AN INTRODUCTION TO RANDOM WALKS

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Abstract. In this paper, we investigate simple random walks in n-dimensional Euclidean Space. We begin by defining simple random walk; we give particular attention to the symmetric random walk on the d-dimensional integer lattice $\mathbb{Z}^d$. We proceed to consider returns to the origin, recurrence, the level-crossing phenomenon, and the Gambler's Ruin.

Contents

1. Introduction

Informally, a random walk is a path that is created by some stochastic process. As a simple example, consider a person standing on the integer line who flips a coin and moves one unit to the right if it lands on heads, and one unit to the left if it lands on tails. The path that is created by the random movements of the walker is a random walk. For this paper, the random walks being considered are Markov chains. A Markov chain is any system that observes the Markov property, which means that the conditional probability of being in a future state, given all past states, is dependent only on the present state.

In short, Section 2 formalizes the definition of a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$, since most of this paper will deal with random walks of this sort. Section 3 considers returns to the origin, first returns to the origin, and the probability of an eventual return to the origin. Section 4 considers the number of returns to the origin that will occur on a random walk of infinite length. Section 5 focuses on the level-crossing phenomenon. Section 6 examines the Gambler's Ruin problem, which involves 1-dimensional random walks that have imposed boundary conditions.

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2. Simple random walk on $\mathbb{Z}^d$

Consider the d-dimensional integer lattice $\mathbb{Z}^d$. Let e_i denote the d-dimensional standard basis vector with 1 in its i^{th} coordinate and 0 elsewhere. Define X_j to be a random vector with image $\pm e_i$ for some $i \in \{1, ..., d\}$. Assume $X_1, X_2, X_3, ...$ are independent and identically distributed. The *simple random walk* of n steps, denoted by S_n, is defined by

$$S_n = x + \sum_{i=1}^{n} X_i. \tag{2.1}$$

Here, x denotes the position on the lattice at time $n = 0$, and X_j represents the movement from time j to time $j + 1$. In particular, if

$$\Pr(X_i = e_i) = \Pr(X_i = -e_i) = \frac{1}{2d}, \quad i = 1, 2, \ldots, d,$$

then the random walk is called *symmetric*.

In other words, on a *symmetric simple random walk*, the walker can move one unit in any one of the $2d$ possible directions, and is equally likely to move in any one direction. Unless otherwise indicated, the initial position x will be the origin on $\mathbb{Z}^d$, denoted by 0.

3. Returns to the Origin

One of the earliest questions that arises in the study of random walks concerns the probability of returning to the initial position. How likely is it for the walker to return to the origin? We begin by giving a rigorous definition of a return to the origin.

Definition 3.1. Consider a simple random walk S_n on $\mathbb{Z}^d$. A return to the origin, often referred to as an *equalization*, occurs when S_n equals 0 for some n greater than 0. If an infinite number of equalizations occur, then the walk is called *recurrent*. If only a finite number of equalizations occur, then the walk is called *transient*.

We will first consider a simple random walk on $\mathbb{Z}$, and develop a number of ideas that we will generalize for higher dimensions later.

Lemma 3.2. *For a random walk on* $\mathbb{Z}$*,*

$$Pr(S_{2n+1} = 0) = 0$$

and

$$Pr(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$$

Proof. The second equation follows from the fact that the random variables X_i are *i.i.d.* for all i, so that each possible path is equally likely, and by the fact that in order to reach the origin, the walker must take an equal number of positive and negative steps in each direction [3]. □

Now we develop some important relationships between first-returns and equalizations.

Definition 3.3. For a random walk on $\mathbb{Z}$, define the event f_{2n} to be the event that the first equalization occurs at time $2n$. That is, f_{2n} occurs if $S_{2n} = 0$, and $S_{2k} \neq 0$ for all $k = 1, ..., n - 1$. For notational convenience, we write $Pr(f_0) = 0$.

Lemma 3.4. *For $n \geq 1$,*

$$Pr(S_{2n} = 0) = \sum_{k=0}^{n} Pr(f_{2k}) Pr(S_{2(n-k)} = 0) \tag{3.5}$$

Lemma 3.4 is proved in [4, p. 3].

