# *Reverse Mathematics of Topology*

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*Abstract.* This paper develops the Reverse Mathematics of second countable topologies, where the elements of the topological space exist. The notion of topology, effective topology (Dorais 2011), and basis will be defined. Several concepts and results that appear in the practice of general topology will be developed in various subsystems of second order arithmetic, $Z_2$. Several of the ideas and results of point set topology will be proved to be equivalent to subsystems of second arithmetic, primarily $\mathrm{ACA}_0$, over the base system $\mathrm{RCA}_0$. Various open questions about basic notions from general topologies are interspersed throughtout.

Topologies are given by a sequence of sets $(B_n)_{n\in\mathbb{N}}$ which satisfy all the conditions of being a basis with the additional condition that $B_0$ is the entire topological space. Thus all the topologies considered are such that their points really exist, that is the points are elements of $\mathbb{N}$, and the entire topological space is a set that exists. A set is open if and only if all its points are interior points. It is shown that $\mathrm{RCA}_0$ can prove that arbitrary union of basic open sets are open whenever this union exists. However, $\mathrm{RCA}_0$ can not prove that arbitrary union of basic open sets exists. It is proved (Theorem 3.12) that over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to the existence of the union of basic open sets. Another familiar result from general topology is that a set is open if and only if it is the union of basic open sets. An open code (Dorais 2011) for an open set $U$ is, informally, a set $X$ such that $\bigcup_X B_n = U$. A open set that has an open code is called effectively open. It is proved (Theorem 3.27) that over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is exactly required to prove that there is a set $X$ which witnesses how an open set $U$ can be written as a union of basic open sets (i.e. all open sets are effectively open). Furthermore, it is shown that $\mathrm{ACA}_0$ is equivalent, over $\mathrm{RCA}_0$, to the existence of the closure and interior of any subset of any topological space.

An effective topology is a topology where there is a function $f$ such that given a point $x$ and basic open sets $B_m$ and $B_n$ such that $x$ is in both sets, then $x \in B_{f(x,m,n)}$ and $B_{f(x,m,n)} \subset B_m \cap B_n$. A open set $U$ is l-effectively open, if there is a function $h$ such that $x \in B_{h(n)}$ and $B_{h(n)} \subset U$. An advantage of effective topology is that $\mathrm{RCA}_0$ can prove the intersection of two $l$-effectively open sets is $l$-effectively open. It is proved that not all topologies are effectively topologies. In fact, over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to the condition that every topology is an effective topology (Theorem 4.8).

Topological ideas that appear to require greater set comprehension axioms are defined and considered. Example includes continuous functions, homomorphism, and connectedness. The ability to determine which topologies are homeomorphic to a given topology or which topologies are connected is considered. They are shown to be proveable in $\Pi^1_1$-$\mathrm{CA}_0$. However, their exact strengths are given as open questions. Moreover, the existence of a connected component for every topology seemingly requires set existence axioms stronger than $\Pi^1_1$-$\mathrm{CA}_0$. The exact stength of this statement over $\mathrm{RCA}_0$ is given as an open question of possibly great interest. There are not many results equivalent to $\Pi^1_2$-$\mathrm{CA}_0$ over $\mathrm{RCA}_0$. If the existence of connected components is equivalent to $\Pi^1_2$-$\mathrm{CA}_0$ over $\mathrm{RCA}_0$, it would be a very natural result - one frequently used in actual mathematical practice - which is equivalent to $\Pi^1_2$-$\mathrm{CA}_0$. Even if the answer happens to be below $\Pi^1_2$-$\mathrm{CA}_0$, the result would of interests since connected components appear to require a $\Sigma^1_2$ definition.

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## 1. Subsystems of Second Order Arithmetic

**Definition 1.1** The first order language of Second Order Arithmetic is the following :

$$L_2 = \{0, 1, +, \cdot, N, S, <, \in\}$$

where 0 and 1 are constants, $+$ and $\cdot$ are 2-ary functions, $<$ and $\in$ are 2-ary relations, and $N$ and $S$ are unary functions.

By convention, one uses infix notation for $+$ and $\cdot$. Logical parenthesis and operation parenthesis will be distinguished by context.

**Definition 1.2** The theory of Second Order Arithmetic, $Z_2$, include the following formulas :

(1) Basic Axiom

$$(\forall x)((N(x) \vee S(x)) \wedge \neg(N(x) \wedge S(x)))$$

$$(\forall m)(\forall x)((m \in x) \Rightarrow (N(m) \wedge S(x)))$$

$$(\forall m)(\forall n)((m < n) \Rightarrow (N(m) \wedge N(n)))$$

$$(\forall x)(\forall y)((\forall n)(n \in X \Leftrightarrow n \in Y) \Rightarrow (X = Y))$$

$$(\forall m)(\forall n)((N(m) \wedge N(n)) \Rightarrow N(m + n))$$

$$(\forall m)(\forall n)((N(m) \wedge N(n)) \Rightarrow N(m \cdot n))$$

$$N(0)$$

$$N(1)$$

$$(\forall n)((N(n) \Rightarrow (n + 1 \neq 0)))$$

$$(\forall m)(\forall n)((N(m) \wedge N(n)) \Rightarrow ((m + 1) = n + 1) \Rightarrow (m = n))$$

$$(\forall m)(N(m) \Rightarrow m + 0 = m)$$

$$(\forall m)(\forall n)((N(m) \wedge N(m)) \Rightarrow m + (n + 1) = (m + n) + 1)$$

$$(\forall m)(N(m) \Rightarrow m \cdot 0 = 0)$$
$$(\forall m)(\forall n)((N(m) \wedge N(n)) \Rightarrow m \cdot (n+1) = (m \cdot n) + m)$$
$$(\forall m)(\neg(m < 0))$$
$$(\forall m)(\forall n)((m < n+1) \Leftrightarrow (m < n \wedge m = n))$$

(2) Induction Axiom

$$(\forall x)((0 \in x \wedge (\forall n)(n \in x \Rightarrow n+1 \in x)) \Rightarrow (\forall n)(n \in x))$$

(3) Comprehension Scheme

$$(\exists x)(\forall n)(N(n) \Rightarrow (n \in x \Leftrightarrow \varphi(n)))$$

where $\varphi(n)$ is a $L_2$ formula in which $X$ does not occur freely.

For any $L_2$ theory in which the basic axiom (1) holds, there is the following convention : Variables represented by lower case letters such as $x$, $y$, $z$, $a$, $b$, $c$, $i$, $j$, $k$, $m$, and $n$ tacitly imply $N(x)$, $N(y)$, ... holds. If $N(x)$ holds, then $x$ is called a number. Similarly, variables represented by upper case letters such as $X$, $Y$, $Z$, $L$, and $W$ tacitly imply $S(X)$, ... holds. If $S(X)$ holds, then $X$ is called a set.

Furthermore, let $\mathcal{N} = (Z, 0^{\mathcal{N}}, 1^{\mathcal{N}}, +^{\mathcal{N}}, -^{\mathcal{N}}, N^{\mathcal{N}}, S^{\mathcal{N}}, <^{\mathcal{N}}, \in^{\mathcal{N}})$ be any $L_2$ structure, where $Z$ is the domain. If $\mathcal{N}$ satisfies the basic axioms (1), then one often writes $\mathcal{N} = (N^{\mathcal{N}}, S^{\mathcal{N}}, 1^{\mathcal{N}}, +^{\mathcal{N}}, -^{\mathcal{N}}, N^{\mathcal{N}}, <^{\mathcal{N}}, \in^{\mathcal{N}})$. This emphazies that the number variable range over $N^{\mathcal{N}}$ and the set variable ranges over $S^{\mathcal{N}}$.

**Definition 1.3** Let $n$ be a number variable and $t$ be a term that does not contain $n$. Let $\varphi$ be a formula. $(\exists n < t)(\varphi(n))$ means $(\exists n)(n < t \wedge \varphi(n))$. $(\forall n < t)(\varphi(n))$ means $(\forall n)(n < t \Rightarrow \varphi(n))$. $(\exists n < t)$ and $(\forall n < t)$ are called bounded quantifiers.

**Definition 1.4** In these definitions, $n \in \omega$.

A formula $\varphi$ is $\Sigma_0^0$ and $\Pi_0^0$ if and only if it has only bounded quantifiers.

A formula $\varphi$ is $\Sigma_{n+1}^0$ if and only if it is logically equivalent to a formula of the form $(\exists x)\theta$ where $\theta$ is $\Pi_n^0$. A formula $\varphi$ is $\Pi_{n+1}^0$ if and only if it is logically equivalent to a formula of the form $(\forall x)\theta$ where $\theta$ is $\Sigma_n^0$.

A formula $\varphi$ is arithmetical if it contains no set quantifiers.

A formula $\varphi$ is $\Sigma_1^1$ if and only if it is logically equivalent to a formula of the form $(\exists X)\theta$ where $\theta$ is arithmetical. A formula $\varphi$ is $\Pi_1^1$ if and only if it is logically equivalent to a formula of the form $(\forall X)\theta$ where $\theta$ is arithmetical. For $n > 0$, a formula $\varphi$ is $\Sigma_{n+1}^1$ if and only if it is logically equivalent to a formula of the form $(\exists X)\theta$ where $\theta$ is $\Pi_1^0$. For $n > 0$, a formula $\varphi$ is $\Pi_{n+1}^1$ if and only if it is logically equivalent to a formula of the form $(\forall X)\theta$ where $\theta$ is $\Sigma_1^0$.

A formula $\varphi$ is $\Delta_n^0$ if and only $\varphi$ is $\Sigma_n^0$ and $\Pi_n^0$. For $n > 0$, a formula $\varphi$ is $\Delta_n^1$ if and only $\varphi$ is $\Sigma_n^1$ and $\Pi_n^1$.

**Definition 1.5** Let $\Gamma$ be some collection of formulas with one free variable in the language $L_2$. The scheme of $\Gamma$-induction, $\Gamma$-IND, consists of all axioms of the form

$$(\varphi(0) \wedge (\forall n)(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow (\forall n)\varphi(n)$$

where $\varphi(n) \in \Gamma$.

