JORDAN NORMAL FORM

KATAYUN KAMDIN

ABSTRACT. This paper outlines a proof of the Jordan Normal Form Theorem. First we show that a complex, finite dimensional vector space can be decomposed into a direct sum of invariant subspaces. Then, using induction, we show the Jordan Normal Form is represented by several cyclic, nilpotent matrices each plus an eigenvalue times the identity matrix – these are the Jordan Blocks.

CONTENTS

INTRODUCTION

Any matrix over an algebraically closed field (we will work over $\mathbb{C}$ here) is similar to an upper triangular matrix, but is not necessarily similar to a diagonal matrix. The motivation behind Jordan Normal Form is the latter. Since not all matrices can be diagonalized, we demand that any matrix still has a 'nice' form; this form is the Jordan Normal Form. Jordan Normal Form has applications in solving problems in physics, engineering, computer science, and applied mathematics. For example, systems of differential equations that describe anything from population growth to the stock market can be represented by matrices – putting this matrix into Jordan Normal Form simplifies the task of finding solutions to the system. In this paper we will outline a proof of the Jordan Normal Form Theorem.

JORDAN NORMAL FORM

Theorem 0.1. *Jordan Normal Form: Let T be a linear mapping of a finite-dimensional vector space U to itself, and assume either that U is a complex vector space or else that all the eigenvalues to T are real. Then U can be decomposed into a direct sum of subspaces each of which is mapped into itself by T, the action of T in that subspace being described, with respect to a suitable basis, by a cyclic matrix plus an eigenvalue, λ_i, times an identity matrix of the appropriate size; in other words, by a matrix of the form:*

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$\begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix}$ $m_j \times m_j$, *where* m_j *is the multiplicity of eigenvalue* λ_j

First, we must show U can be decomposed into a direct sum of subspaces, i.e. U = $U_1 \bigoplus \cdots \bigoplus U_r$ Let U_j = Ker(T-λ_jI)^{m_j} where T is the matrix associated with T with respect to some basis and m_j is the multiplicity of λ_j. These are the characteristic subspaces of T.

Next, define the polynomial g_j(x) = $\frac{f_T(x)}{(x-\lambda_j)^{m_j}}$ j= (1, $\cdots$, r) where $f_T(x)$ is the characteristic polynomial of the matrix of T. Note that the g_j have no non-constant factors common to all the g_j since we divide out a term corresponding to a different λ_j for each j. Therefore, there exists h_j(x) j = (1, $\cdots$, r) such that $g_1(x)h_1(x) + \cdots + g_r(x)h_r(x) = 1$.

We now define the linear mappings $P_j = g_j(T)h_j(T) = h_j(T)g_j(T)$. Note that this commutes because g and h are polynomials. Similarly, $P_1 + \cdots + P_r = I$. Furthermore, $P_j : U \longrightarrow U_j$ by the following: $(T - \lambda_j I)^{m_j} P_j(x) = (T - \lambda_j I)^{m_j}[g_j(T)h_j(T)](x)$ for $x \in U$. Recall g_j(x) = $\frac{f_T(x)}{(x-\lambda_j)^{m_j}}$; g_j(T) = $\frac{f_T(T)}{(T-\lambda_j I)^{m_j}}$ so $(T - \lambda_j I)^{m_j} P_j(x) = f_T(T)h_j(T)(x)$ but $f_T(T) = 0$ by the Cayley-Hamilton Theorem. Therefore, $P_j(x) \in Ker(T - \lambda_j I)^{m_j} = U_j$.

We can also show that $P_j(u) = \begin{cases} x & \text{if } x \in U_j \\ 0 & \text{if } x \in U_k, j \neq k \end{cases}$

If x $\in U_k, j \neq k$, then $P_j = g_j(T)h_j(T)$ contains $(T - \lambda_k I)^{m_k}$. This factor maps everything in U_k to 0. If $x \in U_j$, then $x = P_1(x) + \cdots P_r(x)$, and by the previous, the only non-zero term on the right is $P_j(x) \Longrightarrow P_j(x) = x$

Recall $U = U_1 \bigoplus \cdots \bigoplus U_r \Longleftrightarrow$ for all $x \in U$ there exists a unique $x_1 \cdots x_r$ such that $x = x_1 + \cdots + x_r$ where $x_j \in U_j$ for $j = 1, \cdots, r$. So let $x_j = P_j(x).x = x_1 + \cdots + x_r$.