Proof. Partition the collection of paths into n sets, depending on when the first equalization occurs. Now the number of paths that have the first equalization at time $2k$ and another equalization at time $2n$ is given by $Pr(f_{2k})2^{2k}Pr(S_{2n-2k} = 0)2^{2n-2k}$, since it amounts to considering a path that has its first equalization at time $2k$ followed by a path that has an equalization at time 2^{2n-2k}. Here we have used the independence of X_i.

Summing over $k = 1, \ldots, n$, we have the union of the n sets, which gives the total number of paths that have an equalization at time $2n$. Therefore,

$$Pr(S_{2n} = 0)2^{2n} = \sum_{k=0}^{n} Pr(f_{2k})2^{2k} Pr(S_{2n-2k} = 0)2^{2n-2k}.$$

Dividing by 2^{2n} finishes the proof. □

The following lemma establishes a formula for $Pr(f_{2n})$.

Lemma 3.6. *For $n \geq 1$,*

$$Pr(f_{2n}) = \frac{Pr(S_{2n} = 0)}{2n - 1} \tag{3.7}$$

Lemma 3.6 is proved in [4, p. 4].

Proof. Define the functions

$$S(x) = \sum_{n=0}^{\infty} Pr(S_{2n} = 0)x^n$$

$$F(x) = \sum_{n=0}^{\infty} Pr(f_{2n})x^n$$

defined on the interval $x \in (-1, 1)$. Note that the coefficients in the series are in the interval $[0, 1]$, so that the sums converge absolutely and the functions are well-defined. Thus, Lemma 3.4 shows that

$$S(x) = 1 + S(x)F(x) \tag{3.8}$$

The first term on the right hand side follows from $\Pr(S_0 = 0) = 1$. Therefore,

$$F(x) = \frac{S(x) - 1}{S(x)}$$

Note that these manipulations are justified by absolute convergence.

From Lemma 3.2 and the definition of $S(x)$, we know

$$S(x) = \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-2n} x^n,$$

which can be rewritten as

$$S(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{x}{4}\right)^n.$$

Using the Binomial Theorem, it can be shown that

$$\sum_{n=0}^{\infty} \binom{2n}{n} r^n = \frac{1}{\sqrt{1-4r}}$$

so that $S(x) = \frac{1}{\sqrt{1-x}}$.

Therefore,

$$F(x) = \frac{S(x) - 1}{S(x)} = 1 - \frac{1}{S(x)} = 1 - \sqrt{1-x}.$$

By taking the derivitive of $F(x)$, we obtain

$$F'(x) = (1/2)(1-x)^{-1/2} = (1/2)S(x).$$

In order to find the coefficients of the series for $F(x)$, we integrate the series of $\frac{1}{2}S(x)$. We find

$$Pr(f_{2n}) = \frac{Pr(S_{2n-2} = 0)}{2m}$$

Finally, it follows from Lemma 3.2 that $\frac{Pr(S_{2n-2}=0)}{2m} = \frac{Pr(S_{2n}=0)}{(2m-1)}$, which completes the proof. □

We are almost ready to investigate equalization probabilties on $\mathbb{Z}^d$, but before we begin, we must acknowledge *Stirling's Formula*, which states that as $n \to \infty$,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$$

where $\sim$ means that the ratio of the two sides tends to 1. We will not derive this formula here, but a detailed derivation of the formula can be found in Lawler's book [3, p.13].

Let f_{2n}^d be the event that the first equalization of a random walk on $\mathbb{Z}^d$ occurs at time $2n$. Also, let S_{2n}^d denote the position of the walker on $\mathbb{Z}^d$ at time $2n$. For all $d \geq 1$, we have $Pr(S_0^d = 0) = 1$, and $Pr(f_0^d) = 0$ as before. Furthermore, define the functions

$$S^d(x) = \sum_{n=0}^{\infty} P(S_{2n}^d = 0)x^n,$$

$$F^d(x) = \sum_{n=0}^{\infty} P(f_{2n}^d)x^n.$$

on the interval $(-1, 1)$.

