THE ZETA FUNCTION AND ITS RELATION TO THE PRIME NUMBER THEOREM

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ABSTRACT. The zeta function is an important function in mathematics. In this paper, I will demonstrate an important fact about the zeros of the zeta function, and how it relates to the prime number theorem.

CONTENTS

1. IMPORTANCE OF THE ZETA FUNCTION

The Zeta function is a very important function in mathematics. While it was not created by Riemann, it is named after him because he was able to prove an important relationship between its zeros and the distribution of the prime numbers. His result is critical to the proof of the prime number theorem.

There are several functions that will be used frequently throughout this paper. They are defined below.

Definition 1.1. We define the **Riemann zeta funtion** as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

when $|z| \geq 1$.

Definition 1.2. We define the **gamma function** as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

over the complex plane.

Definition 1.3. We define the **xi function** for all z with $\text{Re}(z) > 1$ as

$$\xi(z) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Lemma 1.1. *Suppose $\{a_n\}$ is a series. If $\sum_{n=1}^{\infty} a_n < \infty$, then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Further, the product converges to 0 if and only if one if its factors is 0.*

In the later proofs we will find a product form of the zeta function more useful. The next lemma will use the fundamental theorem of arithmetic which states that every positive integer, with the exception of the number 1, can be written as a unique product of primes. This was first proven by Euler showing, for the first time, that there is a relationship between the prime numbers and the zeta function.

Lemma 1.2.

$$\zeta(z) = \prod_p \frac{1}{1-p^{-z}}.$$

Further, $\zeta(z)$ converges for all z with Re(z) >1.

Proof. First we notice that $\sum_{n=0}^{\infty} \frac{1}{p^{nz}}$ converges absolutely for all p greater than 1. Through a simple manipulation, we see that $\sum_{n=0}^{\infty} \frac{1}{p^{nz}}$ can be rewritten as $\frac{1}{1-p^{-z}}$. This is shown below.

$$\sum_{n=0}^{\infty} \frac{1}{p^{nz}} = 1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots$$
$$\frac{1}{p^z} \sum_{n=0}^{\infty} \frac{1}{p^{nz}} = \frac{1}{p^z} + \frac{1}{p^{2z}} + \frac{1}{p^{3z}} + \cdots$$
$$\frac{1}{p^z} \sum_{n=0}^{\infty} \frac{1}{p^{nz}} = \sum_{n=0}^{\infty} \frac{1}{p^{nz}} - 1$$
$$\sum_{n=0}^{\infty} \frac{1}{p^{nz}} \left(\frac{1}{p^z} - 1 \right) = -1$$
$$\sum_{n=0}^{\infty} \frac{1}{p^{nz}} = \frac{1}{1 - p^{-z}}.$$

Let M and N be positive integers such that $M > N$. Any positive integer n with $n \leq N$ can be uniquely written as a product of primes by the fundamental theorem of arithmetic. We know that each prime number in this product must be less than or equal to N, else $n > N$ as all primes in the product are positive integers. Further, any prime in this product is repeated less than M times. We can conclude this by observing that the smallest prime is 2 and $2^M > N$ as $2M < 2^M$ since $2M > N$.

We now wish to show that $\sum\limits_{n=1}^{\infty} \frac{1}{n^z} \leq \prod\limits_{p} \left(\frac{1}{1-p^{-z}} \right)$.

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^z} &\leq \prod_{p \leq N} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots + \frac{1}{p^{Mz}} \right) \text{ needs justification} \\
&= \prod_{p \leq N} \left(\frac{1}{1 - p^{-z}} - \sum_{n=Mz+1}^{\infty} \frac{1}{p^{nz}} \right) \\
&\leq \prod_{p \leq N} \left(\frac{1}{1 - p^{-z}} \right) \\
&\leq \prod_{p} \left(\frac{1}{1 - p^{-z}} \right).
\end{aligned}$$

Now it is enough to prove that $\lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} \frac{1}{n^z} \leq \prod\limits_{p} \left(\frac{1}{1-p^{-z}} \right)$. Suppose this is not true. There exists an $\epsilon > 0$ such that, for every integer L, there exists an integer $s > L$ with $\sum_{n=1}^{s} \frac{1}{n^z} > \prod_p \left(\frac{1}{1-p^{-z}}\right)$. But, this is a contradiction, as our choice of N was arbitrary.

