SOBOLEV SPACES WITH APPLICATION TO SECOND-ORDER ELLIPTIC PDE

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Abstract. This paper will motivate and define the Sobolev Space $W^{k,p}(\Omega)$ and then examine this space from a functional analytic perspective. This theory will then be applied to second-order elliptic partial differential equations, concluding with a proof of the First Existence Theorem for solutions to this class of PDE.

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1. Introduction and Motivation

When solving partial differential equations, most analysts are not interested in 'messy' solutions. That is, a solution formed by piecing together solutions which have discontinous derivatives for small time windows is not exactly optimal. Rather, a more useful solution is one which has some nicer topological and analytic properties, such as continuity, infinite differentiblity, continuous derivatives, etc. These nicer solutions often exist and can be quite easy to find, but require that we loosen the conditions for one function to be the derivative of another. The theory of Sobolev Spaces is based on this sort of redefinition of differentibility. Specifically, Sobolev Spaces give what are called *weak* solutions to differential equations - and do so because they are based on the notion of the *weak derivative*. We will see that by moving away from the classical, or *strong*, derivative we will be able to prove some quite powerful theorems about the existence of solutions to uniformly elliptic equations. Though this will require some advanced machinery from functional analysis, the proofs themselves will not be incredibly complex. Ultimately, this is why Sobolev Spaces are so useful: they allow for the use of functional analysis. We do not need to build an entirely new branch of analysis in order to solve these linear partial differential equations. Rather, we appeal to one which is well developed and exploit the powerful tools it has to offer.

In this paper we will assume knowledge of almost all functional analytic tools used. We will invoke the Hahn-Banach Theorem, the Arzela-Ascoli Compactness

Criterion, and the Riesz Representation Theorem, as well as various smaller results. There is also a fair amount of basic background knowledge required, such as the basic theories of mollification and of Banach, Hilbert, and L^p Spaces. Standard results, such as the ones mentioned above, will not be proven, but can be found in any standard analysis text. For the most part, I will assume that the functional analytic notation I use is standard and will not define specific function spaces.

The paper itself is divided into three sections excluding this one. The following section provides the initial definitions and concepts, hopefully familiarizing the reader with the notation, which can occasionally be quite dense. Basic properties of the weak derivative will be discussed, but not too much can be elaborated upon until the Sobolev Spaces themselves are expounded and explored. The second section does exactly this via proving the functional analytic properties of the Sobolev Spaces. We present the standard results: completeness, density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$, the Extension Theorem, the Rellich-Kondrachov Compactness Theorem, and the dual space of $W^{1,p}(\Omega)$ will be characterized as a subspace of $L^p(\Omega)$. The third section will begin the discussion of uniformly elliptic equations from the most basic level, and really no background in differential equations is needed. The First Existence Theorem for uniformly elliptic equations will be proven, following a brief introduction to elliptic PDE and the proofs of the Lax-Milgram Theorem and an inequality allowing us to use it.

2. Initial Definitions and Theory

As stated in the introduction, the study of Sobolev Spaces revolves around the concept of the weak derivative. Hence, we begin by setting up notation for this derivative and motivating its definition.

Definition 2.1. A *multiindex* α is an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$, used to concisely denote the partial differential operator

$$D^\alpha(u) := \frac{d^\alpha}{dx_1^{\alpha_1}...dx_n^{\alpha_n}}(u).$$

We define $|\alpha| = \alpha_1 + ... + \alpha_n$ to be the *degree* of α.

Definition 2.2. For two multiindices α, β we define the following:

(1) $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$, for all $1 \leq i \leq n$
(2) If $\alpha \leq \beta$, we define $\alpha - \beta := \gamma$, where $\gamma = (\alpha_i - \beta_i, ..., \alpha_n - \beta_n)$
(3) $\alpha! := \alpha_1!\alpha_2!...\alpha_n!$

Notation 2.3. For the remainder of the paper, let Ω be a bounded open subset of $\mathbb{R}^n$.

The weak derivative for some function $u \in \mathcal{C}^1(\Omega)$ is defined by integrating against an arbitrary $\phi \in \mathcal{C}_c^\infty(\Omega)$. Because our function u has continuous derivatives, we can integrate by parts to obtain the following:

$$\begin{aligned}\int_\Omega u\frac{\partial \phi}{\partial x_i}\, dx &= u\phi\Big|_{\partial\Omega} - \int_\Omega \frac{\partial u}{\partial x_i}\phi\, dx \\ &= -\int_\Omega \frac{\partial u}{\partial x_i}\phi\, dx \quad \text{(because } \phi \text{ vanishes on the boundary)}\end{aligned}$$

We see immediately that by repeated integration by parts we can generalize this result to a partial differential operator α of arbirary degree, so long as we take $u \in \mathcal{C}^{|\alpha|}(\Omega)$ and account for the parity dependence of the minus sign:

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi \, dx \tag{1}$$

If $u \in \mathcal{C}^{|\alpha|}(\Omega)$, the above formula is valid for every $\phi \in \mathcal{C}_c^\infty(\Omega)$. The notion of the weak derivative asks if this formula is valid when u is not in $\mathcal{C}^{|\alpha|}(\Omega)$. We insist that u be a locally integrable function (that is, it be integrable on compact sets), because otherwise the left hand side of the above equality is meaningless. The right hand side of the equality poses an even bigger problem. How can we define $D^\alpha u$ if u is not in $\mathcal{C}^{|\alpha|}(\Omega)$? We can no longer use the traditional analytic tools to derive the 'correct' solution to this equation, so instead we use this apparent ambiguity to define the weak derivative. In the definition, $D^\alpha u$ is replaced by $v \in L^p(\Omega)$ and we say that any v satisfying the equality for every $\phi \in \mathcal{C}_c^\infty(\Omega)$ is the weak derivative of u.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$, $u, v \in L^p(\Omega)$, and α be a multi-index. Then v is the *weak α-th partial derivative* of u if:

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v \phi \, dx$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega)$.

