# Fractal Geometry: The Mandelbrot and Julia Sets

Stephanie Avalos-Bock

July, 2009

## 1 Introduction

The Mandelbrot set is a set of values $c \in \mathbb{C}$ with certain important properties. We will examine the formal definition of the set as well as many of its interesting, strange, and beautiful properties. The Mandelbrot set is most well known outside of mathematics as a set of beautiful images of fractals; this is partially thanks to the work of Heinz-Otto Peitgen and Peter Richter. The Mandelbrot set is relevant to the fields of complex dynamics and chaos theory, as well as the study of fractals. Here we will delve into the mathematics behind the phenomenon of the Mandelbrot set.

## 2 Formal Definitions

First we will give some definitions of the Mandelbrot set as a whole, and then of some related concepts and characteristics.

**Definition 2.1.** First we must define a set of complex quadratic polynomials $P_c : z \rightarrow z^2 + c$. *The Mandelbrot set* $M$ is the set of all $c$-values such that the sequence $0, P_c(0), P_c(P_c(0))$,... does not escape to infinity, $z, c \in \mathbb{C}$.

### 2.1 Alternate definitions of the Mandelbrot set

**Definition 2.2.** *The Mandelbrot set* is the set $M = \{c \in \mathbb{C} | \exists s \in \mathbb{R}, \forall n \in \mathbb{N}, |P_c^n(0)| \leq s\}$ where $P_c^n(z)$ is the $n$th iterate of $P_c(z)$. (Note that the same result is achieved by replacing $s$ with 2 because the Mandelbrot set falls entirely within a circle of radius 2 and centered at the origin on the complex plane.)

**Definition 2.3.** For a given Julia set (see sect), *the Mandelbrot set* is the subset of the complex plane such that the Julia set of $P_c(z)$ is connected.

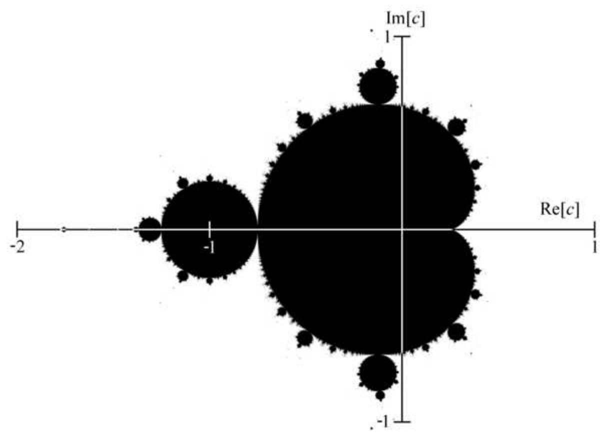


Figure 1: *The Mandelbrot Set.* Black points are in the Mandelbrot set; white points are not. Re[$c$] and Im[$c$] denote the real and imaginary parts of $c$.

## 2.2 Other Related Definitions

**Definition 2.4.** The *orbit* is the sequence $z_0, z_1, z_2$, ... where $z_{n+1} = P_c(z_n)$. The critical orbit is the orbit with $z = 0$. The fate of the critical orbit (whether it escapes to infinity or not) determines whether $c$ is in the Mandelbrot set.

**Definition 2.5.** The *seed* is the starting point for the orbit: $z_0$.

**Definition 2.6.** *The Escape Criterion.* Suppose $|c| \leq 2$. If the orbit of 0 under $z^2 + c$ ever lands outside of the circle of radius 2 centered at the origin, then this orbit escapes to infinity.

The Escape Criterion is used to determine whether a given $c$ is in the Mandelbrot set or not. The most basic picture of the Mandelbrot set, shown in Figure 1, has points black if that point $\in M$ and white if that point $\notin M$.

