CYCLIC HOMOLOGY

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1. Introduction

Many geometric objects associated to a manifold M can be expressed in terms of an appropriate algebra A of functions on M (measurable, continuous, smooth, holomorphic, algebraic, ...). Very often those objects can be defined in a way that is applicable to any algebra A, commutative or not. Study of associative algebras by means of such objects of geometric origin is the subject of non-commutative geometry [C], [M1]. The Hochschild and cyclic (co)homology theory is the part of non-commutative geometry which generalizes the classical differential and integral calculus. The geometric objects being generalized to the non-commutative setting are differential forms, densities, multi-vector fields, etc.

In our exposition, the primary object is the negative cyclic complex $CC_\bullet^-(A)$. Other complexes, namely the Hochschild chain complex $C_\bullet(A)$, the periodic cyclic complex $CC_\bullet^{\mathrm{per}}(A)$, and the cyclic complex $CC_\bullet(A)$, are defined as results of some natural procedure applied to $CC_\bullet(A)$. The cyclic homology is the homology of the cyclic complex $CC_\bullet(A)$. It was originally defined using another standard complex which we denote by $C_\bullet^\lambda(A)$. The study of this latter complex has a distinctly different flavor, mainly coming from the fact that it is related to the Lie algebra homology.

The above complexes are non-commutative versions of the space of differential forms (the) and of the De Rham complex. One also defines the Hochschild cochain complex $C^\bullet(A,A)$ which is a non-commutative analogue of the space of multi-vector fields.

In the first three sections we study rather systematically various algebraic structures on the above complexes. These structures are supposed to generalize the classical algebraic structures arising in calculus. Sections 2 and 4.3 are devoted to the structures on the complex $CC_\bullet^-(A)$ and related complexes; section 3 deals with the cyclic complex $C_\bullet^\lambda(A)$.

These algebraic structures can be studied on three levels of complexity. One can start by defining basic algebaric operations generalizing

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the product and the bracket of multivector fields and various pairings between multivectors and forms; these are operations at the level of complexes. They acquire the standard prorerties from calculus only if one passes from complexes to their cohomology. When the algebra A is the algebra of functions on a manifold, then the homology of the Hochschild chain complex is the space of forms, the cohomology of the Hochschild cochain complex is the space of multivectors, etc. So, in a more naive sense, even at the first level of non-commutative calculus we already achieve our goal: to generalize the basic definitions and structures of calculus to the non-commutative setting (subsections 2.3, 2.4). But this generalization is not interesting when the algebra becomes non-commutative. For example, for the algebra of differential operators on $\mathbb{R}^n$ both the spaces of "non-commutative forms" and "non-commutative multivectors" are one-dimensional.

To make non-commutative calculus useful for applications, one has to pass to the second level of complexity. Namely, one tries to reproduce more of the standard algebraic structure from the classical calculus not at the level of cohomology but rather at the level of complexes. This plan succeeds partially, in the following sense: one can define certain algebraic structures on Hochschild chains and cochains. These structures are only part of what one could expect from the classical calculus. The advantage of these structures is that they are defined by explicit, canonical constructions (2.5, 3.5). They are also adequate for some applications, namely for the index theorems for symplectic deformations.

To construct a richer algebraic structure on Hochschild chains and cochains, one that generalizes a fuller version of the classical calculus, one has to go to a different level of complexity. The work in this direction was started by Tamarkin in [T] and then continued in [TT1], [TT2], [TT3]. We discuss these methods in Section 4.3. They provide a considerable refinement of the results of 2.5, 3.5, but there is no canonical and explicit construction anymore. The "non-comutative differential calculus" can be constructed using some inexplicit formulas; a choice of coefficients in these formulas depends on a choice of a Drinfeld associator [Dr]. In particular, the Grothendieck-Teichmüller group acts on the space of all such calculi. This version of non-commutative calculus allows one to generalize the index theorem form symplectic deformations to arbitrary deformations.

Note that in [C], [CM],[CM1],[CM2] a different, though clearly related, version of non-commutative calculus is used. In particular, there the renormalization group appears as a hidden group of symmetries of the calculus, whereas in our construction the Grothendieck-Teichmüller group acts on the space of universal formulas defining a calculus. This group is closely related to the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that finding a unified symmetry group incorporating both the renormalization group and the Galois group of $\mathbb{Q}$ is one of the important aims of Connes' non-commutative geometry program. In light of this, it seems to be an interesting problem to find a unified framework to the two approaches to non-commutative calculus (the author would like to thank Alain Connes for his remarks on this subject).

In section 6, we study various examples of computations of the Hochschild and cyclic homology. The examples that we choose in this presentation revolve mostly around several related classes of algebras: functions on manifolds or on algebraic varieties; operators on functions; deformed algebras of functions; group algebras. Many interesting examples, in particular the ones related to algebraic topology, are not considered here. We compute the Hochschild homology and various versions of the cyclic homology of any formal deformation of the algebra of smooth functions on a C^∞ manifold. This computation, as well as the classification of all such deformations, is based on the non-commutative differential calculus from Section 4.3. The constructions that we use are neither explicit nor canonical; they depend on a choice of a Drinfeld associator.

In Section 7, index theorems for formal deformations are discussed. An index theorem is an equality that compares two maps from the (periodic) cyclic complex of a deformed algebra of functions on a manifold to the De Rham complex of this manifold. One of these maps is the principal symbol; it is, essentially, reduction modulo deformation parameter, followed by the Connes isomorphism between the cyclic homology and the De Rham cohomology. The other is the trace density map. It is given by an explicit and canonical construction in the case when the quasi-classical limit of the deformation is symplectic; in the general case, its definition depends on a choice of an associator.

In Section 8 we outline an application of the above theorem (in the symplectic case) to a general index theorem. This theorem, conjectured in [SS] and proven in [BNT], contains many classical index theorems as partial cases, such as the Atiyah-Singer theorem for real analytic elliptic operators, the Boutet de Monvel index theorem for boundary value problems, the Kashiwara index theorem for holonomic D-modules, and others.

2. Hochschild and cyclic homology of algebras

Let k denote a commutative algebra over a field of characteristic zero and let A be a flat k-algebra with unit, not necessarily commutative. Let $\overline{A} = A/k \cdot 1$, and let $C_p(A) \stackrel{def}{=} A \otimes_k \overline{A}^{\otimes_k p}$. Define

$$\begin{aligned} b : C_p(A) &\to C_{p-1}(A) \\ a_0 \otimes \cdots \otimes a_p &\mapsto (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \\ &\sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \ . \end{aligned} \tag{2.1}$$

Then $b^2 = 0$ and one gets the complex $(C_\bullet, b)$, called *the standard Hochschild complex of* A. The homology of this complex is denoted by $H_\bullet(A, A)$, or by $HH_\bullet(A)$.