Note that because this definition relies on integration we can only talk about the weak derivative of a function up to sets of measure 0. It is easy to show, however, that the weak derivative is unique if we consider equivalence classes of functions up to sets of measure 0.

Before we prove some elementary properties of these weak derivatives, we introduce Sobolev spaces in order to make notation much easier in the statements and proofs, and because they are the topic of the paper.

Definition 2.5. Let $1 \leq p \leq \infty$ and k be a nonnegative integer. The *Sobolev Space* $W^{k,p}(\Omega)$ consists of all functions $u : \Omega \to \mathbb{R}$, $u \in L^p_{loc}(\Omega)$ such that each weak derivative $D^\alpha u$ with $|\alpha| \leq k$ exists and belongs to L^p. That is,

$$W^{k,p}(\Omega) = \{u \in L^p_{loc}(\Omega) | \ \|D^\alpha u\|_{L^p(\Omega)} < \infty, \ \forall \ |\alpha| \leq k\} \tag{2}$$

Similarly, the space $W^{k,p}_{loc}(\Omega)$ consists of the functions u as above for which $D^\alpha u$ with $|\alpha| \leq k$ exists and belongs to $L^p_{loc}(V)$, where V is an arbitrary compact subset of Ω.

We now explore the basic properties of weak derivatives to see that they behave in some familiar ways.

Theorem 2.6. *(Elementary facts about weak derivatives) Assume $u, v \in W^{k,p}(\Omega)$, $|\alpha| \leq k$. Then*

1. *$D^\alpha(\Omega) \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multi-indices α, β satisfying $|\alpha| + |\beta| \leq k$.*
2. *For each λ, $\mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \leq k$.*

Proof. (1) The first part of the statement follows directly from the definition of being k-times weak differentiable. For the second part, choose $\phi \in \mathcal{C}_c^\infty(\Omega)$. Then, of course, $D^\beta \phi \in \mathcal{C}_c^\infty(\Omega)$. So,

$$\begin{aligned}\int_\Omega D^\alpha u D^\beta \phi \, dx &= (-1)^{|\alpha|} \int_\Omega u D^{\alpha+\beta} \phi \, dx \\ &= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_\Omega D^{\alpha+\beta} u\phi \, dx \\ &= (-1)^{|\beta|} \int_\Omega D^{\alpha+\beta} u\phi \, dx\end{aligned}$$

This shows that $D^\beta(D^\alpha(u)) = D^{\alpha+\beta}u$. The same proof with α and β switched gives the other equality in the statement.

(2) These proofs follow directly from the linearity of partial differentiation. This shows that $W^{k,p}(\Omega)$ is a vector space over $\mathbb{R}$.

□

As with any function space, it is usually more fruitful to consider $W^{k,p}(\Omega)$ as a normed linear space than just as a linear space. The following norm will be used throughout the remainder of the paper. We leave it to the reader to verify that it satisfies the required properties of a norm.

Definition 2.7. We define the following norm on $u \in W^{k,p}(\Omega)$:

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} (\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p \, dx)^{1/p} & \text{if } 1 \le p < \infty \\ \sum_{|\alpha|\le k} \text{ess sup}_\Omega \, |D^\alpha u| & \text{if } p = \infty \end{cases} \tag{3}$$

3. Functional Analytic Properties of $W^{k,p}(\Omega)$

We now begin an exposition of the properties of Sobolev spaces. We will see that they have just about every nice functional analytic property that we could ask for, which is the main reason they are good to work with. First, we show that all Sobolev spaces are Banach and that a special one is Hilbert.

Theorem 3.1. *(Sobolev spaces are complete) Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $1 \le p \le \infty$. Then*

(1) the space $W^{k,p}(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{W^{k,p}}$
(2) the space $H^1(\Omega) := W^{1,2}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle := \int_\Omega uv \, dx + \sum_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \, dx.$$

Proof. (1) Take a Cauchy sequence $\{u_m\}_{m\in\mathbb{N}}$ in $W^{k,p}(\Omega)$. Then, because the norm on $W^{k,p}(\Omega)$ is defined in terms of the norm on $L^p(\Omega)$, for each $|\alpha| \le k$, $\{D^\alpha u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$. We know that $L^p(\Omega)$ is complete, so $\exists u_\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_m \to u_\alpha$$

in $L^p(\Omega)$ for each $|\alpha| \le k$. In particular, we can find some $u \in L^p(\Omega)$ such that for $\beta = (0, ..., 0)$,

$$u_m \to u_\beta = u$$

in $L^p(\Omega)$. We must show that this function u is a limit of $\{u_m\}_{m\in\mathbb{N}}$ in the $W^{k,p}(\Omega)$ sense. That is we must show that

$$u \in W^{k,p}, D^\alpha u = u_\alpha, \tag{4}$$

with the equality in the $W^{k,p}$ sense. To this end, note that because Ω is a bounded measure space, convergence in $L^p(\Omega)$ implies convergence in $L^1(\Omega)$ by Hölder's Inequality. As a result, if $\phi \in \mathcal{C}_c^\infty(\Omega)$, we can exchange limits and integrals in the following way:

$$\begin{aligned}\int_\Omega uD^\alpha\phi \, dx &= \lim_{m\to\infty}\int_\Omega u_m D^\alpha\phi \, dx \\ &= \lim_{m\to\infty}(-1)^\alpha\int_\Omega D^\alpha u_m\phi \, dx \\ &= (-1)^\alpha\int_\Omega u_\alpha\phi \, dx\end{aligned}$$

This proves equation 5. The above analysis shows that $u_m \to u$ in $W^{k,p}$ and, thus that the space is complete.

