AN INTUITIVE INTRODUCTION TO SPECTRAL SEQUENCES

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Abstract. In this paper, we provide an introduction to spectral sequences for those who are comfortable with the basics of homology and cohomology. We avoid beginning with the dense notation associated with spectral sequences and instead aim to give an intuitive approach through a descriptive application of the Serre spectral sequence. We then give a formal introduction and end with two more concise computations.

Contents

1. Introduction

Spectral sequences are a useful tool for computing the homology and cohomology of topological spaces. A spectral sequence is a bit like a sequence of two-dimensional chain complexes, with each successive element of the sequence arising by taking the homology of the previous object. There is a notion of convergence for spectral sequences, and by carefully choosing how we construct one, we can ensure our spectral sequence converges to something useful.

Spectral sequences are notoriously difficult to understand. Both [2] and [5] identify the topic as one which is extremely hard to grasp. This extends back to the subject's beginnings; when Leray presented the Cartan-Leray spectral sequence in 1948 (just two years after he published the first notes on spectral sequences), Whitney reportedly rose to say that he no longer understood algebraic topology, and, if homology was going to be like this, he would have to study other parts of mathematics. It was not until Serre's thesis in 1952 on the Serre spectral sequence

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that the importance of the subject was understood. For more about the history of spectral sequences, see [6].

In this paper, we attempt to introduce spectral sequences in a very accessible manner. To do this, we approach the subject by first giving a carefully detailed, nonrigorous computation using the main theorem of the Serre spectral sequence and only later introducing spectral sequences formally. The notation of spectral sequences involves many indices, and therefore we try to delay a formal introduction as long as we reasonably can.

In section 2, we review the basic constructions of chain complexes and homology.

In section 3, we give an intuitive introduction to spectral sequences. In 3.1, we introduce a general form[1] of a spectral sequence. In 3.2, we introduce fibrations and the Serre spectral sequence which arises from them. In 3.3, we give a careful, nonrigorous computation for the homology groups of $\mathbb{C}P^\infty$ which can be done using the homological Serre spectral sequence.

In section 4, we introduce spectral sequences as collections of bigraded abelian groups along with bigraded maps and give a few definitions regarding them.

In section 5, we define filtrations and exact couples, structures which naturally give rise to spectral sequences.

In section 6, we show how the Serre spectral sequences arises from exact couples. We also give the formal statements of the theorems for the homological and cohomological variants of the Serre spectral sequence.

In section 7, we conclude with two concise computations that make use of the Serre spectral sequence. We compute the homology groups of ΩS^n (the loop space of S^n), as well as the cohomology ring of $\mathbb{C}P^\infty$.

2. Chain Complexes

Recall the following definitions regarding chain complexes.

Definitions 2.1. A *chain complex* is a sequence of abelian groups and homomorphisms

$$\dots \xrightarrow{d_4} A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} \dots$$

such that $d_n d_{n+1} = 0$.

Dually, a *cochain complex* is a sequence of abelian groups and homomorphisms

$$\dots \xrightarrow{d^{-1}} A_0 \xrightarrow{d^0} A_1 \xrightarrow{d^1} A_2 \xrightarrow{d^2} A_3 \xrightarrow{d^3} \dots$$

such that $d^{n+1} d^n = 0$.

In both cases, the maps d_n and d^n are called *differentials*.

Note that the condition $d_n d_{n+1} = 0$ implies $\operatorname{im} d_{n+1} \subset \ker d_n$. In the case that this inclusion is an equality, i.e. $\operatorname{im} d_{n+1} = \ker d_n$, we say the chain complex is *exact at* A_n. If the chain complex is exact at every A_n, we say it is an *exact sequence* (or an *exact* chain complex).

As a shorthand, we refer to a chain complex $\dots \to A_1 \xrightarrow{d_1} A_0 \to \dots$ by (A_*, d_*), or simply A_*.

Examples 2.2. i. The sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \to 0,$$

[1] There are many different formulations of a general spectral sequence. In this paper we give one possible generalization of the notion which we then specialize to the Serre spectral sequence.

where $\times 2$ is multiplication by 2 and q is the quotient map, is exact. In general, an exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is called a *short exact sequence*.[2]

ii. The sequence

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z} \to 0,$$

where $\hookrightarrow$ denotes the inclusion into the first coordinate and $\twoheadrightarrow$ denotes the projection onto the second coordinate, is exact. This is an example of what is called a *split exact sequence*, a special kind of short exact sequence where $B \cong A \oplus C$.

The following proposition gives three easy-to-deduce properties of exact sequences.

Proposition 2.3. *a. If $0 \to A \xrightarrow{f} B \to \dots$ is an exact sequence, f is injective.*

b. If $\dots \to B \xrightarrow{g} C \to 0$ is an exact sequence, g is surjective.

c. If $0 \to A \xrightarrow{h} C \to 0$ is an exact sequence, h is an isomorphism.

Proof. Obvious. □

Definition 2.4. Given a chain complex A_*, for each $n \in \mathbb{N}_0$, we define its *nth homology group* by

$$H_n(A_*) := \ker d_n / \operatorname{im} d_{n+1}.$$

If A_* is an exact sequence, $\ker d_n = \operatorname{im} d_{n+1}$ for all n, and therefore

$$H_n(A_*) = \ker d_n / \operatorname{im} d_{n+1} = \ker d_n / \ker d_n = 0.$$

The other direction holds as well, giving the following proposition.

Proposition 2.5. *A chain complex is exact if, and only if, all of its homology groups are trivial.*

Proof. The forward direction is already shown. For the reverse direction, it follows from the fact that $\operatorname{im} d_{n+1} \subset \ker d_n$ that

$$\ker d_n / \operatorname{im} d_{n+1} = 0 \implies \ker d_n = \operatorname{im} d_{n+1}.$$

Thus, if all of its homology groups are trivial, the given chain complex is exact. □

This proposition gives the intuitive explanation of the homology groups of a chain complex as a way of measuring how "close" it is to being exact.

Chain complexes arise naturally in algebraic topology. Simplicial, singular, and cellular homology constructions (with coefficients in some group G) all involve creating a chain complex

$$\dots \xrightarrow{d_4} C_3(X) \xrightarrow{d_3} C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \to 0$$

and taking its homology,

$$H_n(C_*(X)) = \ker d_n / \operatorname{im} d_{n+1}.$$

[2]These sequences are given a special name due to their significance in algebra, being central to the snake lemma, which derives long exact sequences from short ones. Furthermore, these exact sequences are undeniably short.

We then define these to be the homology groups of the space X with coefficients in G, i.e.

$$H_n(X; G) \coloneqq H_n(C_*(X)),$$

and regardless of whether we chose to use simplicial, singular, or cellular chains, the resulting homology groups end up the same.

Recall that we can also construct relative homology groups of pairs (X, A) with $A \subset X$ by first quotienting the chain groups to create a new chain complex

$$\dots \xrightarrow{d_4} C_3(X)/C_3(A) \xrightarrow{d_3} C_2(X)/C_2(A) \xrightarrow{d_2} C_1(X)/C_1(A) \xrightarrow{d_1} C_0(X)/C_0(A) \to 0,$$

where differentials are induced by the differentials of the chain complex $C_*(X)$. We then define

$$H_n(X, A; G) \coloneqq H_n(C_*(X)/C_*(A)) = \ker d_n / \operatorname{im} d_{n+1}.$$

Relative homology interacts nicely with the homology of X and A, and we get the following theorem, which we state without proof.

Theorem 2.6. [3, Thm. 2.16] *For $A \subset X$, there exists a long exact sequence*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \dots \xrightarrow{\partial} H_0(X, A) \to 0$$

where i_* is induced by the inclusion $i\colon C_n(A) \to C_n(X)$ and j_* is induced by the quotient $j\colon C_n(X) \to C_n(X)/C_n(A)$.

The boundary map ∂ is defined by "diagram chasing" on the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X, A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\ 0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{j} & C_{n-1}(X, A) & \longrightarrow & 0. \end{array}$$

Given $\gamma \in H_n(X, A)$, we define $\partial(\gamma) \in H_{n-1}(A)$ by the following steps:

i. Because γ is an equivalence class, pick a representative $c \in \gamma \subset C_n(X, A)$.
ii. Pick $b \in C_n(X)$ such that $j(b) = c$.
iii. Pick $a \in C_n(A)$ such that $i(a) = d_n(b)$.
iv. Define $\partial(\gamma) = [a] \in H_{n-1}(A)$.

The map ∂ is well-defined by the construction of homology and the fact that the sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

is exact.

Example 2.7. We will now briefly compute the homology groups for $X = \mathbb{C}P^\infty$ and $G = \mathbb{Z}$ using cellular homology. Recall that it is possible to construct $\mathbb{C}P^\infty$ as a CW-complex with a cell in every even dimension and no cells in odd dimensions. This gives rise to the cellular chain complex

$$\dots \xrightarrow{d_4} 0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \to 0.$$

It is easy to see that every differential must be the 0 map. Therefore, we can compute the homology of $\mathbb{C}P^\infty$,

$$H_n(\mathbb{C}P^\infty; \mathbb{Z}) = \ker d_n / \operatorname{im} d_{n+1} = \begin{cases} \mathbb{Z} & n \text{ is even,} \\ 0 & n \text{ is odd.} \end{cases}$$

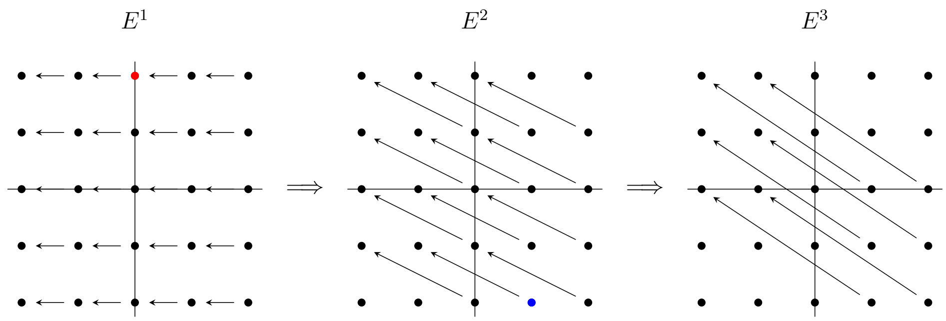

FIGURE 3.1. A visual representation of the first three pages of the Serre spectral sequence with all of the differentials whose domain and codomain are both pictured. Dots represent groups and arrows represent differentials. The red dot represents $E^1_{0,2}$ and the blue dot represents $E^2_{1,-2}$.