**Definition 1.6** Let $\Gamma$ be some collection of formulas with one free variable in the language $L_2$. The scheme of $\Gamma$-comprehension, $\Gamma$-CA, consists of all axioms of the form

$$(\exists X)(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi \in \Gamma$.

**Definition 1.7** $\mathrm{RCA}_0$ is the $L_2$ theory which consiste of (1) Basic Axioms, $\Sigma^0_1$-IND, and $\Delta^0_1$-CA.

**Definition 1.8** $\mathrm{ACA}_0$ is the $L_2$ theory which consist of (1) Basic Axioms, (2) Induction, and Arithmetical-CA.

**Definition 1.9** $\Pi^1_1$-$\mathrm{CA}_0$ is the $L_2$ theory which consist of (1) Basic Axioms, (2) Induction, and $\Pi^1_1$-CA.

## 2. Basics of Reverse Mathematics

**Lemma 2.1** Within $\mathrm{RCA}_0$, the pairing function $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $\langle m, n \rangle = (m+n)^2 + m$. This function is injective.

Proof : Refer to *Subsystems of Second Order Arithmetic* by Simpson page 66. □

**Lemma 2.2** The following can be proved in $\mathrm{RCA}_0$:
(1) $m \leq \langle m, n \rangle$ and $n \leq \langle m, n \rangle$.
(2) $\langle m, n \rangle = \langle m', n' \rangle$ implies $m = m'$ and $n = n'$.

Proof : Refer to *Subsystems of Second Order Arithmetic* by Simpson page 66. □

**Lemma 2.3** For all $X$ and $n$, $\mathrm{RCA}_0 \vdash (\exists Y)(\forall y)(y \in Y \Leftrightarrow \langle y, n \rangle \in X)$.
This set $Y$ is called the $n^{\text{th}}$ row of $X$ and is denoted $X^{[n]}$.

Proof : "$\langle y, n \rangle \in X$" is a $\Delta^0_1$ formula in the free variable $y$ since it has no quantifiers. The result follows from recursive comprehension. □

**Lemma 2.4** Over $\mathrm{RCA}_0$, if $f : \mathbb{N} \to \mathbb{N}$ is such that $(\forall m)(\forall n)(f(m) \leq f(n) \Leftrightarrow m \leq n)$, then $(\exists X)(x \in X \Leftrightarrow (\exists m)(f(m) = x))$.

Proof : $(\exists m)(f(m) = n)$ is equivalent to $(\forall m)(f(m) < n \vee (\forall k < m)(f(k) \neq n))$. Thus $(\exists m)(f(m) = n)$ is a $\Delta^0_1$ formula in the free variable $n$. By $\Delta^0_1$ comprehension, the range of $f$ exists. □

**Lemma 2.5** $\mathrm{RCA}_0$ can prove primitive recursion. Given $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$, there exists an $h : \mathbb{N}^{k+1} \to \mathbb{N}$ such that

$$h(0, n_1, ..., n_k) = f(n_1, ..., n_k)$$
$$h(m+1, n_1, ..., n_k) = g(h(m, n_1, ..., n_k), m, n_1, ..., n_k)$$

and $h$ is unique

Proof : See *Subsystems of Second Order Arithmetic* by Simpson page 69. □

**Lemma 2.6** $\mathrm{RCA}_0$ proves minimization. Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $\langle n_1, ..., n_k \rangle$ there exists $m$ such that $f(m, n_1, ..., n_k) = 1$, then there exists a $g : \mathbb{N}^k \to \mathbb{N}$ such that $g(n_1, ..., n_k) = m'$ such that $f(m', n_1, ..., n_k) = 1$ and for all $m < m'$, $f(m, n_1, ..., n_k) \neq 1$.

Proof : See *Subsystems of Second Order Arithmetics* by Simpson on page 70. □

**Theorem 2.7** Let $\mathrm{REC} = \{X \subseteq \omega : X \leq_T \emptyset\}$, where $\leq_T$ is Turing Reducibility. REC is the unique minimal $\omega$-model of $\mathrm{RCA}_0$.

Proof : Refer to *Subsystems of Second Order Arithmetic* by Stephen Simpson. □

**Proposition 2.8** The basic axiom of $Z_2$ and $\Sigma^0_1$ induction can prove,

$$(\forall m)(\exists n)(\forall i < m)(i \in X \Rightarrow f(i) < n)$$

for all functions $f$ and $X$, a set.

Proof : Let $\varphi(m)$ be the formula $(\exists n)(\forall i < m)(f(i) < n)$. $\varphi(m)$ is $\Sigma^0_1$. Suppose $\varphi(m)$ holds with witness $n$. Then $\max\{n, f(m) + 1\}$ witnesses $\varphi(m + 1)$. By $\Sigma^0_1$ induction, the result follows. □

**Definition 2.9** For each $k \in \omega$, bounded $\Sigma^0_k$ comprehension is the statement

$$(\forall n)(\exists X)(\forall i)(i \in X \Leftrightarrow (i < n \wedge \varphi(i)))$$

where $\varphi(i)$ is any $\Sigma^0_k$ formula in which $X$ does not occur free.

**Proposition 2.10** $\mathrm{RCA}_0$ proves bounded $\Sigma^0_1$ comprehension.

Proof : Refer to *Subsystems of Second Order Arithmetic* by Simpson page 71. □

**Theorem 2.11** The following are equivalent over $\mathrm{RCA}_0$:
(1) $\mathrm{ACA}_0$
(2) $\Sigma^0_1$-CA
(3) If $f : \mathbb{N} \to \mathbb{N}$ is injective, then there exists $X$ such that $(\forall n)(n \in X \Leftrightarrow (\exists m)(f(m) = n))$.

Proof : See *Subsystems of Second Order Arithmetics* by Simpson, page 105. □

**Definition 2.12** Using the pairing function of Definition 2.1, a binary relation $R$ on a set $X$ is a subset of $X \times X$.

A binary relation $R$ on $X$ is well-founded if there exists no injective function $f : \mathbb{N} \to X$ such that

$$(\forall n)(f(n + 1)\ R\ f(n))$$

A binary relation $R$ on $X$ is *well-founded if

$$(\forall U)(U \neq \emptyset \Rightarrow (\exists i)(i \in U \wedge (\forall j)(j \in U \Rightarrow (j = x \vee \neg(jRi)))))$$

A linear ordering is a pair $X$ and $<_X$, which is a binary relation on $X$ such that

$$(\forall i)(\neg(i <_X i))$$

$$(\forall i)(\forall j)(\forall k)((i <_X j \wedge j <_X k) \Rightarrow (i <_X k))$$

$$(\forall i)(\forall j)(i <_X j \vee j <_X i)$$

A linear ordering is a well-ordering if it is well-founded. A linear ordering is *well-ordering if it is *well-founded.

Let $LO(X, <_X)$ be the formula asserting that $<_X$ is a linear ordering on $X$. Let $WO(X, <_X)$ be the formula asserting that $<_X$ is a well-ordering on $<_X$.

**Proposition 2.13** Suppose $<_X$ is a linear ordering on the set $X$. If $f : \mathbb{N} \to X$ is injective with $f(n+1) <_X f(n)$ for all $n$, then $\mathrm{RCA}_0$ can prove that if $m < n$, then $f(n) <_X f(m)$.

Proof : Consider the following formula $\varphi(n)$ in free variable $n$ :

$$(\forall m < n)(f(n) <_X f(m))$$

$\varphi(0)$ holds. Suppose $\varphi(n)$ holds. Then $\varphi(n+1)$ holds. If $m < n$, then since $f(n+1) <_X f(n) < f(m)$ by the property of $f$ and transitivity of $< X$, $f(n+1) < f(m)$. If $m = n$, then $f(n+1) <_X f(n) = f(m+1)$ by property of $f$. Thus $\varphi(n)$ follows. By $\Sigma^0_1$ induction in $\mathrm{RCA}_0$, $(\forall n)(\varphi(n))$ follows. □

**Lemma 2.14** Suppose $<_X$ is a linear ordering on the set $X$. Suppose $f : \mathbb{N} \to X$ is a injective function such that $f(n+1) <_X f(n)$ for all $n$, then $\mathrm{RCA}_0$ can prove $(\exists X)(x \in X \Leftrightarrow (\exists n)(f(n) = x))$.

Proof : $(\exists n)(f(n) = x)$ is equivalent to $\neg((\exists n)(f(n) <_X x \wedge (\forall k)((k < n) \Rightarrow f(k) \neq x)))$. Thus $(\exists n)(f(n) = x)$ is $\Delta^0_1$. It exists in $\mathrm{RCA}_0$. □

**Proposition 2.15** Over $\mathrm{RCA}_0$, a linear ordering is a well-ordering if and only if it is a *well-ordering.

Proof : Suppose the pair $X$ and $R$ is not well-founded. There exists an injective function $f : \mathbb{N} \to X$ such that $f(n+1) <_X f(n)$ for all $n$. By Lemma 2.14, the range $Y$ of $f$ exists and is a subset of $X$. $Y$ is a subset with no least element.

Suppose that $Y \subset X$ has no least element. Consider the formula in free variable $z$ :

$$(\exists a \leq z)(\exists b \leq z)(\exists c \leq z)(\exists i < 2)((z = \langle a, b, c, i\rangle \wedge a \in Y \wedge c \in Y)$$

$$\wedge (i = 1 \vee i = 0) \wedge (i = 1 \Rightarrow c <_X a) \wedge (i = 0 \Rightarrow \neg(c <_X a)))$$

Thus the set defined by the above formula exists by $\Delta^0_1$ comprehension in $\mathrm{RCA}_0$. This defines a function $g' : Y \times \mathbb{N} \times Y \to \{0, 1\}$ with the property that $g'(a, b, c) = 1$ if and only $c <_X a$. Since $Y$ has no least element, $g'$ has the property that for all $a$ and $b$, there exists a $c$ such that $g'(a, b, c) = 1$. Lemma 2.6 give a function $g$ that minimizes $g'$. Let $l \in Y$. Let $f : \mathbb{N}^0 \to \mathbb{N}$ be the constant function that takes on value $l$. By Lemma 2.5, let $h : \mathbb{N} \to Y$ defined by primitive recursion using $f$ and $g$. (Note that $g : Y \times \mathbb{N} \times Y \to \mathbb{N}$ but Lemma 2.5 can be easily modified to hold in this case.) Thus $h : \mathbb{N} \to Y \subset X$ is a infinite decreasing function. □

**Definition 2.16** An isomorphism of linearly order sets $(X, <_X)$ and $(Y, <_Y)$ is a function $f : X \to Y$ such that

$$(\forall i)(\forall j)((i \in X \wedge j \in X) \Rightarrow (i <_X j \Leftrightarrow f(i) <_Y f(j))) \wedge (\forall k)(k \in Y \Rightarrow (\exists i)(i \in X \wedge f(i) = k))$$

If $f : (X, <_X) \to (Y, <_Y)$ is an isomorphism, it will denoted $f : |X| = |Y|$.