(2) To show that $H^1(\Omega)$ has the additional structure of a Hilbert space, we must show the proposed inner product above satisfies the required properties. The proofs are direct, and we omit them here.

□

In general, it is true that $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space, though the inner product is slightly different.

We now begin the process of determining a dense subspace of $W^{k,p}(\Omega)$. This will require a bit of creativity, arising in the form of the mollification process. Specifically, we will need the result that for a standard mollifier η_ϵ and $u \in L^p(\Omega)$, the sequence $\{\eta_\epsilon * u\} \to u$ in $L^p(\Omega)$. We show density of $\mathcal{C}_c^\infty(\Omega)$ in $W^{k,p}_{loc}(\Omega)$ and then extrapolate that result to all of $W^{k,p}(\Omega)$. First, however, we present a lemma we will need for the general case.

Lemma 3.2. *Given $u \in W^{k,p}(\Omega)$ and $v \in \mathcal{C}_c^\infty(\Omega)$, $uv \in W^{k,p}(\Omega)$.*

Proof. Unfortunately, the easiest way to determine a function is in a Sobolev space is to explicitly compute the function's weak derivative. To this end, we guess that the derivative will have the following form as a basis for induction on $|\alpha|$:

$$D^\alpha(uv) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta v D^{\alpha-\beta}u$$

Fix $\phi \in \mathcal{C}_c^\infty(\Omega)$. Note that because ϕ and v are already differentiable, their weak derivative necessarily obeys the product rule for classical derivatives. So if $|\alpha| = 1$,

$$\begin{aligned}\int_\Omega uvD^\alpha(\phi) \, dx &= \int_\Omega uD^(v\phi) - uD^\alpha(v)\phi \, dx \\ &= -\int_\Omega (D^\alpha(u)v + uD^\alpha(v))\phi \, dx,\end{aligned}$$

where we have used integration by parts and the fact that $v\phi$ is still compactly supported only on Ω to obtain the final equality. We know that $D^\alpha u$ exists (as $u \in W^{k,p}$) and $D^\alpha(v)$ exists (as $v \in \mathcal{C}_c^\infty(\Omega)$). Thus, the result holds in the base case. Assume $l < k$ and that the result holds for all $|\alpha| \le l$ and all functions

$v \in \mathcal{C}_c^\infty$. Choose a multiindex α with $|\alpha| = l + 1$. Then, we can decompose $|\alpha|$ in terms of some other multiindicies β, γ satisfying $\alpha = \beta + \gamma$ with $|\beta| = l, |\gamma| = 1$. Then, for any $v, \phi \in \mathcal{C}_c^\infty$,

$$\begin{aligned}\int_\Omega uvD^\alpha\phi \; dx &= \int_\Omega uvD^\beta(D^\gamma\phi)) \; dx \\ &= (-1)^{|\beta|}\int_\Omega \sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^\sigma vD^{\beta-\sigma}uD^\gamma\phi \; dx \\ &= (-1)^{|\beta|+|\gamma|}\int_\Omega \sum_{\sigma\leq\gamma}\binom{\beta}{\sigma}D^\gamma(D^\sigma vD^{\beta-\sigma}u)\phi \; dx \\ &= (-1)^{|\alpha|}\int_\Omega \sum_{\sigma\leq\alpha}\binom{\alpha}{\sigma}D^\sigma vD^{\alpha-\sigma}u\phi \; dx\end{aligned}$$

which is the desired result. □

Theorem 3.3. *(Local approximation by smooth functions) Let $u \in W^{k,p}(\Omega)$ for $1 \leq p < \infty$ and let*

$$u^\epsilon := \eta_\epsilon * u$$

in $U_\epsilon := \{x \in U |\ dist(x, \partial U) > \epsilon\}$. Then

(1) $u^\epsilon \in C^\infty(U_\epsilon)$ for all $\epsilon > 0$, and
(2) $u^\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$.

Proof. The first assertion of the theorem is known generally for all integrable functions u convolved against a standard mollifier. To prove the second assertion, we first prove that the derivative of a mollification is equal to the mollification of a derivative. That is, for any $|\alpha| \leq k$,

$$D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$$

within our domain U_ϵ. This is easily shown through explicit computation:

$$\begin{aligned}D^\alpha u^\epsilon(x) &= D_x^\alpha \int_\Omega \eta_\epsilon(x-y)u(y) \; dy \\ &= \int_\Omega D_x^\alpha(\eta_\epsilon(x-y))u(y)dy \text{ (differentiation under the integral sign)} \\ &= (-1)^{|\alpha|}\int_\Omega D_y^\alpha(\eta_\epsilon(x-y))u(x) \; dx\end{aligned}$$

Now, as a function of y, $\eta_\epsilon(x-y) \in \mathcal{C}_c^\infty(\Omega)$, so we are free to apply the definition of the weak derivative and move the derivatives from η to u, at the cost of multiplication by $(-1)^{|\alpha|}$. We conclude that

$$\begin{aligned}D^\alpha u^\epsilon(x) &= (-1)^{2|\alpha|}\int_\Omega \eta_\epsilon(x-y)D_y^\alpha(u(x)) \; dx \\ &= (\eta_\epsilon * D^\alpha u)(x).\end{aligned}$$

With this result, we apply the definition of convergence in $W^{k,p}_{loc}(\Omega)$. Choose $V \subset\subset U$. Then

$$\|u - u^\epsilon\|_{W^{k,p}(V)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x) - D^\alpha u^\epsilon(x)|^p \, dx \right)^{\frac{1}{p}}$$

Now, each $D^\alpha u$ for $|\alpha| \leq k$ is in $L^p(V)$ and we know from the properties of convolution that for all $f \in L^p(\Omega)$, $f^\epsilon \to f$ in the $L^p(V)$ sense. Hence, each term in the above sum goes to zero, $\|u - u^\epsilon\|_{W^{k,p}(V)} \to 0$, and the result is proven.