In the next section, we introduce spectral sequences in analogy to chain complexes. We then compute the homology of $\mathbb{C}P^\infty$ using a spectral sequence. Later on, we will use spectral sequences to compute new results, but it is worthwhile to see that a new tool gives familiar results as well.

3. Spectral Sequences

In this section, we provide an intuitive introduction to spectral sequences, attempting to avoid cumbersome notation until it is more natural to introduce. We begin by explaining the structure of a general spectral sequence, then specialize to the Serre spectral sequence and gradually work up to the computation of the homology of $\mathbb{C}P^\infty$.

3.1. **General Structure.** Recall from the previous section that, given a chain complex

$$\dots \xrightarrow{d_4} A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} \dots$$

we define its homology by

$$H_n(A_*) = \ker d_n / \operatorname{im} d_{n+1}.$$

Another way we can think about this definition of $H_n(A_*)$ is as "the homology of A_* at A_n." At every A_n, there is *exactly one* differential coming "in" and *exactly one* differential going "out." This means that, if we call the differential coming in d_{IN} and the differential going out d_{OUT}, the homology at any A_n is given by $\ker d_{\mathrm{OUT}} / \operatorname{im} d_{\mathrm{IN}}$. In the case of a chain complex, these ideas are equally simple. However, as we now introduce the more complicated structures of spectral sequences, the latter way of thinking will be useful.

A spectral sequence is like a book with infinitely many pages. Each page is a 2-dimensional lattice of groups with (fairly uniform) homomorphisms connecting them called *differentials*. We apply an operation to get from one page to the next, and, ideally, the pages stabilize at an infinite limiting page. Figure 3.1 is a visual representation for the pages of the Serre spectral sequence, a particular spectral sequence which always stabilizes.

The notation we introduce for each group in this structure is $E^r_{p,q}$, where r is the page number, p is the horizontal index, and q is the vertical index. In the Serre spectral sequence, the differentials take each $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$. In other words, for each group $E^r_{p,q}$ there is a homomorphism going from $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$ and a homomorphism from $E^r_{p+r,q-r+1}$ to $E^r_{p,q}$, and no others involving $E^r_{p,q}$. This can be observed in Figure 3.1, where the differentials on the first page go from $E^1_{p,q}$ to $E^1_{p-1,q}$, on the second page go from $E^2_{p,q}$ to $E^2_{p-2,q+1}$, and on the third page go from $E^3_{p,q}$ to $E^3_{p-3,q+2}$, all for every pair (p, q).

In a way, each page of a spectral sequence is like a 2-dimensional chain complex. Groups are indexed by two parameters instead of one and at every group there is exactly one differential coming in and exactly one differential going out. Furthermore, these differentials satisfy the property that applying one and then another always yields 0, just as in the case of a chain complex. We have yet to describe what the operation is which takes one page to the next, but by now it should be quite natural.

Given the rth page E^r of the spectral sequence, we define the group $E^{r+1}_{p,q}$ to be the homology of E^r at $E^r_{p,q}$. More explicitly, if d_{IN} is the differential coming into $E^r_{p,q}$ and d_{OUT} is the differential going out of $E^r_{p,q}$, we define

$$E^{r+1}_{p,q} := \ker d_{\text{OUT}} / \operatorname{im} d_{\text{IN}}.$$

It is more complicated to describe what exactly the differential maps on each page are, and they depend on the context in which the spectral sequence arises. For now, we will treat spectral sequences as mysterious machines which we can use to compute homology.

A spectral sequence works as a machine in the following way. We set our initial conditions by defining each group $E^1_{p,q}$ on the first page. Then, we allow the machine to apply the aforementioned operation infinitely many times. The way the Serre spectral sequence is constructed causes this process to stabilize at each group, i.e. for every p and q, there is an R such that

$$r > R \implies E^r_{p,q} = E^R_{p,q}.$$

Our machine collects all of the stabilized groups $E^\infty_{p,q}$ and outputs them on the stable page E^∞.

We have yet to explain what we should be putting in a spectral sequence, and, just as importantly, what we should expect out of it. Spectral sequences are a very general structure which can be arise in a variety of ways. In this paper, the instance of a spectral sequence we care about most is the Serre spectral sequence.

3.2. **The Serre Spectral Sequence of a Fibration.** The Serre spectral sequence is built to relate the homology groups of the different pieces of something called a *fibration*. We postpone a formal introduction to fibrations briefly to state the following theorem which describes how we can use the Serre spectral sequence to compute homology groups. For now, a fibration $F \to X \to B$ should be thought of as a map $\pi\colon X \to B$ such that all of the preimages $\pi^{-1}(b)$ for $b \in B$ are homotopy equivalent to F.

Suppose $\pi\colon X \to B$ is a fibration, where, for simplicity, we assume B is a path-connected CW-complex. For all p, let B^p be the p-dimensional skeleton of B, and define $X_p = \pi^{-1}(B^p)$. The X_p together form a *filtration* of X, which is something we will return to later on. For a given group G, we define the first page of the Serre spectral sequence by

$$E^1_{p,q} := H_{p+q}(X_p, X_{p-1}; G).$$

After we have defined the first page, the Serre spectral sequence runs its course, eventually stabilizing at the E^∞ page in a form very closely related to $H_*(X; G)$. More explicitly, we have the following theorem.[3]

Theorem 3.1. *Let $F \to X \to B$ be a fibration with B a simply-connected CW-complex. Then, there is a spectral sequence $E^r_{p,q}$ such that:*

a. *differentials on the rth page map $E^r_{p,q} \to E^r_{p-r,q+r-1}$,*
b. *$E^{r+1}_{p,q}$ is the homology[4] of E^r at $E^r_{p,q}$,*
c. *$E^2_{p,q} \cong H_p(B; H_q(F; G))$,*
d. *the stable terms $E^\infty_{p,n-p} \cong F^p_n / F^{p-1}_n$ where $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X; G)$.*

Parts *a* and *b* of Theorem 3.1 restate what we have already said more formally. Parts *c* and *d*, however, give new, extremely useful information which can be used to relate the homologies of F, X, and B. The best way to understand this theorem is to see it in action, so we will eventually use it to compute the homology of $\mathbb{C}P^\infty$ (with coefficients in $\mathbb{Z}$). First, however, we formally introduce the notion of a fibration.

Definition 3.2. Let $\pi \colon X \to B$ be a map.

Recall that π is said to satisfy the *homotopy lifting property* with respect to a space Y if, for every homotopy $h \colon Y \times [0,1] \to B$ and map $\tilde{h}_0$ such that $h(y,0) = (\pi\tilde{h}_0)(y)$, there exists a homotopy $\tilde{h} \colon Y \times [0,1] \to X$ such that $h = \pi\tilde{h}$. In other words, for every such h and $\tilde{h}_0$, there exists $\tilde{h}$ such that the following diagram commutes:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\tilde{h}_0} & X \\ \big\downarrow & \overset{\tilde{h}}{\nearrow} & \big\downarrow \pi \\ Y \times [0,1] & \xrightarrow{h} & B. \end{array}$$

We say that π is a *fibration* if it satisfies the homotopy lifting property with respect to every space Y. In this case, we call B the *base space*, X the *total space*, and $\pi^{-1}(b)$ the *fiber over* b for each $b \in B$.

The following are examples of fibrations.

Example 3.3. a. Given spaces B and F, the projection $\pi \colon F \times B \to B$ is a fibration with fibers each homeomorphic to F.
b. The quotient map $\pi \colon S^n \to \mathbb{R}P^n$ sending $x \mapsto [x]$ is a fibration with fibers each homeomorphic to S^0.
c. The quotient map $\pi \colon S^{2n+1} \to \mathbb{C}P^n$ sending $x \mapsto [x]$ is a fibration with fibers each homeomorphic to S^1.

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[3] This theorem is given in a slightly simplified form here, but we will give the full theorem later in the paper.

[4] Recall that by this we mean that, if d_{IN} is the differential coming into $E^r_{p,q}$ and d_{OUT} is the differential going out of $E^r_{p,q}$, then $E^{r+1}_{p,q} = \ker d_{\mathrm{OUT}} / \operatorname{im} d_{\mathrm{IN}}$.

To see why (a) is a fibration, let $h\colon Y \times [0,1] \to B$ be a homotopy with $\tilde{h}_0\colon Y \times \{0\} \to F \times B$ such that $(\pi\tilde{h}_0)(y) = h_0(y) = h(y,0)$. Then, let $\tilde{h}\colon Y \times [0,1] \to F \times B$ be defined by

$$\tilde{h}(y,t) = ((p_F\tilde{h}_0)(y,t), h(y,t))$$

where $p_F\colon F \times B \to F$ is the projection onto the first component. In other words, define $\tilde{h}$ so that it agrees with $\tilde{h}_0$ in the F component and h in the B component.