$X$ is an initial segmentof $Y$ if and only if there exists a $k \in Y$ such that

$$(\forall i)(\forall j)(i <_X j \Leftrightarrow (i <_Y j \wedge j <_Y k))$$

$X$ said to be the initial segment of $Y$ determined by $k$. $X$ will also be denoted $Y \upharpoonright k$.

$|X| < |Y|$ will mean that there exists an $f : |X| = |Y \upharpoonright k|$ for some $k \in Y$. The function $f$ that witness this will be denoted $f : |X| < |Y|$.

$|X| > |Y$ if and only if there exists a $k \in X$ and an $f : |X \upharpoonright k| = |Y|$. The function $f$ witnessing this will be denoted $f : |X| > |Y|$.

$|X| \leq |Y|$ will mean $|X| < |Y|$ or $|X| = |Y|$. $f : |X| \leq |Y|$ will denote a function that witness this. Similarly, $|X| \geq |Y|$ and $f : |X| \geq |Y|$ is defined.

Given linear orderings $(X, <_X)$ and $(Y, <_Y)$, $f : X \to Y$ is a comparision map if $f : |X| \geq |Y|$ or $f : |X| \leq |Y|$. $(X, <_X)$ and $(Y, <_Y)$ are comparable if there exists a comparison map between the two.

**Proposition 2.17** Let $(X, <_X)$ be a linear order and $k \in X$. The initial segment determined by $k$ exists.

Proof : Consider the formula in free variable $z$ :

$$z \in X \wedge z <_X k$$

This formula is $\Delta_0^0$. Hence in $\mathrm{RCA}_0$, $X \upharpoonright k$ exists. □

**Definition 2.18** CWO is the following statement :

$$(\forall X)(\forall Y)((WO(X) \wedge WO(Y)) \Rightarrow (|X| \leq |Y| \vee |X| \geq |Y|))$$

that is, any two well-orderings are comparable.

**Proposition 2.19** $\mathrm{RCA}_0$ proves if $(X, <_X)$ is well-ordered and $(Y, <_Y)$ is a linear ordering and $X$ and $Y$ are comparable, then the comparison map is unique.

Proof : See Simpson's *Subsystems of Second Order Arithmetic*, page 177. □

**Definition 2.20** For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ (or $2^{<\mathbb{N}}$). $\sigma$ is an initial segment of $\tau$ (or $\tau$ is an extension of $\sigma$), denoted $\sigma \preceq \tau$, if and only if

$$(|\sigma| \leq |\tau|) \wedge (\forall i)(i < |\sigma| \Rightarrow \sigma(i) = \tau(i))$$

$\sigma$ and $\tau$ are incomparable, denoted $\sigma | \tau$ if and only if

$$(\exists i)(i < \min(|\sigma|, |\tau|) \wedge \sigma(i) \neq \tau(i))$$

Let $T \subset \mathbb{N}^{<\mathbb{N}}$ (or $2^{<\mathbb{N}}$). $T$ is perfect if and only if

$$(\forall \sigma)(\sigma \in T \Rightarrow ((\exists \tau_1)(\exists \tau_2)(\tau_1 \in T \wedge \tau_2 \in T \wedge \sigma \preceq \tau_1 \wedge \sigma \preceq \tau_2 \wedge \tau_1 | \tau_2)))$$

The perfect kernel of $T$ is the union of all the perfect subtrees of $T$. If this set exists, denote it $K_T$.

**Definition 2.21** A sequence of trees (of $\mathbb{N}^{<\mathbb{N}}$ or $2^{<\mathbb{N}}$) is a set $\mathcal{T}$ such that for all $n$, $\mathcal{T}^{[n]}$ is a tree. Let

$T_n := \mathcal{T}^{[n]}$. $\mathcal{T}$ encodes the sequence of trees $(T_n)_{n\in\mathbb{N}}$.

**Theorem 2.22** Over $\mathrm{RCA}_0$, the following are equivalent :
(1) $\mathrm{ATR}_0$
(2) CWO
(3) $\Sigma^1_1$ Separation : For any $\Sigma^1_1$ formula $\varphi_0(n)$ and $\varphi_1(n)$ in which $Z$ does not occur freely

$$(\neg(\exists n)(\varphi_0(n) \wedge \varphi_1(n)) \Rightarrow (\exists Z)(\forall n)((\varphi_0(n) \Rightarrow n \in Z) \wedge (\varphi_1(n) \Rightarrow n \notin Z)))$$

(4) For each arithmetic formula $\varphi(i, X)$.

$$(\forall i)(\exists X)(\varphi(i, X) \Rightarrow ((\forall Y)(\varphi(i, Y) \Rightarrow (Y = X)))) \Rightarrow (\exists Z)(\forall i)(i \in Z \Leftrightarrow (\exists X)\varphi(i, X))$$

(5) For each $(T_n)_{n\in\mathbb{N}}$ a sequence of trees,

$$((\forall i)(\exists f)(\forall n)(f \upharpoonright n \in T_i) \Rightarrow ((\forall g)(\forall n)(g \upharpoonright n \in T_i \Rightarrow f = g))) \Rightarrow (\exists Z)(\forall i)(i \in Z \Leftrightarrow (\exists f)(\forall n)(f \upharpoonright n \in T_i))$$

Proof : See Simpson's Subsystems of Second Order Arithmetics pages 190, 191, and 198. □

**Theorem 2.23** The following are equivalent over $\mathrm{RCA}_0$.
(1) $\Pi^1_1$-CA
(2) For any sequence of trees $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{<\mathbb{N}}$, encoded by $\mathcal{T}$,

$$(\exists X)(\forall k)(k \in X \Leftrightarrow (\exists f)(\forall n)(f \upharpoonright n \in T_k))$$

(3) For any tree $T \subset \mathbb{N}^{<\mathbb{N}}$, the perfect kernel $K_T$ exists.
(4) For any tree $T \subset 2^{<\mathbb{N}}$, the perfect kernel $K_T$ exists.

Proof : See *Subsystems of Second Order Arithmetic* by Simpson, page 217 and 218. □

## 3. Definitions and Basic Results

**Definition 3.1** A topology $\mathscr{T}$ is given by a set $\mathscr{B}$ such that

$$(\forall m)(\forall y)(y \in \mathscr{B}^{[m]} \Rightarrow y \in \mathscr{B}^{[0]})$$

$$\wedge\ (\forall m)(\forall n)(\forall x)((x \in \mathscr{B}^{[m]} \wedge x \in \mathscr{B}^{[n]}) \Rightarrow (\exists o)(x \in \mathscr{B}^{[o]} \wedge (\forall y)(y \in \mathscr{B}^{[o]} \Rightarrow (y \in \mathscr{B}^{[m]} \wedge y \in \mathscr{B}^{[n]}))))$$

The underlying set of the topology $\mathscr{T}$ is $\mathscr{B}^{[0]}$. Define $B_n := \mathscr{B}^{[n]}$. $(B_n)_{n\in\omega}$ is the basis for the topology $\mathscr{T}$.

If necessary, $\mathscr{T}_{\mathscr{B}}$ will denote the topology specified by $\mathscr{B}$. $\mathscr{T}$ and $\mathscr{B}$ will be used interchangeable; however, $\mathscr{T}$ will be used to emphasize the topology and $\mathscr{B}$ will emphasize the set encoding the topology.

Note the formula which $\mathscr{B}$ satisfies is $\Pi^0_3$.

**Definition 3.2** (Dorais, 2011) An effective topology $\mathscr{Q}$ is a set $\mathscr{B}$ and a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that

$$(\forall m)(\forall y)(y \in \mathscr{B}^{[m]} \Rightarrow y \in \mathscr{B}^{[0]})$$

$$\wedge\ (\forall m)(\forall n)(\forall x)((x \in \mathscr{B}^{[m]} \wedge x \in \mathscr{B}^{[n]}) \Rightarrow (x \in \mathscr{B}^{[f(x,m,n)]} \wedge (\forall y)(y \in \mathscr{B}^{[f(x,m,n)]} \Rightarrow (y \in \mathscr{B}^{[m]} \wedge y \in \mathscr{B}^{[n]}))))$$

The underlying set of the effective topology is $\mathscr{B}^{[0]}$. Define $B_n := \mathscr{B}^{[n]}$. $(B_n)_{n\in\mathbb{N}}$ is the basis for the effective topology $\mathscr{Q}$.

If necessary, $\mathscr{Q}_{\mathscr{B},f}$ will denote the effective topology specified by $\mathscr{B}$ and $f$.

Note the formula for the effective topology which $\mathscr{B}$ and $f$ satisfies is $\Pi^0_1$.

**Definition 3.3** The following formula

$$(\forall z)(z \in X \Rightarrow z \in Y)$$

will be represented as $X \subset Y$.

Moreover, given topology $\mathscr{T}$ encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$, the formula with free variable $n$, $\mathscr{B}$, and $Y$

$$(\forall z)(z \in B_n \Rightarrow y \in Y)$$

denote $B_n \subset Y$. Hence, using this notation, Definition 3.1 and Definition 3.2 can be written more concisely.

**Definition 3.4** Let $\mathscr{T}$ be a topology given by $\mathscr{B}$, $U$ is open if and only if

$$(\forall x)((x \in U) \Rightarrow (\exists n)(x \in B_n \wedge B_n \subset U))$$

$U$ is open is denoted $U \in \mathscr{T}$. Similarly for $U \in \mathscr{Q}$ where $\mathscr{Q}$ is an effective topology.