To show uniqueness, suppose $x = x_1' + \cdots + x_r'$ for $x_j \in U_j$. Then the difference $x_j - x_j' = y_j \in U_j$. Therefore, $y_1 + \cdots + y_r = 0$. Now applying P_j to both sides we get $y_j = 0$ for each j so $x_j = x_j'$. Hence the first assertion.

The next assertion, T maps $U_j \longrightarrow U_j$ is easily shown. Clearly, T commutes with $(T - \lambda_j I)^{m_j}$. Let $x \in U_j$. $(T - \lambda_j I)^{m_j} T(x) = T(T - \lambda_j I)^{m_j}(x) = 0$ So for $x \in U_j T(x) \in Ker(T - \lambda_j I)^{m_j} = U_j$

The last assertion of Jordan Normal Form theorem, that the restriction of T to U_j, call it T_j, has a matrix with respect to a suitable basis, which in fact does not depend of the choice of basis for U_j, and that can be represented by $T_j = D_j + N_j$ where D_j is a diagonal matrix with λ_j on the diagonal and N_j is a nilpotent matrix. We need to show that on any U_j, the Jordan normal form of any nilpotent mapping can be represented by a matrix, or sum of matrices, of the following form in a suitable basis $\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

Definition 0.2. Given a linear transformation T from V to itself, we call $(v_0, v_1, \ldots, v_d)$ a d-string if $T(v_i) = v_{i-1}$ for all $0 < i \leq d$.

From this point on we assume that we have fixed a transformation and any d-string is relative to this fixed transformation.

Definition 0.3. If $(v_0, \ldots, v_d)$ is a d-string, we will say that it has length l if there are precisely l non-zero vectors in the string.

Lemma 0.4. *If $s = (v_0, \ldots, v_d)$ is a string for T of length $l < d+1$ then $T^l(v_i) = 0$ for all $1 \leq i \leq d$.*

Proof. This lemma follows directly from the definition of string and length. □

Proposition 0.5. *If $N^d = 0$ then there is a collection of M d-strings, each of which has its first vector 0, such that the non-zero vectors in the strings form a basis of V.*

We prove this proposition by induction on d. Base Case $d = 1$. Here we have $N = 0$ So let $v_1, \ldots, v_n$ be a basis of V. The 1-strings $(0, v_i)$ satisfy the proposition. We will assume the proposition is true for $d-1$ and produce the result for d, i.e., assume $N^d = 0$ and $N^k \neq 0$ for all $k < d$. Let $W = \ker(N^{d-1})$. Then by the induction hypothesis, there exists $\tilde{M}$ $(d-1)$-strings such that the non-zero vectors in these strings are a basis for W. Let us assume that M of these strings $\{s_1, \ldots, s_M\}$ have length $d-1$. We define the set B_k to be the collection of non-zero vectors contained in any of the M strings $\{s_1, \ldots, s_M\}$ that are in the k-th position. For example, if we had two strings $s_1 = (0, v_1, v_2)$ and $s_2 = (0, 0, w_2)$ then $B_1 = \{v_1\}$ and $B_2 = \{v_2, w_2\}$. Let us also denote by B the basis of W appearing as the non-zero vectors in all $\tilde{M}$ strings. We now prove a lemma.

Lemma 0.6. *If $v \notin W$ then $N(v) \notin$ Span$(B - B_{d-1})$.*

Proof. Note that for every vector w in $B - B_{d-1}$ we have that $N^{d-2}(w) = 0$. Indeed, either w is in one of the M strings $\{s_1, \ldots, s_M\}$ in the k-th position where $k < d-1$ implying $N^{d-2}(w) = 0$ (since $v_0 = 0$ for each string), or w is in a string whose length is less than $d-1$, implying again that $N^{d-2}(w) = 0$ by the previous lemma. Thus if $N(v)$ were in the span of $B - B_{d-1}$ then $N^{d-2}(N(v)) = 0$ implying that $v \in \ker(N^{d-1}) = W$. This is a contradiction and proves the lemma. □

Let us now extend the basis B of W to a basis of V by adding the vectors $\{u_1, \ldots, u_m\}$

Corollary 0.7. $m \leq M$.

