2$, gives the desired manifold.

**Proposition 2.1.4** (Cayley)**.** *The orthogonal group, $O_n$, is a smooth, compact $\frac{n(n-1)}{2}$ manifold in $M_{n,n}(\mathbb{R})$.*

*Proof.* [S95] Note that $O_n$ is bounded since $\|T\| = 1$ for all $T \in O_n$. Moreover, $O_n = F^{-1}(Id)$ where $F(X) = XX^T$, and is therefore a bounded preimage of a closed/singleton set. The simplest approach to compute the dimension is to relate $O_n$ to the space of skew symmetric matrices, $\Sigma_n \equiv \{A \in M_{n,n}(\mathbb{R}) : A = -A^T\}$. Observe that the diagonal entries of $\Sigma_n$ must be zero and that the entries below the diagonal are fixed by those above the diagonal, resulting in $\frac{n(n-1)}{2}$ degrees of freedom (i.e. $\Sigma_n \cong \mathbb{R}^{\frac{n(n-1)}{2}}$).

Let $U \equiv \{A \in O_n(\mathbb{R}) \mid \det(A + Id) \neq 0\} \subset O_n(\mathbb{R})$ and let $\varphi : U \to M_{n,n}(\mathbb{R})$ be the map $A \mapsto (A - Id)(A + Id)^{-1}$. We claim that the image of $\varphi$ is $\Sigma_n$.

$$\begin{aligned}\varphi(A) = (A - Id)(A + Id)^{-1} &= (A^T + Id)^{-1}(A^T + Id)(A - Id)(A + Id)^{-1} \\ &= -(A^T + Id)^{-1}(A^T - Id)(A + Id)(A + Id)^{-1} = -\varphi(A)^T\end{aligned}$$

Likewise, one can show by computation that $\varphi^{-1}(X) = (Id+X)(Id-X)^{-1}$ is the inverse map from $\Sigma_n \to U$.

To complete the proof, we claim that the orbits of $U$ span $O_n$ (i.e. $O_n = \bigcup_{A \in O_n} AU$), and thus the dimension of the manifold is the dimension of $U$. One can show that it suffices to use only the diagonal elements of $O_n$ (i.e. those with diagonal entries $\pm 1$), though this proof is somewhat laborious. Since the charts are smooth, we know that $O_n$ satisfies the definition of a smooth manifold. □

**Remark.** An analogous (but slightly more complicated) statement holds for the unitary group.

*Definition* 2.1.5. Let $M$ be a smooth manifold, and consider the set of smooth curves, $\gamma : [-\varepsilon, \varepsilon] = I_\varepsilon \to M$ such that $\gamma(0) = x \in M$. The tangent space at $x$ is defined by

$$T_x(M) \equiv \{\gamma'(0) \mid \gamma \in C^\infty(I_\varepsilon, M)\} \tag{2.1.6}$$

*Example.* If $M = F^{-1}(0)$ for some smooth function, $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$, such that $DF$ is surjective then the tangent space is given $\ker(DF)$. To see this, if $\eta \in T_x(M)$ corresponds to $\gamma : I_\varepsilon \to M$ then $(F \circ \gamma)(t) = 0$ for all $t \in I_\varepsilon$ and thus, $DF_{\gamma(0)}(\gamma'(0)) = DF_x(\eta) = 0$ as desired.

In the familiar case of a hypersurface (like a sphere) in 3-dimensions, this is the all-too-familiar tangent plane.

**Proposition 2.1.7.** *The tangent space of the orthogonal group at each $X \in O_n$ is isomorphic to $\Sigma_n$.*

*Proof.* Recall that $O_n = F^{-1}(Id)$ for $F(X) = XX^T$, and therefore it suffices to find $\ker(DF)$. Note that

$$F(X + H) - F(X) = XH^T + H^TX + HH^T = XH^T + H^TX + O(H^2)$$

$$DF_X(A) = XA^T + AX^T \implies \ker(DF_X) = \{A \in M_{n,n} : A^T = -X^{-1}AX^T\} \cong \Sigma_n$$

□

**Remark.** Much of this content is a review of multivariate calculus. However, the group structure ingrained in $O_n$ makes it possible to execute these computations using an alternative, more efficient method.

*Alternate "Proof" Sketch.* Consider the set of smooth homomorphisms, $\mathbb{R} \to O_n$ (i.e. $C^\infty$ functions, $f$ passing through $f(0) = Id$, such that $f(a + b) = f(a)f(b)$). Observe that the derivative

satisfies

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\left(\lim_{h\to\infty}\frac{f(h)-Id}{h}\right)f(t)=Af(t),\ \ A=f'(0)\in T_{Id}(O_n)$$

This is precisely the definition of the exponential map, $f(t)=e^{tA}$, and thus $e^A\in O_n$ for each $A\in T_{Id}(G)$ (in other words, $\exp(T_{Id}(O_n))\subset O_n$). Since the exponential is a smooth map, we deduce that for each $A\in T_{Id}(O_n)$ in a neighborhood of the identity, the inverse function theorem gives a log function and thus

$$A\in T_{Id}(O_n)\implies e^A\in O_n\iff Id=e^A(e^A)^T=e^Ae^{A^T}=e^{A+A^T}\ (*)$$
$$\log(e^{A+A^T})=\log(Id)=0\iff A+A^T=0\iff A\in\Sigma_n$$

Note that showing $(*)$ is somewhat more challenging and relies upon the fact that $e^Ae^{A^T}=e^{A^T}e^A=Id$ (the simplest approach is to use Theorem 2.3.1). We now require the lemma below to generalize beyond the identity.