**Definition 3.5** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$. If $\mathscr{T}$ has the property that

$$(\forall m \in B_0)(\exists n)(m \in B_n \wedge (\forall y)(y \in B_n \Rightarrow y = m))$$

then $\mathscr{T}$ is called a discrete topology on $B_0$.

**Lemma 3.6** $\mathrm{RCA}_0$ proves if $\mathscr{T}$ is a discrete topology then $(\forall X)(X \in \mathscr{T})$.

Proof : Suppose this was not true. Then

$$(\exists x)(x \in X \wedge (\forall n)(x \notin B_n \vee (\exists y)(y \in B_n \wedge y \notin X)))$$

Thus

$$(\forall n)(x \in B_n \Rightarrow (\exists y)(y \in B_n \wedge y \notin X))$$

must hold. Since $x \in X$, $x \in B_n$, and $y \notin X$, one has that for all $n$, $B_n \neq \{x\}$. This contradicts the fact that $\mathscr{T}$ is discrete. □

**Lemma 3.7** Let $\mathscr{T}$ be a topology. Over $\mathrm{RCA}_0$, if $(\forall X)(X \in \mathscr{T})$, then $\mathscr{T}$ is discrete.

Proof : Suppose $\mathscr{T}$ is not discrete. Then

$$(\exists m \in B_0)(\forall n)(m \notin B_n \vee (\exists y)(y \in B_n \wedge y \neq m))$$

In $\mathrm{RCA}_0$, let $\{m\}$ be the singleton containing $m$. By definition of being open, if $\{m\}$ was open, then it would contradict the formula above. Thus, $(\exists X)(X \notin \mathscr{T})$. □

**Proposition 3.8** Over $\mathrm{RCA}_0$, let $\mathscr{T}$ be a topology given by $\mathscr{B}$, $\mathscr{B}$ is discrete if and only if $(\forall X)(X \in \mathscr{T})$.

Proof : By Lemma 3.6 and Lemma 3.7. □

**Definition 3.9** Let $f \in 2^{\mathbb{N}}$. Let $(B_n)_{n\in\mathbb{N}}$ be the basis for the topology $\mathscr{T}$, which is specified by $\mathscr{B}$. Define the formula $\varphi(y, f, \mathscr{B})$ to be

$$(\exists n)((f(n) = 1) \wedge (y \in B_n))$$

If there exists a $Y$ such that $y \in Y \Leftrightarrow \varphi(y, f, \mathscr{B})$, then denote $Y = \bigcup_f \mathscr{T}$.

Let "Arbitrary Unions Exists" be the statement:
For all functions $f \in 2^{\mathbb{N}}$ and for all topologies $\mathscr{B}$, $(\exists Y)(y \in Y \Leftrightarrow \varphi(y, f, \mathscr{B}))$.

**Lemma 3.10** $\mathrm{ACA}_0$ implies "Arbitrary Unions Exists"

Proof : The formula $\varphi(y, f, \mathscr{B})$ in Definition 3.9 is $\Sigma^0_1$. Hence $Y$ exists by arithmetic comprehension. □

**Lemma 3.11** Over $\mathrm{RCA}_0$, "Arbitrary Unions Exists" implies $\mathrm{ACA}_0$

Proof : Consider the formula in free variable $x$ :

$$(\exists m \leq x)(\exists n \leq x)(x = \langle m, n\rangle) \wedge ((n \neq 0) \Rightarrow (\exists o \leq x)(n = \langle m, o\rangle))$$

Since this formula is $\Delta^0_0$, over RCA, the set $\mathscr{B}$ defined by the formula exists. In particular, $\mathscr{B}^{[0]} = n$ and for all $n \neq 0$, $\mathscr{B}^{[n]}$ is the empty set or a singleton. This follows from the pairing function being injective. Moreover, for each $m$, there exists unboundedly many $n$ such that $B_n = \{m\}$. The precise statement of the latter is

$$(\forall m)(\forall k)(\exists n)(n > k \wedge (\exists x)(n = \langle m, x\rangle))$$

To prove this look at any model of RCA and consider $n = \langle m, k + 1\rangle > k$ by Lemma 2.2.

$\mathrm{RCA}_0$ can prove that $\mathscr{B}$ encodes a topology. This follows from each $B_n$ being either a singleton or empty.

$\mathscr{B}$ is a discrete topology. Suppose not, then

$$(\exists m \in B_0)(\forall n)(m \notin B_n \vee (\exists y)(y \in B_n \wedge y \neq m))$$

Therefore

$$(\exists m \in B_0)(\forall n)(m \in B_n \Rightarrow (\exists y)(y \in B_n \wedge y \neq m))$$

Thus, for all $B_n$ such that $m \in B_n$, there exists a $y \neq m$. Therefore $B_n$ is not a singleton or the empty set. This contradicts what was proved above.

Let $g : \mathbb{N} \to \mathbb{N}$ be an injective function. Consider $f : \mathbb{N} \to 2$ defined by the formula in free variable $x$ below :

$$\begin{aligned}(\exists j \leq x)(\exists k \leq x)(\exists n \leq x)(\exists i \leq 2)((x = \langle n, i\rangle)\\ \wedge (n = \langle j, k\rangle)\\ \wedge (i = 1 \Rightarrow (\exists m \leq n)(g(m) = j))\\ \wedge (i = 0 \Rightarrow (\forall m \leq n)(g(m) \neq j)))\end{aligned}$$

$f$ exists in $\mathrm{RCA}_0$ since the above is $\Delta^0_0$. It can be proved that $f$ is an function. Now for any particular $j$, there exists $m$ such that $g(m) = j$ if and only if there exists $\langle n, i\rangle$ where $n = \langle j, k\rangle$ and $n \geq m$ and $i = 1$ if and only if $f(n) = 1$ if and only if $\varphi(j, f, \mathscr{B})$. By "Arbitrary Union Exists", $\bigcup_f \mathscr{T}$ exists. Thus the range of

$g$ exists. By Theorem 2.11, $\mathrm{ACA}_0$ follows. □

**Theorem 3.12** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ if and only if "Arbitrary Unions Exists".

Proof : By Lemma 3.10 and Lemma 3.11. □

**Definition 3.13** For any $f \in 2^{\mathbb{N}}$, let $\bigcap_f \mathscr{T}$ denote (if it exists) the set definable by the formula in free variables $y$, $f$, and $\mathscr{B}$ :

$$(\forall n)(f(n) = 1 \Rightarrow y \in B_n)$$

If $D$ is a finite set, then denote (if it exists) $\bigcup_D \mathscr{T}$ as the set definable by the formula in free variable $y$, $D$, and $\mathscr{B}$ :

$$(\exists n)(n \in D \wedge y \in B_n)$$

Similarly, $\bigcap_D \mathscr{T}$ is defined.

For any $k \in \omega$ and $m_i \in \mathbb{N}$, $i < k$, let $\bigcap_{i<k} B_{m_i}$ denote the set (if it exists) defined by the following formula in free variable $x$ :

$$\bigwedge_{i<k} x \in B_{m_i}$$

**Lemma 3.14** Let $\mathscr{T}$ be a topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $f : \mathbb{N} \to 2$,

$$\mathrm{RCA}_0 \vdash (\forall n)((f(n) = 1) \Rightarrow (\forall y)(y \in B_n \Rightarrow \varphi(y, f, \mathscr{B})))$$

where $\varphi(y, f, \mathscr{B})$ is the formula from Lemma 3.9.

Proof : In any model of $\mathrm{RCA}_0$, let $n$ be any element of the model and suppose $f(n) = 1$. For all $y$, if $y \in B_n$, then $\varphi(y, \mathscr{B}, f)$ holds, where $\varphi$ is the formula from Definition 3.9, since $n$ serves as the witness. Thus $(\forall y)(y \in B_n \Rightarrow \varphi(y, \mathscr{B}, \mathscr{T}))$ follows. Since $n$ arbitrary, the formula above holds in any model of $\mathrm{RCA}_0$. By the Gödel Completeness theorem, this result follows. □

**Proposition 3.15** For any topology $\mathscr{T}$ and function $f \in 2^{\mathbb{N}}$, whenever $\bigcup_f \mathscr{T}$ exists, $\mathrm{RCA}_0 \vdash \bigcup_f \mathscr{T} \in \mathscr{T}$.

Proof : Suppose this does not hold. Then

$$(\exists n)(f(n) = 1 \wedge x \in B_n) \wedge (\forall n)(x \notin B_n \vee (\exists y)(y \in B_n \wedge y \notin \bigcup_f \mathscr{T}))$$

By Lemma 3.14,

$$(\exists n)(x \in B_n \wedge (\forall y)(y \in B_n \Rightarrow y \in \bigcup_f \mathscr{T})) \wedge (\forall n)(x \notin B_n \vee (\exists y)(y \in B_n \wedge y \notin \bigcup_f \mathscr{T}))$$

Writing the above in the more concise notation using $\subset$,

$$(\exists n)(x \in B_n \wedge B_n \subset \bigcup_f \mathscr{T}) \wedge \neg((\exists n)(x \in B_n \wedge B_n \subset \bigcup_f \mathscr{T}))$$

This is a contradiction. □

**Proposition 3.16** For any topology $\mathscr{T}$ and finite set $D$ encoded by $u = \langle k, \langle m, n\rangle\rangle$,

$$\mathrm{RCA}_0 \vdash (\exists Y)(x \in Y \Leftrightarrow (\forall n)(n \in D \Rightarrow x \in B_n))$$

i.e. $\bigcap_D \mathscr{T}$ exists.

Furthermore for any $k \in \omega$ and $m_i \in \mathbb{N}$, $i \leq k$,

$$\mathrm{RCA}_0 \vdash (\exists Y)\left(x \in Y \Leftrightarrow \bigwedge_{i<k} x \in B_{m_i}\right)$$

i.e. $\bigcap_{i<k} B_{m_i}$ exists.

Proof : Refer to *Subsystems of Second Order Arithmetics* by Simpson page 67 for details of the encoding of the finite set $D$ by $u$. However, note that $k$ encodes a bound on the cardinality of the finite set $D$. Furthermore, from $u$, the relation $x \in D$ is $\Delta_0^0$.

