EXISTENCE AND UNIQUENESS OF HAAR MEASURE

JONATHAN GLEASON

ABSTRACT. In this paper, we prove existence and uniqueness of left and right Haar measures on a locally compact topological group, and show how one can relate left and right Haar measure.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to prove existence and uniqueness of Haar measure on locally compact groups. The paper is divided into four sections, including this introduction.

The purpose of the second section is merely to present some basic definitions and notation that will be used throughout the paper. Also included is brief motivation and justification for some of the definitions. The reader may skip this section if they prefer, and return to it later if needed. The reader should note, however (if they decide to skip this section), that a locally compact topological group is, as a topological space, both locally compact *and* T_1 (and hence $T_{3^{1/2}}$).

The third section contains statements and proofs of four lemmas. They have been placed in a separate section because they are not immediately related to the subject matter of the paper: topological groups and Haar measure. They are, however, needed to complete the existence and uniqueness proofs, as well as the theorem relating left and right Haar measures. They are placed in the order in which they are used in Section 4. The reader may also skip this section, and return to it later if interested in the details of a proof. No exposition is included in this section at all.

The fourth section comprises the main body of the paper, and includes the existence and uniqueness proofs, as well as a proposition relating left and right Haar measures. In order to motivate only focusing on left Haar measure, we first show that given a left Haar measure, one immediately obtains a right Haar measure, and vice versa. We then provide a proof of the existence of left Haar measure on a locally compact topological group. Then, after a couple of lemmas, we prove uniqueness

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of left Haar measure on a locally compact topological group. We note here that, by uniqueness, we mean that any two Haar measures on a locally compact topological group are not *exactly* the same, but in fact only differ by a positive multiplicative constant. We then briefly note how the relation between left and right Haar measure immediately also implies existence and uniqueness of right Haar measure.

2. Basic Definitions

We first introduce some basic definitions and notation.

Notation 2.1. Throughout this paper, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Notation 2.2. If X is a set, we shall denote the power set of X by 2^X.

Notation 2.3. Let X be a set and let $\Sigma \subseteq 2^X$. Then, we shall write $\sigma[\Sigma]$ to denote the σ-algebra generated by Σ.

Notation 2.4. If X is a topological case and $A \subseteq X$, then we shall denote the interior of A by A^o and the closure of A by $\overline{A}$.

Definition 2.5 (Borel Subset)**.** Let X be a topological space with topology τ and let $A \subseteq X$. Then, A is a *Borel subset* of X iff $A \in \sigma[\tau]$.

Definition 2.6 (Topological Measure Space)**.** A *topological measure space* is a measure space (X, Σ, μ), where X is the space, Σ is the σ-algebra of measurable subsets, and μ is the measure, such that X is a topological space and Σ is exactly the collection of Borel subsets of X.

Definition 2.7 (Borel Measure)**.** A measure μ on a topological measure space X is called a *Borel measure* iff X is Hausdorff.

The reason we add the extra condition of T_2 instead of doing things in complete generality, is that, first of all, most spaces we care about in practice are going to be Hausdorff anyways, and furthermore, we would like to know that compact subsets are measurable (because in Hausdorff spaces compact subsets are closed), and in general this won't necessarily be the case.

Definition 2.8 (Regular Measure)**.** Let (X, Σ, μ) be a Borel measure space. Then, μ is said to be *regular*, or sometimes a *regular Borel measure*, iff

(1) Whenever $K \subseteq X$ is compact, then $\mu(K) < \infty$.
(2) Whenever $A \in \Sigma$, then[1]

$$\mu(A) = \inf \{\mu(U) | A \subseteq U, U \text{ is open.}\}.$$

(3) Whenever $U \subseteq X$ is open, then[2]

$$\mu(U) = \sup \{\mu(K) | K \subseteq U, K \text{ is compact.}\}.$$

Definition 2.9 (Locally Compact Group)**.** A *locally compact group* is a topological group G that is locally compact and T_1.

[1] This is sometimes referred to as *outer regularity*.
[2] This is sometimes referred to as *inner regularity*.

Here, we add in the extra condition of T_1 because, one, it is a very weak assumption, two, it is an assumption needed to prove the desired result (see Lemma 3.3 for example), and three, by assuming just the fact that single points sets are closed, we get $T_{3^1/_2}$ for free [3], although not necessarily T_4 (for example, an uncountable product of $\mathbb{R}$)[3].

Definition 2.10 (Haar Measure)**.** Let G be a topological group. A *left Haar measure* (resp. *right Haar measure*) on G is a nonzero regular Borel measure μ on G such that $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) for all $g \in G$ and all measurable subsets A of G.

3. Preliminary Results

Lemma 3.1. *Let $f : X \to Y$ and let $E \subseteq 2^Y$. Then, $\sigma\left[f^{-1}(E)\right] = f^{-1}(\sigma[E])$.*

Proof. Step 1: Show that $\sigma\left[f^{-1}(E)\right] \subseteq f^{-1}(\sigma[E])$