Proposition 2.0.1. *The map*

$$\begin{aligned} B : C_p(A) &\to C_{p+1}(A) \\ a_0 \otimes \cdots \otimes a_p &\mapsto \sum_{i=0}^{p} (-1)^{pi} 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1} \end{aligned} \tag{2.2}$$

satisfies $B^2 = 0$ *and* $bB + Bb = 0$ *and therefore defines a map of complexes*

$$B : C_\bullet(A) \to C_\bullet(A)[-1]$$

Definition 2.0.2. *For* $i, j, p \in \mathbb{Z}$ *let*

$$CC_p^-(A) = \prod_{\substack{i \geq p \\ i = p \mod 2}} C_i(A)$$

$$CC_p^{per}(A) = \prod_{i = p \mod 2} C_i(A)$$

$$CC_p(A) = \bigoplus_{\substack{i \leq p \\ i = p \mod 2}} C_i(A)$$

The complex $(CC_\bullet^-(A), B + b)$ (respectively $(CC_\bullet^{per}(A), B + b)$, respectively $(CC_\bullet(A), B + b)$) is called the *negative cyclic* (respectively *periodic cyclic*, respectively *cyclic*) complex of A. The homology of these complexes is denoted by $HC_\bullet^-(A)$, respectively by $HC_\bullet^{per}(A)$, respectively by $HC_\bullet(A)$.

There are inclusions of complexes

$$CC_\bullet^-(A)[-2] \hookrightarrow CC_\bullet^-(A) \hookrightarrow CC_\bullet^{per}(A) \tag{2.3}$$

and the short exact sequences

$$0 \to CC_\bullet^-(A)[-2] \to CC_\bullet^-(A) \to C_\bullet(A) \to 0 \tag{2.4}$$

$$0 \to C_\bullet(A) \to CC_\bullet(A) \xrightarrow{S} CC_\bullet(A)[2] \to 0 \tag{2.5}$$

The operator S is called the Bott periodicity operator. To the double complex $CC_\bullet(A)$ one associates the spectral sequence

$$E^2_{pq} = H_{p-q}(A, A) \tag{2.6}$$

converging to $HC_{p+q}(A)$.

In what follows we will use the notation of Getzler and Jones ([GJ]). Let u denote a variable of degree -2. Then the negative and periodic cyclic complexes are described by the following formulas:

$$\begin{aligned} CC_\bullet^-(A) &= (C_\bullet(A)[[u]], b + uB) &(2.7)\\ CC_\bullet^{\mathrm{per}}(A) &= (C_\bullet(A)[[u, u^{-1}], b + uB) &(2.8)\\ CC_\bullet(A) &= (C_\bullet(A)[[u, u^{-1}]/uC_\bullet(A)[[u]], b + uB) &(2.9)\end{aligned}$$

In this language, the Bott periodicity map S is just multiplication by u.

Remark 2.0.3. For an algebra A without unit, let $\tilde{A} = A + k \cdot 1$ and put

$$CC_\bullet(A) = \mathrm{Ker}(CC_\bullet(\tilde{A}) \to CC_\bullet(k));$$

similarly for the negative and periodic cyclic complexes. If A is a unital algebra then these complexes are quasi-isomorphic to the ones defined above.

2.1. **Homology of differential graded algebras.** One can easily generalize all the above constructions to the case when A is a differential graded algebra (DGA) with the differential δ (i.e A is a graded algebra and δ is a derivation of degree 1 such that $\delta^2 = 0$).

The action of δ extends to an action on Hochschild chains by the Leibniz rule:

$$\delta(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1}(a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p)$$

The maps b, B, τ are modified to include signs:

$$b(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p-1}(-1)^{\sum_{i=0}^{k}(|a_i|+1)+1} a_0 \ldots a_k a_{k+1} \otimes \ldots a_p \tag{2.10}$$
$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes \ldots \otimes a_{p-1}$$

$$B(a_0\otimes\ldots\otimes a_p)=\sum_{k=0}^{p}(-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}1\otimes a_{k+1}\otimes\ldots a_p\otimes \tag{2.11}$$
$$a_0\otimes\ldots\otimes a_k$$

The complex $C_\bullet(A)$ now becomes the total complex of the double complex with the differential $b+\delta$. The negative and the periodic cyclic complexes are defined as before in terms of the new definition of $C_\bullet(A)$. All the results of this section extend to the differential graded case.

2.2. **The Hochschild cochain complex.** Let A be a graded algebra with unit over a commutative unital ring k of characteristic zero. A Hochschild d-cochain is a linear map $A^{\otimes d}\to M$. Put, for $d\ge 0$,

$$C^d(A)=C^d(A,A)=\mathrm{Hom}_k(\overline{A}^{\otimes d},M)$$

where $\overline{A}=A/k\cdot 1$. Put

$$|D|\ =(\textit{degree of the linear map } D)+d$$

Put for cochains D and E from $C^\bullet(A,A)$

$$(D\smile E)(a_1,\ldots,a_{d+e})=(-1)^{|E|\sum_{i\le d}(|a_i|+1)}D(a_1,\ldots,a_d)\times$$
$$\times E(a_{d+1},\ldots,a_{d+e});$$
$$(D\circ E)(a_1,\ldots,a_{d+e-1})=\sum_{j\ge 0}(-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)}\times$$
$$\times D(a_1,\ldots,a_j,E(a_{j+1},\ldots,a_{j+e}),\ldots);$$
$$[D,\ E]=D\circ E-(-1)^{(|D|+1)(|E|+1)}E\circ D$$

These operations define the graded associative algebra $(C^\bullet(A,A)\ ,\smile)$ and the graded Lie algebra $(C^{\bullet+1}(A,A),\ [\ ,\ ])$ (cf. [CE]; [G]). Let

$$m(a_1,a_2)=(-1)^{\deg a_1}\ a_1a_2;$$

this is a 2-cochain of A (not in C^2). Put

$$\delta D=[m,D];$$
$$(\delta D)(a_1,\ldots,a_{d+1})=(-1)^{|a_1||D|+|D|+1}\times$$
$$\times a_1D(a_2,\ldots,a_{d+1})+$$
$$+\sum_{j=1}^{d}(-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)}D(a_1,\ldots,a_ja_{j+1},\ldots,a_{d+1})$$
$$+(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)}D(a_1,\ldots,a_d)a_{d+1}$$

One has

$$\delta^2=0;\quad \delta(D\smile E)=\delta D\smile E+(-1)^{|degD|}D\smile\delta E$$
$$\delta[D,E]=[\delta D,E]+(-1)^{|D|+1}\ [D,\delta E]$$

($\delta^2 = 0$ follows from $[m, m] = 0$).

Thus $C^\bullet(A, A)$ becomes a complex; we will denote it also by $C^\bullet(A)$. The cohomology of this complex is $H^\bullet(A, A)$ or the Hochschild cohomology. We denote it also by $H^\bullet(A)$. The $\smile$ product induces the Yoneda product on $H^\bullet(A, A) = Ext^\bullet_{A \otimes A^0}(A, A)$. The operation $[\ ,\ ]$ is the Gerstenhaber bracket [G].