The set $\bigcap_D \mathscr{T}$ is defined by the formula

$$(\exists k \leq u)(\exists w \leq u)(u = \langle k, w\rangle \wedge (\forall i \leq k)(i \in D \Rightarrow y \in B_i))$$

This formula is $\Delta_0^0$ in the free variable $y$. Thus it exists by $\Delta_1^0$ comprehension.

The second claims follows from $\Delta_1^0$ Comprehension since the formula defining it in Definition 3.13 is $\Delta_0^0$. □

**Proposition 3.17** For any topology $\mathscr{T}$ given by the set $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$

$$\mathrm{RCA}_0 \vdash \bigcap_{i<k} B_{m_i} \in \mathscr{T}$$

for all $k \in \omega$ and all $m_i \in \mathbb{N}$, $i < k$.

Proof : By Proposition 3.16, $\bigcap_{i<k} B_{m_i}$ exists. First, one proves that for all $k \in \omega$ and $m_i \in \mathbb{N}$, $i < k$, $(\forall x)(\exists n)(x \in \bigcap_{i<k} B_{m_i} \Rightarrow (x \in B_m \wedge B_m \subset \bigcap_{i<k} B_{m_i}))$. This is proved by induction (in metamathematics). Suppose this held for $k$. Suppose $m_i \in \mathbb{N}$, $i < k+1$. For any $x \in \bigcap_{i<k+1} B_{m_i}$, $x \in \bigcap_{i<k} B_{m_i}$. By induction there exists a $n_1$ such that $x \in B_{n_1}$ and $B_{n_1} \subset \bigcap_{i<k} B_{m_i}$. Therefore, $x \in B_{n_1}$ and $x \in B_k$. By definition of a topology, there exists $n_2$ such that $x \in B_{n_2}$ and $B_{n_2} \subset B_{n_1} \cap B_{m_k}$. Thus $B_{n_2} \subset \bigcap_{i<k+1} B_{m_i}$, as one can show. Thus $\bigcap_{i<k+1} B_{m_i}$ is open. By induction, the result follows. □

**Proposition 3.18** $\Pi_3^0$-IND proves $(\forall D)(\bigcap_D \mathscr{T} \in \mathscr{T})$, where $D$ ranges over finite sets.

Proof : From Definition 3.4, a set is open is $\Pi_3^0$. The result then follows from $\Pi_3^0$ induction. □

***Open Question* 3.19** Can $\mathrm{RCA}_0$ prove that finite intersections of basic open sets are open? How much induction is necessary to prove this result?

**Definition 3.20** A sequence of subsets $(S_n)_{n\in\mathbb{N}}$ is encoded by a set $\mathscr{S}$ such that $S_n = \mathscr{S}^{[n]}$.

For any $f \in 2^{\mathbb{N}}$, let the formula $\psi(y, \mathscr{S}, f)$ be the formula

$$(\exists n)(y \in S_n \wedge f(n) = 1)$$

Let $\bigcup_f \mathscr{S}$ denote the set defined by the formula $\varphi(y, \mathscr{S}, f)$ in free variable $y$, if it exists.

The "Union Property" is the statement : For all sets $X$, if $(\forall x)(x \in X \Rightarrow (\exists n)(x \in S_n \wedge S_n \subset X))$, then there exists a function $f \in 2^{\mathbb{N}}$ such that $z \in X \Leftrightarrow \psi(z, \mathscr{S}, f)$.

**Lemma 3.21** $\mathrm{ACA}_0$ proves "Union Property".

Proof : Let $X$ be some set such that $(\forall x)(x \in X \Rightarrow (\exists n)(x \in S_n \wedge S_n \subset X))$. Define $f$ by the following formula in free variable $x$ :

$$(\exists m)(\exists i)(i = 0 \vee i = 1) \wedge (x = \langle m, i \rangle)$$

$$\wedge\ (S_m \subset X \Rightarrow i = 1) \wedge (S_m \not\subset X \Rightarrow i = 0)$$

$f$ exists by arithmetic comprehension.

Suppose $x \in X$, then there exists $n$ such that $x \in S_n$ and $S_n \subset X$ by assumption. Thus $n$ witnesses $\psi(x, \mathscr{S}, f)$. $x \in \bigcup_f \mathscr{S}$. Suppose $\psi(x, \mathscr{S}, f)$. In particular there exists $n$ such that $f(n) = 1$ and $x \in S_n$. However, by definition, $f(n) = 1$ implies $S_n \subset X$. Thus $x \in X$. □

**Lemma 3.22** Over $\mathrm{RCA}_0$, "Union Property" proves $\mathrm{ACA}_0$.

Proof : (Math Overflow community; this proof due to Carl Mummert) Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Define $\mathscr{S}$ by the following formula in free variable $x$ :

$$(\exists m \leq x)(\exists i \leq x)(\exists j \leq x)((x = \langle m, \langle i, j \rangle \rangle)$$

$$\wedge\ (\exists k \leq m)(m = 2i \vee (m = 2k + 1 \wedge j < k \wedge f(k) < i)))$$

This exists by $\Delta^0_1$ comprehension. Therefore, $S_{\langle i,j \rangle} = \{2i\} \cup \{2k + 1 : j < k \wedge f(k) < i\}$ and $S_n = \emptyset$ if there does not exists $i, j$ such that $n = \langle i, j \rangle$.

For each $i$, apply Proposition 2.10 to $i$ and the following formula in free variable $x$ :

$$(\exists n)(f(n) = x)$$

to obtain a set which is denoted $\mathrm{rgn}(f) \upharpoonright i$, i.e. the range up to $i$.

Next, define $f^{-1} \upharpoonright i$ by the formula in free variable $x$ :

$$(\exists m \leq x)(\exists n \leq x)(x = \langle m, n \rangle \wedge \langle n, m \rangle \in f \wedge m < i)$$

This exists by $\Delta^0_1$ comprehension. In $\mathrm{RCA}_0$, $f^{-1} \upharpoonright i$ can be proved to be a function since $f$ is injective.

For each $i$, applying Lemma 2.8 to $i$, one has

$$(\exists n)(\forall k < i)(k \in (\mathrm{rgn}(f) \upharpoonright i) \Rightarrow (f^{-1} \upharpoonright i)(k) < n)$$

Thus (in $\mathrm{RCA}_0$) since all elements of the range of $f$ that are less that $i$ lie in $\mathrm{rgn}(f) \upharpoonright i$, $n$ serves as a bound for the set $\{k : f(k) < i\}$, which exists by $\Delta^0_1$ comprehension.

Next, there exists $j$ such that $x \in S_{\langle i,j \rangle} \Leftrightarrow x = 2i$. In particular $j = n$ (where $n$ is the one found above) witnesses this property. Thus, for every $i$ there exists a $j$ such that $S_{\langle i,j \rangle} = \{2i\}$.

Let $X$ be the set defined by the following formula in free variable $x$ :

$$(\exists i \leq x)(x = 2i)$$

$X$ exists by $\Delta^0_1$ comprehension. Note, $X$ is the even numbers. Now for each $x \in X$, $x = 2i$ for some $i$. Thus there exists a $\langle i, j \rangle$ such that $S_{\langle i,j \rangle} = \{2i\}$, by the above. Thus for all $x \in X$ there exists a $m$ such that $x \in S_m$ and $S_m \subset X$. By the Union Property, there exists a function $g$ such that $\bigcup_g \mathscr{S} = X$.

Note that if $g(\langle i,j\rangle)=1$, then $S_{\langle i,j\rangle}=\{2i\}$ since $X$ consists of only even numbers. Define the function $h$ by the formula in free variable $x$ as follows :

$$(\exists i\leq x)(\exists j\leq x)(x=\langle i,j\rangle\wedge g(\langle i,j\rangle)=1\wedge(\forall k<j)g(\langle i,k\rangle)=0)$$

$h$ exists by $\Delta_1^0$ comprehension. Note that $h(i)$ gives the least $j$ such that $g(\langle i,j\rangle)=1$. Now for all $n$, it can proved in $\mathrm{RCA}_0$ that

$$(\exists m)(f(m)=n)\Leftrightarrow(\exists k<h(n+1))(f(k)=n)$$

by the definition of $S_{\langle n,h(n+1)\rangle}=\{2n\}$. The formula in free variable $n$ is $\Delta_0^0$. Thus, the set defined by this formula in free variable $n$ exists by $\Delta_1^0$ comprehension and is the range of $f$. By Theorem 2.11, $\mathrm{ACA}_0$ follows. □

**Theorem 3.23** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Union Property".

Proof : By Lemma 3.21 and Lemma 3.22. □

**Definition 3.24** Let "Every Open Set is a Union" be the statement:
For all topologies $\mathscr{T}$, if $U\in\mathscr{T}$, then there exists $f\in 2^{\mathbb{N}}$ such that $z\in U\Leftrightarrow\varphi(z,f,\mathscr{B})$.
That is $U=\bigcup_f\mathscr{T}$.

**Lemma 3.25** $\mathrm{ACA}_0$ proves "Every Open Set is a Union".

Proof : Suppose $U\in\mathscr{T}$. Define $f$ to be the set defined by the following formula in free variable $x$ :

$$(\exists m)(\exists i)((i=0\vee i=1)\wedge(x=\langle m,i\rangle)$$

$$\wedge\ (B_m\subset U)\Rightarrow(i=1))\wedge(B_m\not\subset U)\Rightarrow(i=0)$$

$f$ exists by arithmetic comprehension.