**Lemma 2.1.8.** *The tangent space of any matrix in $O_n$ is isomorphic to $T_{Id}(O_n)$.*

*Proof.* Let $V\in O_n$ be an orthogonal matrix and $\phi_V:M_{n,n}\to M_{n,n}$ be the smooth diffeomorphism given by $\phi_V(A)=VA$. If $\gamma:I_\varepsilon\to O_n$ is a smooth path with $\gamma(0)=Id\in O_n$ then $\phi_V(\gamma)$ is a smooth path passing through $V$ (moreover, this produces all such curves). The differential, $D\phi_V$, evaluated at $V$ therefore induces an isomorphism, $T_{Id}(O_n)\mapsto T_V(O_n)$ as desired. □

## 2.2. **Lie Groups and Exponential Maps.**

**Remark.** We can replicate the arguments used for $O_n$ to other matrix groups. For instance, in the special linear group, $SL_n\equiv\{A\in GL_n(\mathbb{R}):\det(A)=1\}$, we can show that the tangent space is given by

$$X\in T_{Id}(SL_n)\implies e^X\in SL_n\iff\det(e^X)=e^{Tr(X)}=1\iff Tr(X)=0$$

And thus the tangent space at each point is isomorphic to the space of matrices with trace 0. More generally, we can apply these arguments to any smooth manifold equipped with a group structure.

*Definition* 2.2.1. A smooth manifold, $G$, forms a Lie Group if there is a smooth map, $(\cdot):G\times G\to G$, satisfying the axioms of a group i.e.

$$\exists Id\in G,\forall g\in G:g\cdot Id=Id\cdot g=g \tag{2.2.2}$$
$$\forall g\in G,\exists g^{-1}\in G:g\cdot g^{-1}=g^{-1}\cdot g=Id \tag{2.2.3}$$
$$\forall g_1,g_2,g_3\in G:(g_1\cdot g_2)\cdot g_3=g_1\cdot(g_2\cdot g_3) \tag{2.2.4}$$

*Definition* 2.2.5. Let $M$ be a smooth manifold. A tangent vector field, $\psi:M\to TM$ (where $TM$ is the tangent bundle), is a mapping from each point, $p\in M$, to $\eta\in T_p(M)$

**Proposition 2.2.6.** *Let $G$ be a Lie Group. For each $\alpha\in T_{Id}(G)$ there is a tangent vector field given by $g\mapsto\psi_\alpha(g)$. Moreover, the tangent space at each point, $T_g(G)$, is isomorphic to $T_{Id}(G)$.*

*Proof.* Let $\phi_g:G\to G$ denote the left multiplication map by $g\in G$. As in Lemma 2.1.8, $D\phi_g$ induces an isomorphism, $T_{Id}(G)\mapsto T_g(G)$. Thus, the map $g\mapsto D\phi_g(\alpha)$ gives a valid vector field. □

*Definition* 2.2.7. Let $G$ be a Lie Group. A one-parameter subgroup is a smooth homomorphism, $f:\mathbb{R}\to G$.

*Example.* As in the alternate proof of Proposition 2.1.7, the exponential map is the only one-parameter subgroup on $O_n$.

**Lemma 2.2.8.** [A69] *Let $M$ be a smooth manifold and $\psi : M \to TM$ be a smooth tangent vector field. For any closed interval, $V \subseteq \mathbb{R}$, any curve, $\gamma : V \to M$, satisfying (for some $t_0 \in V, g_0 \in G$)*

$$\gamma'(t) = \psi(\gamma(t)), \quad \gamma(t_0) = g_0 \tag{2.2.9}$$

*is unique. Moreover, for some $\varepsilon$-neighborhood of the origin, such a curve is guaranteed to exist.*

*Proof.* This follows from the theory of differential equations. □

**Theorem 2.2.10.** *Let $G$ be a Lie Group. There is a unique one-parameter subgroup induced by each vector, $\alpha \in T_{Id}(G)$.*

*Proof.* [A69] Let $\alpha \in T_{Id}(M)$ be a tangent vector which yields a tangent vector field, $\psi_\alpha$, as per Proposition 2.2.6. By Lemma 2.2.8, there is a unique curve $\gamma : I_\varepsilon \to G$ in some neighborhood of the origin such that $\gamma(0) = 0$ and $\gamma'(t) = \psi_\alpha(\gamma(t))$. Suppose $s, t \in I_{\frac{1}{2}\varepsilon}$ and fix $s$ so that

$$\psi_\alpha(\gamma(t)\gamma(s)) = \gamma(s)\gamma'(t) = \big[\gamma(s)\gamma(t)\big]', \quad \psi_\alpha(\gamma(t+s)) = \gamma'(t+s)$$
$$\implies \gamma(t)\gamma(s) = \gamma(t+s) \implies \gamma \text{ is a homomorphism from } I_\varepsilon \to M$$

where $\gamma(t)\gamma(s)$ and $\gamma(t+s)$ must be equal due to the uniqueness established in Lemma 2.2.8. We can define a 'natural extension' of sorts, $\varphi(t) : \mathbb{R} \to G$, that maps $t \mapsto \gamma\big(\frac{t}{N}\big)^N$ for $N > \frac{2t}{\varepsilon}$ (notice that this is unique since $\gamma$ is a homomorphism). This gives the desired one-parameter subgroup since

$$\varphi(s+t) = \gamma\left(\frac{s+t}{N}\right)^N = \gamma\left(\frac{s}{N}\right)^N \gamma\left(\frac{t}{N}\right)^N = \varphi(s)\varphi(t)$$

The uniqueness of $\varphi$ follows from Lemma 2.2.8. □

**Remark.** Recall as before (in the alternate proof of Proposition 2.1.7) that if $f : \mathbb{R} \to G$ is a one-parameter-subgroup with $f(0) = Id$ then

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \left(\lim_{h\to 0} \frac{f(h) - Id}{h}\right) f(t) = Af(t), \quad A \in T_{Id}(G)$$

which gave the notion of an exponential map, $e^{tA}$, in the case of matrices. To find this map explicitly, we took $X \mapsto e^{1\cdot X}$ to be the exponential. Here, we define this more generally:

*Definition* 2.2.11. For each $\alpha \in T_{Id}(G)$ let $\varphi_\alpha$ be the induced one-parameter-subgroup and define $\exp : T_{Id}(G) \to G$ by $\alpha \mapsto \psi_\alpha(1)$ so that $\exp(t\alpha) = \psi_\alpha(t)$ for all $t \in \mathbb{R}$.