Finally we must show that the reverse inequality holds. We use the fundamental theorem of arithmetic to see that

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^z} &\geq \prod_{p\leq N} \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots + \frac{1}{p^{Mz}}\right) \\ &= \prod_{p\leq N} \left(\frac{1}{1-p^{-z}} - \sum_{n=Mz+1}^{\infty} \frac{1}{p^{nz}}\right) \\ &\geq \prod_{p\leq N} \left(\frac{1}{1-p^{-z}}\right) \\ &\geq \prod_{p} \left(\frac{1}{1-p^{-z}}\right).\end{aligned}$$

As before, we see that $\sum_{n=1}^{\infty} \frac{1}{n^z} \geq \prod_p \left(\frac{1}{1-p^{-z}}\right)$ in the limit. Together, these two inequalities prove that $\zeta(z) = \prod_p \frac{1}{1-p^{-z}}$.

Letting $a_n = \frac{1}{n^z}$, it is clear that $\zeta(z)$ converges for all z with $\text{Re}(z) > 1$ by Lemma 1. Since none of the factors are 0 here, we see that $\zeta(z) \neq 0$ when $\text{Re}(z) > 1$ by lemma1 concluding the proof. □

2. Trivial Zeros

The region $0 \leq \text{Re}(z) \leq 1$ is called the **critical strip**. We will soon turn to the study of the zeros in the critical strip. The zeros of the zeta function which lie outside the critical strip are called the **trivial zeros**, and they are easily expressed in a simple formula. We derive this formula in the lemma and theorem below. Before this, we will state without proof, an essential Theorem about the Gamma function.

Theorem 2.1. *The gamma function has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the negative integers.*

Theorem 2.2. *For all $z \in \mathbb{C}$*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Lemma 2.1. $1/\Gamma(z)$ *is an entire function of z with simple zeros (zeros having a multiplicity of one) at $z = 0, -1, -2, \ldots$, and it has zeros nowhere else.*

Proof. Theorem 2.2 allows us to write $\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin \pi z}{\pi}$. Since, by theorem 2.1 we know that the simple poles of $\Gamma(1-z)$ are any $z \in \{n | n \in \mathbb{N}\}$, we see that these cancel with the zeros of $\sin \pi z$. Thus $\frac{1}{\Gamma(z)}$ is entire with simple zeros in $\{0, -1, -2, \ldots\}$. □

Before the proof of the trivial zeros, we will state a theorem known as the fundamental relation.

Theorem 2.3. $\xi(z) = \xi(1-z)$ *for all* $z \in \mathbb{C}$.

Theorem 2.4. *The only zeros of the zeta function outside of the critical strip are the negative even integers.*

Proof. We have already seen, in lemma 2.2 that the zeta function has no zeros when $\text{Re}(z) > 1$. So, we will restrict our consideration to $z < 0$. Using theorem 3.3 we see that

$$\zeta(z) = \pi^{z-\frac{1}{2}} \frac{\Gamma\left((1-z)/2\right)}{\Gamma(z/2)} \zeta(1-z)$$

By lemma 2.2 we see that $\zeta(1-z)$ has no zeros when $\text{Re}(z) < 0$. Since $\Gamma(z)$ has no zeros, $\Gamma\left((1-z)/2\right)$ clearly has no zeros. By lemma 3.1, $\frac{1}{\Gamma(z/2)}$ has zeros at $z = -2, -4, \ldots$ concluding the proof. □

3. Important Observations

Before we can prove a crucial theorem about the zeros of the zeta function on the critical strip, we must prove a few more properties about the zeta function.

Lemma 3.1. *If* $Re(z) > 1$, *then*

$$\log \zeta(z) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

for some $c_n \geq 0$.