We shall show that $f^{-1}(\sigma[E])$ is a σ-algebra containing $f^{-1}(E)$. Let $A \in f^{-1}(E)$. Then, there is some $B \in E$ such that $A = f^{-1}(B)$. Trivially, $B \in E$, so $B \in \sigma[E]$, so $A \in f^{-1}(\sigma[E])$, so $f^{-1}(E)$ is contained in $f^{-1}(\sigma[E])$. Now, we wish to show that $f^{-1}(\sigma[E])$ is a sigma algebra, so let $\{A_n | n \in \mathbb{N}\} \subseteq f^{-1}(\sigma[E])$. Then, for each A_n, there is some $B_n \in \sigma[E]$ such that $A_n = f^{-1}(B_n)$. Now $B = \bigcup_{n\in\mathbb{N}} B_n \in \sigma[E]$, so $f^{-1}(B) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) = \bigcup_{n\in\mathbb{N}} A_n \in f^{-1}(\sigma[E])$. Thus, $f^{-1}(\sigma[E])$ is closed under countable union. Similarly, $B_0^C \in \sigma[E]$, so $f^{-1}(B_0^C) = f^{-1}(B_0)^C = A_0^C \in f^{-1}(\sigma[E])$, so $f^{-1}(\sigma[E])$ is closed under complementation. $Y \in \sigma[E]$, so $f^{-1}(Y) = X \in f^{-1}(\sigma[E])$. Thus, $f^{-1}(\sigma[E])$ is a σ-algebra containing $f^{-1}(E)$, so $\sigma\left[f^{-1}(E)\right] \subseteq f^{-1}(\sigma[E])$.

Step 2: Show that $f^{-1}(\sigma[E]) \subseteq \sigma\left[f^{-1}(E)\right]$.

First, define $\Sigma = \left\{A \subseteq Y | f^{-1}(A) \in \sigma\left[f^{-1}(E)\right]\right\}$. We wish to show that Σ is a σ-algebra containing E. Then, we will have shown that $\sigma[E] \subseteq \Sigma$. Let $A \in E$. Of course, $f^{-1}(A) \in f^{-1}(E)$, so trivially $f^{-1}(A) \in \sigma\left[f^{-1}(E)\right]$, and hence $E \subseteq \Sigma$. Now, let $\{A_n | n \in \mathbb{N}\} \subseteq \Sigma$. Then, $f^{-1}(A_n) = B_n$ for some $B_n \in \sigma\left[f^{-1}(E)\right]$. But then, $f^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) = \bigcup_{n\in\mathbb{N}} B_n \in \sigma\left[f^{-1}(E)\right]$, so Σ is closed under countable union. Now, $f^{-1}(A_0^C) = f^{-1}(A_0)^C = B_0^C \in \sigma\left[f^{-1}(E)\right]$, so Σ is closed under complementation. And of course, $f^{-1}(Y) = X \in \sigma\left[f^{-1}(E)\right]$, so $Y \in \Sigma$. Thus, Σ is a σ-algebra containing E, and so $\sigma[E] \subseteq \Sigma$. Now, let $A \in f^{-1}(\sigma[E])$. Then, $A = f^{-1}(B)$ for some $B \in \sigma[E]$. But then, $B \in \Sigma$, so $f^{-1}(B) = A \in \sigma\left[f^{-1}(E)\right]$. Thus, $f^{-1}(\sigma[E]) \subseteq \sigma\left[f^{-1}(E)\right]$, and hence $\sigma\left[f^{-1}(E)\right] = f^{-1}(\sigma[E])$. $\square$

Lemma 3.2. *Let (X, Σ, μ) be a topological measure space and let $f : X \to X$ be a homeomorphism. Then, the following are equivalent:*

(1) $A \in \Sigma$.
(2) $f(A) \in \Sigma$.
(3) $f^{-1}(A) \in \Sigma$.

Proof. Let τ be the topology on X. Then, by definition, $\Sigma = \sigma[\tau]$.

((1) $\Rightarrow$ (2)) Suppose $A \in \Sigma$. Then, by the above lemma,

$$f(A) \in f(\Sigma) = f(\sigma[\tau]) = \sigma[f(\tau)] = \sigma[\tau] = \Sigma,$$

[3] This holds for a general topological group. The assumption of locally compact is not needed.

where we have used the fact that $f(\tau) = \tau$ because f is a homeomorphism.

$((2) \Rightarrow (3))$ Suppose $f(A) \in \Sigma$. Then, similarly as before,

$$A \in f^{-1}(\Sigma) = f^{-1}(\sigma[\tau]) = \sigma\left[f^{-1}(\tau)\right] = \sigma[\tau] = \Sigma.$$

Now, by $(1) \Rightarrow (2)$ with the homeomorphism f^{-1} (instead of f as before), we have that $f^{-1}(A) \in \Sigma$.

$((3) \Rightarrow (1))$ Suppose $f^{-1}(A) \in \Sigma$. Then, similarly as before,

$$A \in f(\Sigma) = f(\sigma[\tau]) = \sigma[f(\tau)] = \sigma[\tau] = \Sigma.$$

□

Lemma 3.3. *Let X be a Hausdorff space, let K be a compact subset of X, and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then, there are compact sets K_1 and K_2 of X such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and $K = K_1 \cup K_2$.*

Proof. Define $L_1 = K - U_1$ and $L_2 = K - U_2$. K is closed because X is Hausdorff, so each L_i is closed. Because each L_i is a closed subspace of K and K is compact, it follows that each L_i is also compact. Furthermore, because $K \subseteq U_1 \cup U_2$, $L_1 \cap L_2 = \emptyset$. Because L_1 and L_2 are disjoint compact subsets of a Hausdorff space, we can separate them with disjoint open sets, say V_1 and V_2 respectively. Define $K_1 = K - V_1$ and $K_2 = K - V_2$. Similarly as before, both K_1 and K_2 are compact.

$$K_1 = K - V_1 \subseteq K - L_1 = K - (K - U_1) = K \cap (K \cap U_1^C)^C = K \cap (K^C \cup U_1) \subseteq U_1.$$