**Proposition 2.2.12.** *The exponential map, $\exp : T_{Id}(G) \to G$, is $C^\infty$ (i.e. smooth).*

*Proof.* It suffices to show that exp is smooth in some neighborhood of each $\alpha \in T_{Id}(G)$. Propositions 2.2.6 and 2.2.8 give a smooth tangent vector field, $g \mapsto \psi_\alpha(g)$, with smooth curve, $\gamma(t)$, and thus the map, $\rho : I_\varepsilon \times V_\alpha \to G$ taking $(v, t) \mapsto \psi_{\gamma(t)}(v)$ is smooth for some $I_\varepsilon \times V_\alpha$ where $V_\alpha$ is a neighborhood of $\alpha$. For $N$ sufficiently large, we can define $\exp(v) \equiv \rho\big(\frac{1}{N}, v\big)^N$ for $v \in V$, which we know to be a smooth map. □

**Corollary 2.2.13.** *There exists a smooth logarithmic function, $\log : G \to T_{Id}(G)$, from some neighborhood of the identity to the tangent space.*

*Proof.* This follows from the inverse function theorem of multivariate calculus. □

**Remark.** Since we showed that the exponential function (and consequently, the logarithmic function) is unique, and has the property that $\exp'(v) = \exp(v)$, we can compute the Maclaurin Expansions

$$\exp(v) = 1 + v + \frac{v^2}{2!} + \frac{v^3}{3!} + \frac{v^4}{4!} + \dots \tag{2.2.14}$$

$$\log(Id + g) = g - \frac{g^2}{2} + \frac{g^3}{3} - \frac{g^4}{4} + \dots \tag{2.2.15}$$

where $v, g$ are chosen within the respective neighborhoods of $T_{Id}(G), Id$ over which these expressions are well defined.

**Remark.** Any closed subgroup of $O_n$ (or any matrix group) will produce a submanifold with tangent spaces coming from the same logarithmic/exponential function. We will show a more general/precise version of this statement in Theorem 2.2.21.

*Definition* 2.2.16. Let $M$ be a smooth $n$-manifold and $N \subseteq M$. $N$ is said to be a $k$-submanifold if for all $x \in N$ there is a neighborhood, $U_x \subseteq M$, such that for some chart, $\varphi$, in the atlas

$$\varphi(U_x \cap N) = \varphi(U_x) \cap \mathbb{R}^k \tag{2.2.17}$$

where above we embed $\mathbb{R}^k$ in $\mathbb{R}^n$ (i.e. the set of points $(x_1, x_2, \dots, x_{k-1}, x_k, 0, 0, \dots, 0) \in \mathbb{R}^n$).

*Example.* The subset of orthogonal matrices with determinant one, $SO_n = O_n \cap SL_n$, forms a submanifold.

**Lemma 2.2.18.** *Let $H \leq G$ be a closed subgroup of a Lie Group and assume $(G, \|.\|)$ is a norm. Suppose $\{h_n\}_{n\in\mathbb{N}} \subset H$ is a sequence of nonzero points such that $\exp(h_n) \in H$, $h_n \to 0$, and $\frac{h_n}{\|h_n\|} \to v \in T_{Id}(G)$. Thus, $\exp(tv) \in H$ for all $t \in \mathbb{R}$.*

*Proof.* Notice that $t\frac{h_n}{\|h_n\|} \to tv$. Since $\frac{t}{\|h_n\|} << 1$ as $n \to +\infty$, we can pick the nearest integer, $m_n \approx \frac{t}{\|h_n\|} \in \mathbb{Z}$.

Thus, $m_n$ is an integer sequence for which $\|h_n\| \to t$. Thus, $\exp(m_n h_n) = \exp(h_n)^{m_n} \in H$ and $\exp(h_n)^{m_n} \to \exp(tv) \in H$ as desired. □

**Lemma 2.2.19.** *Let $W \subset T_{Id}(G)$ be the subset of vectors in the tangent space for which $\exp(tw) \in H$ for all $t \in \mathbb{R}$. In other words, $W \equiv \{w \in T_{Id}(G) : \forall t \in \mathbb{R}, \exp(tw) \in H\} \subset T_{Id}(G)$, and it forms a subspace of $T_{Id}(G)$*

*Proof.* [A69] By definition, if $\lambda \in \mathbb{R}$ then $\lambda w \in W$. It remains to show that if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$. Let $f(t) = \log\big(\exp(tw_1)\exp(tw_2)\big)$ which is well defined for some neighborhood, $I_\varepsilon$, where $f(0) = 0$. Likewise, let $g(w_1, w_2) = \log\big(\exp(w_1)\exp(w_2)\big)$ so that since $g$ is differentiable, we can linearize $g(w_1, w_2) = a + bw_1 + cw_2 + O(\|w_1 - Id\|^2 + \|w_2 - Id\|^2)$ if we choose $w_1, w_2$ in some neighborhood of the identity. By plugging in $g(w_1, 0) = w_1$ and $g(0, w_2) = w_2$, we get that $a = 0, b = Id, c = Id$ and so $g(w_1, w_2) = w_1 + w_2 + O(\|w_1 - Id\|^2 + \|w_2 - Id\|^2)$.