Proof. First we will consider the case where $\text{Im}(s) = 0$. First we observe that, through a Taylor expansion, $\log\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m}$ for $0 \leq x < 1$. Combining this fact with lemma 2.2 and a property of the logarithm we see that

$$\log \zeta(z) = \log \prod_p \frac{1}{1-p^{-z}} = \sum_p \log\left(\frac{1}{1-p^{-z}}\right) = \sum_{p,m} \frac{p^{-mz}}{m}.$$

We will now show that the above double sum converges absolutely. It suffices to show that $\sum_p \frac{p^{-mz}}{m}$ and $\sum_m \frac{p^{-mz}}{m}$ both converge absolutely when $m, p \geq 1$.

To show that $\sum_m \frac{p^{-mz}}{m}$ converges absolutely we apply the ratio test.

$$\begin{aligned}\lim_{m\to\infty} \frac{p^{-(m+1)z}}{m+1}\frac{m}{p^{-mz}} &= \lim_{m\to\infty}\frac{m}{m+1}p^{-z}\\ &= p^{-z}\\ &\leq 1 \text{ as } p \geq 1 \text{ and } z \geq 1.\end{aligned}$$

To show that $\sum_p \frac{p^{-mz}}{m}$ converges absolutely we observe that the quantity mz is greater than one since m is at least one and z is greater than one. Since the double sum converges absolutely the order of the summation does not matter. The double sum converges absolutely, so it converges whenever $\mathrm{Re}(z) > 1$ giving us another analytic function. By analytic continuation, we now have $\log\zeta(z) = \sum_{p,m} \frac{p^{-mz}}{m}$ whenever $\mathrm{Re}(z) > 1$. Now, if we let $c_n = \frac{1}{m}$ if $n = p^m$ and $c_n = 0$ otherwise, we have completed the proof.

□

Lemma 3.2. *If* $\sigma > 1$ *and* $Im(t) = 0$ *then*

$$\log|\zeta^3(\sigma)\,\zeta^4(\sigma+it)\,\zeta(\sigma+2it)| \geq 0.$$

Proof. We can let $z = \sigma + it$ and observe

$$\begin{aligned}Re(n^{-z}) &= Re(n^{-\sigma-it})\\ &= Re(e^{(-\sigma-it)\log n})\\ &= Re(e^{-\sigma\log n}e^{-it\log n})\\ &= Re(e^{-\sigma\log n}(\cos(t\log n) + i\sin(-t\log n))\\ &= e^{-\sigma\log n}\cos(t\log n)\\ &= n^{-\sigma}\cos(t\log n).\end{aligned}$$

It now follows that

$$\begin{aligned}\log|\zeta^3(\sigma)\,\zeta^4(\sigma+it)\,\zeta(\sigma+2it)| &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|\\ &= 3\mathrm{Re}(\log\zeta(\sigma)) + 4\mathrm{Re}(\log\zeta(\sigma+it)) + \mathrm{Re}(\log\zeta(\sigma+2it))\\ &= \sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos(t\log n) + \cos 2t\log n)\end{aligned}$$

By lemma 3.1. It follows by calculation that this is positive. □

4. Zeros on Re(z)=1

We are almost ready to prove an important theorem about the zeros of the zeta function in the critical strip. Two important concepts used in this proof are those of zeros and poles. A **zero** of a function is a point at which that function vanishes. A **pole** of a function$f$ is a point at which, if $1/f$ is defined to be zero, then $1/f$ is holomorphic at this point. We will now state two important theorems about zeros and poles.

Theorem 4.1. *Let f be holomorphic in a connected open set* Ω *such that it has a zero at* $z_0 \in \Omega$, *and* $f(z) \neq 0$ *for all* $z \in \Omega$. *Then there exists a neighborhood* $U \subset \Omega$ *of* z_0, *a non-vanishing holomorphic function g on* U, *and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \text{ for all } z \in U.$$

Theorem 4.2. *If f has a pole at* $z_0 \in \Omega$ *then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that*

$$f(z) = (z - z_0)^{-n} h(z).$$

We are now ready to prove our theorem.