□

Theorem 3.4. *(Approximation by smooth functions) Assume $\Omega \subset \mathbb{R}^n$ is bounded, and that $u \in W^{k,p}(\Omega)$ for $1 \leq p < \infty$. Then there exist functions $u_m \in \mathcal{C}^\infty_c(\Omega)$ such that*

$$u_m \to u \quad \text{in } W^{k,p}(\Omega).$$

Proof. Decompose Ω into the following subsets:

$$\Omega_i := \{x \in \Omega \mid d(x, \partial\Omega) > 1/i\}, \ i \in \mathbb{N}\backslash\{0\}$$

and define $V_i = \Omega_{i+3} - \bar{\Omega}_{i+1}$. Choose any $V_0 \subset\subset \Omega$ such that $U = \bigcup_{i=0}^\infty V_i$. Let $\{\phi_i\}_{i=0}^\infty$ be defined such that

$$\begin{cases} 0 \leq \phi_i \leq 1 & \phi_i \in \mathcal{C}^\infty_c(V_i) \\ \sum_{i=0}^\infty \phi_i = 1 & \text{on } \Omega \end{cases}$$

Such a collection of functions is called a *smooth partition of unity*. Now choose any $u \in W^{k,p}(\Omega)$. By the previous lemma, we know that, for each i, $\phi_i u \in W^{k,p}(\Omega)$ and $\phi_i u$ has support contained in V_i. The rest of the proof is application of the mollification process to move the set of functions $\{\phi_i u\} \in W^{k,p}(\Omega)$ to a sequence $u^i \in \mathcal{C}^\infty_c(\Omega)$, which proves density. Choose η_ϵ to be a standard mollifying function and choose some $\delta > 0$. Then choose some corresponding $\epsilon_i > 0$ such that $u^i := \eta_i * (\phi_i u)$ satisfies the following relations for $W_i := \Omega_{i+4} - \bar{\Omega}_i, i = 1, 2, ...$:

$$\begin{cases} \|u^i - \phi_i u\|_{W^{k,p}(\Omega)} \leq \frac{\delta}{2^{i+1}} & i = 0, 1, ... \\ supp(u^i) \subset W^i & i = 1, 2, ... \end{cases}$$

Define $v(x) := \sum_{i=0}^\infty u^i(x)$. Because each u^i is $\mathcal{C}^\infty$ and $u^i(x)$ can be nonzero for only a finite number of i, we see that v belongs to $\mathcal{C}^\infty(\Omega)$. And, because $u = \sum_{i=0}^\infty \phi_i u$, for each open set $V \subset\subset U$,

$$\begin{aligned} \|v - u\| &\leq \sum_{i=0}^\infty \|u^i - \phi_i u\|_{W^{k,p}(\Omega)} \\ &\leq \delta \sum_{i=0}^\infty \frac{1}{2^{i+1}} \text{ (by the above mollification process)} \\ &= \delta \end{aligned}$$

Taking a supremum over the sets $V \subset\subset U$ gives that $\|v - u\|_{W^{k,p}(\Omega)} \leq \delta$. □

In the following theorem, we characterize the dual of $W^{1,p}(\Omega)$, for all $1 \leq p < \infty$. Though this process is difficult for many function spaces, clever application of the Riesz Representation Theorem and use of functional analysis make the following characterization quite direct.

Theorem 3.5. *(Riesz's Representation Theorem in $W^{1,p}$) Let $1 \leq p < \infty$ and let $p' = \frac{p}{p-1}$. Then every $L \in (W^{1,p}(\Omega))^*$ can be characterized in the following way: There exist $f_0, ...f_N \in L^{p'}(\Omega)$ such that*

$$L(u) = \int_\Omega f_0 u + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} f_i \, dx$$

for all $u \in W^{1,p}$ and

$$\|L\|_{(W^{1,p})^*} = (\sum_{i=0}^{\infty} \|f_i\|_{L^{p'}(\Omega)}^{p'})^{\frac{1}{p'}}.$$

Proof. Define a function

$$T : W^{1,p} \to L^p(\Omega, \mathbb{R}^{N+1})$$
$$u \mapsto (u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$$

Because the first coordinate of the image is exactly the corresponding element of the domain, we see that this map is injective and that the preimage of an open set will be open, so that the function is continuous. Also,

$$\begin{aligned} \|T(u)\|_{L^p(\Omega,\mathbb{R}^{N+1})} &= \left(\int_{\mathbb{R}^{N+1}} |u|^p + \sum_{i=1}^{N} |\frac{\partial u}{\partial x_i}|^p \, dx \right)^{\frac{1}{p}} \\ &= \|u\|_{W^{1,p}} \end{aligned}$$

for all $u \in W^{1,p}$, so T is norm preserving. We know that $W^{1,p}$ is a Banach space, so it must be closed. Hence, by the norm preserving property, $Y := T(W^{1,p}(\Omega))$ must be closed in $L^p(\Omega, \mathbb{R}^{N+1})$. Given a linear functional $L \in W^{1,p}$, we now define a linear functional on Y:

$$L_1 : Y \to \mathbb{R}$$
$$g = (g_0, ..., g_N) \mapsto L(T^{-1}((g_0, ...g_n)))$$