By the same argument as before, we have

$$Pr(S_{2n}^d = 0) = \sum_{k=0}^{n} \Pr(f_{2k}^d) Pr(S_{2n-2k}^d = 0)$$

and

$$S^d(x) = 1 + S^d(x)F^d(x)$$

We would like an easy way to describe the probability of an eventual return using the coefficients of $S(x)$ and $F(x)$, and this desire motivates the following definition.

Definition 3.9. Define r_{2n}^d to be the probability that an equalization on $\mathbb{Z}^d$ occurs by time $2n$. The probability that a walker eventually returns to the origin is denoted by

$$r_{\infty}^d = \lim_{n \to \infty} r_{2n}^d.$$

The series

$$F^d(x) = \sum_{n=0}^{\infty} Pr(f_{2n}^d)x^n$$

converges for $x \in (-1, 1]$ by the Comparison Test, and by the fact that the coefficients are non-negative and sum to at most 1. Therefore,

$$\lim_{x \to 1^-} F^d(x) = F^d(1)$$

Now

$$r_\infty^d = \sum_{n=0}^{\infty} Pr(f_{2n}^d),$$

which is equal to $F(1)$. So

$$r_\infty^d = \lim_{x \to 1^-} F^d(x) = \lim_{x \to 1^-} \frac{S^d(x) - 1}{S^d(x)} = 1 - \lim_{x \to 1^-} \frac{1}{S^d(x)}.$$

We state the following lemma:

Lemma 3.10. *For the simple random walk on $\mathbb{Z}^d$,*

$$\lim_{x \to 1^-} S^d(x) = \sum_{n=0}^{\infty} Pr(S_{2n}^d = 0).$$

Lemma 3.10 is proved in [4, p.6].

Proof. Since the coefficients of $S(x)$ are probabilities and therefore non-negative, the power series increases monotonically for $x \in [0, 1]$. Therefore,

$$\lim_{x \to 1^-} S^d(x) \le \sum_{n=0}^{\infty} Pr(S_{2n}^d = 0)$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{N} Pr(S_{2n}^d = 0) &= \lim_{x \to 1^-} \sum_{n=0}^{N} Pr(S_{2n}^d = 0)x^n \\
&\le \lim_{x \to 1^-} \sum_{n=0}^{\infty} Pr(S_{2n}^d = 0)x^n \\
&= \lim_{x \to 1^-} S^d(x)
\end{aligned}$$

Letting $N \to \infty$, the Squeeze Theorem concludes the proof. □

If the series is finite, the probability of an eventual return is

$$r_\infty^d = 1 - \frac{1}{\sum_{n=0}^{\infty} Pr(S_{2n}^d = 0)}.$$

If the series is infinite, then $r_\infty^d = 1$.

Theorem 3.11. *Suppose S_n is a simple random walk on $\mathbb{Z}^d$. If d equals 1 or 2, the probability that an equalization occurs is 1.*

Proof. For $d = 1$, we can refer to the proof of Lemma 3.6, which shows that

$$F^1(x) = F(x) = 1 - \sqrt{(1-x)}.$$

Now

$$\sum_{n=0}^{\infty} Pr(f_{2n}^d) \leq 1,$$

so

$$g_N(x) =: \sum_{n=0}^{N} Pr(f_{2n}^d) x^n$$

converges uniformly in $(-1, 1)$. So we have

$$\begin{aligned}
\sum_{n=0}^{\infty} Pr(f_{2n}^d) &= \lim_{N\to\infty} \lim_{x\to 1^-} g_N(x) \\
&= \lim_{x\to 1^-} \lim_{N\to\infty} g_N(x) \\
&= \lim_{x\to 1^-} F(x) \\
&= 1
\end{aligned}$$

So, with probability 1, the walker will return to the origin on $\mathbb{Z}$.

For $d = 2$, we have

$$Pr(S_{2n}^2 = 0) = \frac{1}{4^{2n}} \binom{2n}{n}^2$$

Using *Stirling's Formula*, it can be shown that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$$

Combining the two preceding statements, we see that

$$Pr(S_{2n}^2 = 0) \sim \frac{1}{\pi n}$$

Now by the divergence of the harmonic series,

$$\sum_{n=0}^{\infty} Pr(S_{2n}^2 = 0) = \infty$$

and so $r_\infty^2 = 1$. □

For higher dimensions, Theorem 3.11 does not hold.