If (A, ∂) is a differential graded algebra then one can define the differential ∂ acting on $C^\bullet(A)$ by:

$$\partial D = [\partial, D]$$

Theorem 2.2.1. [G] *The cup product and the Gerstenhaber bracket induce a Gerstenhaber algebra structure on $H^\bullet(A)$.*

For cochains D and D_i define a new Hochschild cochain by the following formula of Gerstenhaber ([G]) and Getzler ([Ge1]):

$$D_0\{D_1, \ldots, D_m\}(a_1, \ldots, a_n) =$$

$$= \sum (-1)^{\sum_{k \le i_p} (|a_k|+1)(|D_p|+1)} D_0(a_1, \ldots, a_{i_1}, D_1(a_{i_1+1}, \ldots), \ldots,$$

$$D_m(a_{i_m+1}, \ldots), \ldots)$$

Proposition 2.2.2. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q \le i_p} (|E_p|+1)(|F_q|+1)} \times$$

$$\times D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots,\}, \ldots, E_k\{F_{i_k+1}, \ldots,\}, \ldots,\}$$

The above proposition can be restated as follows. For a cochain D let $D^{(k)}$ be the following k-cochain of $C^\bullet(A)$:

$$D^{(k)}(D_1, \ldots, D_k) = D\{D_1, \ldots, D_k\}$$

Proposition 2.2.3. *The map*

$$D \mapsto \sum_{k \ge 0} D^{(k)}$$

is a morphism of differential graded algebras

$$C^\bullet(A) \to C^\bullet(C^\bullet(A))$$

2.3. **Products on Hochschild and cyclic complexes.**

2.3.1. *Product and coproduct; the Künneth exact sequence.* For an algebra A define the shuffle product

$$\mathrm{sh} : C_p(A) \otimes C_q(A) \to C_{p+q}(A) \tag{2.12}$$

as follows.

$$(a_0 \otimes \ldots \otimes a_p) \otimes (c_0 \otimes \ldots \otimes c_q) = a_0 c_0 \otimes \mathrm{sh}_{pq}(a_1, \ldots, a_p, c_1, \ldots, c_q) \tag{2.13}$$

where

$$\mathrm{sh}_{pq}(x_1, \ldots, x_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma) x_{\sigma^{-1}1} \otimes \ldots \otimes x_{\sigma^{-1}(p+q)} \tag{2.14}$$

and

$$\mathrm{Sh}(p, q) = \{\sigma \in \Sigma_{p+q} \,|\sigma 1 < \ldots < \sigma p;\ \sigma(p+1) < \ldots < \sigma(p+q)\}$$

In the graded case, $\mathrm{sgn}(\sigma)$ gets replaced by the sign computed by the following rule: in all transpositions, the parity of a_i is equal to $|a_i|+1$ if $i \neq 0$ and $|a_0|$ if $i = 0$, and similarly for c_i. A transposition contributes a product of parities.

The shuffle product is not a morphism of complexes unless A is commutative. It defines, however, an exterior product as shown in the following theorem. For two unital algebras A and C, let i_A, i_C be the embeddings $a \mapsto a \otimes 1$, resp. $c \mapsto 1 \otimes c$ of A, resp. C, to $A \otimes C$. We will use the same notation for the embeddings that i_A, i_C induce on all the chain complexes considered by us.

Theorem 2.3.1. *For two unital algebras A and C the composition*

$$C_p(A) \otimes C_q(C) \xrightarrow{i_A \otimes i_C} C_p(A \otimes C) \otimes C_q(A \otimes C) \xrightarrow{\mathrm{sh}} C_{p+q}(A \otimes C)$$

defines a quasi-isomorphism

$$\mathrm{sh} : C_\bullet(A) \otimes C_\bullet(C) \to C_\bullet(A \otimes C)$$

To extend this theorem to cyclic complexes, define

$$\mathrm{sh}' : C_p(A) \otimes C_q(A) \to C_{p+q+2}(A) \tag{2.15}$$

as follows.

$$(a_0 \otimes \ldots \otimes a_p) \otimes (c_0 \otimes \ldots \otimes c_q) \mapsto 1 \otimes \mathrm{sh}'_{p+1,\,q+1}(a_0, \ldots, a_p, c_0, \ldots, c_q) \tag{2.16}$$

where

$$\mathrm{sh}'_{p+1,q+1}(x_0, \ldots, x_{p+q+1}) = \sum_{\sigma \in \mathrm{Sh}'(p+1,q+1)} \mathrm{sgn}(\sigma) x_{\sigma^{-1}0} \otimes \ldots \otimes x_{\sigma^{-1}(p+q+1)} \tag{2.17}$$

and $\mathrm{Sh}'(p+1,q+1)$ is the set of all permutations $\sigma \in \Sigma_{p+q+2}$ such that $\sigma 0 < \ldots < \sigma p$, $\sigma(p+1) < \ldots < \sigma(p+q+1)$, and $\sigma 0 < \sigma(p+1)$

$$\mathrm{sh}' : C_\bullet(A) \otimes C_\bullet(C) \to C_{\bullet+2}(A \otimes C)$$

Now define (2.15) to be the composition

$$C_p(A) \otimes C_q(C) \xrightarrow{i_A \otimes i_C} C_p(A \otimes C) \otimes C_q(A \otimes C) \xrightarrow{\mathrm{sh}'} C_{p+q+2}(A \otimes C)$$

In the graded case, the sign rule is as follows: any a_i has parity $|a_i|+1$, and similarly for c_i.

Theorem 2.3.2. *The map* $\mathrm{sh}+u\mathrm{sh}'$ *defines a* $k[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$(C_\bullet(A) \otimes C_\bullet(C))[[u]] \to CC^-_\bullet(A \otimes C)$$

as well as

$$(C_\bullet(A) \otimes C_\bullet(C))[[u^{-1}, u]] \to CC^{\mathrm{per}}_\bullet(A \otimes C)$$

$$(C_\bullet(A) \otimes C_\bullet(C))[[u^{-1}, u]]/u(C_\bullet(A) \otimes C_\bullet(C))[[u]] \to CC_\bullet(A \otimes C)$$

(the differentials on the left hand sides are equal to $b \otimes 1 + 1 \otimes b + u(B \otimes 1 + 1 \otimes B)$*).*

Note that the left hand side of the last formula maps to the tensor product $CC_\bullet(A) \otimes CC_\bullet(C)$: $\Delta(u^{-p} c \otimes c') = (u^{-1} \otimes 1 + 1 \otimes u^{-1})^p c \otimes c'$. One checks that this map is an embedding whose cokernel is the kernel of the map $u \otimes 1 - 1 \otimes u$, or $S \otimes 1 - 1 \otimes S$ where S is the Bott periodicity operator. ¿From this we get

Theorem 2.3.3. *There is a long exact sequence*

$$\to HC_n(A \otimes C) \xrightarrow{\Delta} \bigoplus_{p+q=n} HC_p(A) \otimes HC_q(C) \xrightarrow{S \otimes 1 - 1 \otimes S}$$

$$\bigoplus_{p+q=n-2} HC_p(A) \otimes HC_q(C) \xrightarrow{\times} HC_{n-1}(A \otimes C) \xrightarrow{\Delta}$$

2.4. **Pairings between chains and cochains.** For a graded algebra A, for $D \in C^d(A, A)$, define

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D| \sum_{i \leq d}(|a_i|+1)} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n \tag{2.18}$$

Proposition 2.4.1.