Putting this altogether, we have that $f(t) = g(tw_1, tw_2) = tw_1 + tw_2 + O(t^2)$ and thus $\frac{f(t)}{t} \to w_1 + w_2$ as $t \to 0$. This yields a sequence, $h_n = f\big(\frac{1}{n}\big)$, satisfying the hypothesis of Lemma 2.2.18 and thus $h_n \to v = \frac{w_1+w_2}{\|w_1+w_2\|} \in H$. This suffices to show that $w_1 + w_2 \in W$ as desired. □

**Lemma 2.2.20.** *The image, $\exp(W)$, is a neighborhood of the identity in $H$.*

*Proof.* [A69] Decompose the tangent space, $T_{Id}(G) = W' \oplus W$. The idea is to show that $W' = 0$. If $W'$ is a nontrivial vector space, then we can extract a bounded sequence, $(w'_n, w_n) \to (w', w)$ (i.e.

first extract a bounded sequence, then a convergent subsequence by Bolzano-Weistrass), such that $\exp(w_n')\exp(w_n) \to Id$ but $w_n' \neq 0$. We have that $\frac{w_n'}{\|w_n'\|} \to \overline{w'}$ and $\|\overline{w'}\| = 1$, but by Lemma 2.2.18 this implies $\overline{w'} \in W$ which is a contradiction. □

**Theorem 2.2.21** (Cartan)**.** [A69] *A closed subgroup of a Lie Group is a submanifold.*

*Proof.* [A69] Let $H \leq G$ be a closed subgroup of Lie Group, $G$. By Lemma 2.2.20, the logarithm maps from a neighborhood of the identity, $U_{Id}$, to a subspace of $T_{Id}(G)$ and therefore

$$\log(U_{Id} \cap H) = W = T_{Id}(G) \cap W \tag{2.2.22}$$

as desired. To generalize beyond the identity, simply compose the logarithm with the isomorphism from Proposition 2.2.6 for each point $g \in G$. □

## 2.3. **The Baker Campbell Hausdorff Formula.**

**Remark.** In the proof of Lemma 2.2.19, we considered the logarithm of the product, $\log\big(\exp(tA)\exp(tB)\big) = tA + tB + O(t^2)$. A standard course in matrix theory provides the identity $O(t^2) = 0$ iff $AB = BA$. This seems to motivate some ties from this remainder to the commutator of the two elements, $[A, B] = AB - BA$. If we wanted to compute the next term, for instance, we could use the expansion in 2.2.14 to yield

$$\exp(tA)\exp(tB) = 1 + tA + tB + \frac{1}{2!}t^2(A^2 + B^2 + 2AB) + O(t^3)$$

$$\exp(tA + tB) = 1 + tA + tB + \frac{1}{2!}t^2(A^2 + AB + BA + B^2) + O(t^3)$$

$$\exp(tA + tB + \frac{1}{2}t^2(AB - BA) + O(t^3)) = \exp(tA + tB + \frac{1}{2}t^2[A, B] + O(t^3)) = \exp(tA)\exp(tB)$$

Likewise, the next term can be shown to be $\frac{1}{12}\big([A,[A,B]]+[B,[B,A]]\big)$. But what of the remaining terms? As it turns out, they are simply the iterates of this commutator.

**Theorem 2.3.1** (Baker-Campbell-Hausdorff)**.** *One can express*

$$\log(e^A e^B) = A + B + \frac{1}{2}\rho(A, B) + \frac{1}{12}\Big(\rho(A, \rho(A, B)) + \rho(B, \rho(B, A))\Big) + O(\rho^3) \tag{2.3.2}$$

*where $\rho$ is the bilinear commutator map i.e. $\rho(A, B) = [AB] = AB - BA$. In essence, each successive term in the expansion is given by combinations of iterates of the $\rho$ map applied to $A, B$ (these are also known as Lie Polynomials).*

*Proof.* [E67] Let $\exp(A)\exp(B) = \exp\big(\sum_{n\in\mathbb{N}} F_n(A, B)\big)$ be the expansion, and assume by induction that $F_m$ is generated by iterates of $\rho$ for $1 < m < n$ ($F_m$ for $m < n$ are Lie Polynomials). Note that the base case was proved in the remark above where $F_1 = A + B$ is not a Lie Polynomial, and $F_2 = \frac{1}{2}\rho(A, B)$ is. Moreover, since $e^A e^B = e^{A+B}$ if $A, B$ commute, we have that $F_n(\lambda_1 A, \lambda_2 A) = 0$ if $\lambda_1, \lambda_2$ are commuting variables (since the expansion cannot include a term of order $n$).

Looking at the three variable expansion we obtain by associativity

$$\log\big(\exp(A)\exp(B)\exp(C)\big) = \sum_{n\in\mathbb{N}} F_n\Big(\sum_{m\in\mathbb{N}} F_m(A, B), C\Big) = \sum_{n\in\mathbb{N}} F_n\Big(A, \sum_{m\in\mathbb{N}} F_m(B, C)\Big) \tag{2.3.3}$$

Note that if $P_1, P_2$ are Lie Polynomials then $P_1 \circ P_2$ is also a Lie Polynomial. The term of degree $\leq n$ on the left hand side is comprised of compositions of degree $< n$ terms and $F_n(A, B) + F_n(F_1(A, B), C) = F_n(A, B) + F_n(A + B, C)$ (and $F_n(B, C) + F_n(A, B + C)$ on the right hand side).

By induction, this means that all of the other terms in the $n$th term expansion are Lie Polynomials, and therefore

$$F_n(A,B) + F_n(A+B,C) - F_n(B,C) - F_n(A,B+C) = [Lie] - [Lie] = [Lie]$$

is a Lie Polynomial. We write concisely that

$$F_n(A,B) + F_n(A+B,C) \sim F_n(B,C) + F_n(A,B+C) \tag{2.3.4}$$

are equivalent modulo the space of Lie Polynomials.