Similarly, $K_2 \subseteq U_2$. Furthermore, $K_1 \cup K_2 = K - (V_1 \cap V_2) = K$. □

Lemma 3.4. *Let (X, μ) be a measure space, let $f : X \to \mathbb{R}$ be measurable, and let $A \subseteq X$ be measurable. Then, if $A = \{x \in X | f(x) > 0\}$ and $\mu(A) > 0$, there is some $a > 0$ such that $\mu(\{x \in A | f(x) \geq a\}) > 0$.*

Proof. Suppose $A = \{x \in X | f(x) > 0\}$ and $\mu(A) > 0$. We proceed by contradiction: suppose that, for all $a > 0$, $\mu(\{x \in A | f(x) \geq a\}) = 0$. Write $S_n = \left\{x \in A | f(x) \geq \frac{1}{2^n}\right\}$. Then,

$$A = \bigcup_{n \in \mathbb{N}} S_n,$$

so

$$\mu(A) \leq \sum_{n \in \mathbb{N}} \mu(S_n) = 0,$$

so $\mu(A) = 0$: a contradiction. Thus, there is some $a > 0$ such that

$$\mu(\{x \in A | f(x) \geq a\}) > 0.$$

□

4. Existence and Uniqueness

Before we prove anything about existence and uniqueness, we first show how to obtain left Haar measure from right Haar measure, and vice versa.

Proposition 4.1. *Let G be a topological group, let μ be a Haar measure on G, and define $\mu'(A) = \mu'\left(A^{-1}\right)$. Then, μ is a left (resp. right) Haar measure iff μ' is a right (resp. left) Haar measure on G.*

Proof. ($\Rightarrow$) Suppose that μ is a left Haar measure on G.

Step 1: Show that μ' is a Borel measure on G.

We first note that, because inversion is a homeomorphism of G, by Lemma 3.2, μ' is defined exactly on the Borel subsets of G. Trivially, μ' is nonnegative and $\mu'(\emptyset) = 0$. Let $\{A_n | n \in \mathbb{N}\}$ be a collection of pairwise disjoint measurable subsets of G. We would like to know that $\left\{A_n^{-1} | n \in \mathbb{N}\right\}$ is also a collection of pairwise disjoint measurable subsets. Once again, by Lemma 3.2, they are all measurable. Suppose there is some $x \in A_m^{-1} \cap A_n^{-1}$ for $m \neq n$. Then ,$x = a^{-1} = b^{-1}$ for some $a \in A_m$ and some $b \in A_n$, so that $a = b \in A_m \cap A_n$: a contradiction. Thus, $\left\{A_n^{-1} | n \in \mathbb{N}\right\}$ is a collection of pairwise disjoint measurable subsets, and hence

$$\mu'\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\left(\bigcup_{n\in\mathbb{N}} A_n\right)^{-1}\right) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n^{-1}\right) = \sum_{n\in\mathbb{N}} \mu\left(A_n^{-1}\right) = \sum_{n\in\mathbb{N}} \mu'(A_n).$$

Thus, μ' is a Borel measure on G.

Step 2: Show that μ' is regular.

Let $K \subseteq G$ be compact. Then, K^{-1} is also compact, so

$$\mu'(K) = \mu\left(K^{-1}\right) < \infty.$$

Let A be a measurable subset of G. $A^{-1} \subseteq U$ and U is open iff $A \subseteq U^{-1}$ and U^{-1} is open, so that $\left\{\mu(U) | A^{-1} \subseteq U, U \text{ is open.}\right\} = \left\{\mu\left(U^{-1}\right) | A \subseteq U, U \text{ is open.}\right\}$. Then,

$$\begin{aligned}\mu'(A) = \mu\left(A^{-1}\right) &= \inf\left\{\mu(U) | A^{-1} \subseteq U, U \text{ is open.}\right\} \\ &= \inf\left\{\mu\left(U^{-1}\right) | A \subseteq U, U \text{ is open.}\right\} = \inf\left\{\mu'(U) | A \subseteq U, U \text{ is open.}\right\}.\end{aligned}$$

Similarly, for A open,

$$\mu'(A) = \sup\left\{\mu'(K) | K \subseteq A, K \text{ is compact.}\right\},$$

and so μ' is regular.

Step 3: Show that μ' is a right Haar measure.

Trivially, μ' is nonzero. Also,

$$\mu'(Ag) = \mu\left((Ag)^{-1}\right) = \mu\left(g^{-1}A^{-1}\right) = \mu\left(A^{-1}\right) = \mu'(A).$$

Thus, μ' is a right Haar measure on G.

The other directions are essentially identical. □

This proposition tells us that, while left and right Haar measure on a group may be different, they are related in a simple manner, and so we may as well simply concern ourselves with the study of one or the other. Because of convention, we shall restrict ourselves to proving existence and uniqueness of *left* Haar measure.

Before we prove existence of left Haar measure, however, one lemma is needed.

Lemma 4.2. *Let G be a topological group, let K be a compact subset of G, and let U be an open subset of G such that $K \subseteq U$. Then, there is an open set V containing the identity such that $KV \subseteq U$.*

Proof. For each $x \in K$, define $W_x = x^{-1}U$. Because $x \in U$, W_x is an open neighborhood of the identity. Then, pick V_x to be an open neighborhood of the identity such that $V_xV_x \subseteq W_x$. Then, the collection $\{xV_x | x \in K\}$ is an open cover of K, so there is a finite collection of points $x_1, \ldots, x_n$ such that $K \subseteq \bigcup_{k=1}^n x_kV_{x_k}$.