**Example 2.7.** Let $c = 0$ and $z_0 = 0$. Then,
$z_1 = 0^2 + 0 = 0$
$z_2 = 0^2 + 0 = 0$
$z_3 = 0^2 + 0 = 0$
...

Since this orbit *remains fixed for all iterations*, $c = 0$ is in the Mandelbrot set.

**Example 2.8.** Let $c = 1$ and $z_0 = 0$. Then,
$z_1 = 0^2 + 1 = 1$
$z_2 = 1^2 + 1 = 2$
$z_3 = 2^2 + 1 = 5$

$z_4 = 5^2 + 1 = 26$
$z_5 = 26^2 + 1 = 677$
$z_5 = 26^2 + 1 = 456,977$
...

Since this orbit *escapes to infinity*, $c = 1$ is not in the Mandelbrot set.

**Example 2.9.** Let $c = -1$ for seed 0. Then,
$z_1 = 0^2 - 1 = -1$
$z_2 = -1^2 - 1 = 0$
$z_3 = 0^2 - 1 = -1$
$z_4 = -1^2 - 1 = 0$
$z_5 = 0^2 - 1 = -1$
...

We say that this orbit is a *cycle with period 2*.

# 3 Main Properties

One of the most salient features of the Mandelbrot set is that its boundary forms a fractal and its inside a main cardioid with several bulbs attached. The main cardioid and the bulbs attached to it have certain invariable characteristics.

**Definition 3.1.** *The main cardioid* is the subset of the complex plane such that $c = \frac{1-(\mu-1)^2}{4}$ for some $\mu$ in the unit disk ($\mu \leq 1$).

The *bulb* at $c = -3/4$ is a circle with radius 1/4 around -1. The bulb is the subset of the complex plane such that $P_c$ has a cycle of period 2. There are also bulbs such that $c_{\frac{p}{q}} = \frac{1}{4}(1 - \left(e^{2\pi i \frac{p}{q}} - 1\right)^2)$, for every rational number $\frac{p}{q}$ with $p, q$ co-prime. (This number $\frac{p}{q}$ is known as the *rotation number.)* This means that there are infinitely many bulbs attached to the main cardioid; each bulb is called the $\frac{p}{q}$ bulb of the Mandelbrot set. Each bulb or decoration has an infinite number of antennae on it; each antenna in turn has an infinite number of spokes.

## 3.1 Determining the Period of a Bulb

The main antenna is the antenna connecting the decoration or hub of spokes to the nearest large bulb. The number of spokes on this antenna (including the main antenna) gives the period of the bulb. The period is always a value $n \in \mathbb{N}$ and is the period of the attracting cycle (orbit) for all values $c$ in that bulb. By counting the number of spokes, we can determine the period of a bulb (meaning, the period of all $c$ inside that bulb), just by looking at the picture. Examples for practice are given in Figures 2 and 3.

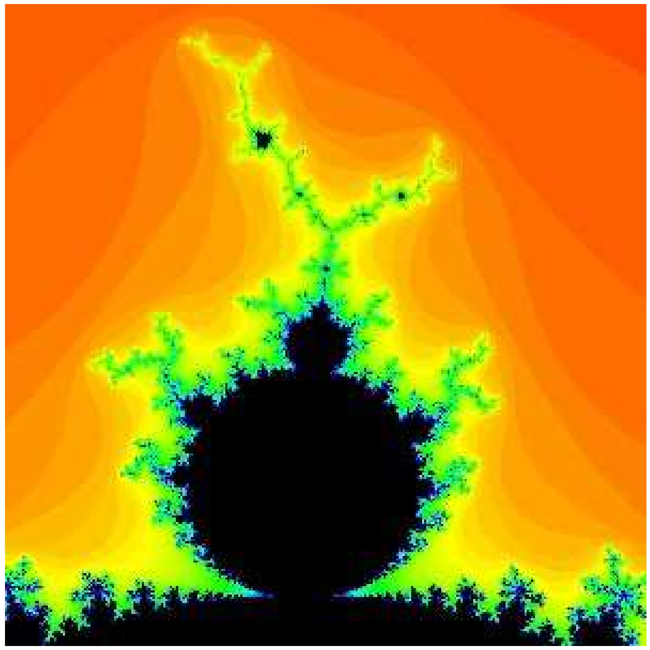

Figure 2: *A Period 3 Bulb.* The main antenna is that stretching straight up off of the circle bulb. It is connected to two other antennae, giving a period of 3.