Now T^{-1} is obviously linear, and L is linear by definition, so as a composition of linear maps L_1 is linear; and because T^{-1} and L are bounded, their composition L_1 must be bounded and thus is continuous (a linear functional is bounded if and only if it is continuous). Finally, because T preserves the norm, we have that:

$$\|L_1\|_{Y'} = \|L\|_{(W^{1,p})^*}$$

We are now free to apply the Hahn-Banach theorem and extend L_1 as a continuous linear operator to:

$$\tilde{L_1} : L^p(\Omega, \mathbb{R}^{N+1}) \to \mathbb{R}$$

satisfying

$$\|\tilde{L_1}\|_{(L^p(\Omega;\mathbb{R}^{N+1}))} = \|L_1\|_{Y'} = \|L\|_{(W^{1,p})^*}$$

We now apply the Riesz Representation Theorem in $L^p(\Omega; \mathbb{R}^{N+1})$ and conclude that there exist functions $f_0, ...f_N \in L^p(\Omega)$ such that

$$L_1(g) = \int_\Omega f_0(x) g_0(x) + \sum_{i=1}^{N} f_i(x) g_i(x) \, dx$$

for all $g \in L^p(\Omega; \mathbb{R}^{N+1})$ and that

$$\|L\|_{(W^{1,p}(\Omega))^*} = \|\tilde{L_1}\|_{(L^p(\Omega;\mathbb{R}^{N+1}))} = (\sum_{i=0}^{N+1} \|f_i\|^p_{L^{p'}(\Omega)})^{\frac{1}{p'}}$$

By identifying the above functions $g_0, ...g_N$ with an $N+1$-tuple of the form $(u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$, we obtain the desired result:

$$L(u) = \int_\Omega f_0(x)u(x) + \sum_{i=1}^{N} f_i(x)\frac{\partial u}{\partial x_i}\, dx$$

□

Up to this point, all results have been formulated on the basis of an open bounded subset $\Omega \subset \mathbb{R}^n$. The following theorem shows that when $k = 1$, we could just as well be working with all of $\mathbb{R}^n$. This is not immediately obvious, because it is quite possible that by constructing the obvious extension of $u \in W^{1,p}(\Omega)$ by setting $u(x) := 0$ for $x \in \mathbb{R}^n \setminus \Omega$ we create a discontinuity so bad that we lose the property of weak differentibility and are no longer in $W^{1,p}(\Omega)$. The proof of the theorem follows the standard argument of flattening the boundary and reflecting the function in question in such a way that the derivatives are preserved. For full details, consult [2].

Theorem 3.6. *(Extension Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded subset with a $\mathcal{C}^1$ boundary, $\partial\Omega$, and $1 \leq p < \infty$. Choose a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(\Omega)$,

(1) $Eu = u$ a.e. in U
(2) Eu has support contained in V
(3) $\|Eu\|_{W^{1,p}(\mathbb{R})} \leq C\|u\|_{W^{1,p}(\Omega)}$, with $C = C(p, U, V)$ (not a function of u).

Such an operator E is called an extension of u to $\mathbb{R}^n$.

Definition 3.7. If $1 \leq p < n$, the *Sobolev conjugate* of p is defined to be

$$p^* := \frac{np}{n-p}$$

We see that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},\ p^* > p.$$

We now turn to a very powerful tool in the study of Sobolev spaces - the Sobolev Inequalities. These inequalities give relations comparing either the norm of a function to the norm of its weak derivatives or different norms of the same function. While the results can be quite useful, the proofs can become incredibly dense and are not particularly enlightening. For this reason, we state only the results we will need later in the paper and do so without proof.

(1) (Gagliardo-Nirenberg-Sobolev Inequality) Assume $1 \leq p < n$. There exists a constant C depending only on p and n such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in \mathcal{C}_c^1(\mathbb{R}^n)$.

(2) Let Ω be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial\Omega$ is $\mathcal{C}^1$. Let $1 \leq p < n$, and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ satisfies the estimate

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

with the constant, C, dependent only on p, n, and Ω.

(3) (Poincare's Inequality) Let Ω be a bounded, open subset of $\mathbb{R}^n$, $1 \leq p < n$, and $u \in W_0^{1,p}(\Omega)$. Then

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

for $q \in [1, p^*]$, and the constant C dependent only on p, q, n, and Ω. In particular, we can set $q = p$ and bound the norm of an L^p function by the norm of its gradient.

Definition 3.8. Let X and Y be Banach Spaces with $X \subset Y$. Then X is *compactly embedded* in Y, denoted

$$X \subset\subset Y$$

provided that

(1) $\|x\|_Y \leq C\|x\|_X$, for some constant C, and
(2) each bounded sequence in X is precompact in Y. That is, every bounded sequence in X must have a convergent subsequence in Y.

Theorem 3.9. *(Rellich-Kondrachov Compactness) Take $\Omega \subset \mathbb{R}^n$ a bounded, open set with $\mathcal{C}^1$ boundary $\partial\Omega$ and $1 \leq p < n$. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

Proof. Fix some $1 \leq q < p^*$ and note that since Ω is bounded, we can apply the second of the above Sobolev Inequalities to conclude that

$$W^{1,p}(\Omega) \subset L^q(\Omega), \|u\|_{L^q}(\Omega) \leq \|u\|_{W^{1,p}(\Omega)}.$$

This is exactly the first condition for compact embedding. Hence, all that remains to be shown is the precompactness of bounded sequences in $W^{1,p}(\Omega)$. That is, we must show that if $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,p}(\Omega)$, there is a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ which converges in $L^q(\Omega)$.

Now, we may apply the Extension Theorem and assume that $\Omega = \mathbb{R}^n$ with no loss of generality. In doing so, we are free to take the functions $\{u_m\}$ with compact support in some bounded open set $V \subset \mathbb{R}^n$ satisfying:

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty.$$

We first work with the smoothed functions

$$u_m^\epsilon := \eta_\epsilon * u_m$$

where η_ϵ is the usual mollifier. Later, we will use a density argument to remove this assumption. Of course, we may take the functions $\{u_m^\epsilon\}$ to have support in the set V as well.