To see that this must be true, we observe that N maps the span of $\{u_1, \ldots, u_m\}$ to W. Composing N with the projection to the span of B_{d-1} we have a map from an m-dimensional space to an M-dimensional space. If this map has a non-trivial kernel, then there is a $v \notin W$ (i.e. in the span of $\{u_1, \ldots, u_m\}$ such that $N(v)$ is in the span of $B - B_{d-1}$ contradicting the lemma. Thus this map has a trivial kernel implying that $m \leq M$.

Lemma 0.8. *Suppose $A = (a_{ij})$ is an invertible $r \times r$-matrix and $\{t_1, \ldots, t_r\}$ are strings of equal length l whose non-zero entries are linearly independent. Let $\tilde{t}_i = \sum_{j=1}^{r} a_{ij} t_j$ where we simply sum each k-th entry when we add strings. Then $\{\tilde{t}_1, \ldots, \tilde{t}_r\}$ are strings of equal length whose non-zero entries are linearly independent.*

The proof of this lemma is clear. A maps the span of the non-zero vectors in the strings (of dimension $r \cdot l$) to itself and A^{-1} is an inverse map, implying the spaces have the same dimension. Multiplying by matrices is not the only operation we can perform on strings with linearly independent entries. Given a string $s = (0, v_1, \ldots, v_l)$ we can shift k terms to the right to obtain $Sh_k(s) = (0, \ldots, 0, v_1, \ldots, v_{l-k})$.

Lemma 0.9. *Suppose s is a string of length l and $\{t_1, \ldots, t_p\}$ are strings whose length are less than or equal to l and all of the non-zero entries in the strings are linearly independent. If T is a linear combination of the t_i along with any shifts $Sh_k(t_i)$ or $Sh_k(s)$ (so long as $k > 0$), then $\{s + T, t_1, \ldots, t_p\}$ is a set of strings whose non-zero entries are linearly independent.*

Proof. One only needs to see that the span of the non-zero entries of $\{s, t_1, \ldots, t_p\}$ is equal to that of the span of the non-zero entries of $\{T + s, t_1, \ldots, t_p\}$. But this is clear. Any non-zero vector in $T + s$ is a linear combination of one vector in s and those appearing in $\{t_1, \ldots, t_p\}$. Subtracting off this combination implies that each vector in s is contained in the span of the non-zero entries of $\{T + s, t_1, \ldots, t_p\}$. Reversing this argument we see that the spans are equal. As the length of s is maximal, we see that the number of non-zero vectors in each of these collections is equal, and since the first is assumed to be linearly independent, a dimension count shows the second must be as well. □

These last two lemmas are the basic ingredients in the proof of our proposition, which we now complete.

Proof. Recall that $\{u_1, \ldots, u_m\}$ was an extension of our basis of W to one for V. Let us denote by U the span of these vectors. By the proof of the corollary, we saw that $\pi_{d-1} \circ N : U \to \operatorname{Span}(B_{d-1})$ was injective, where π_{d-1} is the projection in W to the span of B_{d-1}. Since this is an injective map, we have $\{\pi_{d-1} \circ N(u_1), \ldots, \pi_{d-1} \circ N(u_m)\}$ are linearly independent vectors in the span of B_{d-1}. Recall that B_{d-1} is the collection of the last vector entries in the M strings $\{s_1, \ldots, s_M\}$. Now we extend the collection $\{\pi_{d-1} \circ N(u_1), \ldots, \pi_{d-1} \circ N(u_m), w_1, \ldots, w_{M-m}\}$ to a basis of the span of B_{d-1} and examine the change of basis matrix $A = (a_{ij})$ from B_{d-1} to this basis. By Lemma 0.7, we obtain a new collection of strings $\{\tilde{s}_1, \ldots, \tilde{s}_M\}$ where $\tilde{s}_i = \sum_{j=1}^{M} a_{ij} s_j$, and whose non-zero vectors span the same space as those of $\{s_1, \ldots, s_M\}$ and are linearly independent. By the proof of Lemma 0.7, we also see that the span of each B_k is equal to the span of the new $\tilde{B}_k$ (here the $\tilde{B}_k$ is defined as before with these new strings). Now, as this process does not affect any of the strings whose length is less than $d - 1$, we have a new collection, $\{\tilde{s}_1, \ldots, \tilde{s}_M, t_1, \ldots, t_{\tilde{M}-M}\}$ which satisfies our proposition for W. Furthermore, $\pi_{d-1} \circ N(u_i) = w_i$ where w_i is the last entry in $\tilde{s}_i$.