Theorem 3.12. *Suppose S_n is a simple random walk on $\mathbb{Z}^d$. If $d = 3$, the probability of an equalization is less than 1.*

Proof. Keep the definitions and strategies employed in the previous theorem. For $d = 3$, a combinatorial argument in [4, p.8] shows that for j, k with $j, k \geq 0, j+k \leq n$

$$Pr(S_{2n}^3 = 0) = \frac{1}{2^{2n}} \binom{2n}{n} \sum_{j,k} \frac{1}{3^n} \frac{n!}{j!k!(n-j-k)!}$$

Let M denote the maximum value of

$$\frac{n!}{j!k!(n-j-k)!}$$

so that

$$Pr(S^3_{2n} = 0) \leq \frac{1}{2^{2n}} \binom{2n}{n} \sum_{j,k} \frac{M}{3^n} \frac{n!}{j!k!(n-j-k)!} \tag{3.13}$$

Using Stirling's Formula, it can be shown that

$$M \sim \frac{c}{n} \tag{3.14}$$

for some constant c. Now, noting that

$$\sum_{j,k} \frac{1}{3^n} \frac{n!}{j!k!(n-j-k)!} = 1$$

and using (2.13), the right side of (2.12) is bounded above by $d/n^{3/2}$, for some constant d. This is a convergent p-series. Therefore $\sum\limits_{n=0}^{\infty} Pr(S^d_{2n} = 0)$ is finite and $r^3_\infty < 1$. □

Intuitively, because a higher dimensional lattice has more possible moves, it seems that for any integer lattice $\mathbb{Z}^d$, $d > 3$, the probability of returning to the origin should be less than one.

Theorem 3.15. *For any integer lattice $\mathbb{Z}^d$, $d \geq 3$, the probability of returning to the origin is less than one.*

Proof. Normally, we want to think of the position S^d_n as a vector P with d components. However, if we let the vector P have $d+2$ components, and note that the $(d+1)$-component and $(d+2)$-component are always 0, nothing changes in terms of the position on the lattice or the probability of an equalization. With this vector P^{d+2} in mind, we can develop an inductive argument. First define E^d to be the probability of an eventual equalization on $\mathbb{Z}^d$. If we prove that

$$Pr(E^{d_i}) \leq Pr(E^{d_j}), i = j + 1$$

we are done. Note that this inequality is true for $i = 1$ by Theorem 3.11. Now, assuming it is true for $j = n - 1$, we show that

$$Pr(E^{n+1}) \leq Pr(E^n).$$

Since S^n_{2n} is a vector with $n+2$ components, define Z_n to be the event that the first n components are 0. Notice that Z_n can be thought of as the event that the walker returns to the origin on $\mathbb{Z}^n, n \leq n+2$. For S^{n+1}_{2n}, which can be represented by a vector with $n+2$ componments, define the event Z_{n+1} to be the event that the first $n+1$ components are 0. Again, Z_{n+1} is the event that the walker returns to the origin on $\mathbb{Z}^{n+1}, n \leq n+1$. Now $Z_{n+1} \subseteq Z_n$, and so

$$Pr(E_{n+1}) \leq Pr(E_n).$$

This completes the proof. In particular, because the probability of an eventual equalization on $\mathbb{Z}^3$ is less than 1, the probability of an eventual equalization in $\mathbb{Z}^d, d > 3$, is also less than 1. □

4. Number of Equalizations on $\mathbb{Z}^d$

The probablility of an eventual equalization on $\mathbb{Z}^d$ is 1 for $d = 1, 2$, and less than 1 for $d \geq 3$. We now use this information to investigate the number of times that a random walk on $\mathbb{Z}^d$ will have an equalization.