## 3.2 Determining the Rotation Number of a Bulb

As stated above, each bulb has a characteristic number $\frac{p}{q}$. The denominator is simply the period of the bulb (the number of spokes). There are many methods for determining $p$. Only one will be given at this time because the remainder require Julia sets, which have not yet been introduced.

Method 1: Determine the period as described above. Counting spokes in the counterclockwise direction, the shortest spoke will be p spokes away from the main antenna/spoke. See Figure 4 for an example.

## 3.3 Properties of Rotation Numbers and Hidden Sequences

There are several amazing features of the Mandelbrot set in relation to the rotation numbers of bulbs. First, we must introduce Farey addition. This is a special operation that allows us to add fractions the easy way, by adding numerator to numerator and denominator to denominator, without the work of first finding a common denominator. The two fractions being added are called the Farey parents and their sum is known as the Farey child. Using Farey addition, we can determine the rotation number of the largest bulb between two given bulbs:

**Theorem 3.2.** *By performing Farey addition on all of the rotation numbers*

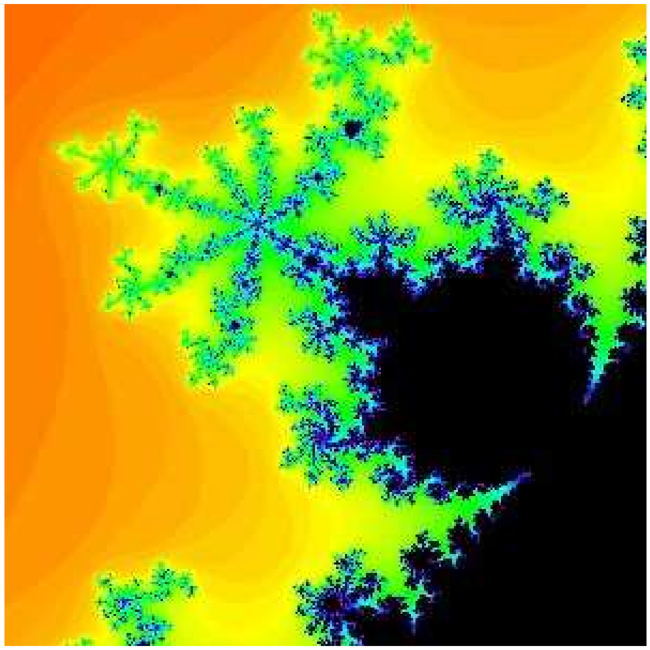

Figure 3: *A Period 7 Bulb.* The main antenna is that stretching northwest from the circle bulb. It is connected to six other antennae, giving a period of 7.

*of the primary bulbs, all the rational numbers can be constructed. That is,* $\mathbb{Q} = \sum_{n=1}^{\infty} \frac{p_i}{q_i}$ *where* $p_i/q_i$ *is a rotation number of a primary bulb.*

*Proof.* Use induction or see [4] and [5]. □

Another amazing fact is the presence of the Fibonacci sequence in the Mandelbrot set. It is found in a similar way as in the previous characteristic, but using period instead of rotation number. For any two bulbs, the sum of their period is the period of the largest bulb between them. By taking bulbs closer and closer to each other, the Fibonacci sequence is generated.

Finally, by traveling around the main cardioid and taking the period of the next largest bulb, the sequence of natural numbers is generated (see Figure 7).