$$[b, i_D] = i_{\delta D}$$

$$i_D i_E = (-1)^{|D||E|} i_{E \smile D}$$

Now, put

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{k=1}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \tag{2.19}$$

$$\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k$$

(The second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D). The signs are given by

$$\epsilon_k = (|D| + 1) \sum_{i=0}^{k} (|a_i| + 1)$$

and

$$\eta_k = |D| + 1 + \sum_{i \le k} (|a_i| + 1) \sum_{i \ge k} (|a_i| + 1)$$

Proposition 2.4.2.

$$[L_D, L_E] = L_{[D,E]}$$

$$[b, L_D] + L_{\delta D} = 0$$

$$[L_D, B] = 0$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j \ge 0;\, k \ge j+d} \epsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes \tag{2.20}$$

$$D(a_{j+1}, \ldots, a_{j+d}) \otimes \ldots \otimes a_k$$

(The sum is taken over all cyclic permutations for which a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = |D|(|a_0| + \sum_{i=1}^{n} (|a_i|+1)) + (|D|+1) \sum_{j+1}^{k} (|a_i|+1) + \sum_{i \le k} (|a_i|+1) \sum_{i \ge k} (|a_i|+1)$$

As we will see later, all the above operations are partial cases of a unified algebraic structure for chains and cochains; the sign rule for this unified construction will be explained in 2.5.

Proposition 2.4.3. *(*[R]*)*

$$[b + uB, i_D + uS_D] - i_{\delta D} - uS_{\delta D} = L_D$$

Proposition 2.4.4. *([DGT]) There exists a linear transformation $T(D,E)$ of the Hochschild chain complex, bilinear in $D,\ E\in C^\bullet(A,A)$, such that*

$$[b+uB,T(D,E)]-T(\delta D,E)-(-1)^{|D|}T(D,\delta E)=$$

$$=[L_D,i_E+uS_E]-(-1)^{|D|+1}(i_{[D,E]}+uS_{[D,E]})$$

2.5. A_∞ **structure on** $C_\bullet(C^\bullet(A))$**.** All the pairings described above are in fact different parts of the same algebraic structure that we are going to describe next.

Recall that an A_∞ algebra is a graded vector space $\mathcal{C}$ together with a Hochschild cochain m of total degree 1,

$$m=\sum_{n=1}^{\infty}m_n$$

where $m_n\in C^n(\mathcal{C})$ and

$$[m,m]=0$$

To define A_∞ modules over A_∞ algebras, note that for a graded space $\mathcal{M}$, the Gerstenhaber bracket $[\ ,\ ]$ can be extended to the space

$$\mathrm{Hom}(\mathcal{C}^{\otimes\bullet},\mathcal{C})\oplus\mathrm{Hom}(\mathcal{M}\otimes\mathcal{C}^{\otimes\bullet},\mathcal{M})$$

For a graded space $\mathcal{M}$, a structure of an A_∞ module over an A_∞ algebra $\mathcal{C}$ on $\mathcal{M}$ is a cochain

$$\mu=\sum_{n=1}^{\infty}\mu_n$$

$$\mu_n\in Hom(\mathcal{M}\otimes\mathcal{C}^{\otimes n-1},\mathcal{M})$$

such that

$$[m+\mu,m+\mu]=0$$

Consider the Hochschild cochain complex of a graded algebra A as a differential graded associative algebra $(C^\bullet(A),\ \smile,\ \delta)$. Consider the Hochschild *chain* complex of this differential graded algebra. The total differential in this complex is $b+\delta$; the degree of a chain is given by

$$|D_0\otimes\ldots\otimes D_n|=|D_0|+\sum_{i=1}^{n}(|D_i|+1)$$

where D_i are Hochschild cochains.

The complex $C_\bullet(C^\bullet(A))$ contains the Hochschild cochain complex $C^\bullet(A)$ as a subcomplex (of zero-chains) and has the Hochschild chain complex $C_\bullet(A)$ as a quotient complex:

$$C^\bullet(A)\xrightarrow{i}C_\bullet(C^\bullet(A))\xrightarrow{\pi}C_\bullet(A)$$

(this sequence is not by any means exact). The projection on the right splits if A is commutative. If not, $C_\bullet(A)$ is naturally a graded subspace but not a subcomplex.

Theorem 2.5.1. *There is an A_∞ structure on $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All m_n are $k[[u]]$-linear, (u)-adically continuous*
- $m_1 = b + \delta + uB$
  For $x,\ y \in C_\bullet(A)$:
- $(-1)^{|x|} m_2(x, y) = (\mathrm{sh} + u\,\mathrm{sh}')(x, y)$
  For $D,\ E \in C^\bullet(A)$:
- $(-1)^{|D|} m_2(D, E) = D \smile E$
- $m_2(1 \otimes D,\ 1 \otimes E) + (-1)^{|D||E|} m_2(1 \otimes E,\ 1 \otimes D) = (-1)^{|D|} 1 \otimes [D,\ E]$
- $m_2(D,\ 1 \otimes E) + (-1)^{(|D|+1)|E|} m_2(1 \otimes E,\ D) = (-1)^{|D|+1} [D,\ E]$

Theorem 2.5.2. *On $C_\bullet(A)[[u]]$, there exists a structure of an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$ such that:*

- *All μ_n are $k[[u]]$-linear, (u)-adically continuous*
- $\mu_1 = b + uB$ *on* $C_\bullet(A)[[u]]$
  For $a \in C_\bullet(A)[[u]]$:
- $\mu_2(a, D) = (-1)^{|a||D|+|a|} (i_D + uS_D) a$
- $\mu_2(a, 1 \otimes D) = (-1)^{|a||D|} L_D a$
  For $a,\ x \in C_\bullet(A)[[u]]$: $(-1)^{|a|} \mu_2(a, x) = (\mathrm{sh} + u\,\mathrm{sh}')(a, x)$

Here is an explicit description of the above A_∞ structures. We define for $n \geq 2$

$$m_n = m_n^{(1)} + m_n^{(2)} + u m_n^{(3)}$$

where, for

$$a^{(k)} = D_0^{(k)} \otimes \ldots \otimes D_{N_k}^{(k)},$$

$$m_n^{(1)} = 0$$

for $n \geq 3$;

$$m_2^{(1)}(a^{(1)}, a^{(2)}) = (-1)^{|a^{(1)}|} \sum \pm D_0^{(1)} \smile D_0^{(2)} \underline{\{\ \}} \otimes \underline{\ldots} \otimes$$

$$\otimes D_1^{(2)} \underline{\{\ \}} \otimes \underline{\ldots} \otimes D_{N_2}^{(2)} \underline{\{\ \}} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_1^{(1)}, \ldots, D_{N_1}^{(1)}$ whose order is preserved. The sign rule is as follows: the parity of $D_j^{(i)}$ is $|D_j^{(i)}|$ for $j = 0$ and $|D_j^{(i)}| + 1$ otherwise.

$$m_n^{(2)}(a^{(1)}, \ldots, a^{(n)}) =$$

$$(-1)^{\sum_{i=1}^{n-1}|a_1|+n}\sum \pm D^{(n)}_{N_n}\{\underline{\ldots}, D^{(1)}_0, \underline{\ldots}, D^{(n-1)}_0, \underline{\ldots}\} \smile D^{(n)}_0$$
$$\otimes \underline{\ldots} \otimes D^{(n)}_1\underline{\{\ \}} \otimes \underline{\ldots} \otimes D^{(n)}_{N_n-1}\underline{\{\ \}} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_i^{(j)}$ for $j < n$ in such a way that:

- the cyclic order of each group $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ is preserved
- $D_0^{(1)}, \ldots, D_0^{(n-1)}$ are all inside the braces in $D_{N_n}^{(n)}\{\ \}$
- $D_0^{(i)}$ is to the left of $D_0^{(j)}$ for $i < j$
- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p < i$

The parity of $D_j^{(i)}$ is $|D_j^{(i)}|$ if $i = n$ and $j = 0$; it is $|D_j^{(i)}| + 1$ otherwise.

$$m_n^{(3)}(a^{(1)}, \ldots, a^{(n)}) = (-1)^{n+1}\sum \pm 1 \otimes \underline{\ldots} \otimes D_0^{(0)} \otimes \underline{\ldots} \otimes D_0^{(n)} \otimes \underline{\ldots}$$

The space designated by $\_$ is filled with $D_i^{(j)}$ for in such a way that:

- the cyclic order of each group $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ is preserved
- $D_0^{(i)}$ is to the left of $D_0^{(j)}$ for $i < j$
- any cochain $D_j^{(i)}$ may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form $D_q^{(p)}$ with $p < i$. The parity of $D_j^{(i)}$ is always $|D_j^{(i)}| + 1$.

The above formulas provide an A_∞ structure from Theorem 2.5.1. To obtain a structure of an A_∞ module from Theorem 2.5.2, one has to assume that all $D_j^{(1)}$ are elements of A and to replace braces $\{\ \}$ by the usual parentheses ( ) symbolizing evaluation of a multi-linear map at elements of A.

Remark 2.5.3. Let A be a commutative algebra. Then $C_\bullet(A)[[u]]$ is not only a subcomplex but an A_∞ subalgebra of $C_\bullet(C^\bullet(A))[[u]]$. This A_∞ structure on $C_\bullet(A)[[u]]$ was introduced in [GJ].

The corresponding binary product was defined by Hood and Jones [HJ]. One can define it for any algebra A, commutative or not. If A is not commutative then this product is not compatible with the differential. Nevertheless, we will use it in 3.5.

3. The cyclic complex $C_\bullet^\lambda$

3.1. **Definition.** Recall the original definition of the cyclic complex from [C1], [Ts1]. Let $\tau = \tau_p$ denote the endomorphism of $A^{\otimes_k(p+1)}$ given by the formula

$$\tau(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p \otimes a_0 \cdots \otimes a_{p-1}$$

Let

$$C^\lambda_p(A) = A^{\otimes_k (p+1)}/\mathrm{Im}(\mathrm{id} - \tau)\ . \tag{3.1}$$

Define

$$\begin{aligned} b' : C_p(A) &\rightarrow C_{p-1}(A) \\ a_0 \otimes \cdots \otimes a_p &\mapsto \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \end{aligned}$$

and

$$\begin{aligned} N : C_p(A) &\rightarrow C_{p-1}(A) \\ N = \mathrm{id} + \tau + \ldots \tau^{p-1} \end{aligned}$$

One has

$$b(\mathrm{id} - \tau) = (\mathrm{id} - \tau) b'; \;\; b' N = N b. \tag{3.2}$$

Therefore the differential b descends to a map

$$b : C^\lambda_p(A) \rightarrow C^\lambda_{p-1}(A)\ .$$

The complex $C^\lambda_\bullet(A)$ is quasi-isomorphic to $CC_\bullet(A)$. (Sketch of the proof: in the cyclic double complex $CC_\bullet(A)$ one can replace $C_\bullet(A)$ by the cone of $\mathrm{id} - \tau$ and B by N. One gets a double complex which is quasi-isomorphic to $CC_\bullet$ and which is in turn quasi-isomorphic to $C^\lambda_\bullet$. Cf. [L] for more detail).

Lemma 3.1.1. *For algebras without unit, the cyclic homology defined in Remark 2.0.3 is isomorphic to the homology of the complex $C^\lambda_\bullet(A)$.*

3.2. **The reduced cyclic complex.** Let $\overline{A} = A/k$ and let

$$\overline{C}^\lambda_p(A) = \overline{A}^{\otimes p+1}/\mathrm{Im}(\mathrm{id} - \tau)\ . \tag{3.3}$$

It is easy to see that the diferential b descends to $\overline{C}^\lambda_\bullet(A)$. We denote the homology of the complex $\overline{C}^\lambda_\bullet(A)$ by $\overline{HC}_\bullet(A)$.

Proposition 3.2.1. *There is an exact triangle*

$$C^\lambda_\bullet(k) \rightarrow C^\lambda_\bullet(A) \rightarrow \overline{C}^\lambda_\bullet(A) \rightarrow C^\lambda_\bullet(k)[1]\ . \tag{3.4}$$

Cf. [L] for the proof. (Note that the above proposition could be deduced from the Hochschild-Serre spectral sequence

$$E^2_{pq} = H_p(\mathfrak{gl}(A), \mathfrak{gl}(k)) \otimes H_q(\mathfrak{gl}(k))$$

converging to $H_{p+q}(\mathfrak{gl}(A))$ and from theorem 3.3.1).

3.3. **Relation to Lie algebra homology.** For an algebra (A,δ) over k let $\mathfrak{gl}(A)=\varinjlim_n \mathfrak{gl}_n(A)$ denote the Lie algebra of finite matrices. Note that $\mathfrak{gl}(k)$ is a DGL-subalgebra of $\mathfrak{gl}(A)$. Theorem 3.3.1 below identifies their respective subcomplexes of primitive elements of the DG coalgebras $C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}$ and $C_\bullet(\mathfrak{gl}(A),\mathfrak{gl}(k))$.

Let E^a_{pq} denote the elementary matrix with $\left(E^a_{pq}\right)_{pq}=a$ and other entries equal to zero.

Theorem 3.3.1. *The map*

$$\begin{array}{rcl} A^{\otimes p+1} & \to & \bigwedge^{p+1}\mathfrak{gl}(A) \\ a_0\otimes\cdots\otimes a_p & \mapsto & E^{a_0}_{01}\wedge E^{a_1}_{12}\wedge\cdots\wedge E^{a_p}_{p-1,0} \end{array}$$

induces isomorphisms of complexes

$$\begin{array}{rcl} C^\lambda_\bullet(A) & \to & \operatorname{Prim} C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}[1] \\ \overline{C}^\lambda_\bullet(A) & \to & \operatorname{Prim} C_\bullet(\mathfrak{gl}(A),\mathfrak{gl}(k))[1] \end{array}$$

The proof is contained in [Ts], [LQ], [L]

3.4. **The connecting morphism.** Here we give an explicit formula for the connecting morphism ∂ of the exact triangle from Proposition 3.2.1. For a DGA (A,δ) we denote by $T(A)$ the tensor algebra of A and by j a k-linear map $A\to k$ such that $j(1)=1$.