Define $V = \bigcap_{k=1}^{n} V_{x_k}$. Let $x \in K$. Then, there is some x_k such that $x \in x_k V_{x_k}$, so that

$$xV \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k W_{x_k} = U.$$

Thus, $KV \subseteq U$. □

Theorem 4.3 (Existence)**.** *Let G be a locally compact group. Then, there exists a left Haar measure on G.*

Proof. Step 1: Define $(K : V)$.

Let K be a compact subset of G and let V be a subset of G with nonempty interior. Then, $\{gV^o | g \in G\}$ is an open cover of K, so there are a finite number of elements of G, $g_1, \ldots, g_n$, such that $K \subseteq \bigcup_{k=1}^{n} g_k V^o$. Let $(K : V)$ denote the smallest nonnegative integer for which such a sequence exists.

Step 2: Define μ_U.

Let $\mathcal{K}$ denote the collection of compact subsets of G and let $\mathcal{U}$ denote the collection of open subsets of G containing the identity. Because G is locally compact, there is a compact subset of G with nonempty interior: call it K_0. For each $U \in \mathcal{U}$, define a function $\mu_U : \mathcal{K} \to \mathbb{R}$ such that

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}.$$

Because K_0 is nonempty, $(K_0 : U) \neq 0$, and so this is well-defined.

Step 3: Show that $0 \leq \mu_U(K) \leq (K : K_0)$.

As $(K : U)$ is always a nonnegative integer, μ_U is clearly always nonnegative. We now show that $(K : U) \leq (K : K_0)(K_0 : U)$ for $K \in \mathcal{K}$ and $U \in \mathcal{U}$. For the remainder of this paragraph, let us write $m = (K : K_0)$ and $n = (K_0 : U)$. Then, let $g_1, \ldots, g_m \in G$ and let $h_1, \ldots, h_n \in G$ be such that $K \subseteq \bigcup_{k=1}^{m} g_k K_0^o$ and $K_0 \subseteq \bigcup_{k=1}^{n} h_k U$. Then,

$$K \subseteq \bigcup_{i=1}^{m} \left[\bigcup_{j=1}^{n} g_i h_j U \right],$$

so that K can be covered by mn cosets of U, so that $(K : U) \leq mn = (K : K_0)(K_0 : U)$. It follows that

$$0 \leq \mu_U(K) \leq (K : K_0).$$

Step 4: Construct the Haar measure on $\mathcal{K}$.

Define $X = \prod_{K \in \mathcal{K}} [0, (K : K_0)]$. Because $0 \leq \mu_U(K) \leq (K : K_0)$, each μ_U may be thought of as a point in X. Thinking of each μ_U as a point in X, for each $V \in \mathcal{U}$, define $C(V) = \overline{\{\mu_U | U \in \mathcal{U}, U \subseteq V\}}$. We wish to show that the collection $\{C(V) | V \in \mathcal{U}\}$ possess the finite intersection property, so let $V_1, \ldots, V_n \in \mathcal{U}$. Then, $\mu_{\bigcap_{k=1}^{n} V_k} \in \bigcap_{k=1}^{n} C(V_k)$, so that $\bigcap_{k=1}^{n} C(V_k)$ is nonempty. Thus, $\{C(V) | V \in \mathcal{U}\}$ satisfies the finite intersection property, and because X is compact by Tychonoff's Theorem, it follows that $\bigcap_{V \in \mathcal{U}} C(V)$ is nonempty, so we may pick some $\mu \in \bigcap_{V \in \mathcal{U}} C(V)$.

Step 5: Show that $\mu(K_1) \leq \mu(K_2)$ if $K_1 \subseteq K_2$.

Let $K_1, K_2 \in \mathcal{K}$ be such that $K_1 \subseteq K_2$. We first show that, for each $U \in \mathcal{U}$, $\mu_U(K_1) \leq \mu_U(K_2)$. But this is trivial, because the covering of K_2 with $(K_2 : U)$ cosets of U is also a covering of K_1 with $(K_2 : U)$ cosets of U, so that $(K_1 : U) \leq (K_2 : U)$, and hence $\mu_U(K_1) \leq \mu_U(K_2)$.

Thinking of elements f of X as functions from $\mathcal{K}$ to $\mathbb{R}$, consider the map that sends $f \in X$ to $f(K_2) - f(K_1)$. This is a composition of continuous functions, and hence continuous.[4]. This map is also nonnegative on each $C(V)$ because $\mu_U(K_1) \leq \mu_U(K_2)$ for each $U \in \mathcal{U}$ (we need continuity so that we know it is nonnegative on the entire *closure*.). It follows that this map is also nonnegative at μ, so that $\mu(K_2) - \mu(K_1) \geq 0$, so that $\mu(K_1) \leq \mu(K_2)$.

STEP 6: SHOW THAT $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$.

Let $K_1, K_2 \in \mathcal{K}$. We first show that $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$ for each $U \in \mathcal{U}$. Thus this is trivial, because a covering of K_1 with $(K_1 : U)$ cosets of U together with a covering of K_2 with $(K_2 : U)$ cosets of U, is a cover of $K_1 \cup K_2$ with $(K_1 : U) + (K_2 : U)$ cosets of U, so that $(K_1 \cup K_2 : U) \leq (K_1 : U) + (K_2 : U)$. It follows that $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$.

Proceeding similarly as in step 5, the map that sends $f \in X$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and nonnegative on each $C(V)$, and hence is nonnegative for $\mu \in X$. Thus, $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$.

STEP 7: SHOW THAT $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ IF $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$.