Note that by plugging $C = -B$ into (2.3.4), the right hand side becomes $F_n(B,-B) + F_n(A,0) = 0$ (since $e^B e^{-B} = e^{B-B}$ and $e^A e^0 = e^{A+0}$) and thus $F_n(A,B) \sim -F_n(A+B,-B)$. Likewise, plugging in $A = -B$ gives $F_n(A,B) \sim -F_n(-A,A+B)$ after changing variables. Applying these two equivalences successively gives

$$F_n(A,B) \sim -F_n(-A,A+B) \sim F_n(B,-A-B) \sim -F_n(-B,-A) = (-1)^{n+1} F_n(B,A) \tag{2.3.5}$$

where the last identity comes from the fact that $F_n$ is homogeneous of degree $n$ (this comes from the properties of the exponential, $\exp(\lambda x) = \exp(x)^\lambda$).

Here, we employ a final trick by applying (2.3.4) with $C = -\frac{1}{2}B$ and $A = -\frac{1}{2}B$ to give (respectively)

$$F_n(A,B) \sim F_n\left(\frac{1}{2}A, B\right) - F_n\left(-\frac{1}{2}A, A+B\right) \tag{2.3.6}$$

$$F_n(A,B) \sim F_n\left(A, \frac{1}{2}B\right) - F_n\left(A+B, -\frac{1}{2}B\right) \tag{2.3.7}$$

and applying (2.3.7) to the left hand side of (2.3.6) and (2.3.5) (as well as homogeneity) to give

$$F_n\left(\frac{1}{2}A, B\right) \sim F_n\left(\frac{1}{2}A, \frac{1}{2}B\right) - F_n\left(\frac{1}{2}A+B, -\frac{1}{2}B\right) \sim \frac{1}{2^n}F_n(A,B) - \frac{1}{2^n}F_n(A+B,B)$$

and likewise to the right hand side,

$$F_n\left(-\frac{1}{2}A, A+B\right) \sim F_n\left(\frac{1}{2}A, \frac{1}{2}A+\frac{1}{2}B\right) - F_n\left(\frac{1}{2}A+B, -\frac{1}{2}A-\frac{1}{2}B\right) \sim \frac{1}{2^n}F_n(A,B) - \frac{1}{2^n}F_n(B,A+B)$$

and thus,

$$F_n(A,B) \sim \frac{1}{2^{n-1}}F_n(A,B) + \frac{1}{2^n}F_n(A+B,B) - \frac{1}{2^n}F_n(B,A+B)$$

$$\left(1 - \frac{1}{2^{n-1}}\right)F_n(A,B) \sim \frac{1}{2^n}\left(1+(-1)^n\right)F_n(A+B,B) \tag{2.3.8}$$

If $n$ is odd then the proof is already done since $F_n(A,B) \sim 0(\dots) = 0$. If $n$ is even then apply (2.3.8) and homogeneity to give

$$F(A-B,B) \sim -F(A,-B) \sim \frac{1}{2^n}\left(\frac{1}{1-\frac{1}{2^{1-n}}}\right)\left(1+(-1)^n\right)F_n(A,B) = \frac{1}{2^{n-1}}\left(\frac{1}{1-\frac{1}{2^{1-n}}}\right)F_n(A,B)$$

$$F_n(A,B) \sim \frac{1}{2^{2n-2}}\left(\frac{1}{1-\frac{1}{2^{1-n}}}\right)^2 F_n(A,B) \implies F_n(A,B) \sim 0$$

□

## 3. Lie Theory and Peter Weyl

3.1. **The Ado/Iwasawa Theorem.** The elements generated by applying the commutator to the tangent space of a Lie Group gave an Algebra whose terms we observed in the Baker Campbell Hausdorff Expansion. Moreover, we establish the identity below:

**Proposition 3.1.1** (Jacobi)**.** *Let* $[\,]$ *be the commutator operation of an associative algebra,* $A$*.*

$$\forall X, Y, Z \in A: \ [[XY]Z] + [[YZ]X] + [[ZX]Y] = 0$$

We can use Jacobi's Identity to motivate a more general class of spaces with similar structure to these associative algebras.

*Definition* 3.1.2. An vector space, $L$, with a field, $F$, is an Algebra over $F$ if there is a bilinear product, $(\cdot): L \times L \to L$, such that for all $x, y, z \in L$ and $\alpha, \beta \in F$ it holds that

$$x \cdot (y + z) = x \cdot y + x \cdot z, \ (y + z) \cdot x = y \cdot x + z \cdot x, \ \alpha x \cdot \beta y = (\alpha\beta)(x \cdot y)$$

$L$ is a Lie Algebra if there is a map $[\,]: L \times L \to L$ satisfying

$$\forall X, Y \in L: [XY] \text{ is a bilinear mapping in } L \tag{3.1.3}$$

$$X = Y \in L \implies [XY] = 0 \tag{3.1.4}$$

$$\forall X, Y, Z \in L: \ [[XY]Z] + [[YZ]X] + [[ZX]Y] = 0 \tag{3.1.5}$$

**Proposition 3.1.6.** *The* $[\,]$ *operator on a Lie Algebra,* $L$*, is anti-commutative i.e.* $[XY] = -[YX]$ *for all* $X, Y \in L$*.*

*Proof.* This is a simple computation using the properties described above,

$$0 = [X + Y, X + Y] = [X, X + Y] + [Y, X + Y] = [XY] + [YX] = 0$$

which suffices to prove the result. □

**Theorem 3.1.7** (Lie's Third Theorem)**.** *The commutator gives a Lie Algebra structure to the tangent space,* $T_{Id}$*, of a Lie Group. Moreover, the functor sending* $G \mapsto T_{Id}(G)$ *is an equivalence between the category of connected, simply connected Lie Groups and the category of Lie Algebras.*

**Remark.** The proof essentially amounts to extracting the bilinear map from the Baker-Campbell-Hausdorff identity and extending it to the entire group. This is a rather difficult task, but it can be done by making use of the two connectedness conditions (i.e. the space has no holes and cannot be partitioned into disjoint open sets).