# 4 Filled Julia Sets

**Definition 4.1.** The *filled Julia set*, $J_c = \{n \in \mathbb{N} | \exists s \in \mathbb{R}, \forall c \in \mathbb{C}, |P_c^n(0)| \leq s\}$ where $P_c^n(z)$ is the $n$th iterate of $P_c(z)$. That is, instead of taking the orbit of 0 for each $c$, here we choose one $c$ and take every orbit. Bounded orbits, those that do not escape to infinity, are inside the filled Julia set.

**Definition 4.2.** A *Julia set* is the set of points that constitutes the border of a filled Julia set.

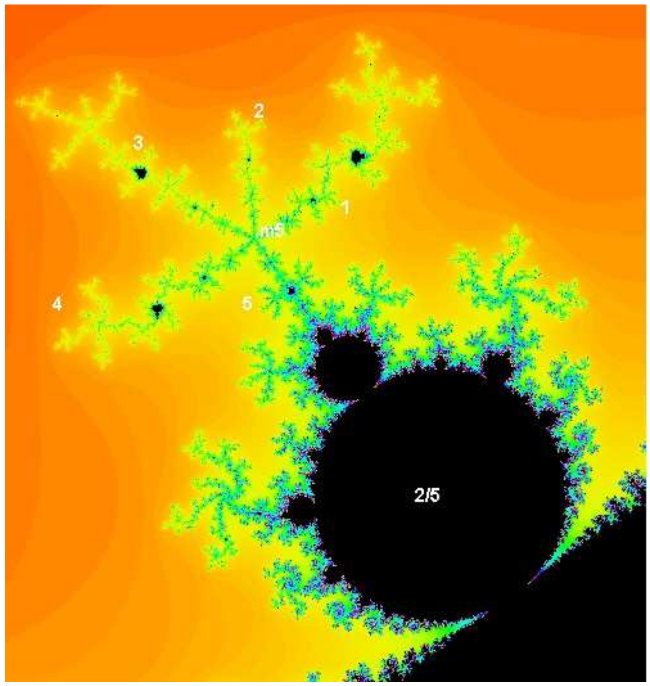


Figure 4: *A 2/5 Bulb.* The period of the bulb is 5 because there are 5 spokes, including the main antenna. The shortest spoke is located at position 2 from the main antenna, so the rotation number is 2/5.

## Summary of the Difference between Mandelbrot and Julia Sets:

The Mandelbrot set:

- is a picture in parameter space
- records the fate of the orbit of 0 for all $c$

The filled Julia set:

- is a picture in dynamical plane
- records the fate of all orbits for a given $c$

Filled Julia sets are intimately related to Mandelbrot sets and are useful for determining the peroid and rotation number of a bulb. Here we have some other methods:

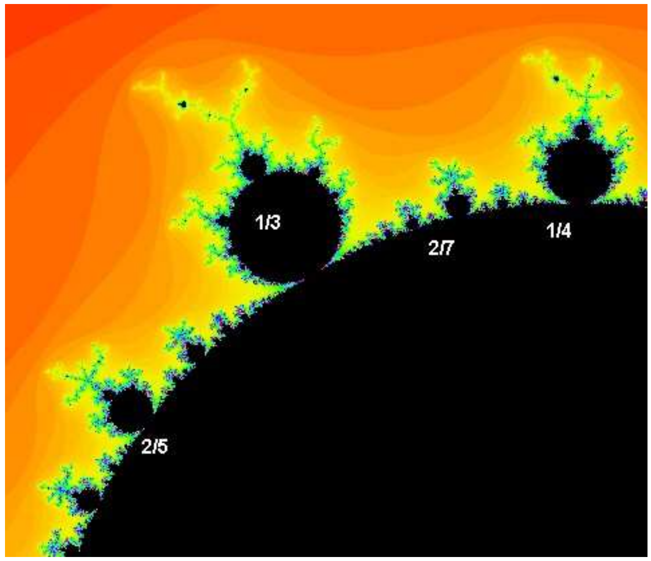


Figure 5: *Farey Addition on Rotation Numbers of a Primary Bulb.*The largest bulb between any two other bulbs has a rotation number equal to the Farey sum of the rotation numbers of the chosen neighbor bulbs.