Let $\varphi(x,\mathscr{B},f)$ be the formula that defined $\bigcup_f\mathscr{T}$ from Definition 3.9. If $x\in U$, then by definition of $U\in\mathscr{T}$, there exists $m$ such that $x\in B_m$ and $B_m\subseteq U$. Thus $f(m)=1$ and hence $\varphi(x,\mathscr{B},f)$. If $\varphi(x,\mathscr{B},f)$, then there exists $m$ such that $f(m)=1$ and $x\in B_m$. By definition of $f$, $f(m)=1$ implies that $B_m\subseteq U$. Thus $x\in U$. Thus, $U=\bigcup_f\mathscr{T}$. □

**Lemma 3.26** Over $\mathrm{RCA}_0$, "Every Open Set is a Union" proves $\mathrm{ACA}_0$.

Proof : Let $f:\mathbb{N}\to\mathbb{N}$ be an injective function. Define $\mathscr{B}$ by the following formula in free variable $x$ :

$$(\exists m\leq x)(\exists i\leq x)(\exists j\leq x)((x=\langle m,\langle i,j\rangle\rangle)$$

$$(\exists k\leq m)(m=\langle 2i,0\rangle\vee(m=\langle k,\langle i,j\rangle+1\rangle)\wedge j<k\wedge f(k)<i))$$

This set exists by $\Delta_1^0$ comprehension. Note that $B_{\langle i,j\rangle}=\{\langle 2i,0\rangle\}\cup\{\langle k,\langle i,j\rangle+1\rangle : j<k\wedge f(k)<i\}$. (Note that for simplicity, the underlying set was not included in $B_0$. One could have easily made $B_0=\mathbb{N}$ by shifting the index; however, notation would have become cumbersome).

In much the same way as in Lemma 3.22, for each $i$ there exists an $j$ such that $B_{\langle i,j\rangle}=\{\langle 2i,0\rangle\}$.

Within $\mathrm{RCA}_0$, one can prove that $\mathscr{B}$ is a topology over $\mathbb{N}$ (although strictly, one should have defined $B_0=\mathbb{N}$). This is because $B_{\langle i,j\rangle}\cap B_{\langle a,b\rangle}$ intersect if and only if $i=a$ and their intersection is $\{\langle 2i,0\rangle\}$. By the above, it was proved that there exists some $j'$ such that $B_{\langle i,j'\rangle}=\{\langle 2i,0\rangle\}$. Thus $\mathscr{B}$ encodes a topology

$\mathscr{T}$.

Let $X$ be the set defined by the following formula in free variable $x$ :

$$(\exists i \leq x)(x = \langle 2i, 0\rangle)$$

$X$ exists by $\Delta^0_1$ comprehension. $X \in \mathscr{T}$, since above, it was proved that for all $\langle 2i, 0\rangle \in X$, there exists a $j$ such that $B_{\langle i,j\rangle} = \{\langle 2i, 0\rangle\}$. Thus $x \in B_{\langle i,j\rangle}$ and $B_{\langle i,j\rangle} \subset X$.

By "Every Open Set is a Union", there exists a function $g$ such that $\bigcup_g \mathscr{T} = X$. Using this function, one can produce the range of $f$ very much like in Lemma 3.22. By Theorem 2.11, $\mathrm{ACA}_0$ follows. □

**Theorem 3.27** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Every Open Set is a Union".

Proof : By Lemma 3.25 and Lemma 3.26. □

**Definition 3.28** (Dorais 2011) Given a topology $\mathscr{T}_{\mathscr{B}}$ or a effective topology $\mathscr{Q}_{\mathscr{B},f}$, an open code in the topology is a subset of $\mathbb{N}$.

Given a topology or effective topology and $U$ an open set, the open code for $U$ (if it exists) is the set $A$ such that $U = \bigcup_{f_A} \mathscr{B}$ where $f_A$ is the characteristic function of $A$.

If an open set has an open code, then the open set is effectively open.

From this point forth, $\bigcup_A \mathscr{B}$ denotes $\bigcup_{f_A} \mathscr{B}$.

*Remark 3.29* François G. Dorais in *Reverse Mathematics of Compact Countable Second-Countable Spaces* refers to the open code and effectively open in the context of effective topologies. This paper broadens the term to topologies as well.

Note that "Arbitrary Union Exists" is the statement that in any topology the open set determined by any open code exists. "Every Open Set is a Union" is the statement that in any topology any open set has an open code, i.e every open set is effectively open.

**Definition 3.30** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. An open set $U$ is l-effectively open if and only if there exists a function $f : U \to \mathbb{N}$ such that $(\forall u)(u \in U \Rightarrow (u \in B_{f(u)} \wedge B_{f(u)} \subset U))$.

## 4. Effective Topologies

**Proposition 4.1** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. $\mathrm{RCA}_0$ proves that if $U$, a subset of $B_0$, is effectively open, then $U$ is l-effectively open.

Proof : Let $X$ be an open code for $U$. Define the function $f$ using the following formula in free variable $z$ :

$$(\exists x \leq z)(\exists n \leq z)(\exists i < 2)(z = \langle x, n, i\rangle \wedge x \in U \wedge (i = 1 \Rightarrow (x \in B_n \wedge n \in X)))$$

This formula has only bounded quantifier; therefore, the set $f$ defined by this formula exists in $\mathrm{RCA}_0$ by $\Delta^0_0$-CA. It can be proved that $f$ is a function from $U$ to $\mathbb{N}$.

Since $X$ is an open code for $U$, for every $x$ there exists a $n$ such that $f(x, n) = 1$. Applying Lemma 2.6 (minimization) to $f$, one obtains $g$, which is the desired function which witnesses that $U$ is l-effectively open. □

***Open Question* 4.2** Does $\mathrm{RCA}_0$ prove that every open set is l-effectively open? If not, over $\mathrm{RCA}_0$, is it equivalent to any well known subsystems of $Z_2$? The conjecture is no and that this property is equivalent to $\mathrm{ACA}_0$, over $\mathrm{RCA}_0$. What about the same two questions for effective topologies?

Does $\mathrm{RCA}_0$ prove that every l-effectively open set is effectively open? If not, over $\mathrm{RCA}_0$, is this property equivalent to any well known subsystems of $Z_2$? Again, the conjecture is no and that it is equivalent to $\mathrm{ACA}_0$ over $\mathrm{RCA}_0$. What about the same questions for effective topologies?

**Proposition 4.3** Let $\mathscr{Q}$ be an effective topology encoded by $\mathscr{B}$ and $f$ with basis $(B_n)_{n\in\mathbb{N}}$. $\mathrm{RCA}_0$ proves that if $U$ and $V$ are l-effectively open, then $U \cap V$ is l-effectively open.

Proof : Suppose $g$ and $h$ witness the fact that $U$ and $V$ are l-effectively open. Then define the function $p$ by the following formula in free variable $z$ :

$$(\exists x \leq z)(\exists n \leq z)(z = \langle x, n\rangle \wedge x \in U \cap V \wedge n = f(x, g(x), h(x))$$

$p$ witnesses that $U \cap V$ is l-effectively open. □

***Open Question* 4.4** Can $\mathrm{RCA}_0$ prove that for any topology $\mathscr{T}$, $U \cap V$ is l-effectively open when both $U$ and $V$ are l-effectively open?

**Definition 4.5** Let "Every Topology is Effective" be the following statement :

For every topology $\mathscr{T}_{\mathscr{B}}$ encoded by $\mathscr{B}$, there exists a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that $\mathscr{Q}_{\mathscr{B},f}$, encoded by $\mathscr{B}$ and $f$, is an effective topology.

**Lemma 4.6** $\mathrm{ACA}_0$ proves "Every Topology is Effective".

Proof : Given a topology $\mathscr{T}$ encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$, consider the formula in free variable $z$ given below :

$$(\exists x)(\exists m)(\exists n)(\exists y)(z = \langle x, m, n, y\rangle$$
$$\wedge\ ((x \in B_m) \wedge (x \in B_n)) \Rightarrow (x \in B_y \wedge B_y \subset B_m \cap B_n \wedge (\forall i)(i < y \Rightarrow \neg(B_i \subset B_m \cap B_m) \vee\ \neg x \in B_i))$$
$$\vee\ (\neg((x \in B_m) \wedge (x \in B_m)) \Rightarrow (y = 0)))$$

Clearly the set defined by this formula exists in $\mathrm{ACA}_0$ since the formula is arithmetical. Informally this is $f(x, m, n) = y$ where $y = 0$ if $\neg((x \in B_m) \wedge (x \in B_n))$ and else $y$ is least such that $x \in B_y \wedge B_y \subset B_m \cap B_n$, which exists by definition of a topology. Thus $\mathscr{B}$ and $f$ turns $\mathscr{T}$ into an effective topology. □

**Lemma 4.7** Over $\mathrm{RCA}_0$, "Every Topology is Effective" proves $\mathrm{ACA}_0$.

Proof : Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Suppose that

$$(\exists a)(\exists b)(\forall z)(a \neq b \wedge f(z) \neq a \wedge f(z) \neq b)$$

Certainly if the range of $f$ is $\mathbb{N}$ or $\mathbb{N}$ without a single point, then it exists in $\mathrm{RCA}_0$.

Consider the following formula in free variable $x$ :

$$(\exists m \leq x)(\exists n \leq x)(\exists p \leq x)(x = \langle m, \langle n, p\rangle\rangle$$

$$\wedge\ ((p=0)\vee(p=1\wedge n=0\wedge m=a)\vee(p=1\wedge n=1\wedge m=b)$$
$$\vee\ (p=2\wedge(m=a\vee m=n))\vee(p=3\wedge(m=b\vee m=n))$$
$$\vee\ (p=4\wedge(m=n\vee(\exists k\leq x)(f(k)=n\wedge m=n+k+1)))\vee(p=5\wedge(m=f(n)))))$$

This formula is $\Delta_0^0$; hence, the set $\mathscr{B}$ defined by it exists in $\mathrm{RCA}_0$. Informally, the basic open sets are

$$B_0=\mathbb{N}$$
$$B_{\langle 0,1\rangle}=\{a\}\quad B_{\langle 1,1\rangle}=\{b\}$$
$$B_{\langle n,2\rangle}=\{a,n\}\quad B_{\langle n,3\rangle}=\{b,n\}$$
$$B_{\langle n,4\rangle}=\{n\}\cup\{n+k+1:f(k)=n\}$$
$$B_{\langle n,5\rangle}=\{f(n)\}$$

and $B_k=\emptyset$ if $k$ is otherwise. $\mathrm{RCA}_0$ can prove that $\mathscr{B}$ encodes a topology. Note that $B_{\langle i,2\rangle}\cap B_{\langle j,2\rangle}=\{a\}$, which is $B_{\langle 0,1\rangle}$. Similarly for $B_{\langle i,3\rangle}\cap B_{\langle j,3\rangle}$. Note that $B_{\langle n,2\rangle}\cap B_{\langle n,3\rangle}=\{n\}$. Now if $n$ is in the range, then there exists a $n'$ such that $B_{\langle n',5\rangle}=\{n\}$. If $n$ is not in the range, then $B_{\langle n,4\rangle}=\{n\}$. Several of the other cases are considered. (In fact it can be proved that this topology is discrete since all the singletons occur either as $B_{\langle n,4\rangle}$ or $B_{\langle n,5\rangle}$.) Thus it has been proved that $\mathscr{T}$ encodes a topology $\mathscr{T}$.