We must first show that

$$u_m^\epsilon \to u_m \text{ in } L^q(V) \text{ as } \epsilon \to 0$$

This inequality will later be instrumental to our use of the Arzela-Ascoli Compactness Criterion. To this end, note that if u_m is smooth, then

$$\begin{aligned}
u_m^\epsilon(x) - u_m(x) &= \int_\Omega \eta_\epsilon(x-y)(u_m(y) - u_m(x))\ dy \\
&= \frac{1}{\epsilon^n}\int_{B(0,\epsilon)} \eta(\frac{x-y}{\epsilon})(u_m(y) - u_m(x))\ dy \\
&= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))\ dy \text{ (by change of variables)} \\
&= \int_{B(0,1)} \eta(y)\int_0^1 \frac{\partial}{\partial t}u(x-\epsilon t y)\ dtdy \\
&= -\epsilon\int_{B(0,1)} \eta(y)\int_0^1 Du_m(x-\epsilon t y)\cdot y\ dtdy
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_V |u_m^\epsilon(x) - u_m(x)|\ dx &\le \epsilon\int_{B(0,1)}\eta(y)\int_0^1\int_V |Du_m(x-\epsilon t y)|\ dxdt\ dy \\
&\le \epsilon\int_V |Du_m(z)|\ dz.
\end{aligned}$$

By the density of smooth functions in $W^{1,p}(\Omega)$, approximation shows that this inequality will hold for all $u_m \in W^{1,p}$. Hence,

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \epsilon\|Du_m\|_{L^1(V)} \le \epsilon C\|Du\|_{L^p(V)},$$

with the final inequality holding because V is bounded, so $L^1(V)$ norms will be bounded by $L^p(V)$ norms. Now, because we have taken $\sup\|u_m\|_{W^{1,p}(V)} < \infty$, for all m, we see that,

$$u_m^\epsilon \to u_m$$

in $L^1(V)$ uniformly in m as $\epsilon \to 0$. Because $1 \le q < p^*$, we can apply the interpolation inequality for L^p-norms to see that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$. However, we can now apply the Gagliardo-Nirenberg-Sobolev Inequality and utilize again the fact that each u_m is uniformly bounded in $W^{1,p}(V)$:

$$\begin{aligned}
\|u_m^\epsilon - u_m\|_{L^q(V)} &\le C\|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|D(u_m^\epsilon - u_m)\|_{L^p(V)}^{1-\theta} \\
&\le C\|u_m^\epsilon - u_m\|_{L^1(V)}^\theta,
\end{aligned}$$

for some appropriate constant C. But then, we know that the $L^1(V)$ norm on the right converges from above as $\epsilon \to 0$, so we conclude that

$$u_m^\epsilon \to u_m \text{ uniformly in m, in the } L^p(V) \text{ sense.}$$

We now begin the process of extracting a convergent subsequence in L^q of a bounded sequence in $W^{1,p}(V)$. Ultimately we will need to invoke the Arzela-Ascoli

Compactness Criterion, so we begin with the following claim: For each $\epsilon > 0$, the sequence $\{u_m^\epsilon\}_{m=1}^\infty$ is uniformly bounded and equicontinuous. Given $x \in \mathbb{R}^n$,

$$\begin{aligned}|u_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|\ dy \\ &\leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^n} < \infty\end{aligned}$$

for $m = 1, 2, ...$ This proves uniform boundedness. To prove equicontinuity, it is enough to show that the derivatives of the u_m are uniformly bounded by the same bounds. To this end, we perform the same computation as above:

$$\begin{aligned}|Du_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|\ dy \\ &\leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\epsilon^{n+1}} < \infty.\end{aligned}$$

This proves equicontinuity.

Choose $\delta > 0$. We will now show that there is a subsequence $\{u_{m_j}^\epsilon\}_{j=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

To prove this, we first use the fact that the u_m^ϵ converge uniformly to u_m as $\epsilon \to 0$ to choose an $\epsilon > 0$ sufficiently small that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2}$$

for $m = 1, 2, ...$

Now, because $\{u_m\}_{m=1}^\infty$ all have support contained in the fixed bounded set $V \subset \mathbb{R}^n$, so do the functions $\{u_m^\epsilon\}_{m=1}^\infty$. With this result and the uniform boundedness and equicontinuity proven above, we can use the Arzela-Ascoli Compactness Criterion to extract a subsequence $\{u_{m_j}^\epsilon\}_{j=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ which converges uniformly on V. In particular, we note that

$$\limsup_{j,k\to\infty} \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} = 0.$$

But, by the triangle inequality,

$$\begin{aligned}\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} &\leq \limsup_{j,k\to\infty} (\|u_{m_j} - u_{m_j}^\epsilon\|_{L^q(V)} + \|u_{m_j}^\epsilon - u_{m_k}^\epsilon\|_{L^q(V)} + \|u_{m_k}^\epsilon - u_{m_k}\|_{L^q(V)}) \\ &\leq \frac{\delta}{2} + 0 + \frac{\delta}{2} \\ &\leq \delta,\end{aligned}$$

which implies uniform convergence on V.

Finally, we use this result with $\delta = 1, \frac{1}{2}, \frac{1}{3}, ...$ along with a diagonal argument to extract a subsequence $\{u_{m_l}^\epsilon\}_{l=1}^\infty \subset \{u_m^\epsilon\}_{m=1}^\infty$ such that

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

Choose the subsequence where the nth term is the nth term of the subsequence obtained with $\delta = \frac{1}{n}$. This completes the proof. □

4. Application to Second-Order Elliptic Partial Differential Equations

We will now begin a discussion of how to apply the theory of Sobolev Spaces to the study of differential equations. Specifically, we will talk about the existence of solutions to boundary value problems of the form:

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where L is is a second-order partial differential operator in *divergence form*, that is,

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

Further, we will assume that the functions $\{a^{ij}\}$ are symmetric. That is, $a^{ij} = a^{ji}$ for all $i, j \leq n$. The final condition we impose is that the operator L be *uniformly elliptic*, as defined below.