*Anecdote.* Upon first studying this material, I naturally asked if there were any worked examples of compact Lie Groups that were not Matrix Groups. The natural reply from my mentor was surprising; the issue is not that such a group is too inconvenient to work out, but rather that no such group exists. This will become clear in Theorem 3.2.15.

*Definition* 3.1.8. A representation of a Lie Algebra, $L$ over a field $F$, is a homomorphism, $\rho: L \to \mathfrak{gl}_n(F)$, where $\mathfrak{gl}_n(F)$ is the Lie Algebra of the general linear group with the standard commutator map. More generally, a representation of an associative algebra, $A$ over a field $F$, is a map, $\rho: A \to \mathrm{End}(V)$ where $V$ is a vector space.

*Definition* 3.1.9. Two representations, $\rho_1, \rho_2$, of a Lie Algebra, $L$, are equivalent if there is a matrix, $T \in GL_n(F)$, such that

$$\forall x \in L: \ \rho_2(x) = T\rho_1(x)T^{-1} \tag{3.1.10}$$

*Definition* 3.1.11. Let $(\rho, V)$ be a representation of an associative algebra, $A$. The subspace, $W \subseteq V$, forms a subrepresentation if $W$ is invariant under the image of $\rho$ (i.e. each $\rho(a)[W] \subset W$ for each $a \in A$). A representation is irreducible if it only admits trivial subrepresentations, namely $\{0\}$ and itself.

*Definition* 3.1.12. Let $(V_1, \rho_1)$ and $(V_2, \rho_2)$ be two representations of $A$. A homomorphism of Lie Algebras is defined as a map, $\phi : V_1 \to V_2$, such that

$$\forall a \in A, v \in V_1 : \quad \phi(\rho_1(a)[v]) = \rho_2(a)[\phi(v)] \tag{3.1.13}$$

Since this notation is rather tedious, we often abbreviate using the notation of group action, $\phi(av) = a\phi(v)$.

**Lemma 3.1.14** (Schur)**.** *Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations.*

- *If $V_1$ is irreducible then $\phi$ is injective*
- *If $V_2$ is irreducible then $\phi$ is surjective*

*Proof.* If $V_1$ is irreducible then $\ker(\phi)$ is a subrepresentation of $V_1$, and thus $\ker(\phi) = 0$ is forced. Likewise, if $V_2$ is irreducible then $\mathrm{Im}(\phi) = V_2$ is forced. □

**Theorem 3.1.15** (Schur)**.** *Let $V$ be a finite dimensional representation of an algebra over $\mathbb{C}$ (or any algebraically closed field). The only homomorphisms from $V \to V$ are scalar multiples of the identity.*

*Proof.* We can extract one of the eigenvalues of $\phi$, $\lambda \in \mathbb{C}$, to yield a homomorphism, $\phi - \lambda Id$, that has a nontrivial kernel. By Lemma 3.1.14, $\phi - \lambda Id$ must be zero and $\phi = \lambda Id$ as desired. □

**Remark.** We close with two foundational results that tie together Lie Groups and Lie Algebras.

**Theorem 3.1.16** (Ado/Iwasawa)**.** *Any finite dimensional Lie Algebra, $L$ over $F$, admits a faithful representation.*

**Corollary 3.1.17.** *Any finite dimensional connected, simply connected Lie Group, $G$, is isomorphic to a Matrix Group.*

**Remark.** It is worth mentioning that Theorem 3.1.17 can also be used to prove Theorem 3.1.7. The combination of these two theorems marks an equivalence between the abstract Lie Groups and concrete Matrix Groups. Furthermore, this sends a fairly explicit signal that the study of representations will be an integral part of resolving problems in Lie Theory.

## 3.2. **Peter Weyl Theorem.**

*Definition* 3.2.1. Let $G$ be a compact Lie Group, $G$ (with a metric), and $f : G \to \mathbb{C}$ be a continuous, complex valued function. More generally, let $C(G)$ be the space of all continuous functions from $G \to \mathbb{C}$. We define the integral as a map, $\int_G : C(G, \mathbb{C}) \to \mathbb{C}$, such that

$$\int_G f(g)dg = z \in \mathbb{C} \text{ if } \forall g \in G, f(g) = z \tag{3.2.2}$$

$$\forall h \in G, f \in C[G, \mathbb{C}] : \quad \int_G f(hg)dg = \int_G f(g)dg \tag{3.2.3}$$

$$\text{If } U \text{ is a convex subset of } \mathbb{C} \text{ such that } f(G) \subseteq U \text{ then } \int_G f dg \in U \tag{3.2.4}$$

Note that (3.2.3) is known as the Haar Property. The proof of the existence of this integral is omitted on the grounds of being too Haar to handle.

**Proposition 3.2.5.** *For any compact Lie Group, $G$, with representation, $(V, \rho)$, there is a natural unitary representation, $(\mathcal{V}, \psi)$; in other words, $\psi(g)\psi(g)^* = Id$ for all $\nu \in \mathcal{V}$.*

*Proof.* [S95] One simply defines $\mathcal{V}$ to be the same space with a new inner product, $\langle . \rangle_*$ defined by

$$\forall \nu_1, \nu_2 \in \mathcal{V} : \langle \nu_1, \nu_2 \rangle_* \equiv \int_G \langle g\nu_1, g\nu_2 \rangle dg$$

□

*Definition* 3.2.6. $C(G)^{\text{fin}}$ is the subspace of $C(G, \mathbb{C}) = C(G)$ for which each $f \in C(G)^{\text{fin}}$ is contained in a finite-dimensional $G$-invariant subspace of $C(G)$. Note that $G$ acts on $C(G)$ by left translation i.e. $gf(x) = f(gx)$.