Theorem 4.3. *The zeta function has no zeros on the line Re*$(z) = 1$.

Proof. Suppose that there exists a point $z_0 = 1 + it_0$ with $t_0 \neq 0$ such that $\zeta(z_0) = 0$. ζ is homomorphic at z_0, so, by theorem 4.1, there exists a $C > 0$ such that

$$\lim_{\sigma \to 1} \frac{|\zeta(\sigma + it_0)|^4}{C(\sigma - 1)^4} \leq 1.$$

Similarly, by theorem 4.2 (as z=1 is a simple pole), we know there exists a $C' > 0$ such that

$$\lim_{\sigma \to \infty} \frac{|\zeta(\sigma)|^3}{C'(\sigma - 1)^{-3}} \leq 1.$$

Since ζ is holomorphic at $\sigma + 2it_0$ for all σ, $|\zeta(\sigma + 2it_0)$ stays bounded as σ approaches 1 by continuity. Thus

$$\lim_{\sigma \to 1} |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| = 0.$$

But this contradicts lemma 4.3 as the logarithm of a number less than one is negative concluding the proof. □

5. Estimating $1/\zeta$ and ζ'

The proof of the prime number requires knowledge of the rates of growth of $1/\zeta$ and ζ'. We will now prove theorems that will deal with this. First we will state a theorem given in *Complex Analysis.*[1] handling the growth of ζ'.

Theorem 5.1. *Suppose $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Then for each $\sigma_0, 0 \leq \sigma_0 \leq 1$, and every $\epsilon > 0$, there exists a constant c_ϵ, so that*

(1) $|\zeta(z)| \leq c_\epsilon |t|^{1-\sigma_0+\epsilon}$, *if* $\sigma_0 \leq \sigma$ *and* $|t| \geq 1$
(2) $|\zeta'(z)| \leq c_\epsilon |t|^{1-\sigma_0+\epsilon}$, *if* $1 \leq \sigma$ *and* $|t| \geq 1$

Now we prove a theorem dealing with the growth of $1/\zeta$. With this theorem and the ones preceding it we will finally be able to analyse the theorems which eventually lead to the prime number theorem.

Theorem 5.2. *For every $\epsilon > 0$, we have $1/|\zeta(z)| \leq c_\epsilon |t|^\epsilon$ when $z = \sigma + it, \sigma \geq 1$, and $|t| \geq 1$.*

Proof. From the poof of Theorem 5.3 we know that $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1$ when $\sigma \geq 1$ else we end up with a contradiction. Applying theorem 6.1 we get

$$|\zeta^4(\sigma + it)| \geq c|\zeta^{-3}(\sigma)||t|^{-\epsilon} \geq c'(\sigma - 1)^3|t|^\epsilon \text{ for all } \sigma \geq 1 \text{ and } |t| \geq 1.$$

By taking the a fourth root we arrive at the equation

$$|\zeta(\sigma + it)| \geq c'(\sigma - 1)^{3/4}|t|^{\epsilon/4} \text{ when } \sigma \geq 1 \text{ and } |t| \geq 1. \tag{1}$$

We will choose a constant A later in the proof. Now We look at two cases. The first is the case where $\sigma - 1 \geq A|t|^{-5\epsilon}$ holds. (1) now gives us $|\zeta(\sigma + it)| \geq A'|t|^{-4\epsilon}$.

If the inequality does not hold, then we take $\sigma' > \sigma$ with $\sigma' - 1 = A|t|^{-5\epsilon}$. This now implies

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - |\zeta(\sigma' + it) - \zeta(\sigma + it)|.$$

Now we use the mean value theorem with theorem 6.1 to get

$$|\zeta(\sigma + it) - \zeta(\sigma + it)| \leq c''|\sigma' - \sigma||t|^\epsilon \leq c''|\sigma' - 1||t|^\epsilon.$$

using (1) and setting $\sigma = \sigma'$ we see that

$$\zeta(\sigma + it)| \geq c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} - c''(\sigma - 1)|t|^\epsilon.$$

Now choosing $A = \left(\frac{c'}{2c''}\right)^4$ we get $c'(\sigma' - 1)^(3/4)|t|^{-\epsilon/4} = 2c''(\sigma' - 1)|t|^\epsilon$. Thus, we have $|\zeta(\sigma + it)| \geq A''|t|^{-4\epsilon}$ completing the proof.