Let $K_1, K_2 \in \mathcal{K}$ be such that $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$. Let $g_1, \ldots, g_n$ be such that $n = (K_1 \cup K_2 : U)$ and $K_1 \cup K_2 \subseteq \bigcup_{k=1}^n g_k U$. If some $g_k U$ intersects both K_1 and K_2, then $g_k \in K_1 U^{-1} \cap K_2 U^{-1}$: a contradiction. Thus, each $g_k U$ intersects *either* K_1 *or* K_2, but not both. Thus, we may find some natural number m with $0 \leq m \leq n$ and reindex the g_ks so that $K_1 \subseteq \bigcup_{k=1}^m g_k U$ and $K_2 \subseteq \bigcup_{k=m+1}^n g_k U$. Thus, $(K_1 : U) + (K_2 : U) \leq (K_1 \cup K_2 : U)$. Combining this result with the previous step, it follows that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ for each $U \in \mathcal{U}$.

STEP 8: SHOW THAT $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ IF $K_1 \cap K_2 = \emptyset$.

Let $K_1, K_2 \in \mathcal{K}$ be such that $K_1 \cap K_2 = \emptyset$. Then, we may find disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. By Lemma 4.2, there are open neighborhoods of the identity V_1 and V_2 such that $K_1 V_1 \subseteq U_1$ and $K_2 V_2 \subseteq U_2$. Define $V = V_1 \cap V_2$. Then, $K_1 V$ and $K_2 V$ are disjoint because U_1 and U_2 are disjoint. Thus, for any $U \in \mathcal{U}$ with $U \subseteq V^{-1}$, we have that $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$, so that, by the previous step, $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Thus, the continuous map from X to $\mathbb{R}$ that sends $f \in X$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is 0 for each $f \in S\left(V^{-1}\right)$. In particular, $\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2)$.

STEP 9: EXTEND μ TO ALL SUBSETS OF G.

For $U \subseteq G$ open, define

$$\mu(U) = \sup\{\mu(K) | K \subseteq U, K \in \mathcal{K}\},$$

We must show that if K is compact and open, these two definitions of $\mu(K)$ agree. That is, we must show that

$$\mu(K) = \sup\{\mu(K') | K' \subseteq K, K' \in \mathcal{K}\},$$

where here the LHS is the original definition of μ as a point in $\bigcap_{U \in \mathcal{U}} C(U)$. Trivially, since $\mu(K) \in \{\mu(K') | K' \subseteq K, K' \in \mathcal{K}\}$, $\mu(K) \leq \sup\{\mu(K') | K' \subseteq K, K' \in \mathcal{K}\}$. On the other hand, by step 5, the set $\{\mu(K') | K' \subseteq K, K' \in \mathcal{K}\}$ is bounded above by $\mu(K)$, so that $\sup\{\mu(K') | K' \subseteq K, K' \in \mathcal{K}\} \leq \mu(K)$. Thus, this definition agrees

[4]The first map from X into $\mathbb{R} \times \mathbb{R}$ is the projection of $f \in X$ onto the K_1^{th} coordinate in the first coordinate and the projection of $f \in X$ onto the K_2^{th} coordinate in the second coordinate. This map is continuous because it is continuous in each coordinate. Each coordinate is continuous by definition of the product topology. This first map is followed by the map that subtracts the second coordinate from the first, which is well-known to be a continuous map from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

with the previous. It follows trivially that this extension still satisfies the property $\mu(U_1) \leq \mu(U_2)$ if $U_1 \subseteq U_2$.

Now, for an arbitrary subset A of G, define

$$\mu(A) = \inf \{\mu(U) | A \subseteq U, U \text{ is open.}\}.$$

Similarly as before, this indeed is an extension of our previous definition of μ to all subsets of G. It again follows trivially that this extension still satisfies the property that $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subseteq A_2$.

Step 10: Show that μ is an outer measure on G.

Trivially, $\mu(\emptyset) = 0$ because $(\emptyset : U) = 0$ for every $U \in \mathcal{U}$. Furthermore, to show that μ is nonnegative, because of the definitions of the extensions, it suffices to show that μ is nonnegative on $\mathcal{K}$. For a fixed K, the map that sends $f \in X$ to $f(K)$ is continuous (by similar reasoning as before). Furthermore, because this map is nonnegative at each μ_U, it follows that this map is nonnegative on each $C(V)$. Thus, this map is nonnegative at μ, so that $\mu(K) \geq 0$.

To show countable subadditivity, we first show that for each countable collection of *open* sets $\{U_n | n \in \mathbb{N}\}$, we have that

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) \leq \sum_{n\in\mathbb{N}} \mu(U_n).$$

Let $\{U_n | n \in \mathbb{N}\}$ be a countable collection of open subsets of G. Let K be a compact subset of $\bigcup_{n\in\mathbb{N}} U_n$. Then, $K \subseteq \bigcup_{k=1}^n U_k$ for some $n \in \mathbb{N}$. Applying Lemma 3.3 inductively, we may find compact sets $K_1, \ldots, K_n$ such that $K = \bigcup_{k=1}^n K_k$ and $K_k \subseteq U_k$ for $1 \leq k \leq n$. Then, applying step 6 inductively,

$$\mu(K) \leq \sum_{k=1}^n \mu(K_k) \leq \sum_{k=1}^n \mu(U_k) \leq \sum_{n\in\mathbb{N}} \mu(U_n).$$

It follows that

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \sup\left\{\mu(K) | K \subseteq \bigcup_{n\in\mathbb{N}} U_n, K \in \mathcal{K}\right\} \leq \sum_{n\in\mathbb{N}} \mu(U_n).$$