By "Every Topology is Effective", there exists a function $f$ such that $\mathscr{B}$ and $f$ encode an effective topology. Consider the following formula in free variable $x$ :

$$(\exists n\leq f(x,\langle x,2\rangle,\langle x,3\rangle))(f(x,\langle x,2\rangle,\langle x,3\rangle)=\langle n,5\rangle)$$

This formula is $\Delta_0^0$. The set $R$ which is defined by this formula exists. The claim is that $R$ is the range of $f$. In $\mathrm{RCA}_0$, it can be proved that if $(\exists n)(f(n)=x)$, then $B_{\langle x,4\rangle}$ is not a singleton. However, $B_{\langle n,5\rangle}=\{x\}$ where $n$ is such that $f(n)=x$. Since $a$ and $b$ are not part of the range, it is clear that the only possible value of $f(x,\langle x,2\rangle,\langle x,3\rangle)=\langle n,5\rangle$. Hence, $x$ satisfies the condition of the formula. $x\in R$. Suppose $(\forall n)(f(n)\neq x)$. Then $B_{\langle n,4\rangle}\neq\{x\}$ for all $n$. Thus $f(x,\langle x,2\rangle,\langle x,3\rangle)$ can not be $\langle n,5\rangle$ for any $n$. $x$ does not satisfies the formula. $x\notin R$. Thus $R$ is the range of $f$. By Theorem 2.11, $\mathrm{ACA}_0$ follows. □

**Theorem 4.8** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Every Topology is Effective".

Proof : By Lemma 4.6 and Lemma 4.7. □

## 5. Interior and Closure

**Definition 5.1** Let $\mathscr{T}$ be a topology on $B_0$ given by the set $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. Let $A\subset B_0$. "$x$ is an interior point of X" is the formula in free variable $x$

$$(\exists n)(x\in B_0\wedge x\in B_n\wedge B_n\subset A)$$

If the set exists, let $A^\circ$ be the set defined by the formula above. This set is called the interior of $A$.

Let "Interior Exists" be the statement : for all $A$, $A^\circ$ exists.

**Theorem 5.2** $\mathrm{RCA}_0$ does not prove "Interior Exists"

Proof : By Theorem 2.7, REC is a $\omega$ model of $\mathrm{RCA}_0$. $\emptyset'$ is not computable; hence, there exists a $k$ such that $k\notin\emptyset'$.

Define $\mathscr{B}$ such that $B_0 = \omega$ and $B_s = \emptyset'_s$ for all $s > 0$. It is clear that $\mathscr{B}$ is computable. It is clear that $(B_n)$ is a basis for the a topology on $\omega$ since $B_n \subseteq B_{n+1}$ for all $n$. Finally, letting $A = \omega - \{k\}$, one has that the set of points that are interior point of $A$ is $\emptyset'$ which does not exists in REC since $\emptyset'$ is not computable. Thus REC $\not\models$ "Interior Exists". □

**Lemma 5.3** $\mathrm{ACA}_0$ proves "Interior Exists".

Proof : The formula in Definition 5.1 is arithmetic. □

**Lemma 5.4** Over $\mathrm{RCA}_0$, "Interior Exists" proves $\mathrm{ACA}_0$.

Proof : Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Define $\mathscr{B}$ by the $\Delta^0_0$ formula in the free variable $x$ :

$$(\exists m \leq x)(\exists n \leq x)(x = \langle m, n\rangle \wedge n = 0) \vee (x = \langle m, n+1\rangle \wedge f(n) = m)$$

$\mathscr{B}$ is a topology. Note $B_0 = \mathbb{N}$. Consider any model of $\mathrm{RCA}_0$. Since $f$ is injective, $B_{n+1}$ is a singleton and $B_{a+1} \cap B_{b+1} = \emptyset$ for all $a \neq b$. From this, one can show that $\mathscr{B}$ is a topology over the underlying set $B_0 = \mathbb{N}$.

If $f : \mathbb{N} \to \mathbb{N}$ is surjective, it is clear that the range exists. Suppose $f$ is not surjective. Suppose $k$ is such that $(\forall n)(f(n) \neq k)$. Over $\mathrm{RCA}_0$, $\mathbb{N} - \{k\}$ exists for any $k$. By "Interior Exists", $\mathbb{N} - \{k\}^\circ$ exists. The claim is that $(\mathbb{N} - \{k\})^\circ$ is the range of $f$. $x \in (\mathbb{N} - \{k\})^0$ if and only if $(\exists n)(x \in B_0 \wedge x \in B_n \wedge B_n \subset A)$. $x \in B_n$ implies there exists $j$ such that $n = j + 1$ and $f(j) = x$. Suppose there exists a $j$ such that $f(j) = x$, then $x \in B_{j+1}$. Hence $x \in B_0 \wedge x \in B_{j+1} \wedge B_{j+1} \subset A$ (since $x \neq k$ since $k$ is not in the range). Thus $j + 1$ witness, $(\exists n)(x \in B_0 \wedge x \in B_n \wedge B_n \subset A)$. Thus $x \in (\mathbb{N} - \{k\})^\circ$. By Theorem 2.11, $\mathrm{ACA}_0$ follows. □

**Theorem 5.5** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Interior Exists".

Proof : By Lemma 5.3 and Lemma 5.4. □

**Definition 5.6** Let $\mathscr{T}$ be a topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $X \subset B_0$. Any element that satisfies the following formula in free variable $x$ and X :

$$(\forall n)((x \in B_n) \Rightarrow (\exists y)((y \in B_n) \wedge (y \in X) \wedge (y \neq x)))$$

is called a limit point of $X$. Denote the formula above as "$x$ is a limit point of $X$".

Fix $X$. If it exists, denote $X'$ to be the set of limit points of $X$.

Let "Set of Limit Points Exists" be the statement that for any topology $\mathscr{T}$ and $X \subset B_0$, $X'$ exists.

**Definition 5.7** Let $\mathscr{T}$ be a topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $X \subset B_0$. If it exists, let $\bar{X}$ be defined by the formula with free variable $x$ :

$$(x \in X) \vee (\text{"}x \text{ is a limit point of } X\text{"})$$

$\bar{X}$ is called the closure of $X$.

Let "Closure Exists" be the statement that for all topology $\mathscr{T}$, if $X \subset B_0$ then $\bar{X}$ exists.

**Lemma 5.8** Let $\mathscr{T}$ be a topology. Let $X \subset B_0$. $\mathrm{RCA}_0$ proves

$$(x \in B_0 \wedge \neg\text{"}x \text{ is interior point of } X\text{"}) \Leftrightarrow (x \in B_0 \wedge (x \in B_0 - X \vee \text{"}x \text{ is limit point of } B_0 - X\text{"}))$$

Note this is the familiar statement that the complement of the interior is the closure of the complement.

Proof : Suppose $x \in B_0$ and $\neg$"$x$ is interior point of $X$". This implies that

$$x \in B_0 \wedge (\forall n)(x \notin B_n \vee B_n \not\subset X)$$

Assuming $x \in B_0$, this is equivalent to $(\forall n)((x \in B_n) \Rightarrow (\exists y)(y \in B_n \wedge y \notin X))$. Now either $x \in X$ or $x \notin X$. However the latter implies $(\forall n)((x \in B_n) \Rightarrow (y \in B_n \wedge y \in B_0 - X \wedge y \neq x))$.

Conversely, suppose the righthand side held. Then $x \in B_0$ and $x \in B_0 - X$ or "$x$ is a limit point of $B_0 - X$". Now suppose that $x \notin B_0 - X$. This implies $(\forall n)((x \in B_n) \Rightarrow (\exists y)((y \in B_n) \wedge (y \in B_0 - X) \wedge (y \neq x)))$. This is

$$\neg((\exists n)(x \in B_n \wedge (\forall y)((y \notin B_n) \vee (y \in B_0 - X) \vee (y = x))))$$

This implies

$$\neg((\exists n)(x \in B_n \wedge (\forall y)((y \in B_n) \Rightarrow (y \in B_0 - X))))$$

since it was assumed $x \notin B_0 - X$. Finally this is equivalent to $\neg$("$x$ is an interior point of $X$"). □

**Lemma 5.9** $\mathrm{ACA}_0$ proves "Closure Exists".

Proof : The formula in Definition 5.7 is arithmetical. □

**Lemma 5.10** Over $\mathrm{RCA}_0$, "Closure Exists" proves $\mathrm{ACA}_0$.

Proof : Let $\mathscr{T}$ be any topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $X \subset B_0$. In $\mathrm{RCA}_0$, $B_0 - X$ exists. By "Closure Exists", $\overline{B_0 - X}$ exists. Then $B_0 - \overline{B_0 - X}$ is $X^\circ$ by Lemma 5.8. By Theorem 5.5, $\mathrm{ACA}_0$ follows. □

**Theorem 5.11** $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Closure Exists".

Proof : By Lemma 5.9 and Lemma 5.10. □

**Lemma 5.12** $\mathrm{ACA}_0$ proves "Set of Limit Points Exists".

Proof : The formula defining limit points is arithmetical. □

**Lemma 5.13** Over RCA, "Set of Limit Points Exists" proves $\mathrm{ACA}_0$.

Proof : Let $\mathscr{T}$ be some topology given by the encoding $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. Let $X \subset B_0$. By "Set of Limit Points Exists", $X'$ exists. In $\mathrm{RCA}_0$, $X \cup X'$ is defined by the following $\Delta_0$ formula

$$x \in X \vee x \in X'$$

Hence, this set exists in $\mathrm{RCA}_0$. By definition, this is the closure of $X$. By Theorem 5.11, $\mathrm{ACA}_0$ follows. □

**Theorem 5.14** Over $\mathrm{RCA}_0$, $\mathrm{ACA}_0$ is equivalent to "Set of Limit Points Exists".

Proof : By Lemma 5.12 and Lemma 5.13. □

## 6. Various Topologies

**Definition 6.1** Suppose $\mathscr{T}$ is a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. Let $A \subset B_0$. Consider the following formula with free variable $x$ :

$$x \in \mathscr{B} \wedge (\exists m \leq x)(\exists n \leq x)(x = \langle m, n \wedge m \in A)$$

Within $\mathrm{RCA}_0$, let $\mathscr{B}_A$ be the set defined by the above $\Delta^0_0$ formula. $\mathrm{RCA}_0$ can prove that $\mathscr{B}_A$ encodes a topology $\mathscr{T}_A$ over the underlying set $\mathscr{B}^{[0]} = A$. Let $B^A_n := \mathscr{B}^{[n]}$. $(B^A_n)_{n\in\mathbb{N}}$ is the basis for the topology.