**Theorem 3.2.7** (Peter-Weyl)**.** *For any compact Lie Group, $G$, $C(G)^{fin}$ is dense in $C(G)$.*

*Proof.* [S95] Here we present a sketch of the result. For each $f \in C(G)$, consider the linear operator, $T_f(\psi)[g] : C(G) \to C(G)$, given by

$$T_f(\psi)[x] = \int_G f(g)\psi(g^{-1}x)dg \tag{3.2.8}$$

$$= \int_G f(gx^{-1})\psi(g^{-1})dg = \int_G f(xg^{-1})\psi(g)dg \tag{3.2.9}$$

where (3.2.10) follows from the Haar Property (3.2.3). Note that $T_f$ is a compact operator if $f$ is bounded (in norm), the proof can be found here [B72]. Next, we construct a bounded sequence of functions in $C(G)$, $\delta_n \to \delta$, for which

$$\lim_{n\to\infty} \delta_n(g) = \begin{cases} 0 & g \neq Id \\ +\infty & g = Id \end{cases}, \quad \int_G \delta = 1, \quad \forall x, g \in G : \delta_n(gxg^{-1}) = \delta_n(x), \delta_n(g^{-1}) = \overline{\delta_n(g)} \tag{3.2.10}$$

We claim that $T_{\delta_n}$ is symmetric under the inner product, $\langle f, g \rangle = \int_G f(x)\overline{g(x)}dx$. Note that

$$\langle \phi, T_{\delta_n}(\psi) \rangle = \int_G \int_G \phi(x)\overline{\delta_n(g)\psi(g^{-1}x)}dgdx = \int_G \int_G \phi(x)\delta_n(g^{-1})\overline{\psi(g^{-1}x)}dgdx$$

$$= \int_G \int_G \phi(x^{-1})\delta_n(xg)\overline{\psi(g^{-1})}dxdg = \int_G \overline{\psi(g^{-1})} \int_G \phi(x^{-1})\delta_n(xg)dxdg = \langle T_{\delta_n}(\phi), \psi \rangle$$

and thus $T^*_{\delta_n} = T_{\delta_n}$ as desired. Using now the conjugation invariance, we conclude that $g \cdot T_{\delta_n}(\psi) = T_{\delta_n}(g \cdot \psi)$. Thus, the eigenfunctions of $T_{\delta_n}$ are in $C(G)^{\text{fin}}$ as desired. Finally, we use the property of the dirac-delta function, $T_\delta(\psi)[x] = \psi(x)$, which gives that each function in $C(G)$ is an eigenfunction. Of course, the $\delta$-function is not in $C(G)$, but since it lies in the closure of the $\delta_n$ sequence, density is proved. □

**Remark.** We can now return to the familiar territory of matrices by reformulating Theorem 3.2.11.

*Definition* 3.2.11. Let $G$ be a Lie Group with a unitary, representation $M$. For each $\eta, \gamma \in M$ the corresponding representative function is the linear functional,

$$f : G \to \mathbb{C} : \quad f_{\eta,\gamma}(g) = \langle \eta, g\gamma \rangle \tag{3.2.12}$$

Note that this definition derives from the Riesz Representation Theorem, and in the case of finite dimensional spaces is simply analogous to one entry of the matrix induced by $g \in G$.

**Theorem 3.2.13.** *The subalgebra given by the space of represenative functions, $C_{alg}(G)$, is equal to $C(G)^{fin}$.*

*Proof.* It suffices to show that $C(G)^{\text{fin}} \subseteq C_{\text{alg}}(G)$. Let $\phi \in C(G)^{\text{fin}}$ be in some finite dimensional $G$-invariant subspace, $V \subseteq G$. Let $\{\phi_i\}_{i \leq n}$ be a finite basis of $W$ with $\phi_1 = \phi$.

$$g\phi_i = \sum \omega_{ji}(g)\phi_j, \quad \phi(x) = (x^{-1}\phi)(Id) = \sum \omega_{j1}x^{-1}\phi_j(Id) = \sum \overline{M_{1j}(x)}\phi_j(Id)$$

and thus $\phi \in C_{\text{alg}}$ as desited. □

**Theorem 3.2.14** (Peter-Weyl)**.** *Any compact Lie Group is isomorphic to a subgroup of $U_n$.*

*Proof.* By Theorem 3.2.8, $C_{\text{alg}}$ is dense in $C(G)$. For each representation, $V$, we know that for any $g \in G/\{Id\}$ there is some finite dimensional representation, $V_g$, on which $g$ acts nontrivially. If we build a chain, $\Omega_n = \bigoplus_{i=1}^{n} V_{g_i}$, using distinct points in $G$ then the kernel of the representation map, $\rho : G \to \text{Aut}(\Omega_n)$, must decrease in size. By compactness, there exists some $n \in \mathbb{N}$ for which $\ker(\rho) = \{Id\}$ and $G \cong \text{Im}(\rho) \subseteq \text{Aut}(\Omega_n) \subseteq U_m$ for some $m \in \mathbb{N}$ as desired. □

*Definition* 3.2.15. Let $G$ be a Lie Group with unitary representation, $V$, and a subspace $P \subseteq V$. The space of $G$-equivariant maps, $P \to V$, is the set of functions obeying the property, $f(gx) = gf(x)$ for all $x \in P, g \in G$. The set of linear $G$ equivariant maps is denoted by $\text{Hom}_G(P, V)$.