Now, let $\{A_n | n \in \mathbb{N}\}$ be an arbitrary collection of subsets of G. If $\sum_{n\in\mathbb{N}} \mu(A_n) = \infty$, then trivially $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$, so suppose $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$. Let $\varepsilon > 0$, and for each $n \in \mathbb{N}$, pick an open set U_n such that $A_n \subseteq U_n$ and $\mu(U_n) \leq \mu(A_n) + \frac{\varepsilon}{2^n}$. Then,

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) \leq \sum_{n\in\mathbb{N}} \mu(U_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n) + \varepsilon \sum_{n\in\mathbb{N}} \frac{1}{2^n} = \sum_{n\in\mathbb{N}} \mu(A_n) + \frac{\varepsilon}{2},$$

but since $\varepsilon > 0$ was arbitrary, we have that

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n),$$

so that μ is an outer measure on G.

Step 11: Show that the collection of Carathedory measurable sets contain the Borel subsets of G.

To show that the collection of Carathedory measurable sets contain the Borel subsets of G, it suffices to show that every open subset of G is measurable (because the collection of measurable sets form a σ-algebra, if this collection contains the topology of G, then it certainly contains the σ-algebra generated by the topology). So let $U \subseteq G$ be open and let $A \subseteq G$. If $\mu(A) = \infty$, then trivially $\mu(A) \geq \mu(A \cap U) + \mu(A \cap U^C)$, so we might as well assume that $\mu(A) < \infty$. Let $\varepsilon > 0$ and pick $V \subseteq G$ open and such that $A \subseteq V$ and $\mu(V) \leq \mu(A) + \varepsilon$. Let K be a compact subset of $V \cap U$ such that $\mu(V \cap U) - \varepsilon \leq \mu(K)$, and let L be a compact subset of $V \cap K^C$ such that $\mu(V \cap K^C) - \varepsilon \leq \mu(L)$. Since $K \subseteq U$, $V \cap U^C \subseteq V \cap K^C$, so

$$\mu(V \cap U^C) - \varepsilon \leq \mu(V \cap K^C) - \varepsilon \leq \mu(L).$$

Thus, by step 8,

$$\begin{aligned} \mu(A \cap U) + \mu(A \cap U^C) - 2\varepsilon &\leq \mu(V \cap U) + \mu(V \cap U^C) - 2\varepsilon \leq \mu(K) + \mu(L) \\ &= \mu(K \cup L) \leq \mu\left((V \cap U) \cup (V \cap K^C)\right) \\ &\leq \mu(V) \leq \mu(A) + \varepsilon. \end{aligned}$$

It follows that

$$\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A) + 3\varepsilon.$$

Since ε is arbitrary, we have that

$$\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A),$$

and hence U is measurable. It follows that μ restricts to a measure on the Borel subsets of G, so that it is a Borel measure (G is completely regular, as mentioned early, and in particular Hausdorff).

Step 12: Show that μ is regular.

Trivially, considering μ as an element of X, μ is finite on compact sets. Furthermore, as by construction $\mu(A) = \inf\{\mu(U) | A \subseteq U, U \text{ is open.}\}$, μ is trivially outer regular. Similarly, μ is trivially inner regular (we showed that the extension agreed with its definition for open sets which is by construction inner regular).

Step 13: Show that μ is nonzero.

$\mu_U(K_0) = 1$ for each $U \in \mathcal{U}$, and the continuous function that maps $f \in X$ to $f(K_0)$ is a constant 1 on each $C(U)$, and in particular $\mu(K_0) = 1$, and hence μ is nonzero.

Step 14: Show that μ is translation invariant.

Fix $g \in G$. The elements $x_1, \ldots, x_n$ generate a cover for K iff the elements $gx_1, \ldots, gx_n$ generate a cover of gK, so that $(K : U) = (gK : U)$ for each $U \in \mathcal{U}$, and hence $\mu_U(K) = \mu_U(gK)$ for each $U \in \mathcal{U}$. It follows that the continuous function that maps $f \in X$ to $f(K) - f(gK)$ is 0 on each $C(U)$, and hence $\mu(K) = \mu(gK)$. Thus, μ is translation invariant, and hence a left Haar measure on G. □

Before we dive into the proof of uniqueness, we first need to prove a couple of lemmas about topological groups.

Lemma 4.4. *Let G be a locally compact group and let $f \in C_c(G)$. Then, for every $\varepsilon > 0$, there is an open neighborhood U of the identity such that whenever $y \in xU$, it follows that $|f(x) - f(y)| < \varepsilon$.*

Proof. Step 1: Construct the neighborhood.

Define $K = \text{supp}[f]$. Let $\varepsilon > 0$. By continuity of f, for each $x \in K$, we may find an open neighborhood U_x of the identity such that whenever $y \in xU_x$, it follows that $|f(y) - f(x)| < \varepsilon$. Then, for each $x \in K$, choose another open neighborhood of the identity V_x such that $V_x V_x \subseteq U_x$. By compactness of K, there is a finite number of $x_1, \ldots, x_n$ such that $K \subseteq \bigcup_{k=1}^n x_k V_{x_k}$. Define $V = \bigcap_{k=1}^n V_{x_k}$ and define

$U = V \cap V^{-1}$. U is clearly an open neighborhood of the identity, and we claim that this neighborhood works.

Step 2: Show that this neighborhood is a correct one.