More generally, the fact that (a), (b), and (c) are fibrations follows from the following theorem which we state without proof. First, recall the following definition of paracompactness.

Definition 3.4. Let X be a space and $\mathcal{U}$ be an open cover of X.

A *refinement* is a new open cover $\mathcal{V}$ of X made up of open sets each contained in some open set in $\mathcal{U}$. In other words, $\mathcal{V}$ is an open cover of X such that, for all $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ with $V \subset U$.

We say a collection of sets $\mathcal{A}$ is *locally finite* if every point covered by $\mathcal{A}$ has a neighborhood intersecting only finitely many elements of $\mathcal{A}$.

We say X is *paracompact* if any open cover of X admits a locally finite refinement.

Note that compact spaces are all paracompact. In general, many of the spaces we usually work with are paracompact. In particular, we have the following result.

Lemma 3.5. *Every CW-complex is paracompact.*

Proof. Let X be a CW-complex with n-skeletons X^n. We can write each X^n as the union

$$X^n = \bigcup K^n_\alpha$$

of finite CW-complexes.

Let $\mathcal{U}$ be an open cover of X. For each n, the set

$$\mathcal{U}^n_\alpha = \{U \cap K^n_\alpha \mid U \in \mathcal{U}\}$$

is an open cover of K^n_α. Because each K^n_α is compact, there exists a refinement $\mathcal{V}^n_\alpha$ of $\mathcal{U}^n_\alpha$. For all open sets $A \in \mathcal{V}^n_\alpha$, there exists some V_A open in X such that $A = V_A \cap K^n_\alpha$. Because $A \subset U_A \cap K^n_\alpha$ for some $U_A \in \mathcal{U}$, taking the intersection of V_A with U_A allows us to, without loss of generality, assume $V_A \subset U_A$.

Let

$$\mathcal{V} = \bigcup_{n,\alpha} \{V_A \mid A \in \mathcal{V}^n_\alpha\}.$$

Because each $\{V_A \mid A \in \mathcal{V}^n_\alpha\}$ is locally finite, $\mathcal{V}$ is a countably locally finite refinement of $\mathcal{U}$ (i.e. $\mathcal{V}$ is a refinement which is the countable union of locally finite collections). Since CW-complexes are regular, by Lemma 41.3 of [7], this suffices to deduce that X is paracompact. □

We now state the aforementioned theorem.

Theorem 3.6. [4, Thm. 1] *If $\pi\colon X \to B$ is a fiber bundle and B is paracompact, then π is a fibration.*

From Lemma 3.5 and Theorem 3.6 the following is immediate.

Corollary 3.7. *Every fiber bundle with base space a CW-complex is a fibration.*

We now mention a useful property of fibrations.

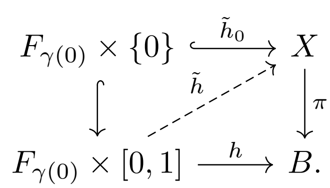
Proposition 3.8. [3, Prop. 4.61] *If $\pi\colon X \to B$ is a fibration, the fibers $F_b = p^{-1}(b)$ over each path-component of B are all homotopy equivalent.*

Proof. We can associate to each path $\gamma\colon [0,1] \to B$ a map between fibers

$$L_\gamma\colon F_{\gamma(0)} \to F_{\gamma(1)}$$

by using the homotopy lifting property with respect to the fiber $F_{\gamma(0)}$.

Let $\gamma\colon [0,1] \to B$ be a path. Define a homotopy $h\colon F_{\gamma(0)} \times [0,1] \to B$ by $h(x,t) = \gamma(t)$. Let $\tilde{h}_0\colon F_{\gamma(0)} \times \{0\} \hookrightarrow X$ be the inclusion map. Note that, for all $x \in F_{\gamma(0)}$, we have $(\pi \circ \tilde{h}_0)(x) = \gamma(0) = h_0(x)$. Thus, since $\pi\colon X \to B$ is a fibration, there exists a map $\tilde{h}\colon F_{\gamma(0)} \times [0,1] \to X$ such that the diagram



commutes. In particular, since $h = \pi\tilde{h}$, $\tilde{h}(x,t) \in F_{\gamma(t)}$ for all t. Thus, when $t = 1$, we have a map $L_\gamma\colon F_{\gamma(0)} \to F_{\gamma(1)}$ defined by $L_\gamma(x) = \tilde{h}(x,1)$.

We would now like to show that the association $\gamma \mapsto L_\gamma$ behaves reasonably well. More precisely, we would like the following two properties:

i. If γ_0 and γ_1 are paths from a to b and $\gamma \simeq \gamma' \operatorname{rel}\{0,1\}$ (i.e. there is a homotopy from γ to γ' constant on basepoints), then $L_\gamma \simeq L_{\gamma'}$.
ii. If γ_0 and γ_1 are paths such that $\gamma_1(0) = \gamma_0(1)$, then $L_{\gamma_0 * \gamma_1} \simeq L_{\gamma_1} \circ L_{\gamma_0}$.

Proving (i) and (ii) would be inefficient for the purposes of this paper. To see a proof of the statements, refer to [3, Prop. 4.61].

We now deduce the proposition from the two properties. Let F_a and F_b be fibers, and let γ be a path from a to b. Consider $L_\gamma\colon F_a \to F_b$ and $L_{\bar{\gamma}}\colon F_b \to F_a$.

Let α be the constant loop which is always a and $\tilde{h}$ be the map used to define $L_\alpha(x)$. Because $h(x,t) = \alpha(x) = a$ for all x, we know that $\tilde{h}(x,t) \in F_a$ for all x and t. Thus, $\tilde{h}$ is a homotopy between $\tilde{h}(x,0) = \mathrm{id}_{F_a}$ and $\tilde{h}(x,1) = L_\alpha$. Thus, $\mathrm{id}_{F_a} \simeq L_\alpha$ and, similarly, $\mathrm{id}_{F_b} \simeq L_\beta$.

By (i) and (ii), we have

$$L_{\bar{\gamma}} \circ L_\gamma \simeq L_{\gamma * \bar{\gamma}} \simeq L_\alpha \simeq \mathrm{id}_{F_a}$$

and

$$L_\gamma \circ L_{\bar{\gamma}} \simeq L_{\bar{\gamma} * \gamma} \simeq L_\beta \simeq \mathrm{id}_{F_b}$$

Thus, $F_a \simeq F_b$. □

This proposition leads to the convention of referring to the fibers of a fibration by a single space to which they are all homotopy equivalent, as long as the base space is path-connected. It is common to write a fibration as $F \to X \to B$, where F is a space homotopy equivalent to all of the fibers.

In this way, we may refer to the fibrations in Example 3.3 by

a. $F \to F \times B \to B$,
b. $S^0 \to S^n \to \mathbb{R}P^n$, and
c. $S^1 \to S^{2n+1} \to \mathbb{C}P^n$.

These three examples all satisfy the stronger restraint that all of the fibers are homeomorphic. A fibration which does not satisfy this property is the *path space fibration* $\Omega X \to PX \to X$, which will consider later on in the paper.

Having introduced fibrations, we can now compute the homology groups of $\mathbb{C}P^\infty$.

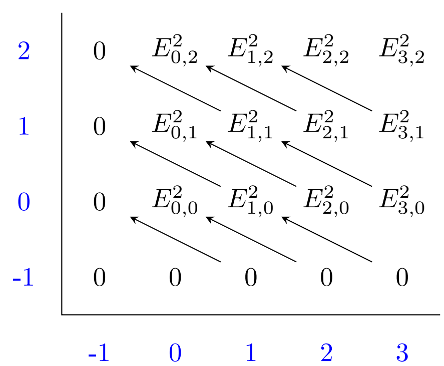
3.3. **Computing the Homology of $\mathbb{C}P^\infty$.** By Corollary 3.7 the fiber bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$ (with $\pi \colon S^\infty \to \mathbb{C}P^\infty$ the quotient map) is a fibration. Because $\mathbb{C}P^\infty$ is a simply-connected CW-complex, we can apply Theorem 3.1 to get a spectral sequence which we can use to compute the homology groups of $\mathbb{C}P^\infty$.

The strange part about using the Serre spectral sequence is that we for the most part ignore the first page E^1 and start looking only from the second page E^2 onward. The reason for this is that we have a very simple but useful formula for the groups on the second page. In the case of our fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$, we can easily write out its entire E^2 page.

Let $B = \mathbb{C}P^\infty$, $X = S^\infty$, and $F = S^1$, so our fibration can be written $F \to X \to B$. By part c of Theorem 3.1, the E^2 terms are

$$E^2_{p,q} = H_p(B; H_q(F; \mathbb{Z})).$$

One immediate observation is that, since homology groups are 0 for all $p < 0$ or $q < 0$, $E^2_{p,q}$ can only be nontrivial if $p \geq 0$ and $q \geq 0$. Thus, the E^2 page near $(p, q) = (0, 0)$ looks like



where the horizontal axis is the p coordinate and the vertical axis is the q coordinate. Note that this is only one small section of the page E^2.