*Definition* 3.2.16. Let $V$ be a unitary representation of a compact Lie Group, $G$. For any subspace, $P \subseteq V$, the $P$-isotropical part of $V$ is given by

$$V_P \equiv \text{Im}(\psi), \quad \psi : P \otimes \text{Hom}_G(P, V) \to V, \quad \forall p \in P, f \in \text{Hom}_G(P, V) : \psi(p, f) = f(p) \tag{3.2.17}$$

**Theorem 3.2.18** (Peter-Weyl)**.** *Let $G$ be a Lie Group with unitary representation, $V$, with finite dimensional irreducible representations, $P$. There is a decomposition (note that the hat indicates that this is a closed sum),*

$$V = \hat{\bigoplus_{P \subseteq V}} V_P \tag{3.2.19}$$

*Proof.* It suffices to show that $V^{\text{fin}}$ is dense in $V$. Define the operator, $T_f : V \to V$, by

$$T_f(v) = \int_G f(g) gv dg \tag{3.2.20}$$

and notice that $T_\delta(v) = v$. If we approximate $\delta$ by functions in $\delta_n \in C_{\text{alg}}(G)$. One can simplify things by considering the group action, $gT_{\delta_n}(v) = T_{g\delta_n}(v)$ which suffices to show that $T_{\delta_n}(v) \in V^{\text{fin}}$. Taking this approximation, $T_{\delta_n}(v) \to v$ as $n \to +\infty$ suffices to prove the density. □

## 3.3. **Decomposing Functions on Compact Lie Groups.**

**Proposition 3.3.1.** *If $V_1, V_2$ are irreducible representations of compact groups, $G_1, G_2$ then $V_1 \otimes V_2$ is an irreducible representation of $G_1 \times G_2$.*

**Theorem 3.3.2.** *There is an isomorphism,*

$$C_{alg}(G) \cong \hat{\bigoplus} P \otimes \tilde{P}, \quad \eta \otimes \nu \mapsto f_{P,\eta,\nu} \tag{3.3.3}$$

$$\langle \eta_1 \otimes \nu_1, \eta_2 \otimes \nu_2 \rangle = \frac{1}{\dim(P)} \overline{\langle \eta_1, \eta_2 \rangle} \langle \nu_1, \nu_2 \rangle \tag{3.3.4}$$

*where the direct sum goes through the irreducible, unitary representations of $G$. Note that $G$ acts on the left on $P$ and on the right on $\tilde{P}$.*

*Proof.* The proofs of these statements can be found on page 98 of Segal [S95]. The guiding principle comes from Theorem 3.2.18 by decomposing the representations into irreducible, finite dimensional representations. □

**Theorem 3.3.5** (Fourier)**.** *Let $S = \{z \in \mathbb{C} : \ |z| = 1\}$ be the unit circle in $\mathbb{C}$. Equivalently, $S = U_1$ is the one-dimensional unitary group. Any continuous function, $f : S \to S$, admits a decomposition*

$$f(x) = \sum_{-\infty}^{+\infty} a_k e^{2\pi i k \lambda} \tag{3.3.6}$$

*Proof.* Now, we demonstrate the power of the Peter Weyl Theorem by proving the Fourier decomposition effortlessly. Note that the only irreducible representations of $S$ are $z \mapsto z^k$ for $k \in \mathbb{Z}$. Moreover, since this is the unit circle, we write equivalently that $e^{2\pi\lambda} \mapsto e^{2\pi\lambda k}$. The tensor product of these simply give vector spaces of the form, $\mathbb{C}e^{2\pi\lambda k}$, and thus the decomposition takes the form

$$C(S) \cong \tilde{\bigoplus_{k\in\mathbb{Z}}} \mathbb{C}e^{2\pi\lambda k} \tag{3.3.7}$$

□

**Remark.** We can replicate this process for any compact Lie Group by following the process above. The next simplest group is $SU(2)$, however a concrete formulation is tedious to produce. I provide a simplification given by Tyler Leaser, who also provides a generalization of the heat equation to $SU(2)$ in his paper. [T12]

Let $P_r(x) \equiv (1 - r^2)\det(Id - rx)^{-2}$ be the Poisson Kernel in $SU(2)$ and $*$ denote the convolution product (as in the proof of Theorem 3.2.8), $(f * \psi)(x) \equiv \int f = \int_G f(g)\psi(g^{-1}x)dg)$. For any $f \in SU(2)$, we can extrapolate

$$f(x) = \lim_{r\to 1^-}(P_r * f)(x) = \lim_{r\to 1^-}\sum_{m\geq 0}(m+1)r^m\lambda_m \tag{3.3.8}$$

where $\lambda_m$ comes from the convolution product.

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## References

[A69] Adams, J. Frank. Lectures on Lie Groups. W.A. Benjamin. New York, 1969.

[B72] Busby, Robert C., Schochetman, Irwin, and Smith, Harvey A. Integral Operators and the Compactness of Induced Representations. American Mathematical Society, 1972.

[S95] Carter, Roger W., Ian G. MacDonald, Graeme B. Segal, and M. Taylor. Lectures on Lie Groups and Lie Algebras. London Mathematical Society Student Texts. Cambridge: Cambridge University Press, 1995.

[C20] Casselman, Bill. Introduction to Admissible Representations of $p$-adic Groups. University of British Columbia, 2020.

[E67] Eichler, Martin. A new proof of the Baker-Campbell-Hausdorff formula. Journal of the Mathematical Society of Japan, J. Math. Soc. Japan, 1967.

[T12] Leaser, Tyler. Fourier Analysis on $SU(2)$. East Carolina University. 2012.