Let $y \in xU$. If $x, y \notin K$, then $|f(x) - f(y)| = 0$, and so there is nothing to worry about, so we may assume that either $x \in K$ or $y \in K$. First suppose that $x \in K$. Because $x \in K$, it follows that $x \in x_k V_{x_k}$ for some $1 \leq k \leq n$, and hence that $x \in x_k U_{x_k}$. On the other hand, because $x \in x_k V_{x_k}$ and $V \subseteq V_{x_k}$, it follows that $y \in xV \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k U_{x_k}$. Thus,

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < 2\varepsilon.$$

Now let us suppose that $y \in K$. $y = xu$ for some $u \in U$, so $x = yu^{-1}$. But $U = V \cap V^{-1}$, so $u^{-1} \in U$, so that $x \in yU$. Then, we have that $y \in K$ and $x \in yU$, so we may apply the same logic as in the previous paragraph (interchanging the roles of x and y). $\square$

Lemma 4.5. *Let G be a topological group and let μ be a Haar measure on G. Then, for every $x \in G$, $\int_G f(xg)d\mu(g) = \int_G f(g)d\mu(g)$ for $f \in L^1(G)$.*

Proof. Let $x \in G$.

Step 1: Prove for characteristics functions.

Let A be measurable and let $f = \chi_A$. Then,

$$\begin{aligned}\int_G f(xg)d\mu(g) &= \int_G \chi_A(xg)d\mu(g) = \int_G \chi_{x^{-1}A}(g)d\mu(g) = \mu\left(x^{-1}A\right) = \mu(A) \\ &= \int_G \chi_A(g)d\mu(g) = \int_G f(g)d\mu(g).\end{aligned}$$

Step 2: Prove for simple functions.

Let f be a simple function. Then, $f = \sum_{k=1}^n a_k \chi_{A_K}$ for some constants $a_1, \ldots, a_n$ and some measurable sets $A_1, \ldots, A_n$. Then,

$$\begin{aligned}\int_G f(xg)d\mu(g) &= \sum_{k=1}^n a_k \int_G \chi_{A_k}(xg)d\mu(g) = \sum_{k=1}^n a_k \int_G \chi_{A_k}(g)d\mu(g) \\ &= \int_G f(xg)d\mu(g).\end{aligned}$$

Step 3: Prove for f nonnegative measurable.

Let f be a nonnegative measurable function on G. Then, there exists a monotonic increasing sequence of simple functions ϕ_n that converges pointwise almost everywhere to f. Thus, by the Monotone Convergence Theorem,

$$\int_G f(xg)d\mu(g) = \lim \int_G \phi_n(xg)d\mu(g) = \lim \int_G \phi_n(g)d\mu(g) = \int_G f(g)d\mu(g).$$

Step 4: Prove for real-valued integrable functions.

Let f be a real-valued integrable function on G. Define $f_+(g) = \begin{cases} f(g) & \text{if } f(g) \geq 0 \\ 0 & \text{otherwise} \end{cases}$ and $f_-(g) = \begin{cases} -f(g) & \text{if } f(g) \leq 0 \\ 0 & \text{otherwise} \end{cases}$. Then, f_+ and f_- are both nonnegative measurable functions, so

$$\begin{aligned} \int_G f(xg)d\mu(g) &= \int_G f_+(xg)d\mu(g) - \int_G f_-(xg)d\mu(g) \\ &= \int_G f_+(g)d\mu(g) - \int_G f_-(g)d\mu(g) = \int_G f(g)d\mu(g). \end{aligned}$$

STEP 5: PROVE FOR $f \in L^1(G)$.

Let $f \in L^1(G)$. Define $R = \Re[f]$ and $I = \Im[f]$. Then, R and I are real-valued integrable functions, so

$$\begin{aligned} \int_G f(xg)d\mu(g) &= \int_G R(xg)d\mu(g) + i\int_G I(xg)d\mu(g) \\ &= \int_G R(g)d\mu(g) + i\int_G I(g)d\mu(G) = \int_G f(g)d\mu(g). \end{aligned}$$

□

Theorem 4.6 (Uniqueness)**.** *Let G be a locally compact group, and let μ and μ' be two left Haar measures on G. Then, $\mu = a\mu'$ for some $a \in \mathbb{R}^+$.*

Proof. STEP 1: FIND A COMPACT SET OF NONZERO MEASURE.

Because μ is nonzero, there is some set of nonzero measure (with respect to μ). It follows by outer regularity that there is some open set (containing this set) that is also of positive measure, and by inner regularity, it follows that there s a compact set of nonzero measure (contained in this open set): call it K.

STEP 2: SHOW THAT $\int_G f d\mu > 0$ FOR $f \in C_c(G)$ NONNEGATIVE AND NOT IDENTICALLY 0.

Let $f \in C_c(G)$ be nonnegative and not identically 0. Define $U = f^{-1}(\mathbb{R}^+)$. U is nonempty because f is not identically 0. By continuity, U is open, so because K is compact and U is nonempty, there is a finite number of elements $g_1, \ldots, g_n$ such that $K \subseteq \bigcup_{k=1}^n g_k U$, so that

$$0 < \mu(K) \leq \sum_{k=1}^n \mu(g_k U) = n\mu(U),$$

so that $\mu(U) > 0$. Then, by Lemma 3.4, it follows that there is some $a > 0$ such that $V = \{g \in G | f(g) \geq a\}$ is of positive measure. It follows that

$$\int_G f d\mu \geq \int_V f d\mu \geq a\mu(V) > 0.$$

STEP 3: DEFINE h.

Let $g \in C_c(G)$ be nonnegative and not identically 0, and let $f \in C_c(G)$ be arbitrary. g will remain the same throughout the remainder of the proof. Define

$$h(x, y) = \frac{f(x)g(yx)}{\int_G g(tx)d\mu'(t)}.$$

By step 2, the denominator never vanishes, and so h is well-defined on all of $G \times G$. Trivially, h is compactly supported because both f and g are.