Let $\rho: T(A)\to k$ denote the map defined by

$$\begin{array}{rcl} \rho(a) & = & j(\delta a) \text{ for } a\in A, \\ \rho(a_1\otimes a_2) & = & j(a_1)j(a_2)-j(a_1a_2) \text{ for } a_1,a_2\in A, \text{ and} \\ \rho & = & 0 \text{ on } A^{\otimes p} \text{ for } p\neq 1,2. \end{array}$$

For $a_i\in A$, $a_0\otimes\cdots\otimes a_p\in A^{\otimes p+1}$ with $(\sum_i(\deg a_i+1))-1=2n+1$ let

$$\operatorname{br}_{2n+1}(a_0\otimes\cdots\otimes a_p)=$$

$$\frac{1}{n!}\sum_{i=0}^{p}(-1)^{\sum_{k<i}(\deg a_k+1)\sum_{k\geq i}(\deg a_k+1)}(\rho\otimes\cdots\otimes\rho)(a_i\otimes\cdots a_0\otimes\cdots\otimes a_{i-1})\,. \tag{3.5}$$

This defines a map $\operatorname{br}_{2n+1}:\overline{C}^\lambda_{2n+1}(A)\to k$ (cf. [Br]). Let $\operatorname{br}_{2n}=0$ on $\overline{C}^\lambda_{2n}(A)$.

Let $1^{(n+1)}=n!(n+1)!\cdot 1^{\otimes 2n+1}\in C^\lambda_{2n}(k)$. Let $\operatorname{Br}^A_{2n+1}=\operatorname{br}_{2n+1}\cdot 1^{(n)}$, $\operatorname{Br}^A_{2n}=0$, $\operatorname{Br}^A=\sum_p \operatorname{Br}_p$.

One gets the map of graded spaces

$$\operatorname{Br}^A:\overline{C}^\lambda_\bullet(A)\to C^\lambda_\bullet(k)[1]$$

which is, in fact, a map of complexes as follows from the alternative description given below.

The splitting j induces the splitting $\boldsymbol{\rho} : \mathfrak{gl}(A) \to \mathfrak{gl}(k)$ of the inclusion $\mathfrak{gl}(k) \hookrightarrow \mathfrak{gl}(A)$. Let P_n denote the invariant polynomial $X \mapsto \frac{1}{n!} tr(X^n)$ on $\mathfrak{gl}(k)$. Put

$$R(\boldsymbol{\rho}) = (\partial^{\mathrm{Lie}} + \delta)R(\boldsymbol{\rho}) + \frac{1}{2}[R(\boldsymbol{\rho}), R(\boldsymbol{\rho})] \tag{3.6}$$

Lemma 3.4.1. *The restriction of the pairing with $P_n(R(\boldsymbol{\rho}))$ to the subspace of all primitive elements (see Theorem 3.3.1) coincides with the map* br_{2n+1}.

The proof is straightforward.

Corollary 3.4.2. *The map* Br^A *is a morphism of complexes.*

Proposition 3.4.3. *The morphism* $\mathrm{Br}^A : \overline{C}^\lambda_\bullet(A) \to C^\lambda_\bullet(k)[1]$ *represents the connecting morphism in the triangle* (3.4).

3.4.1. *Explicit formula for the product $HC_p \otimes HC_q \to HC_{p+q+1}$.* Here we give an explicit formula for the product $\times$ from Theorem 2.3.3 in one of the realizations of the cyclic complex. Note first that, because of (3.2), the map N induces an isomorphism

$$C^\lambda(A) \simeq (\mathrm{Ker}(\mathrm{id} - \tau), b') \tag{3.7}$$

where in the right hand side $\mathrm{id} - \tau$ is considered as an operator on $A^{\otimes(\bullet+1)}$.

Proposition 3.4.4. *If one identifies $C^\lambda_\bullet$ with the right hand side of (3.7), then the shuffle product*

$$(a_0 \otimes \ldots \otimes a_p) \times (c_0 \otimes \ldots \otimes c_q) = \mathrm{sh}_{p+1,q+1}((a_0, \ldots \otimes a_p, c_0, \ldots , c_q)$$

is a morphism of complexes

$$\times : C^\lambda_\bullet(A) \otimes C^\lambda_\bullet(C) \to C^\lambda_{\bullet+1}(A \otimes C) \tag{3.8}$$

which induces on homology the $\times$ product from Theorem 2.3.3.

By the same construction one defines the product

$$\times : \overline{C}^\lambda_\bullet(A) \otimes \overline{C}^\lambda_\bullet(C) \to \overline{C}^\lambda_\bullet(A \otimes C)$$

on the reduced cyclic homology. Cf. [L] for the explicit formula for the $\times$ products at the level of complexes $CC_\bullet$.

3.5. **Other operations on the cyclic complexes.** Here we recall, in a modified version, some results from [NT2].

Let $\mathfrak{g}_A^\bullet[\epsilon] = \mathfrak{g}_A^\bullet + \mathfrak{g}_A^\bullet \epsilon$, $\epsilon^2 = 0$, $\deg\epsilon = 1$. Let u be a formal parameter of degree 2 and consider the graded algebra $U(\mathfrak{g}_A^\bullet[\epsilon])[u]$ with the differential $u \cdot \frac{\partial}{\partial \epsilon} + \delta$ (recall that δ is the differential in $\mathfrak{g}_A^\bullet$).

The motivation for the theorem below is the following. If $A = C^\infty(M)$ then $HC_\bullet^-(A)$ is isomorphic to the cohomology of the complex $\Omega^\bullet(M)[[u]], ud$; the cohomology of $\mathfrak{g}_A^\bullet$ is the graded Lie algebra $\mathfrak{g}_M^\bullet$ of multivector fields on M; one can define an action of $\mathfrak{g}_M^\bullet[\epsilon][u]$ on $HC_\bullet^-(A)$: for two multivector fields X, Y the action of $X + \epsilon Y$ is given by $L_X + \iota_Y$ where ι_Y is the contraction operator and $L_X = [d, \iota_X]$.

We would like to have a non-commutative analog of the above action. In fact, because of Theorem 4.4.1, we know that there is an L_∞ action of $\mathfrak{g}_A^\bullet[\epsilon][u]$ on $CC_\bullet^-(A)$, i.e., a DGLA which is quasi-isomorphic to the former acts on a complex quasi-isomorphic to the latter. However, the proof of Theorem 4.4.1 is inexplicit, and we need explicit operations for index-theoretical applications. Below we provide an explicit formula for an action of the *complex* $U(\mathfrak{g}_A^\bullet[\epsilon])[u]$ on $CC_\bullet^-(A)$.