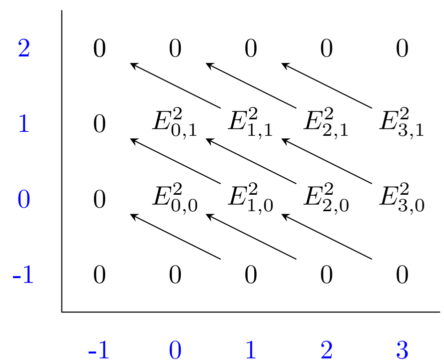
Since $F = S^1$, we know

$$H_q(F; \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

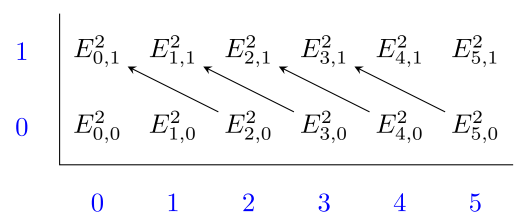
Because a homology group with coefficients in 0 is itself 0, we know that

$$q \neq 0, 1 \implies H_p(B; H_q(F; \mathbb{Z})) = H_p(B; 0) = 0 \implies E^2_{p,q} = 0.$$

Thus, we know the E^2 page can only have nontrivial groups in the two rows $q = 0$ and $q = 1$ with $p \geq 0$. The section of E^2 page can be simplified to



Because the E^2 page is now 0 everywhere except in rows 0 and 1, it is a good idea to redraw our depiction of E^2 as

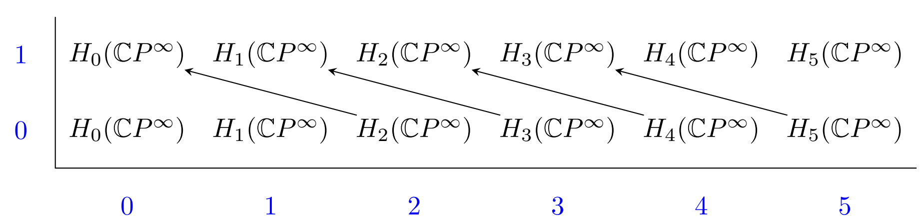


Note that there is still a differential entering and a differential exiting at each $E^r_{p,q}$, but all of the ones not depicted either come from 0 or go to 0 and are therefore the 0 map.

Since $q = 0, 1$ implies $H_q(F; \mathbb{Z}) = \mathbb{Z}$, we know that when $p \geq 0$ and $q = 0, 1$, we have

$$E^2_{p,q} = H_p(B; H_q(F; \mathbb{Z})) = H_p(B; \mathbb{Z}) = H_p(B) = H_p(\mathbb{C}P^\infty).$$

Thus, E^2 looks like



Now that we understand what the E^2 page looks like, it is beneficial to look at what the E^∞ page looks like. A simple observation we can make is the following.

Lemma 3.9. *If $E^2_{p,q} = 0$, then $E^\infty_{p,q} = 0$.*

Proof. Because $E^2_{p,q} = 0$, the differentials entering and exiting $E^2_{p,q}$ are both the 0 map. Therefore, $E^3_{p,q} = 0$ as well. By the same argument, $E^r_{p,q} = 0$ for all $r \geq 2$, so $E^\infty_{p,q} = 0$ as well. □

Lemma 3.9 tells us that the E^∞ page looks like

1	$E^\infty_{0,1}$	$E^\infty_{1,1}$	$E^\infty_{2,1}$	$E^\infty_{3,1}$	$E^\infty_{4,1}$	$E^\infty_{5,1}$
0	$E^\infty_{0,0}$	$E^\infty_{1,0}$	$E^\infty_{2,0}$	$E^\infty_{3,0}$	$E^\infty_{4,0}$	$E^\infty_{5,0}$
	0	1	2	3	4	5

with 0 everywhere except in the rows $q = 0, 1$ when $p \geq 0$.

By part d of Theorem 3.1, for all n, there exist $0 \subset F_n^0 \subset \cdots \subset F_n^n = H_n(X;\mathbb{Z})$ such that $E^\infty_{p,n-p} \cong F_n/F_n^{p-1}$. Because $X = S^\infty$ is contractible, $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$ and $H_0(X;\mathbb{Z}) = \mathbb{Z}$. Thus, $F_n^p = 0$ for all $n > 0$ and $0 \leq p \leq n$, so $E^\infty_{p,n-p} = 0$ for all n, p such that $n \neq 0$. When $n = 0$, we have that $E^\infty_{p,-p} = 0$ whenever $p \neq 0$. Because $E^\infty_{0,0} = F_0^0 = H_0(X;\mathbb{Z}) = \mathbb{Z}$, we know the E^∞ page looks like

1	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

We now reach the final part of our argument. Consider the following observation.

Lemma 3.10. $E^2_{0,0} = E^\infty_{0,0}$.

Proof. Let d_{IN} be the differential going into $E^2_{0,0}$ and d_{OUT} the differential going out of it. Because $d_{\text{IN}}\colon E^2_{2,-1} = 0 \to E^2_{0,0}$ and $d_{\text{OUT}}\colon E^2_{0,0} \to E^2_{-2,1} = 0$, both differentials are 0, so we have

$$E^3_{0,0} = \ker d_{\text{OUT}} / \operatorname{im} d_{\text{IN}} = E^2_{0,0}/0 = E^2_{0,0}.$$

From the proof of Lemma 3.9, we know that, for either $p < 0$ or $q \neq 0, 1$, $E^r_{p,q} = 0$ for all $r \geq 2$. On the rth page, the differential going into $E^r_{0,0}$ comes from $E^r_{r,-r+1}$ and the differential going out of $E^r_{0,0}$ maps to $E^r_{-r,r-1}$. Since $-r+1 < 0$ and $-r < 0$, both groups are zero, so both differentials are zero maps. Thus, $E^{r+1}_{0,0} = E^r_{0,0}$.

Therefore, for all $r \geq 2$ we have that $E^r_{0,0} = E^2_{0,0}$, so we conclude $E^\infty_{0,0} = E^2_{0,0}$. □

We can now generalize this argument slightly to have the following.

Lemma 3.11. *For all p, q, $E^3_{p,q} = E^\infty_{p,q}$.*

Proof. From the proof of Lemma 3.9, we know that, for either $p < 0$ or $q \neq 0, 1$, $E^r_{p,q} = 0$ for all $r \geq 2$. Thus, $E^3_{p,q} = E^\infty_{p,q}$ for $p < 0$ or $q \neq 0, 1$.

Suppose $p \geq 0$ and q is either 0 or 1. On the rth page for some $r \geq 3$, the differential going into $E^r_{p,q}$ comes from $E^r_{p+r,q-r+1}$ and the differential going out of $E^r_{p,q}$ maps to $E^r_{p-r,q+r-1}$. Because $r \geq 3$ and $q \leq 1$, $q - r + 1 \leq q - 2 \leq -1$, so $E^r_{p+r,q-r+1} = 0$. Because $r \geq 3$ and $q \geq 0$, $q + r - 1 \geq q + 2 \geq 2$, so $E^r_{p-r,q+r-1} = 0$. Thus, both differentials are 0, so $E^{r+1}_{p,q} = E^r_{p,q}$. Thus, for all $r \geq 3$, $E^r_{p,q} = E^3_{p,q}$, so we conclude $E^3_{p,q} = E^\infty_{p,q}$. □

We now know what both the E^2 and E^3 pages look like, the former in terms of the homology of $\mathbb{C}P^\infty$ and the latter explicitly. To calculate the homology groups of $\mathbb{C}P^\infty$, it suffices to consider how the E^3 page was created from the E^2 page.

By Lemma 3.10, we can already deduce that

$$H_0(\mathbb{C}P^\infty) = E^2_{0,0} = E^3_{0,0} = \mathbb{Z}.$$

However, computing the other groups requires slightly more work.

Theorem 3.12. *For all $k \geq 1$, $H_k(\mathbb{C}P^\infty) \cong H_{k-2}(\mathbb{C}P^\infty)$.*

Proof. Since differentials compose to 0, for each k we can extract from the E^2 page the chain complex

$$E^2_{k+2,-1} = 0 \to E^2_{k,0} \to E^2_{k-2,1} \to E^2_{k-4,2} = 0$$

If $k \geq 1$, then, $E^3_{k,0} = 0$ and $E^3_{k-2,1} = 0$. Thus, the homology groups of the given chain complex are all trivial. By Proposition 2.5, the given chain complex is an exact sequence. By Proposition 2.3, this implies $E^2_{k,0} \cong E^2_{k-2,1}$. Thus, for all $k \geq 1$,

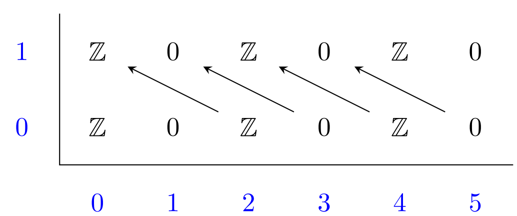
$$H_k(\mathbb{C}P^\infty) \cong H_{k-2}(\mathbb{C}P^\infty).$$

□

Corollary 3.13. $H_n(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & n \text{ is even,} \\ 0 & n \text{ is odd.} \end{cases}$

Proof. It follows from Lemma 3.10 that $H_0(\mathbb{C}P^\infty) = \mathbb{Z}$, and we also know that $H_{-1}(\mathbb{C}P^\infty) = 0$. By Theorem 3.12, a simple proof by induction shows that, for all even n, $H_n(\mathbb{C}P^\infty) = \mathbb{Z}$, and for all odd n, $H_n(\mathbb{C}P^\infty) = 0$. □

Thus, the E^2 page looks like



with all of the differentials (except those involving the $\mathbb{Z}$ at the bottom left corner) isomorphisms.

Later on we will show that we can deduce even more from the Serre spectral sequence with fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$ if we construct it instead with cohomology.

In the next section, we take a more formal approach to spectral sequences.