## 7. Continuous Functions

**Definition 7.1** Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies encoded by $\mathscr{B}$ and $\mathscr{B}'$, respectively. Let $(B_n)_{n\in\mathbb{N}}$ and $(B'_n)_{n\in\mathbb{N}}$ be the basis for $\mathscr{T}$ and $\mathscr{T}'$, respectively. A function $f : B_0 \to B'_0$ is continuous if and only if

$$(\forall m)(\forall x)(f(x) \in B'_m \Rightarrow (\exists n)(x \in B_n \wedge (\forall y)(y \in B_n \Rightarrow f(y) \in B'_m)))$$

The definition of a continuous function between effective topologies is similar.

**Definition 7.2** Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies encoded by $\mathscr{B}$ and $\mathscr{B}'$, respectively, with basis $(B_n)_{n\in\mathbb{N}}$ and $(B'_n)_{n\in\mathbb{N}}$. A function $f : B_0 \to B'_0$ is a homeomorphism if and only if $f$ is a bijection and $f^{-1} : B'_0 \to B_0$ is continous. $\mathscr{T}$ and $\mathscr{T}'$ are said to be homeomorphic if there exists a homeomorphism between them.

***Open Question* 7.3** Given any topology $\mathscr{T}$ and a sequence of topological spaces $(\mathscr{T}_n)_{n\in\mathbb{N}}$, is the statement:

$$(\exists X)(n \in X \Leftrightarrow \mathscr{T} \text{ homeomorphic to } \mathscr{T}_n)$$

equivalent to any well known subsystems of $Z_2$.

The conjecture is that it is equivalent to $\Pi^1_1$-$\mathrm{CA}_0$, over $\mathrm{RCA}_0$.

## 8. Connectedness

**Definition 8.1** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$.

$$\neg((\exists X)(\exists Y)(X \in \mathscr{T} \wedge Y \in \mathscr{T} \wedge (\exists z)(z \in X) \wedge (\exists z)(z \in Y)$$

$$\wedge\ (\forall z)\neg(z \in X \wedge z \in Y) \wedge (\forall z)(z \in B_0 \Leftrightarrow (z \in X \vee z \in Y))))$$

if and only if $\mathscr{T}$ is connected (or say $B_0$ is connected).

**Definition 8.2** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. A sequence of subsets of $B_0$ is

a set $\mathscr{F}$ such that $(\forall n)(\mathscr{F}^{[n]} \subset B_0)$. Let $F_n := \mathscr{F}^{[n]}$. Let $(F_n)_{n\in\mathbb{N}}$ denote the sequence of subsets of $\mathscr{B}$ encoded by $\mathscr{F}$.

**Lemma 8.3** Let $\mathscr{T}$ be a topology encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $B_0$ encoded by $\mathscr{F}$. Over $\mathrm{RCA}_0$,

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \mathscr{T}_{F_n} \text{“is connected”})$$

proves $\mathrm{ACA}_0$.

Proof : Let $f : \mathbb{N} \to \mathbb{N}$ be an injective function. Define a topology $\mathscr{T}$ by the following formula in free variable $x$ :

$$(\exists m \le x)(\exists n \le x)((x = \langle\langle m, 0\rangle, \langle n, 0\rangle + 1\rangle \wedge m < n)$$
$$\vee (x = \langle\langle m, 1\rangle, \langle n, 1\rangle + 1\rangle \wedge m \ge n) \vee (n = 0))$$

This formula is $\Delta^0_0$. Thus let $\mathscr{B}$ be the set defined by this formula, which exists in $\mathrm{RCA}_0$. Thus informally, $B_0 = \mathbb{N}$, $B_{\langle n,0\rangle+1} = \{\langle m, 0\rangle : m < n\}$, and $B_{\langle n,1\rangle+1} = \{\langle m, 1\rangle : m \ge n\}$.

Define a sequence of subset of $\mathbb{N}$ by the following formula in free variable $x$ :

$$(\exists m \le x)(\exists n \le x)((x = \langle\langle m, 0\rangle, n\rangle \wedge (\forall k < m)(f(k) \ne n)) \vee (x = \langle m, 1\rangle, n\rangle \wedge (\exists k < m)(f(k) = n)))$$

This formula is $\Delta^0_0$, so let $\mathscr{F}$ be the set defined by this formula.

Let $X$ be such that $(\forall n)(n \in X \Leftrightarrow \mathscr{T}_{F_n}$‘is connected”). Now suppose $(\exists k)(f(k) = m)$. Since $f$ is injective, there exists only one such witness $k$. Then $F_m = \{\langle i, 0\rangle : i < k + 1\} \cup \{\langle i, 1\rangle : i \ge k + 1\}$. However $B_{\langle k+1,0\rangle+1} = \{\langle i, 0\rangle : i < k + 1\}$ and $B_{\langle k+1,1\rangle+1} = \{\langle i, 1\rangle : i \ge k + 1\}$. These two sets are nonempty and disjoint. They are open in $\mathscr{T}_{F_m}$. Thus $F_m$ is not connected. $m \notin X$. Suppose $(\forall k)(f(k) \ne m)$. Then $F_m = \{\langle m, 0\rangle : m \in \mathbb{N}\}$. $F_m$ is connected. To prove this, first note that $B_{\langle m,0\rangle+1} \subset B_{\langle n,0\rangle+1}$ if and only if $m \le n$. Now suppose that $F_m$ was not connected. Then there exist $X$ and $Y$ such that $X \in \mathscr{T}_{F_m}$ and $Y \in \mathscr{T}_{F_m}$ which are disjoint, nonempty, and union is $F_m$. Since they are disjoint and nonempty, there exists $x \in X$ and $y \in Y$. By definition of open in $\mathscr{T}_{F_m}$, there exist $B_m$ and $B_n$ such that $x \in B_m$ and $y \in B_n$ and $B_m \cap F_m \subset X$ and $B_n \cap F_m \subset Y$. By definition of $F_m$, it can be proved that $m = \langle m', 0\rangle+1$ and $n = \langle n', 0\rangle+1$ for some $m'$ and $n'$. However, this implies that $\neg(B_{\langle m',0\rangle+1} \subset B_{\langle n',0\rangle+1})$ and $\neg(B_{\langle n',0\rangle+1} \subset B_{\langle m',0\rangle+1})$. Contradiction.

Thus $\mathbb{N} - X$ is the range of $f$. $\mathrm{ACA}_0$ follows from Theorem 2.11. □

**Lemma 8.4** For all topologies $\mathscr{T}$, encoded by $\mathscr{B}$ with basis $(B_n)_{n\in\mathbb{N}}$ and $(F_n)$ a sequence of subsets of $B_0$ encoded by $\mathscr{F}$,

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \mathscr{T}_{F_n} \text{“is connected”})$$

is proved by $\Pi^1_1$-CA.

Proof : From Definition 8.1, “$\mathscr{T}_{F_n}$ is conected” is $\Pi^1_1$. The set $X$ exists by $\Pi^1_1$-CA. □

***Open Question* 8.5** Over $\mathrm{RCA}_0$, is

$$(\exists X)(\forall n)(n \in X \Leftrightarrow \mathscr{T}_{F_n} \text{“is connected”})$$

for all topologies and sequences of subsets $(F_n)_{n\in\mathbb{N}}$ equivalent to any well-known subsystems of $Z_2$?

The conjecture is that it is equivalent to $\Pi^1_1$-$\mathrm{CA}_0$, over $\mathrm{RCA}_0$.

**Definition 8.6** Let $\mathscr{T}$ be a topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $y \in B_0$. Consider the following formula in free variable $x$ :

$$(\forall X)(\forall Y)(((X \in \mathscr{T}) \wedge (Y \in \mathscr{T}) \wedge (x \in X) \wedge (y \in Y) \wedge (\forall z)(\neg((z \in X) \wedge (z \in Y))))$$

$$\Rightarrow (\neg(\forall z)(z \in B_0 \Leftrightarrow z \in X \vee z \in Y)))$$

The set defined by this formula, if it exists, is called the quasicomponent containing $x$.

Let the statement "Quasicomponents Exist" assert that for all topologies $\mathscr{T}$ with basis $(B_n)_{n\in\mathbb{N}}$, and for all $x \in B_0$, the quasicomponent containing $x$ exists.

***Open Question* 8.7** Over $\mathrm{RCA}_0$, is the existence of a quasicomponent of any point in any topology equivalent to any well known subsystems of second order arithmetics?

Certainly $\Pi^1_1$-$\mathrm{CA}_0$ can prove this. The conjecture is that it is equivalent to $\Pi^1_1$-$\mathrm{CA}_0$.

**Definition 8.8** Let $\mathscr{T}$ be a topology with basis $(B_n)_{n\in\mathbb{N}}$. Let $y \in B_0$. Consider the following formula in free variable $x$ :

$$(\exists X)(x \in X \wedge y \in X \wedge X \text{ is connected })$$

Note this formula is $\Sigma^1_2$. Let the statement "Connected Components Exist" assert that for all topology $\mathscr{T}$ with basis $(B_n)_{n\in\mathbb{N}}$, and for all $x \in B_0$, the component containing $x$ exists.

***Open Question* 8.9** Can "Connected Components Exist" be proved in $\Pi^1_1$-$\mathrm{CA}_0$? Over $\mathrm{RCA}_0$, is there any known subsystem of $Z_2$ which is equivalent to this statement?

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