Definition 4.1. The partial differential operator L is **uniformly elliptic** if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\rho_i\rho_j \geq \theta|\rho|^2$$

for almost every $x \in \Omega$ and every $\rho \in \mathbb{R}^n$.

Now, before any detailed discussion of how we are going to work with these differential equations, we prove a purely functional analytic result that will be invaluable to us for the remainder of the paper.

Theorem 4.2. *(Lax-Milgram) Let H be a Hilbert Space and $B : H \times H \to \mathbb{R}$ be a bilinear mapping satisfying the following inequalities:*

(1) There exists $\alpha > 0$ such that $|B[u, v]| \leq \alpha\|u\|\|v\|$ for all $u, v \in H$,
(2) There exsits $\beta > 0$ such that $\beta\|u\|^2 \leq B[u, u]$ for all $u \in H$.

Then if $f : H \to \mathbb{R}$ is a bounded linear functional on H, there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v\rangle$$

for all $v \in H$.

Note that we cannot directly apply the Riesz Representation Theorem because we do not know that our bilinear form B is symmetric (i.e. it is possible that $B[u, v] \neq B[v, u]$).

Proof. By the assumed inequalities, we see that for a fixed element $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H. Here we can directly apply the Riesz Representation Theorem to find an element $u \in H$ such that

$$B[u, v] = (w, v) \text{ for } v \in H.$$

We define a mapping $A : H \to H$ by $u \mapsto w$, where w fits the above definition. This allows us to write

$$B[u, v] = (Au, v) \ (v \in H).$$

We claim that A is a bounded linear operator. It is easy to see that the operator is linear: take $\lambda \in \mathbb{R}$, $t, u, v \in H$. Then

$$(A(\lambda t + u), v) = B[\lambda t + u, v] \quad = \lambda B[t, v] + B[u, v] = (\lambda At, v) + (Au, v) \quad = (\lambda At + Au, v),$$

proving linearity. We can apply the first inequality required of the bilinear form B to prove boundedness:

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|$$

So, $\|Au\| \leq \alpha \|u\|$ for all $u \in H$, and so by definition A is bounded. We now apply the other assumed inequality to show that A is injective and has a closed range in H. This second criterion will be used later to show that our function is surjective. Directly applying the second assumed inequality,

$$\beta \|u\|^2 \leq B[u, u] = (Au, u) \leq \|Au\| \|u\|.$$

Hence, $\beta \|u\| \leq \|Au\|$. If A sent some $u \neq 0$ to 0, this inequality would be contradicted. This is enough to show injectivity. To show that the range is closed, choose some convergent sequence $\{Au_n\} \to w$. Now, $\beta \|u_n - u_m\| \leq \|Au_n - Au_m\| \to 0$. So $\{u_n\}$ is a convergent sequence in H so $u_m \to u$ in H, and thus by continuity, $\lim_{n \to \infty} Au_n = w$. Hence, the range of A is closed in H because $Au = w$.

We now show that the range of A, denoted $R(A)$ is in fact all of H and hence that our map A is a bijection. We know that the range of A is closed. Suppose that the range is not all of H. Then there exists a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this implies that

$$\beta \|w\|^2 \leq B[w, w] = (Aw, w) = 0,$$

which is a contradiction.

Again we can apply the Riesz Representation Theorem to find a $u \in H$ such that

$$\langle f, v \rangle = (w, v)$$

for all $v \in w$. But then because A is bijective, we can find $u \in H$ such that $Au = w$. This gives

$$B[u, v] = (Au, v) = (w, v) = \langle f, v \rangle.$$

This is exactly what we needed to show. All that is left is uniqueness and we prove this with a typical contradiction argument. Suppose there exist $t, u \in H$ satisfying the above equation. Then

$$B[u, v] = \langle f, v \rangle = B[t, v].$$

We conclude that $B[u - t, v] = 0$, for all $v \in H$. Setting $v = u - t$ we see that

$$\beta \|u - t\|^2 \leq B[u - t, u - t] = 0.$$

□

So, how are we going to use these Sobolev Spaces of weakly differential functions to find classical solutions to PDE? The answer is that we are not. Instead, we are going to find *weak solutions* to the equations. We define a weak solution of a second-order linear PDE in an analogous way to our formulation of weak derivatives. That is, we integrate the equation $Lu = f$ against $v \in \mathcal{C}_c^{\infty}(\Omega)$ and determine for which

function u the resulting integral equality holds. That is, we wish to find a function u satisfying,

$$\int_\Omega \sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i}v + c(x)uv \; dx = \int_\Omega fv \; dx, \quad \text{for all } v \in \mathcal{C}_c^\infty(\Omega),$$

where we have integrated by parts in the first term and used the fact that v is compactly supported. By density of $\mathcal{C}_c^\infty(\Omega)$ in $H_0^1(\Omega)$, choosing $v \in H_0^1(\Omega)$ in the above process will yield the same result.

Definition 4.3. The bilinear form $B[\cdot,]$ associated with a divergence form linear operator L is defined by

$$B[u,v] := \int_\Omega \sum_{i,j\leq n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i\leq n} b^j(x)u_{x_i}v + c(x)uv \; dx$$

Definition 4.4. We say that $u \in H_0^1(\Omega)$ is a *weak solution* of the boundary value problem

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

if

$$B[u,v] = (f,v)$$

for all $v \in H_0^1(\Omega)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

The following proof will require a clever modification of Cauchy's Inequality ($ab \leq \frac{a^2}{2} + \frac{b^2}{2}$). By setting $ab := ((2\epsilon)^{\frac{1}{2}}a)\left(\frac{b}{(2\epsilon)^{\frac{1}{2}}}\right)$ we obtain Cauchy's Inequality with ϵ:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon},$$

for all $a \in \mathbb{R}$ and $b, \epsilon > 0$.