4. Formal Definitions

Spectral sequences can be defined generally over abelian categories, or less generally over R-modules. Rather than introduce a subject in its utmost generality, it is often more illuminating to introduce it specially and only then attempt to generalize. Therefore, in this paper we only consider spectral sequences over $\mathbb{Z}$-modules, i.e. abelian groups.

$$\begin{array}{cccccc} A & A_{0,0} & A_{1,0} & A_{0,1} & A_{-1,0} & \cdots \\ \downarrow{\scriptstyle f} & \downarrow{\scriptstyle f_{0,0}} & \downarrow{\scriptstyle f_{1,0}} & \downarrow{\scriptstyle f_{0,1}} & \downarrow{\scriptstyle f_{-1,0}} & \\ B & B_{0,0} & B_{1,0} & B_{0,1} & B_{-1,0} & \cdots \end{array}$$

FIGURE 4.1. A depiction of a bigraded map between bigraded abelian groups, considered as many parallel maps.

Definition 4.1. Fix $r_0 \geq 0$. A *homological spectral sequence* $\{E^r_{p,q}, d^r_{p,q}\}$ consists of

i. a collection of abelian groups $\{E^r_{p,q}\}_{p,q\in\mathbb{Z},r\geq r_0}$ and
ii. a collection of groups homomorphisms $\{d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ called *differentials*

such that, for all p, q, r,

a. $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$ and
b. $E^{r+1}_{p,q}$ is the homology of E^r at $E^r_{p,q}$, i.e. $E^{r+1}_{p,q} = \ker d^r_{p,q} / \operatorname{im} d_{p+r,q-r+1}$.

The notation quickly becomes quite cumbersome. A common simplification is done by fixing each r and treating all of the $E^r_{p,q}$ as one object E^r called a *bigraded abelian group*. The differentials $d^r_{p,q}$ are then collectively called a *bigraded map*. These are not too difficult to define and help consolidate notation, so we introduce them.

Definitions 4.2. [9, Defn. 2.2] A *bigraded abelian group* A is a collection of abelian groups $A = \{A_{p,q} \mid p, q \in \mathbb{Z}\}$.

If $A = \{A_{p,q} \mid p, q \in \mathbb{Z}\}$ and $B = \{B_{p,q} \mid p, q \in \mathbb{Z}\}$ are bigraded abelian groups, a *bigraded map of bidegree* (c, d) is a collection of group homomorphisms $f = \{f_{p,q} \colon A_{p,q} \to B_{p+c,q+d} \mid p, q \in \mathbb{Z}\}$.

It is possible to think of a bigraded abelian group $A = \{A_{p,q} \mid p, q \in \mathbb{Z}\}$ as the direct sum of its elements, i.e. $A = \oplus_{p,q\in\mathbb{Z}} A_{p,q}$. However, these two presentations give us the same amount of information, and the direct sum gives a false sense of interaction between the elements of a bigraded abelian group. For our purposes, the pieces of a bigraded abelian groups should be considered as parallel to each other. See Figure 4.1.

Example 4.3. For a ring R, the additive structure on $R[x, x^{-1}, y, y^{-1}]$ is a bigraded abelian group with $A_{p,q} = Rx^p y^q$ for all $p, q \in \mathbb{Z}$.

We can also define the following for graded abelian groups.

Definition 4.4. Let A and B be bigraded abelian groups. We define

i. $A \subset B$ to mean $A_{p,q} \subset B_{p,q}$ for all p, q,
ii. If $A \subset B$, then $B/A \coloneqq \{B_{p,q}/A_{p,q} \mid p, q \in \mathbb{Z}\}$.

Definition 4.5. If $f \colon A \to B$ is a bigraded map of degree (c, d) between bigraded abelian groups, we define

i. $\ker f \coloneqq \{\ker f_{p,q} \mid p, q \in \mathbb{Z}\}$ and
ii. $\operatorname{im} f \coloneqq \{\operatorname{im} f_{p-c,q-d} \mid p, q \in \mathbb{Z}\}$.

With these definitions, we have a new equivalent definition for Definition 4.1.

Definition 4.6. [9, Defn. 3.2] Fix $r_0 \geq 0$. A *homological spectral sequence* $\{E^r, d^r\}$ consists of, for each r,

i. a bigraded abelian group E^r and
ii. a bigraded map $d^r \colon E^r \to E^r$ of degree $(-r, r-1)$ called a *differential*

such that, for all r,

a. $d^r \circ d^r = 0$ and
b. E^{r+1} is the homology of E^r with respect to the differential d^r, i.e.

$$E^r = \ker d_r / \operatorname{im} d_r = \{\ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1} \mid p, q \in \mathbb{Z}\}.$$

For each r, E^r is called the rth *sheet* or *page* of $\{E^r, d^r\}$.

We will use this more concise definition for the remainder of this paper.

Remark 4.7. There is also a notion of a *cohomological spectral sequence* which is defined in exactly the same way a homological spectral sequence is, except for the following:

i. the pages are typically denoted by $E_r = \{E_r^{p,q} \mid p, q \in \mathbb{Z}\}$,
ii. the differentials are typically denoted by $d_r \colon E_r \to E_r$ and are all of degree $(r, -r+1)$.

The reason for this naming convention is very simple: homological spectral sequences are a tool for homology and cohomological spectral sequences are a tool for cohomology.

Definition 4.8. Suppose $\{E^r, d^r\}$ is a homological spectral sequence such that, for all $p, q \in \mathbb{Z}$, there exists $r(p,q) \in \mathbb{N}$ such that, for all $r \geq r(p,q)$, $E^r_{p,q} \cong E^{r(p,q)}_{p,q}$. Then, we say the bigraded abelian group

$$E^\infty := \{E^{r(p,q)}_{p,q} \mid p, q \in \mathbb{Z}\}$$

is the *limit term* of $\{E^r, d^r\}$. Equivalently, we say the spectral sequence *abuts* to E^∞.

A similar definition exists for cohomological spectral sequences with a limit term E_∞.

Definition 4.9. We say a spectral sequence is *first quadrant* if, for any $r \in \mathbb{N}$, entries on the rth page are nonzero only when $p \geq 0$, $q \geq 0$.

In the next section, we show how spectral sequences can come about. We introduce the notions of *filtration* and *exact couple*. In section 6 we describe how the Serre spectral sequence is constructed from these two structures.

5. Filtrations and Exact Couples

Definition 5.1. A *filtration* of a space X is a sequence of increasing subspaces

$$\cdots \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X.$$

In general, a filtration could include nonempty X_p when $p < 0$, but for our purposes these will all be empty.

Suppose we have a space X and a "nice"[5] subspace A. Suppose further that we are already well acquainted with both A and X/A, in the sense that we know all of

[5]By "nice" we mean that there exists a neighborhood U of A in X which deformation retracts onto A. Recall that this is the condition required to have $H_n(X, A) \cong \tilde{H}_n(X/A)$.

their homology groups. The natural step is then to use the long exact sequence of reduced homology for (X, A) given by

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to H_{n-1}(A) \to H_{n-1}(X) \ldots$$

to compute the homology of X.

Suppose instead we have a triple $B \subset A \subset X$ (where the subspaces are assumed to be "nice"), where we know the homologies of B, A/B, and X/A. We would like some generalization of the exact sequence of a pair which works for this situation too. More generally, we have the following question.

Question 5.2. For a filtration

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X,$$

what can we say about the homology of X using what we know about the homology of the X_i and/or the quotients X_i/X_{i-1}?

When we specialize to the case where X is the total space of a fibration, we are able to obtain a satisfying answer.

In a spectral sequence, it is often the case that the E^r and d^r come form another structure called an *exact couple*.

Definition 5.3. An *exact couple* is a pair of bigraded abelian groups A and E along with bigraded maps $i\colon A \to A$, $j\colon A \to E$, and $k\colon E \to A$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & A \\ & {}_{k}\nwarrow \qquad \swarrow_{j} & \\ & E & \end{array}$$

is exact, i.e.

- $\ker i = \operatorname{im} k$,
- $\ker j = \operatorname{im} i$, and
- $\ker k = \operatorname{im} j$.

Given an exact couple, there is a natural way to define a differential $d = jk\colon E \to E$ which satisfies $d^2 = jkjk = j(kj)k = 0$. We can then use this differential to form a new exact couple called the *derived couple*.

Lemma 5.4. *Given an exact couple (A, E, i, j, k), there exists an exact couple (called the* derived couple*) (A', E', i', j', k') with*

$$\begin{array}{ccc} A' & \xrightarrow{\ i'\ } & A' \\ & {}_{k'}\nwarrow \qquad \swarrow_{j'} & \\ & E' & \end{array}$$

defined by

- $A' = i(A')$,
- $E' = \ker(jk)/\operatorname{im}(jk) = \ker d/\operatorname{im} d$,
- $i' = i|_{i(A)}$,
- *for all* $i(a) \in A'$, $j'(i(a)) = [j(a)] \in E'$, *and*
- *for all* $[e] \in E'$, $k'([e]) = k(e)$.

Proof. We first check that j' and k' are well-defined.

For j', suppose $i(a_1) = i(a_2)$ for $a_1, a_2 \in A$. Then, $a_1 - a_2 \in \ker i = \operatorname{im} k$, so $j(a_1) - j(a_2) \in \operatorname{im}(jk) = \operatorname{im} d$. Thus, $[j(a_1)] = [j(a_2)]$ in E'.

For k', first note that for all $e \in \ker d$, we have $k(e) \in \ker j = \operatorname{im} i = A'$. Furthermore, if $[e_1] = [e_2]$ for $e_1, e_2 \in E$, then $e_1 - e_2 \in \operatorname{im} d \subset \operatorname{im} j = \ker k$. Therefore, $k'([e_1] - [e_2]) = k(e_1 - e_2) = 0$, so $k'([e_1]) = k'([e_2])$.