Theorem 4.1. *For a simple random walk on $\mathbb{Z}^d$, $d = 1, 2$, the number of equalizations is infinite. For a simple random walk on $\mathbb{Z}^d$, $d \geq 3$, the number of equalizations is finite.*

Proof. For a random walk on $\mathbb{Z}^d$, let the event E_1 be the first equalization. Similarly, E_n is defined to be the n^{th} equalization. Notice that

$$E_n = E_n \cap E_{n-1} \cap E_{n-2} \cap \ldots \cap E_1$$

since if the n^{th} equalization occurs, the previous equalizations must also occur. Using this and the formula

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

we see that

$$\begin{aligned} Pr(E_n) & \\ &= Pr(E_n|E_{n-1} \cap E_{E-2} \cap \ldots \cap E_1) Pr(E_{n-1} \cap E_{E-2} \cap \ldots \cap E_1) \\ &= Pr(E_n|E_{n-1} \cap E_{E-2} \cap \ldots \cap E_1) \\ &\quad * Pr(E_{n-1}|E_{n-2} \cap \ldots \cap E_1) Pr(E_{n-2} \cap E_{n-3} \cap \ldots \cap E_1) \end{aligned} \tag{4.2}$$

Now because the simple random walk is a Markov chain,

$$Pr(E_n|E_{n-1}) = Pr(E_1) \tag{4.3}$$

Continuing the expansion of 4.2 and noting 4.3, we find that

$$Pr(E_n) = Pr(E_1)^n. \tag{4.4}$$

Thus, for a random walk on the one or two-dimensional integer lattice, the probability that an infinite number of equalizations occurs is 1. However, note that for $d \geq 3$, because $P(E_1) < 1$,

$$\lim_{n \to \infty} Pr(E_n) = 0$$

In other words, the probability of returning to the origin infinitely often on a random walk on $\mathbb{Z}^d$, $d \geq 3$, is 0. Equivilantly, a random walk on $\mathbb{Z}^d, d \geq 3$, has a finite number of equalizations with probability 1. □

5. The Level-Crossing Phenomenon

It can be shown that recurrent random walks reach every point infinitely often. This result is referred to as the level-crossing phenomenon. Before we prove that recurrent random walks do indeed have this property, we must provide several definitions.

Definition 5.1. A random walk on $\mathbb{Z}^d$ is *irreducible* if every point on $\mathbb{Z}^d$ is reachable at some time. Formally, a random walk is irreducible if, for every point P on $\mathbb{Z}^d$,

$$Pr(S^d = P > 0)$$

Notice that since the precise time is not important, we omit the usual subscript beneath S. In all of the previous material, simple random walks were assumed to be irreducible.

Definition 5.2. Define the *period* W of a random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ to be the greatest common divisor of the set $\{k \in \mathbb{Z}^d : Pr(S^d = k) > 0\}$. Since we are working with the simple random walk on $\mathbb{Z}^d$,

$$W = 1$$

Theorem 5.3. *If a simple random walk on $\mathbb{Z}^d$ is recurrent, it visits every point on $\mathbb{Z}^d$ infinitely often.*

While Theorem 5.3 is not directly proved in [2], the majority of the proof comes from elements of this source.

Proof. Let A denote the set of all points that are reachable on a random walk on $\mathbb{Z}^d$. Since we've assumed that the random walks are irreducible, $A = \mathbb{Z}^d$. We know the random walk is recurrent, so by definition it reaches the origin infintely often. Now since $x \in A$,

$$P(S^d = x) > 0$$

An infinite number of equalizations occur, so by the Law of Large numbers the number of times x is reached must also be infinite, since the number of visits approaches a fixed proportion of an infinitely large number. □

Remark 5.4. If a random walk is transient it cannot visit all points infinitely often.

6. Gambler's Ruin

As an example of a more applied problem, consider the following. Suppose a game is being played in which a gambler flips a coin, and gains 1 dollar if it lands on heads, and loses 1 dollar if it lands on tails. Furthermore, suppose that the gambler wins the game if he reaches n dollars, and loses the game if he reaches 0 dollars. This game is represented by a random walk on $\mathbb{Z}$, with the fortune of the gambler at time t given by S_t. By the level-crossing phenomenon, we know the game must terminate, since either point n or 0 is eventually reached. What is the probability of the gambler winning the game, given that he starts with x dollars? In order to answer this question, we first establish several definitions.