Denote by $\mathcal{E}_A^\bullet$ the DGA $C^\bullet(A)$ with the cup product and the differential δ. Let $\star$ be the Hood - Jones product on $CC_\bullet^-(\mathcal{E}_A^\bullet)$ (Remark 2.5.3). In other words,

$$a \star b = (-1)^{|a|}(m_2^{(1)}(a,b) + u m_2^{(3)}(a,b))$$

where $m_i^{(j)}$ are defined in 2.5 and applied here to the subspace $CC_\bullet^-(C^0(\mathcal{E}))$ of $CC_\bullet^-(C^\bullet(\mathcal{E}))$

Theorem 3.5.1. *Formula*

$$(\epsilon D_1 \cdot \dots \cdot \epsilon D_m) \bullet \alpha = I(D_1, \dots, D_m)\alpha =$$
$$(-1)^{\deg\alpha \sum_{i=1}^m deg D_i} \frac{1}{m!} \sum_\sigma \epsilon_\sigma \alpha \bullet (D_{\sigma_1} \star (D_{\sigma_2} \star (\dots \star D_{\sigma_m})) \dots)$$

defines morphisms of complexes of $k[u]$*-modules*

$$U(\mathfrak{g}_A^\bullet[\epsilon])[u] \otimes_{U(\mathfrak{g}_A^\bullet)[u]} CC_\bullet^-(A) \rightarrow CC_\bullet^-(A)$$
$$U(\mathfrak{g}_A^\bullet[\epsilon])[u, u^{-1}] \otimes_{U(\mathfrak{g}_A^\bullet)[u,u^{-1}]} CC_\bullet^{per}(A) \rightarrow CC_\bullet^{per}(A)$$

Let u be a formal parameter of degree two. Consider the differential graded algebra $A[\eta] = A + A\eta$, $\deg\eta = -1$, $\eta^2 = 0$ with the differential $\frac{\partial}{\partial \eta}$. Consider the complex $\overline{C}_\bullet^\lambda(A[\eta])[u]$ with the differential $\frac{\partial}{\partial \eta} + u \cdot b$.

Theorem 3.5.2. *There exist natural pairings of* $k[[u]]$*-modules*

$$\bullet : \overline{C}_{\bullet-1}^\lambda(A[\eta])[[u]] \otimes CC_{-\bullet}^-(A) \rightarrow CC_{-\bullet}^-(A)$$

$$\bullet : \overline{C}^{\lambda}_{\bullet -1}(A[\eta])[[u,u^{-1}] \otimes CC^{per}_{-\bullet}(A) \to CC^{per}_{-\bullet}(A)$$

such that:

(1) $\eta^{\otimes m}\bullet ? = \frac{1}{(m-1)!}Id$ *for* $m>0$.
(2) *For* $x_i \in A$ *the operation* $(x_1 \otimes \cdots \otimes x_p)\bullet ?$ *sends* $C_N(A)$ *to* $\sum_{i,j\geq 0} C_{N-p+i}(A)u^j$.
(3) *The component of* $(x_1 \otimes \cdots \otimes x_p) \bullet (a_0 \otimes \cdots \otimes a_N)$ *in* $C_{N-p}[[u]]$ *is equal to*

$$\frac{1}{p!}\sum_{i=1}^{p}(-1)^{i(p-1)}a_0[x_{i+1},a_1][x_{i+2},a_2]\cdots[x_i,a_p]\otimes a_{p+1}\otimes\cdots\otimes a_N$$

(4) $(x_1 \otimes \cdots \otimes x_p) \bullet 1 = \sum_{i=1}^{p}(-1)^{i(p-1)} 1 \otimes x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_i$.

Note that the formula (4) above defines the map from $\overline{C}^{\lambda}_{\bullet}(A)$ to $Ker(B : C_\bullet(A) \to C_{\bullet+1}(A))$ (one can show that this map is a quasi-isomorphism). Clearly, the kernel above embeds into $CC^{-}_{\bullet}(A)$. The above theorem shows that this embedding extends to a pairing $\bullet$.

The complex $\overline{C}^{\lambda}_{\bullet -1}(A[\eta])[[u]]$ is very simple at the level of homology; it is quasi-isomorphic to $k[[u]]$. Therefore the pairing $\bullet$ does not define any new homological operations. It is, however, very important at the level of chains, as one sees in [BNT].

3.6. **Rigidity of periodic cyclic homology.** Let A be an associative algebra over a ring k of characteristic zero. Let I be a nilpotent two-sided ideal of A. Denote $A_0 = A/I$.

Theorem 3.6.1. *(Goodwillie) The natural map* $CC^{\mathrm{per}}_{\bullet}(A) \to CC^{\mathrm{per}}_{\bullet}(A/I)$ *is a quasi-isomorphism.*

Proof. Fix a k-linear isomorphism $A \simeq A_0 \oplus I$. Denote by $*$ the image of the direct sum multiplication under this isomorphism. Put $\lambda(a_1,a_2) = a_1a_2 - a_1 * a_2$; then λ is an element of $\mathfrak{g}^1_A = C^2(A)$ such that

$$\delta\lambda + \frac{1}{2}[\lambda,\lambda] = 0.$$

Since λ takes values in I, the operation

$$\chi = \sum_{m=0}^{\infty}\frac{1}{m!}I(\lambda,\dots,\lambda) \tag{3.9}$$

is well-defined on $CC^{\mathrm{per}}_{\bullet}(A)$ (where $I(\lambda,\dots,\lambda)$ are the operators from theorem 3.5.1). One checks that χ is indeed a morphism of complexes ([NT2]). It is easy to check that this morphism is invertible. ∎

4. Non-commutative differential calculus

4.1. **Gerstenhaber algebras.** Let k be the ground ring of characteristic zero. A *Gerstenhaber algebra* is a graded space $\mathcal{V}^\bullet$ together with

- A graded commutative associative algebra structure on $\mathcal{V}^\bullet$;
- a graded Lie algebra structure on $\mathcal{V}^{\bullet+1}$ such that

$$[a, bc] = [a, b]c + (-1)^{\deg(a)\deg(b)} b[a, c]$$

Example 4.1.1. Let M be a smooth manifold. Then

$$\mathcal{V}_M^\bullet = \wedge^\bullet T_M$$

is a sheaf of Gerstenhaber algebras.

The product is the exterior product, and the bracket is the Schouten bracket. We denote by $\mathcal{V}^\bullet(M)$ the Gerstenhaber algebra of global sections of this sheaf.

Example 4.1.2. Let $\mathfrak{g}$ be a Lie algebra. Then

$$C_\bullet(\mathfrak{g}) = \wedge^\bullet \mathfrak{g}$$

is a Gerstenhaber algebra.

The product is the exterior product, and the bracket is the unique bracket which turns $C_\bullet(\mathfrak{g})$ into a Gerstenhaber algebra and which is the Lie bracket on $\mathfrak{g} = \wedge^1(\mathfrak{g})$.

4.2. **The Gerstenhaber algebra $\mathcal{V}^\bullet(A)$.** Below is the theorem from [T].