We now show the resulting diagram is exact.

Given $a' \in A'$, there is some $a \in A$ such that $i(a) = a'$. We see

$$(j'i')(a') = j'(i(a)) = [j(a')] = [(ji)(a)] = 0$$

since $ji = 0$. Thus, $j'i' = 0$, so $\operatorname{im} i' \subset \ker j'$.

For $a' \in \ker j'$ with $a' = i(a)$, we have $[j(a)] = 0$, so $j(a) \in \operatorname{im} d$. Thus, there exists $e \in E$ such that $j(a) = (jk)(e)$, and therefore $a - k(e) \in \ker j = \operatorname{im} i$. Thus, there is some $b \in A$ such that $a - k(e) = i(b)$. Because $ik = 0$, we have that $a' = i(a) = i(a) - (ik)(e) = i(a - ke) = i^2(b)$. Thus, we have that $a' = i(i(b))$, so $a' \in \operatorname{im} i'$. Thus, $\ker j' \subset \operatorname{im} i'$, so $\ker j' = \operatorname{im} i'$.

Given $a' = i(a) \in A'$, we see $(k'j')(a') = k'([ja]) = (kj)(a) = 0$. Thus, $\operatorname{im} j' \subset \ker k'$.

Let $[e] \in \ker k'$. Then, $k(e) = 0$, so $e \in \operatorname{im} j$. Thus, there exists $a \in A$ such that $e = j(a)$, and therefore $[e] = [j(a)] = j'(i(a))$. Thus, $[e] \in \operatorname{im} j'$, so we conclude $\ker k' = \operatorname{im} j'$.

Given $[e] \in E'$, we have $(i'k')([e]) = i'(k(e)) = (ik)(e) = 0$ since $ik = 0$, so $\operatorname{im} k' \subset \ker i'$.

Let $a' \in \ker i'$. Then, $a' \in \ker i$, so $a' \in \operatorname{im} k$. If $a' = k(e)$, then $a' = k'([e])$. Thus, we conclude $\ker i' = \operatorname{im} k'$.

We conclude the derived couple is an exact couple. □

The process of producing a derived couple from an exact couple can be repeated an arbitrary number of times, giving us an rth derived couple for all $r \in \mathbb{N}$ which we denote by $(A^r, E^r, i^r, j^r, k^r)$. Letting $d^r = j^r k^r$, it should now be clear how a spectral sequence might come about from such a construction.

Given a filtration $X_0 \subset X_1 \subset \cdots \subset X$, we can construct an exact couple in the following way.

Remark 5.5. (Exact Couples from a Filtration)

For all $p, q \in \mathbb{Z}$, let $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$ and $A^1_{p,q} = H_{p+q}(X_p)$. We define i^1, j^1, and k^1 on each piece of A^1 or E^1 according to the long exact sequences of pairs

$$\dots \xrightarrow{k^1_{p,q+1}} H_{p+q}(X_{p-1}) \xrightarrow{i^1_{p,q}} H_{p+q}(X_p) \xrightarrow{j^1_{p,q}} H_{p+q}(X_p, X_{p-1}) \xrightarrow{k^1_{p,q}}$$
$$\xrightarrow{k^1_{p,q}} H_{p+q-1}(X_{p-1}) \xrightarrow{i^1_{p,q-1}} \dots$$

Thus,

- $i^1 \colon A^1 \to A^1$ is induced on each piece by the inclusion $X_{p-1} \hookrightarrow X_p$,
- $j^1 \colon A^1 \to E^1$ is induced on each piece by the quotient map on chains, and
- $k^1 \colon E^1 \to A^1$ is made up of the boundary maps

$$k^1_{p,q} = \partial \colon H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}.$$

Lemma 5.6. *The construction in Remark 5.5 produces an exact couple.*

Proof. The proof follows immediately from exactness along each long exact sequence of a pair. ☐

Because Remark 5.5 gives us an exact couple, we also obtain infinitely many derived couples. By "ignoring" the A^r pieces of derived couples, we have a spectral sequence.

Theorem 5.7. *Given the derived couples $(A^r, E^r, i^r, j^r, k^r)$ produced from the exact couple in Remark 5.5, there is a homological spectral sequence $\{E^r, d^r\}$ where $d^r = j^r k^r$.*

Proof. We already know that $d^r \circ d^r = 0$ and E^{r+1} is the homology of E^r with respect to d^r. Thus, it suffices to check that $d^r\colon E^r \to E^r$ is of degree $(-r, r-1)$.

On the first page, $d^1 = j^1 k^1$ tells us that, for a given $H_{p+q}(X_p, X_{p-1})$, d^1 maps

$$H_{p+q}(X_p, X_{p-1}) \xrightarrow{k^1} H_{p+q-1}(X_{p-1}) \xrightarrow{j^1} H_{p+q-1}(X_{p-1}, X_{p-2}).$$

Thus, d^1 maps $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$ to $E^1_{p-1,q} = H_{p+q-1}(X_{p-1}, X_{p-2})$. Therefore, d^1 is of degree $(-1, 0)$.

For the general case, we first prove the following.

a. $A^r_{p,q} = i^{r-1}(A^1_{p,q})$.
b. $k^r_{p,q}\colon E^r_{p,q} \to A^r_{p-1,q}$.
c. $j^r_{p,q}\colon A^r_{p,q} \to E^r_{p-r+1,q+r-1}$.

(a) is immediate from the definition of a derived couple. (b) is also immediate because $k^{r+1}_{p,q}$ is simply the map induced by $k^r_{p,q}$ on $\ker d^r / \operatorname{im} d^r$. We know $k^1_{p,q}\colon E^1_{p,q} \to A^1_{p-1,q}$, so a proof by induction gives us (b) for every r.

For (c), let $a \in A^r_{p,q}$. Because $A^r_{p,q} = i^{r-1}(A^1_{p,q})$, there exists $b \in A^1_{p,q}$ such that $i^{r-1}(b) = a$. Thus,

$$j^r(a) = j^r(i^{r-1}(b)) = [j^{r-1}(i^{r-2}(b))] = [[j^{r-2}(i^{r-3}(b))]] = \cdots = [\ldots[j^1(b)]\ldots],$$

where multiple brackets $[[\alpha]]$ for some $\alpha \in E^{r-1}_{p,q}$ denotes first taking the equivalence class $[\alpha] \in E^r_{p,q}$, then taking the equivalence class of that $[[\alpha]] \in E^{r+1}_{p,q}$.

Because $[\ldots[j^1(b)]\ldots]$ and $j^1(b)$ exist in abelian groups which differ only in the r coordinate and have the same p and q coordinates, the p and q coordinates of $j^r(a)$ are the same as those of $j^1(b)$. Because $b \in i^{r-1}(A^r_{p,q}) = A^1_{p-r+1,q+r-1}$, $j^1(b) \in E^r_{p-r+1,q+r-1}$. Thus, $j^r_{p,q}\colon A^r_{p,q} \to E^r_{p-r+1,q+r-1}$.

It follows from (b) and (c) that

$$d^r\colon E^r_{p,q} \xrightarrow{k^r} A^r_{p-1,q} \xrightarrow{j^r} E^r_{p-r,q+r-1}$$

Thus, the degree of the bigraded map d^r is $(-r, r-1)$. ☐

In the next section, we describe how the Serre spectral sequence comes about from a filtration of the total space X. We then state three theorems we can use to make computations using the Serre spectral sequence.

6. Formal Presentation of the Serre Spectral Sequence

Given a fibration $\pi\colon X \to B$ where B is a path-connected CW-complex, we can create a filtration of X by taking the p-skeletons B^p of B and defining $X_p = \pi^{-1}(B^p)$. From this filtration we obtain an exact couple, and from this exact couple we obtain a homological spectral sequence. In this case, the resulting spectral sequence is called the Serre spectral sequence. In section 3.2 of this paper, we stated the existence of the Serre spectral sequence in a slightly simplified form (Theorem 3.1). Once we go over the following notion, we can state the theorem in its proper form.

Throughout this section, G is an arbitrary abelian group.

Recall how in Proposition 3.8 we assigned to each path γ in the base space B a map $L_\gamma\colon F_a \to F_b$. In the case that these paths are loops based at the basepoint $b_0 \in B$, the associated map $L_\gamma\colon F_{b_0} \to F_{b_0}$ induces a map on homology $L_{\gamma *}\colon H_*(F_{b_0}; G) \to H_*(F_{b_0}; G)$. Because homotopic paths give rise to homotopic associated maps, they induce identical maps on homology. Thus, $\pi_1(B)$ acts on $H_*(F_{b_0}; G)$ in this way. When we say $\pi_1(B)$ acts trivially on $H_*(F_{b_0}; G)$, we mean that this induced map is the identity.

Theorem 6.1. *Let $F \to X \to B$ be a fibration with B path-connected. If $\pi_1(B)$ acts trivially on $H_*(F; G)$, then there is a homological spectral sequence $\{E^r, d^r\}$ with*

a. $E^2_{p,q} \cong H_p(B; H_q(F; G))$ *and*
b. stable terms $E^\infty_{p,n-p}$ isomorphic to the successive quotients F^p_n/F^{p-1}_n for a filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X; G)$.

Remark 6.2. Note that B is not required to be a CW-complex due to CW approximation: there exists a CW-complex B' weakly homotopy equivalent to B via a weak homotopy equivalence $\eta\colon B' \to B$. By pulling back along η, we have a new fibration $F \to X \to B'$. Because $H_*(B) = H_*(B')$, the new fibration can be used without loss of generality.

In statement (b), a filtration of a group is simply a nested sequence of subgroups of G, $0 \subset G_0 \subset G_1 \subset \cdots \subset G$, with the additional requirement that each G_i be normal in G (which is satisfied trivially in our case since our groups are abelian).