## 4.1 Determining the Period of a Bulb (with filled Julia sets)

Now we can introduce an alternative method of finding the period of a bulb, using the corresponding filled Julia set.

Method 2: Choose any $c$ in a primary bulb and compute $J_c$. Since removing any point changes the filled Julia set from connected to disconnected, removing this chosen point $c$ separates $J_c$ into $n$ pieces. This number $n$ is the period of the bulb (See Figure 8).

## 4.2 Determining the Rotation Number of a Bulb (with filled Julia sets)

We can also use filled Julia sets to help us read off the rotation number of a primary bulb. Both of these methods start the same way: Choose any $c$ in a primary bulb and compute $J_c$. The number of pieces to the cycle, as described in the above section, is the period, $q$.

Method 2: Find the region of the filled Julia set that contains 0 (this will be the biggest region). Count in a counterclockwise direction to the smallest region. This number, essentially the distance between the largest and the smallest region of the filled Julia set, is $p$.

Method 3: Compute the orbit for $J_c$. The number of regions the cycle "hops"

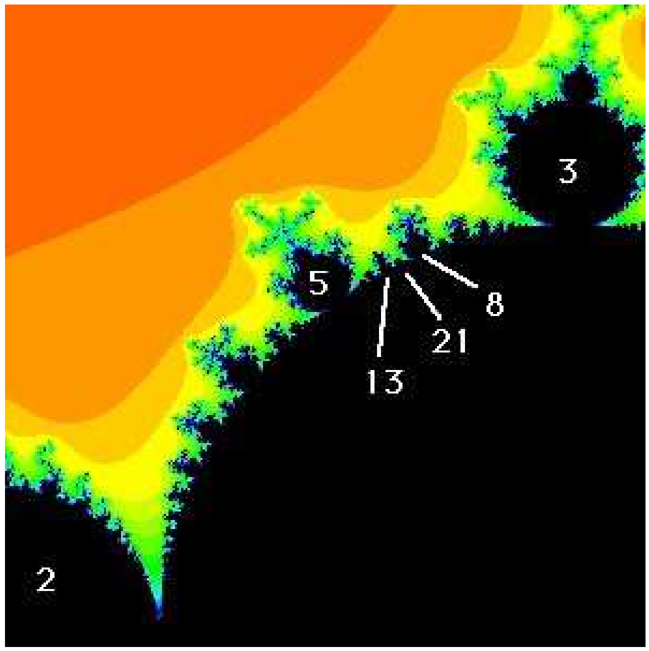


Figure 6: The Fibonacci Sequence in $M$.

around each iteration is $p$.

# 5 Two Theorems for filled Julia sets

## 5.1 The Fundamental Dichotomy

**Theorem 5.1.** *For each c, the filled Julia set is either a connected set or a Cantor set. More precisely, if the orbit of 0 escapes to infinity, that is, if* $c \in M$*,* $J_c$ *is a Cantor set. If the orbit does not escape to infinity, that is,* $c \notin M$*,* $J_c$*, is connected.*

*Proof.* The proof is given in [6]. □

## 5.2 The Unit Disk

**Theorem 5.2.** $J_0$ *is the closed unit disk centered at the origin in the plane.*

*Proof.* First, suppose that $z_0$ is a seed. Using polar coordinates, $z_0 = r^{i\theta}$ where $\theta$ is the polar angle and $r$ is the magnitude of $z_0$. Then, the orbit of $z_0$ under $x^2$ is