□

6. The ψ Function

Finally we will look at a function which imitates the distribution of the primes. It was created by Tchebychev. We define

$$\psi(x) = \sum_{p^m \leq x} \log p.$$

In this notation, p is a prime number and m is a positive integer. We can now create two other definition. We define $\Lambda = \log p$ if $n = p^m$ for some prime p and $m \geq 1$, and 0 otherwise.now we have

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n)$$

and

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p$$

where $[x]$ is the greatest integer function. We will now prove just how closely related the ψ function is to the distribution of prime numbers. In the following theorems $\pi(x)$ denotes the number of primes less than or equal to x.

Theorem 6.1. *If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim x/\log x$ as $x \to \infty$.*

Proof. We will prove this by showing that the following two inequalities hold

$$1 \leq \lim_{x\to\infty} \inf \pi(x)\frac{\log x}{x} \text{ and } \lim_{x\to\infty} \sup \pi(x)\frac{\log x}{x} \leq 1.$$

Now notice the following bound on the ψ function which comes from the definition of the greatest integer function.

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p = \pi(x) \log x.$$

Now we divide by x to find $\frac{\psi(x)}{x} \leq \frac{\pi(x)\log x}{x}$. Since $\psi(x) \sim x$, we have $\lim\limits_{x\to\infty} \frac{\psi(x)}{x} = 1$. Thus, the first inequality is clearly true.

Now, fix $0 < \alpha < 1$, and notice that $\psi(x) \geq \sum\limits_{p \leq x} \log p \geq \sum\limits_{x^\alpha < p \leq x} \log p \geq (\pi(x) - \pi(x^\alpha)) \log x^\alpha$. So, now

$$\psi(x) + \alpha\pi(x^\alpha) \log x \geq \alpha\pi(x) \log x.$$

dividing by x gives

$$\frac{\psi(x)}{x} + \frac{\alpha\pi(x^\alpha)\log x}{x} \geq \frac{\alpha\pi(x)\log x}{x}$$
$$1 + \frac{\alpha\pi(x^\alpha)\log x}{x} \geq \frac{\alpha\pi(x)\log x}{x} \text{ since } \psi(x) \sim x$$
$$1 \geq \alpha\frac{logx(\pi(x) - \pi(x^c antarell\alpha}{x}$$
$$1 \geq \alpha \lim_{x\to\infty} \sup \pi(x)\frac{\log x}{x} \text{ as } \alpha < 1.$$

This concludes the proof. □

It is actually easier to work with the ψ_1 function. We define

$$\psi_1(x) = \int_1^x \psi(u)du.$$

We will now show that the same asymptotic properties applies to this function.

Theorem 6.2. *If $\psi_1(x) \sim x^2/2$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$, and therefore $\pi(x) \sim x/logx$ as $x \to \infty$*

Proof. By theorem 6.1 we only need to prove that $\psi(x) \sim x$ as $x \to \infty$. If $\alpha < 1 < \beta$, then we have the following as ψ is an increasing function

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^x \psi(u)du \leq \psi(x) \leq \frac{1}{x(\beta-1)}\int_x^{\beta x} \psi(u)du.$$

We now immediately have $\psi(x) \leq \frac{1}{(\beta-1)x}(\psi_1(\beta x) - \psi_1(x))$. Thus $\frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)x}(\frac{\psi_1(\beta x)}{(\beta x)^2} - \frac{\psi_1(x)}{x^2})$.And so we have

$$\lim_{x\to\infty} \sup \frac{\psi(x)}{x} \leq 1/2(\beta+1)$$

Thus, we have proved that $\lim_{x\to\infty} \sup \psi(x)/x \leq 1$. Similarly, we can show $\lim_{x\to\infty} \inf \psi(x)/x \geq 1$ concluding the proof. □

We now state a lemma from *Complex Analysis.*[1].