There is also a cohomological version of the Serre spectral sequence, which we can use to compute the cohomology groups of spaces.

Theorem 6.3. *Let $F \to X \to B$ be a fibration with B path-connected. If $\pi_1(B)$ acts trivially on $H_*(F; G)$, then there is a cohomological spectral sequence $\{E_r, d_r\}$ with*

a. $E_2^{p,q} \cong H^p(B; H^q(F; G))$ *and*
b. stable terms $E_\infty^{p,n-p}$ isomorphic to the successive quotients F^n_p/F^n_{p-1} for a filtration $0 \subset F^n_0 \subset \cdots \subset F^n_n = H^n(X; G)$.

There is an additional multiplicative structure present in the cohomological Serre spectral sequence which we can use to deduce the ring structure on the cohomology groups of a space.

Recall that the cohomology groups of a space X form a graded commutative ring under addition and cup product.

Given a fibration $F \to X \to B$, we can easily construct a fibration $F \times F \to X \times X \to B \times B$ by taking $\pi \times \pi$ for the map $X \times X \to B \times B$. If B is a CW-complex, we can filter $X \times X$ by letting $(X \times X)_p$ be the preimage of the p-skeleton of $B \times B$, $(\pi \times \pi)^{-1}(B \times B)^p$.

Let $X_p = \pi^{-1}(B^p)$ be the standard filtration of X. For all a, b, c, d we have that

$$(X_a \times X_b) \cap (X_c \times X_d) = (X_a \cap X_c) \times (X_b \cap X_d).$$

Due to the product CW structure of $B \times B$, we have the congruence

$$H^k((X \times X)_p, (X \times X)_{p-1}) \cong \oplus_{i+j=p} H^k(X_i \times X_j, X_{i-1} \times X_j \cup X_i \times X_{j-1}).$$

Let $\Delta \colon X \to X \times X$ be the diagonal map $x \mapsto (x, x)$. Then, letting $m = p + q$ and $n = s + t$, we define the product structure on the first page of the cohomological Serre spectral sequence $E_1^{p,q} \times E_1^{s,t} \to E_1^{p+s,q+t}$ by the composition

$$\begin{array}{c} H^m(X_p, X_{p-1}) \times H^n(X_s, X_{s-1}) \\ \Big\downarrow \times \\ H^{m+n}(X_p \times X_s, X_{p-1} \times X_s \cup X_p \cup X_{s-1}) \\ \Big\downarrow \\ \oplus_{i+j=p+s} H^{m+n}(X_i \times X_j, X_{i-1} \times X_j \cup X_i \times X_{j-1}) \\ \Big\downarrow \cong \\ H^{m+n}((X \times X)_{p+s}, (X \times X)_{p+s-1}) \\ \Big\downarrow \Delta^* \\ H^{m+n}(X_{p+s}, X_{p+s-1}), \end{array}$$

where $\times$ is the relative cross product map. This product structure can then induce maps on each of the following pages, including the infinite one.

We now state without proof the following theorem which details how we can use this structure for computations.

Theorem 6.4. *i. Each differential d_r is a* derivation*, i.e. satisfies, for $x \in E_r^{p,q}$ and $y \in E_r^{s,t}$,*

$$d_r(xy) = (d_r x)y + (-1)^{p+q} x(d_r y).$$

The product on the rth page induces a product on the $(r+1)$th page which maps $E_{r+1}^{p,q} \times E_{r+1}^{s,t}$ to $E_{r+1}^{p+s,q+t}$ by sending equivalence classes $[\alpha]$ and $[\beta]$ to the equivalence class $[\alpha\beta]$. The above equality ensures this map is well-defined.

ii. The induced product on the E_2 page $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is equal to the standard cup product

$$H^p(B; H^q(F; \mathbb{Z})) \times H^s(B; H^t(F; \mathbb{Z})) \xrightarrow{\smile} H^{p+s}(B; H^{q+t}(F; \mathbb{Z}))$$

where each pair of cocycles (ϕ, ψ) is sent to $\phi \smile \psi$ with coefficients multiplied via the cup product

$$H^q(F; \mathbb{Z}) \times H^t(F; \mathbb{Z}) \to H^{q+t}(F; \mathbb{Z}).$$

iii. The cup product structure of $H^*(X; \mathbb{Z})$ restricts to maps $F_p^m \times F_s^n \to F_{p+s}^{m+n}$, which induce quotient maps $F_p^m / F_{p+1}^m \times F_s^n / F_{s+1}^n \to F_{p+s}^{m+n} / F_{p+s+1}^{m+n}$ that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

Part (ii) tells us that the E_2 page has simple formulas not only for the groups, but for the products.

The best way to understand these theorems is to see how they are applied. In the next section, we give two more examples of computations which can be done using Theorems 6.1, 6.3, and 6.4.

7. More Examples

In this section, we first give a computation using the homological Serre spectral sequence, then give one for the cohomological version.

In our introduction for fibrations in section 3.2, we mentioned a fibration called a *path space fibration*, $\Omega X \to PX \to X$. We first define ΩX and PX, then use the fibration they form to determine the homology groups of ΩS^n.

Definitions 7.1. Let (X, x_0) be a based space, i.e. a space with a specified base-point. The *path space* of X, denoted PX, is the set of all paths in X which begin at x_0 with the compact-open topology.

The *compact-open topology* can be defined generally for the set of maps from any space Y to another space X (denoted X^Y). For a compact subset $K \subset Y$ and an open subset $U \subset X$, let $M(K, U)$ be the set of continuous maps $f\colon Y \to X$ such that $f(K) \subset U$. The compact-open topology is the topology generated by the subbasis

$$\{M(K, U) \mid K \subset Y \text{ is compact}, U \subset X \text{ is open}\}.$$

When $Y = [0, 1]$, the path space PX is the subspace of $X^{[0,1]}$ consistenting of paths beginning at x_0 with the induced subspace topology.

In the case that Y is compact (e.g. $Y = [0, 1]$), the compact-open topology is equivalent to the metric topology induced by the metric

$$d(f, g) = \sup_{y \in Y} d(f(y), g(y)).$$

Thus, it makes sense intuitively to think of open sets in the path space as collections of paths which are consistently "close" to each other.

The *loop space* of X, denoted ΩX, is the subspace of PX consisting of paths which begin *and* end at x_0.

Remark 7.2. For any path-connected space X, the path space PX is contractible. To see this, consider the map which shrinks every path to the constant path, i.e. $F\colon PX \times [0, 1] \to PX$ defined by

$$F(\gamma, s)(t) = \gamma((1 - s)t).$$

Proposition 7.3. *For any based space (X, x_0), there is a fibration $\Omega X \to PX \to X$ called the* path space fibration.

Proof. The map $\pi\colon PX \to X$ is defined by sending paths which begin at x_0 to the point they end at, i.e. if γ is a path from x_0 to x, $\pi(\gamma) = x$. For each $x \in X$, the fiber contains all paths from x_0 to x, i.e.

$$\pi^{-1}(x) = \{\gamma\colon [0, 1] \to X \mid \gamma(0) = x_0, \gamma(1) = x\}.$$

We can show that each $\pi^{-1}(x) \simeq \Omega X$ in the following way. Fix α a path from x_0 to x. Let $f\colon \Omega X \to \pi^{-1}(x)$ be the map which sends a loop γ based at x_0 to the path $\gamma * \alpha$ formed by concatenation. Let $g\colon \pi^{-1}(x) \to \Omega X$ be the map which sends a path η from x_0 to x to the loop $\eta * \bar{\alpha}$. Then, for a loop γ based at x_0,

$$(gf)(\gamma) = g(\gamma * \alpha) = (\gamma * \alpha) * \bar{\alpha}$$

and for a path η from x_0 to x,

$$(fg)(\eta) = f(\eta * \bar{\alpha}) = (\eta * \bar{\alpha}) * \alpha.$$

Both maps can be shown to be homotopic to the identity on ΩX or $\pi^{-1}(x)$.

To show that $\pi\colon PX \to X$ is a fibration, let Y, h, and $\tilde{h}_0$ be such that the diagram

$$\begin{CD} Y \times \{0\} @>{\tilde{h}_0}>> PX \\ @VVV @VV{\pi}V \\ Y \times [0,1] @>{h}>> X. \end{CD}$$

commutes. For all $y \in Y$, there is a path $\gamma_y = \tilde{h}_0(y) \in PX$ from x_0 to $h(y,0)$. For all $(y,s) \in Y \times [0,1]$, there is a path in X from $h(y,0)$ to $h(y,s)$ defined by $h(y,st)$. Define $\tilde{h}(y,s)$ to be the path[6] which first traverses γ_y in the time $[0, 1-\frac{s}{2}]$ then traverses $h(y,st)$ in the time $[1-\frac{s}{2}, 1]$. One can show that $\tilde{h}$ is continuous. Furthermore, we see

$$\tilde{h}(y,0) = \gamma_y = \tilde{h}_0(y)$$

and, since $h(y,st)$ is a path which ends at $h(y,s)$,

$$(\pi \circ \tilde{h})(y,s) = h(y,s).$$

Thus, the diagram

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\tilde{h}_0} & PX \\ \big\downarrow & \overset{\tilde{h}}{\nearrow} & \big\downarrow \pi \\ Y \times [0,1] & \xrightarrow{h} & X. \end{array}$$

commutes, so $\Omega X \to PX \to X$ is indeed a fibration. □

Example 7.4. We can now compute the homology of ΩS^n. When $n = 1$, the loop space ΩS^1 has as path components the equivalence classes of paths[7] in S^1, and each equivalence class can be shown to deformation retract onto the typical representatives of their classes. For $n > 1$, S^n is simply connected, so the $\pi_1(S^n)$ acts trivially on $H_*(\Omega S^n)$. Thus, we can use the homological Serre spectral sequence for the fibration $\Omega S^n \to PS^n \to S^n$.