Theorem 4.5. *There exist constants $\alpha, \beta > 0$ such and $\gamma \geq 0$ such that*

(1) $|B[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \; \|v\|_{H_0^1(\Omega)}$, *and,*
(2) $\|u\|^2_{H_0^1(\Omega)} \leq B[u,u] + \gamma \|u\|^2_{L^2(\Omega)}$

for all $u, v \in H_0^1(\Omega)$

Proof. The first inequality is given by direct computation, although we do need a bit of creativity -

$$\begin{aligned}
|B[u,v]| &= \int_\Omega \sum_{i,j\leq n} a^{ij}(x) u_{x_i} v_{x_j}\ dx + \int_\Omega \sum_{i\leq n} b^j(x) u_{x_i} v\ dx + \int_\Omega c(x) uv\ dx \\
&\leq \|a^{ij}\|_{L^\infty} \sum_{i,j\leq n} \int_\Omega u_{x_i} v_{x_j}\ dx + \sum_{i\leq n} \|b^j\|_{L^\infty} \int_\Omega u_{x_i} v\ dx + \|c\|_{L^\infty} \int_\Omega uv\ dx \\
&\leq C \int_\Omega (|u| + |Du|)(|v| + |Dv|)\ dx \\
&\leq C \int_\Omega (\sum_{|\alpha|\leq 1} |D^\alpha u|)(\sum_{|\alpha|\leq 1} |D^\alpha v|)\ dx \\
&\leq C \left(\sum_{|\alpha|\leq 1} \int_\Omega |D^\alpha u|^2\ dx \right)^{\frac{1}{2}} \left(\sum_{|\alpha|\leq 1} \int_\Omega |D^\alpha v|^2\ dx \right)^{\frac{1}{2}},
\end{aligned}$$

which is the desired result.

For the second inequality, we appeal to the ellipticity condition. Setting $\rho = Du$ and integrating both sides of the inequality, we see that

$$\begin{aligned}
\theta \int_\Omega |Du|^2\ dx &\leq \int_\Omega \sum_{i,j=n}^{n} a^{ij} u_{x_i} u_{x_j}\ dx \\
&= B[u,u] - \int_\Omega \sum_{i=1}^{n} b^i u_{x_i} u + cu^2\ dx \\
&\leq B[u,u] \sum_{i=1}^{n} \|b^i\|_{L^\infty} \int_\Omega |Du|\ |u|\ dx + \|c\|_{L^\infty} \int_\Omega u^2\ dx
\end{aligned}$$

Now, applying Cauchy's Inequality with ϵ and integrating,

$$\int_\Omega |Du|\ |u|\ dx \leq \epsilon \int_\Omega |Du|\ dx + \frac{1}{4\epsilon} \int_\Omega |u|^2\ dx$$

Choosing ϵ small enough that

$$\epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty} \leq \frac{\theta}{2},$$

we can apply this estimate to the above equation to obtain

$$\begin{aligned}
\theta \int_\Omega |Du|^2\ dx &\leq B[u,u] + \sum_{i=1}^{n} \|b^i\|_{L^\infty} \epsilon \int_\Omega |Du|\ dx + \frac{1}{4\epsilon} \int_\Omega |u|^2\ dx + \|c\|_{L^\infty} \int_\Omega u^2\ dx \\
&\leq B[u,u] + \sum_{i=1}^{n} \frac{\theta}{2} \|b^i\|_{L^\infty} \int_\Omega |Du|\ dx + \|c'\|_{L^\infty} \int_\Omega u^2\ dx,
\end{aligned}$$

and finally, that

$$\frac{\theta}{2} \leq B[u,u] + C \int_\Omega u^2\ dx$$

We now call on Poincaré's Inequality which states that

$$\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}.$$

From this it is easy to see that

$$\beta\|u\|^2_{H^1_0} \leq B[u,u] + \gamma\|u\|^2_{L^2(\Omega)}$$

for constants $\beta > 0, \gamma \geq 0$. □

While this is a nice result, if $\gamma \neq 0$ we do not satisfy the hypothesis of the Lax-Milgram Theorem. In the following proof of the existence theorem, we will have to manufacture a new bilinear form which does satisfy the hypothesis.

Theorem 4.6. *(First Existence Theorem for Weak Solutions) There exists a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(\Omega)$ there exsits a unique weak solution $u \in H^1_0(\Omega)$ of the boundary-value problem:*

$$\begin{cases} Lu + \mu u = f \ \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

Proof. Choose γ to be the γ from the second inequality of the energy estimates theorem, and choose any $\mu \geq \gamma$. We define the following bilinear form:

$$B_\mu[u,v] := B[u,v] + \mu(u,v), \quad \text{for } u, v \in H^1_0(\Omega),$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. We see that this biliear form corresponds to the operator $L_\mu u := Lu + \mu u$ and that $B_\mu[\cdot,\cdot]$ satisfies the hypothesis of the Lax-Milgram Theorem. Now fix $f \in L^2(\Omega)$ and set $\langle f, v\rangle := (f,v)_{L^2(\Omega)}$. This is a bounded linear functional on $L^2(\Omega)$ (because the inner product is continuous and linear), and hence is a bounded linear functional on $H^1_0(\Omega)$ as well. We apply the Lax-Milgram Theorem and find a unique $u \in H^1_0(\Omega)$ satisfying

$$B_\mu[u,v] = \langle f, v\rangle$$

for all $v \in H^1_0(\Omega)$. By definition, this makes u a weak solution of the stated boundary value problem. □

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