Lemma 6.1. *For all $c > 1$*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

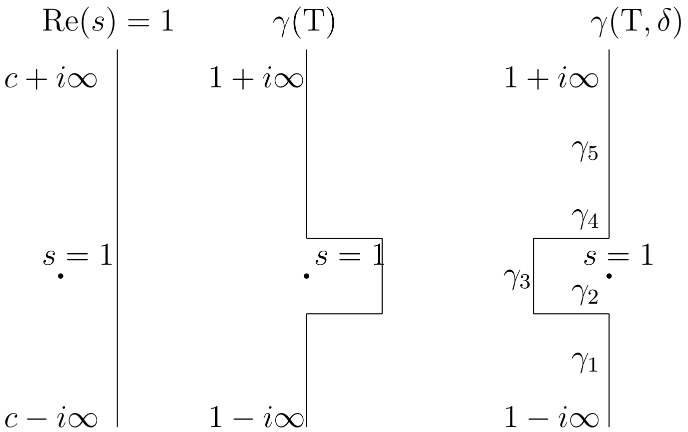
With this theorem we are now ready to prove the fact about the zeta function which lead to the prime number theorem.

Theorem 6.3. *$\psi_1 \sim \frac{x^2}{2}$ as $x \to \infty$.*

Proof. By lemma 6.1 we know $\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$. In this integral, we integrate on the line $\mathrm{Re}(s) = c$. We will now transform this to integrate over $\mathrm{Re}(s) = 1$. when $c > 1$. For now we assume that $x \geq 2$.

$$\text{Let } F(s) = \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right).$$

We define the paths below using T to determine the size of the "box" containing the point s.



An application of Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds.$$

Now we need to show the equivalence of the integral over $\gamma(T, \delta)$. For any T, we can pick a δ greater than zero such that the zeta function has no zeros when $|\mathrm{Im}(s)| \leq T$ and $1 - \delta \leq \mathrm{Re}(s) \leq 1$ as the zeta function has no zeros on $\mathrm{Re}(s) = 1$. $F(s)$ has a simple pole at $s = 1$. By calculation, the residue of $F(s)$ at $s = 1$ is $\frac{x^2}{2}$. Thus we find

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} \frac{x^{s+1}}{s(s+1)}F(s)ds.$$

If we let $s \in \gamma_1$, we see that $|x^{1+s}|\int_{\gamma_1} F(s)ds| \leq \frac{\epsilon}{2}x^2| = x^2$. By theorem 5.2 we can find a C such that

$$|\int_{\gamma_1} F(s)ds| \leq Cx^2\int_T^{\infty} \frac{|t|^{1/2}}{t^2}dt.$$

Since this converges we can find T large enough so that $|\int_{\gamma_1} F(s)ds| \leq \frac{\epsilon}{2}x^2$ and $|\int_{\gamma_5} F(s)ds| \leq \frac{\epsilon}{2}x^2$. We fix a T to satisfy this and pick a δ small enough as well.

Since, on γ_3, we have $|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta}$, we can find a C' such that $|\int_{\gamma_3} F(s)ds| \leq C'x^{2-\delta}$.

We can also approximate γ_2 and γ_4 by

$$|\int_{\gamma_2} F(s)ds| \leq C'' \int_{1-\delta}^{1} x^{1+\sigma} d\sigma \leq C'' \frac{x^2}{\log x}.$$

Thus, we now have

$$|\psi_1(x) - \frac{x^2}{2}| \leq \epsilon x^2 + C'x^{2-\delta} + C'' \frac{x^2}{\log x}.$$

Thus, we have

$$|\frac{2\psi_1(x)}{x^2} - 1| \leq 4\epsilon$$

□

7. Acknowledgements

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References

[1] Elias M. Stein, Rami Shakarchi, *Complex Analysis*, Princeton University Press, New Jersey, 2003.