Definition 6.1. Define L to be a left-hand boundary point on $\mathbb{Z}$, and R to be a right-hand boundary point on $\mathbb{Z}$. For this discussion, the boundary points are *absorbing*, since the game terminates once the gambler's fortune reaches a boundary point. Mathematically, if at time n, $S_n = L, R$, then $S_k = L, R, k \geq n$.

Definition 6.2. For a simple random walk on $\mathbb{Z}$, let T denote the earliest time that the walker is at a boundary point. In other words,

$$T = min\{n : S_n = L, R\}$$

Define the function $F : \{L, \ldots, R\} \to [0, 1]$ by

$$F(x) = Pr(S_T = R | S_0 = x)$$

We assume the gambler uses a fair coin, so

$$F(x) = \frac{1}{2}F(x-1) + \frac{1}{2}F(x+1), x = L+1, \ldots, R-1 \tag{6.3}$$

Furthermore, we know that since the boundary points are absorbing, $F(L) = 0$, and $F(R) = 1$. Because the game terminates once the gambler is either broke or at a fortune of n dollars, we have $L = 0$ and $R = n$. Now consider the function

$G(x) = x/n$. It is clear that $G(x)$ satisfies the boundary conditions and 6.3. The following theorem shows that this function is the unique solution.

Theorem 6.4. *Suppose a,b are real numbers and R is a positive integer. Then the only function* $F : \{L, \dots, R\} \to \mathbb{R}$ *satisfying 6.3 with* $F(L) = a, F(R) = b$ *is*

$$F(x) = a + \frac{x(b-a)}{R}$$

Theorem 6.4 is proved in [3, p.23].

Proof. Suppose F is a solution to 6.3. For each x with $L < x < R$,

$$F(x) \leq max\{F(x-1), F(x+1)\}$$

To see why this is true, notice that if $F(x+1) = F(x-1)$, equivalence obviously holds, and that if $F(x+1) \neq F(x-1)$

$$max\{F(x-1), F(x+1)\} \geq \frac{1}{2}F(x-1) + \frac{1}{2}F(x+1)$$

Similarly

$$F(x) \geq min\{F(x-1), F(x+1)\}$$

Now suppose we're searching for the maximum value of F on $\{L, \dots, R\}$. Since F(x) is always less than or equal to one of its neighbors, we can always move towards one of the boundary points by selecting $max\{F(x-1), F(x+1)\}$. From this, it is clear that the maximum value of F is obtained at either L or R. Similarly, the minimum of F is obtained at either L or R. Now if $F(L) = F(R) = c$, then $F(x) = c, x \in \{L, \dots, R\}$. Now suppose $F(L) = a$ and $F(R) = b$, which satisfies the boundary conditions specified in the theorem. Let

$$G(x) = a + \frac{x(b-a)}{R}$$

G(x) clearly satisfies the boundary conditions as well. Since $H(x) = G(x) - F(x) \equiv 0$, G(x) = F(x), completing the proof. □

Contextually Theorem 6.4 shows that the probability of the gambler winning the game given that he starts with x dollars is x/n, $x = 1, \dots, n-1$. Now suppose that the coin is biased, so that it lands on heads with probability p and tails with probability q, $p \neq 1/2$. The following is based on material found in [1]. Since $F(x) = pF(x+1) + qF(x-1), x = 1, \dots, n-1$, and $p + q = 1$, we have

$$F(x+1) - F(x) = (q/p)(F(x) - F(x-1)) \tag{6.5}$$

Using 6.5 and solving for $F(2), F(3), \dots, F(n-1)$, we find that

$$F(x) = F(1)\frac{1-(q/p)^x}{1-q/p} \tag{6.6}$$

Next, we use this formula and the fact that $F(n) = 1$ to show that

$$F(1) = \frac{1-q/p}{1-(q/p)^n}$$

Plugging this into 6.6 we have

$$F(x) = \frac{1-(q/p)^x}{1-(q/p)^n} \tag{6.7}$$

Thus, we have derived a formula for the Gambler's Ruin problem with non-symmetricity.

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