Let $\tilde{B}$ be the collection of non-zero vectors in these strings. We see that $W = \operatorname{Span}(\tilde{B}_{d-1}) \oplus \operatorname{Span}(\tilde{B} - \tilde{B}_{d-1})$. In this direct sum we have $N(u_i) = w_i + v_i$ where v_i is a linear combination of elements in $\tilde{B} - \tilde{B}_{d-1}$. As such, we can view v_i as the last vector in some string which is a linear combination of the strings $\{t_1, \ldots, t_{\tilde{M}-M}\}$ along with their shifts as well as positive shifts of the strings $\{\tilde{s}_1, \ldots, \tilde{s}_M\}$. This follows since, by shifting enough, we can place any vector in $\tilde{B} - \tilde{B}_{d-1}$ in the $d - 1$ position of a string. Multiplying by constants and adding the resulting string gives a string T_i such that v_i is in the $d-1$-st position. By repeated application of Lemma

0.8, we see that the collection $\{\tilde{s}_1 + T_1, \ldots, \tilde{s}_m + T_m, \tilde{s}_{m+1}, \ldots, \tilde{s}_M, t_1, \ldots, t_{\tilde{M}-M}\}$ spans W and the non-zero vectors in these strings form a basis of W. Finally, by construction we see that $N(u_i) = w_i + v_i$ is the $d-1$-st vector in the i-th string $s_i + T_i$.

After having done the difficult work, we now complete the induction step. For each of the $d-1$-strings, $\{\tilde{s}_1 + T_1, \ldots, \tilde{s}_m + T_m\}$ we will create a d-string S_i by placing u_i in the d-th position of $\tilde{s}_i + T_i$. By our construction we see that this is indeed a string since $\tilde{s}_i + T_i$ is a string and $N(u_i)$ is the $d-1$-st vector in this string. For all other $d-1$-strings $\{\tilde{s}_{m+1}, \ldots, \tilde{s}_M, t_1, \ldots, t_{\tilde{M}-M}\}$ in our collection, we shift them up by one and add a zero in the zeroth position to make them d-strings $\{S_{m+1}, \ldots, S_{\tilde{M}}\}$. I.e., if $d = 3$ and $(0, v_1, v_2)$ is one of our strings, then we replace it with $(0, 0, v_1, v_2)$. Clearly, each of these is still a d-string relative to N. Now, since the non-zero vectors of $\{\tilde{s}_1 + T_1, \ldots, \tilde{s}_m + T_m, \tilde{s}_{m+1}, \ldots, \tilde{s}_M, t_1, \ldots, t_{\tilde{M}-M}\}$ formed a basis for W and the non-zero vectors that were added to form the d-strings $\{S_1, \ldots, S_{\tilde{M}}\}$ were just the extended basis elements $\{u_1, \ldots, u_m\}$, we see that the non-zero elements of these strings form a basis for V and the induction step is complete. □

To connect our proposition to Jordan Normal form is easy. We see that for each string S_i, the non-zero vectors form a basis for a subspace V_i and $V = V_1 \oplus \cdots \oplus V_{\tilde{M}}$. Putting the transformation N into matrix form for each V_i using the basis from S_i we see that N is a direct sum of matrices of the form:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & & \cdots & & 0 \end{bmatrix}$$

Thus if we let $V = U_j$ where U_j is the characteristic eigenspace of λ_j, $d = m_j$ and $N = T - \lambda_j I$ then we see that $T = \lambda_j I + N$ and the matrix form for T on each direct summand of U_j (corresponding to the V_i's in our decomposition of V) is just:

$$\begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ \vdots & \lambda_j & 1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \lambda_j & 1 \\ 0 & & \cdots & & \lambda_j \end{bmatrix}$$

This is precisely the form we wanted for Jordan Normal Form, and thus our proposition yields the result.

Conclusion

The Jordan Normal Form is a powerful tool because it assures us that any matrix over an algebraically closed field has a "nice" form, through which we can find the solutions to complicated equations. For the same reason, Jordan Normal form is also an interesting mathematical object.

References

[1] Mostow and Sampson. Linear Algebra. McGraw Hill. 1969.
[2] Special thanks to Gabriel Kerr.