Step 4: Show that h is continuous.

To show that h is continuous, it suffices to show that $I(x) \equiv \int_G g(tx)d\mu'(t)$ is a continuous function. Define $K = \text{supp}[g]$, let $x_0 \in G$, and let U be an open neighborhood of x_0 whose closure is compact (which exists because G is locally compact). $K \times \overline{U}^{-1}$ is compact by Tychonoff's Theorem, so $K\overline{U}^{-1}$ is compact because this is the image of $K \times \overline{U}^{-1}$ under a continuous function. Let $\varepsilon > 0$, and choose $\delta > 0$ so that $\delta\mu'\left(K\overline{U}^{-1}\right) < \varepsilon$, which we may do because $K\overline{U}^{-1}$ is compact, and hence of finite measure. By Lemma 4.4, there is an open neighborhood V of the identity such that whenever $y \in xV$, it follows that $|g(x) - g(y)| < \delta$.

Then, whenever $x \in U \cap x_0V$, an open neighborhood of x_0, $tx \in tx_0V$, so that

$$|I(x) - I(x_0)| \leq \int_G |g(tx) - g(tx_0)|\, d\mu'(t) \leq \delta\mu'\left(K\overline{U}^{-1}\right) < \varepsilon,$$

where we have used the fact that integrand vanishes for t outside of $K\overline{U}^{-1}$. Thus, I is continuous, and hence h is continuous, and hence $h \in C_c(G \times G)$.

Step 5: Show that $\frac{\int_G f(x)d\mu(x)}{\int_G g(x)d\mu(x)} = C$, where C is some constant independent of μ.

By a generalization of Fubini's Theorem[5], we have that

$$\begin{aligned}
\int_G \left[\int_G h(x,y)d\mu'(y)\right] d\mu(x) &= \int_G \left[\int_G h(x,y)d\mu(x)\right] d\mu'(y) \\
&= \int_G \left[\int_G h\left(y^{-1}x, y\right) d\mu(x)\right] d\mu'(y) \\
&= \int_G \left[\int_G h\left(y^{-1}x, y\right) d\mu'(y)\right] d\mu(x) \\
&= \int_G \left[\int_G h\left(y^{-1}, xy\right) d\mu'(y)\right] d\mu(x),
\end{aligned}$$

where we have applied Lemma 4.5 several times. Thus,

$$\begin{aligned}
\int_G f(x)d\mu(x) &= \int_G \left[f(x)\frac{\int_G g(yx)d\mu'(y)}{\int_G g(tx)d\mu'(t)}\right] d\mu(x) \\
&= \int_G \left[\int_G \frac{f(x)g(yx)}{\int_G g(tx)d\mu'(t)} d\mu'(y)\right] d\mu(x) = \int_G \left[\int_G h(x,y)d\mu'(y)\right] d\mu(x) \\
&= \int_G \left[\int_G h\left(y^{-1}, xy\right) d\mu'(y)\right] d\mu(x) \\
&= \int_G \left[\int_G \frac{f\left(y^{-1}\right) g(x)}{\int_G g\left(ty^{-1}\right) dt} d\mu'(y)\right] d\mu(x) \\
&= \left(\int_G g(x)d\mu(x)\right)\left(\int_G \frac{f\left(y^{-1}\right)}{\int_G g\left(ty^{-1}\right) d\mu'(t)} d\mu'(y)\right)
\end{aligned}$$

Thus, $\frac{\int_G f(x)d\mu(x)}{\int_G g(x)d\mu(x)} = C$, where C is some constant independent of μ.

Step 6: Deduce that $\int_G f d\mu' = a\int_G f d\mu$ for some positive constant a.

[5] See [1], pg. 243–244. Note that this is why we needed $h \in C_c(G \times G)$.

Because this constant does not depend on μ, it must be the case that

$$\frac{\int_G f d\mu}{\int_G g d\mu} = C = \frac{\int_G f d\mu'}{\int_G g d\mu'},$$

and hence that

$$\int_G f d\mu' = a \int_G f d\mu,$$

where $a \equiv \frac{\int_G g d\mu'}{\int_G g d\mu}$.

Step 7: Show that $\mu' = a\mu$.

For $f \in C_c(G)$, define $\phi(f) = \int_G f d\mu$ and $\psi(f) = \int_G f d\nu$, where ν is a measure defined by $\nu = 1/a\mu'$. Both ϕ and ψ are positive linear functions on $C_c(G)$, and

$$\phi(f) = \int_G f d\mu = 1/a \int_G f d\mu' = \int_G f d\nu = \psi(f).$$

Thus, by the Riesz Representation Theorem[6], it follows that $\mu = \nu$, i.e. that $\mu' = a\mu$ with $a \in \mathbb{R}^+$. $\square$

This theorem tells us that left Haar measure on G is "essentially" unique, in the sense that any two left Haar measures differ only by a positive multiplicative constant. Furthermore, if we add the requirement that a certain fixed subset has a specified measure, this completely determines the measure on the group.

This concludes the proof of the existence and uniqueness of left Haar measure on a locally compact topological group. Given existence and uniqueness of left Haar measure, the corresponding results for right Haar measure follow immediately from Proposition 4.5.

References

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[4] E. M. Stein; R. Shakarchi, *Real Analysis: Measure Theory, Integration, & Hilbert Spaces*, (2007), 274.

[6] See [1], pg. 209–210.