$z_0 = re^{i\theta}$
$z_1 = r^2 e^{i2\theta}$
$z_2 = r^4 e^{i4\theta}$

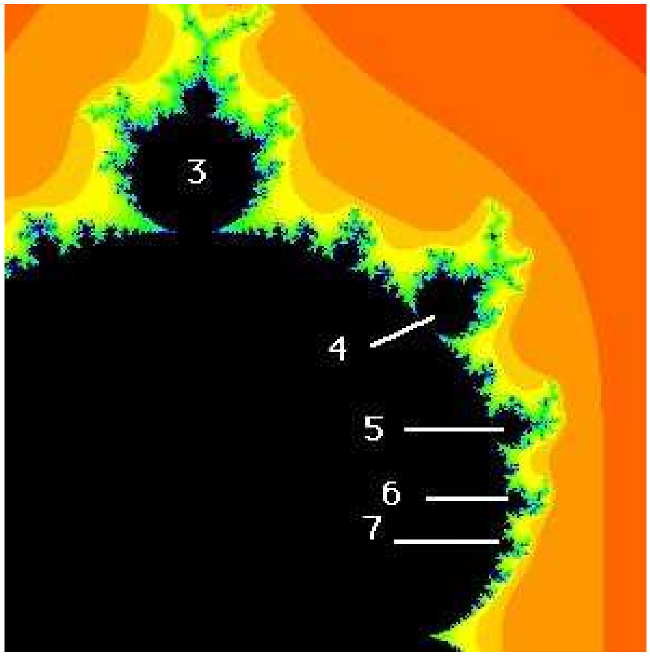


Figure 7: The Natural Numbers in $M$.

$z_3 = r^8 e^{i8\theta}$
...
$z_n = r^{2^n} e^{i2^n\theta}$

Then, the orbit of $x_0$ tends to infinity if $r > 1$ and to 0 if $r < 1$. If $r = 1$, the orbit of the seed remains trapped for all iterations on the unit circle. It follows that any seed on or inside the circle of radius 1 centered at the origin has an orbit that does not escape to infinity. So, $J_0$ consists of all those seeds whose orbits lie on or inside the unit circle centered at the origin. $\square$

## Acknowledgments

Many thanks to my mentors Ian Biringer and Hyomin Choi for their patient explanations and assistance on this paper.

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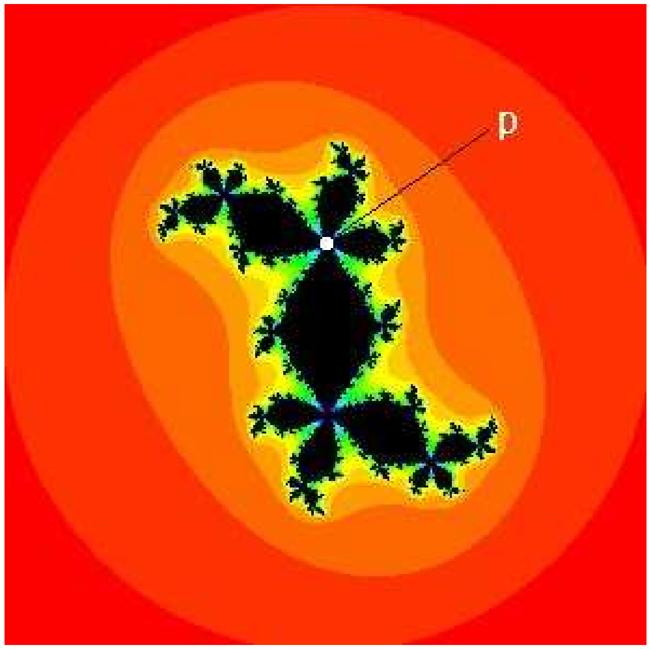


Figure 8: *A disconnected filled Julia set.* Removing a point $p$ from the filled Julia set corresponding to a point $c$ in a primary bulb separates the Julia set into $n$ pieces, where $n$ is the period of that primary bulb. This bulb has period of 4 because there are 4 regions. We begin counting at the biggest region and count counterclockwise. The smallest region is second, so $p = 2$ and the rotation number is 2/4.


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