Recall that

$$H_k(S^n; G) = \begin{cases} G & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

and, since PS^n is contractible,

$$H_k(PS^n; G) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

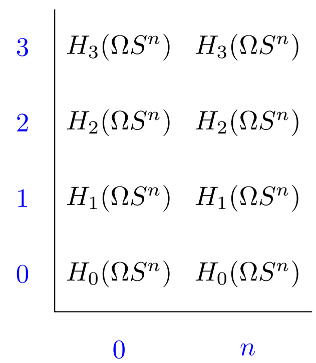
For all p, q, the E^2 page of the spectral sequence has groups

$$E^2_{p,q} = H_p(S^n; H_q(\Omega S^n; \mathbb{Z})) = \begin{cases} H_q(\Omega S^n; \mathbb{Z}) & p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

[6] A first guess might be defining $\tilde{}(y,s)$ to be $\gamma_y * h(y,st)$, but then $\tilde{h}(y,0)$ would not be equal to γ_y. By introducing a changing time interval we fix this problem.

[7] This is because paths between elements of ΩS^1 correspond to homotopies between those elements in S^1.

and (without differentials) looks like



The differentials have domain or codomain zero (and are therefore zero) up until the E^n page, so for all p, q,

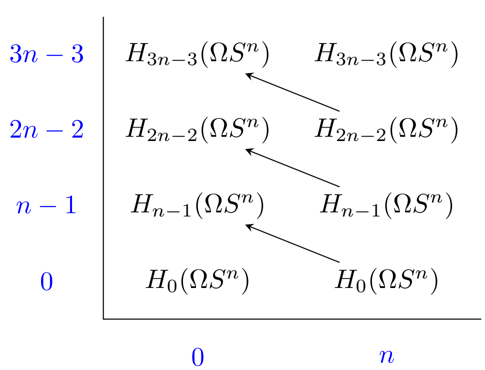
$$E^2_{p,q} = E^3_{p,q} = \cdots = E^n_{p,q}.$$

Differentials are again zero on the E^{n+1} page and zero on all subsequent pages, so for all p, q,

$$E^{n+1}_{p,q} = E^{n+2}_{p,q} = \cdots = E^\infty_{p,q}.$$

The differentials involving $E^r_{0,0}$ are zero even when $r = n$, so we also have that $E^2_{0,0} = E^\infty_{0,0} = \mathbb{Z}$.

With differentials, the E^n page looks like



Note that there are maps from every $E^n_{n,k} = H_k(\Omega S^n)$ to $E^n_{0,k+n-1} = H_{k+n-1}(\Omega S^n)$ and not just the ones pictured.

Since PX is contractible, the E^∞ page looks like

$3n-3$	0	0
$2n-2$	0	0
$n-1$	0	0
0	$\mathbb{Z}$	0
	0	n

Thus, by the same reasoning as that in the proof of Theorem 3.12, the sequences

$$0 \to H_k(\Omega S^n) \to H_{k+n-1}(\Omega S^n) \to 0$$

are exact when $k \neq 1-n$. Therefore, for all $k \neq 1-n$, we have isomorphisms

$$H_k(\Omega S^n) \cong H_{k+n-1}(\Omega S^n).$$

A simple proof by induction shows that $H_k(\Omega S^n) \cong H_j(\Omega S^n)$ whenever $k \equiv j \bmod (n-1)$ and $k, j > 1-n$. Thus, when k is a multiple of $n-1$,

$$H_k(\Omega S^n) \cong H_0(\Omega S^n) = \mathbb{Z},$$

and when $k \equiv m \bmod (n-1)$ for some $m \in [0, n-1)$,

$$H_k(\Omega S^n) \cong H_m(\Omega S^n) \cong H_{m-n+1}(\Omega S^n) \cong 0.$$

Thus,

$$H_k(\Omega S^n) = \begin{cases} \mathbb{Z} & k \mid (n-1) \\ 0 & \text{otherwise.} \end{cases}$$

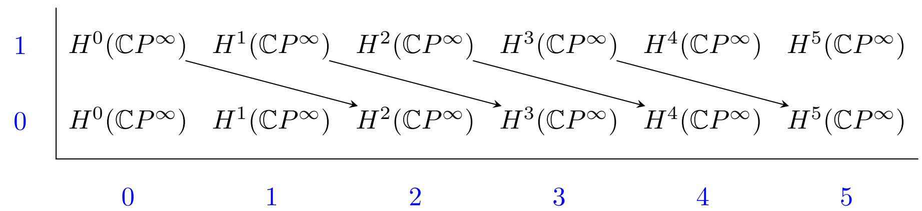
In both of these examples, we used the homological Serre spectral sequence to determine homology groups of spaces. We can do the same with the cohomological Serre spectral sequence and determine cohomology groups, but due to the multiplicative structure present in the spectral sequence, we can go even further and compute the ring structure.

We now demonstrate a method for computing the ring structure of $H^*(\mathbb{C}P^\infty)$. The same can be done for ΩS^n, but the procedure is more complicated and beyond the scope of this paper.

Example 7.5. Recall that $S^1 \to S^\infty \to \mathbb{C}P^\infty$ is a fibration. Let $B = \mathbb{C}P^\infty$, $X = S^\infty$, and $F = S^1$. Because $\mathbb{C}P^\infty$ is simply connected, the condition that $\pi_1(B)$ acts trivially on $H^*(F)$ is satisfied, so we can apply the cohomological Serre spectral sequence to this fibration. Theorem 6.3a tells us that the E_2 page is made of groups

$$E_2^{p,q} = H^p(B; H^q(F; \mathbb{Z})) = \begin{cases} H^p(B; \mathbb{Z}) & q = 0, 1 \\ 0 & q \neq 0, 1 \end{cases}$$

and a differential d_2 of degree $(2, -1)$. Thus, the E_2 page looks like



By Theorem 6.3b, the E_∞ page of this spectral sequence looks like

1	0	0	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

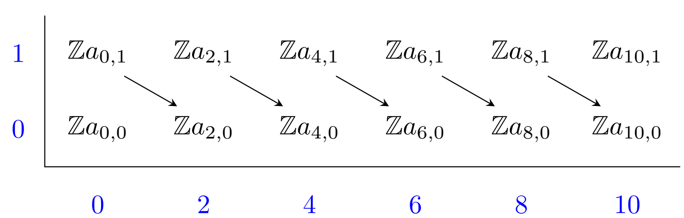
By arguments analogous to those for Lemmas 3.10 and 3.11, $E_2^{0,0} = \mathbb{Z}$ and $E_3^{p,q} = 0$. Thus, for all $k \neq 0$, we have an exact sequence

$$0 \to H^{k-2}(\mathbb{C}P^\infty; \mathbb{Z}) \to H^k(\mathbb{C}P^\infty; \mathbb{Z}) \to 0.$$

It follows likewise that

$$H^k(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & k \text{ is even} \\ 0 & k \text{ is odd.} \end{cases}$$

Thus, the E_2 page looks like

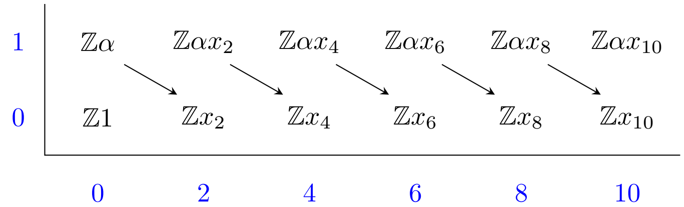


where $\mathbb{Z}a_{p,q}$ denotes that $a_{p,q}$ is a generator of $E_2^{p,q} \cong \mathbb{Z}$.

Let $\alpha = a_{0,1}$. We find that all of the generators in the 1st row are α times the generator from the 0th row, i.e. $a_{p,1} = \alpha \cdot a_{p,0}$. To see this, consider the generator α is really $1 \otimes \alpha$, and a generator $a_{p,0}$ is really $a_{p,0} \otimes 1$. Thus, their product $\alpha \cdot a_{p,0} = (1 \otimes \alpha)(a_{p,0} \otimes 1)$ is a generator of $E_2^{p,1}$.

Because one of the generators of $H^0(F; \mathbb{Z})$ is the multiplicative identity on the cohomology ring $H^*(F; \mathbb{Z})$, without loss of generality let $a_{0,0} = 1$.

Recall that all of the differentials shown are isomorphisms. Therefore, for all $p \geq 0$, we have that $d_2 a_{2p,1}$ is a generator of $E_2^{2p+2,0}$, so without loss of generality we let $a_{2p+2,0} = d_2 a_{2p,1}$. Furthermore, since the product structure is simply multiplication of coefficients, for all $p \geq 0$ we have that $a_{0,1} a_{2p,0}$ is a generator for $E_2^{2p,1}$. Thus, letting $x_i = a_{i,0}$, the E_2 page now looks like



By the derivation property (Theorem 6.4a) of d_2 we have that

$$x_{2i+2} = d_2(\alpha x_{2i}) = d_2(\alpha)x_{2i} \pm \alpha(d_2 x_{2i}) = d_2(\alpha)x_{2i} = x_2 x_{2i}$$

since $d_2(x_{2i}) = 0$. Thus, $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ is $\mathbb{Z}[x_0, x_2, x_4, \dots]$ modulo the relations $x_2 x_{2i} = x_{2i+2}$ for all i. We conclude

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x].$$

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