Theorem 4.2.1. *For every associative algebra A there exists a dg Gerstenhaber algebra $\mathcal{V}^\bullet(A)$ such that:*

- *There is a quasi-isomorphism of dg Lie algebras*

$$\mathcal{V}^\bullet(A)[1] \to C^\bullet(A)[1]$$

- *The above quasi-isomorphism induces an isomorphism of Gerstenhaber algebras*

$$H^\bullet(\mathcal{V}^\bullet(A)) \to H^\bullet(A)$$

  where the Gerstenhaber structure on the right hand side is the standard one from 2.2.
- *For $A = C^\infty(M)$ there is a quasi-isomorphism of dg Gerstenhaber algebras*

$$\mathcal{V}^\bullet(A) \to \mathcal{V}^\bullet(M)$$

4.3. **Calculi.**

Definition 4.3.1. *A precalculus is a pair of a Gerstenhaber algebra* $\mathcal{V}^\bullet$ *and a graded space* $\Omega^\bullet$ *together with*

- *a structure of a graded module over the graded commutative algebra* $\mathcal{V}^\bullet$ *on* $\Omega^{-\bullet}$ *(corresponding action is denoted by* i_a, $a \in \mathcal{V}^\bullet$*);*
- *a structure of a graded module over the graded Lie algebra* $\mathcal{V}^{\bullet+1}$ *on* $\Omega^{-\bullet}$ *(corresponding action is denoted by* L_a, $a \in \mathcal{V}^\bullet$*) such that*
$$[L_a, i_b] = i_{[a,b]}$$
and
$$L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

Definition 4.3.2. *A calculus is a precalculus together with an operator* d *of degree 1 on* $\Omega^\bullet$ *such that* $d^2 = 0$ *and*
$$[d, i_a] = L_a.$$

Example 4.3.3. For any manifold one defines a calculus $\mathrm{Calc}(M)$ with $\mathcal{V}^\bullet$ being the algebra of multivector fields, $\Omega^\bullet$ the space of differential forms, and d the De Rham differential.

Example 4.3.4. For any associative algebra A one defines a calculus $\mathrm{Calc}_0(A)$ by putting $\mathcal{V}^\bullet = H^\bullet(A, A)$ and $\Omega^\bullet = H_\bullet(A, A)$. The five operations from Definition 4.3.2 are the cup product, the Gerstenhaber bracket, the pairings i_D and L_D, and the differential B, as in 2.4. The fact that it is indeed a calculus follows from Theorem 2.4.4.

A differential graded (dg) calculus is a calculus with extra differentials δ of degree 1 on $\mathcal{V}^\bullet$ and b of degree -1 on $\Omega^\bullet$ which are derivations with respect to all the structures.

4.4. **The calculus** $\mathrm{Calc}(A)$**.**

Theorem 4.4.1. [TT1] *For any associative algebra* A *one can define a dg calculus* $\mathrm{Calc}(A)$ *such that:*

1). As dg Lie algebras, $\mathcal{V}^{\bullet+1}(A)$ *is quasi-isomorphic to the Hochschild cochain complex* $C^{\bullet+1}(A)$*. As dg modules over the dg Lie algebra* $\mathcal{V}^{\bullet+1}(A)$*,* $\Omega^\bullet(A)[[u]]$ *with the differential* $\delta + uB$ *is quasi-isomorphic to the negative cyclic complex* $C_\bullet(A)[[u]]$ *with the differential* $b + uB$*.*

2). If $A = C^\infty(M)$ *then there is a quasi-isomorphism of calculi*
$$\mathrm{Calc}(A) \to \mathrm{Calc}(M)$$

3). For any A*, the calculus* $H^\bullet(\mathrm{Calc}(A))$ *is isomorphic to the calculus* $\mathrm{Calc}_0(A)$ *from Example 4.3.4.*

Combining 2) and 3), one gets for $A = C^\infty(M)$ quasi-isomorphisms

$$C_\bullet(A)[[u]] \leftarrow \Omega^\bullet(A)[[u]] \rightarrow \Omega^\bullet(M)[[u]].$$

Let **1** *be the canonical homology class of the left hand side. Then*

4). The image of **1** *in the cohomology of the right hand side is the square root of the Todd class of the tangent bundle TM:*

$$1 \mapsto \sum u^p(\sqrt{\mathrm{Td}(M)})^{2p}$$

4.5. **Enveloping algebra of a Gerstenhaber algebra.** The following construction is motivated by Example 4.1.1. For a Gerstenhaber algebra $\mathcal{V}^\bullet$, let $Y(\mathcal{V}^\bullet)$ be the associative algebra generated by two sets of generators i_a, L_a, $a \in \mathcal{V}^\bullet$, both i and L linear in a,

$$|i_a| = |a|;\ |L_a| = |a| - 1$$

subject to relations

$$i_a i_b = i_{ab};\ \ [L_a, L_b] = L_{[a,b]};$$

$$[L_a, i_b] = i_{[a,b]};\ L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

The algebra $Y(\mathcal{V}^\bullet)$ is equipped with the differential d of degree one which is defined as a derivation sending i_a to L_a and L_a to zero.

For a smooth manifold M one has a homomorphism

$$Y(\mathcal{V}^\bullet(M)) \rightarrow \mathrm{D}(\Omega^\bullet(M))$$

The right hand side is the algebra of differential operators on differential forms on M. It is easy to see that the above map is in fact an isomorphism.

4.6. **Relation to the A_∞ structure on chains.**

Theorem 4.6.1. *There is a A_∞ quasi-isomorphism*

$$Y(\mathcal{V}^\bullet(A)) \rightarrow C_\bullet(C^\bullet(A))$$

which extends to a $k[[u]]$-linear, (u)-adically continuous A_∞ quasi-isomorphism

$$(Y(\mathcal{V}^\bullet(A))[[u]], \delta + ud) \rightarrow C_\bullet(C^\bullet(A))[[u]]$$

5. Cyclic objects

5.1. **Simplicial objects.** Denote by Δ the category whose objects are sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and morphisms are nondecreasing maps. For a category C define a simplicial object of C an a functor from Δ^0 to C where Δ^0 is the category opposite to Δ. Any morphism in Δ can be written as a composition of the following:

1. Embeddings $d_i : [n] \rightarrow [n+1]$, $0 \leq i \leq n+1$ (i is not in the image of d_i);

2. Surjections $s_i : [n+1] \to [n]$, $0 \le i \le n$ (i is the image of two points under s_i).

A simplicial object of C can be equivalently described as a collection of objects X_n of C, $n \ge 0$, together with morphisms

$$d_i : X_{n+1} \to X_n,\ 0 \le i \le n+1 \tag{5.1}$$

$$s_i : X_n \to X_{n+1},\ 0 \le i \le n \tag{5.2}$$

subject to

$$d_i d_j = d_{j-1} d_i,\ i < j \tag{5.3}$$

$$s_i s_j = s_{j+1} s_i,\ i \le j \tag{5.4}$$

$$d_i s_j = s_{j-1} d_j,\ i < j;\ d_i s_i = d_{i+1} s_i = \mathrm{id};\ d_i s_j = s_j d_{i-1},\ i > j+1 \tag{5.5}$$

Example 5.1.1. For a topological space X define $\mathrm{Sing}_n(X)$ to be the set of singular simplices of X. Then $\mathrm{Sing}(X)$ has a natural structure of a simplicial set.

Example 5.1.2. Let A be a graded algebra. Put $A_n^{\#} = A^{\otimes(n+1)}$. Define

$$d_i(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{\sum_{p \le i} |a_p|} a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \tag{5.6}$$

for $i \le n$

$$d_{n+1}(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{|a_{n+1}| \sum_{p \le n} |a_p|} a_{n+1} a_0 \otimes a_1 \otimes \ldots \otimes a_n \tag{5.7}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = (-1)^{\sum_{p \le i} |a_p|} a_0 \otimes \ldots a_i \otimes 1 \otimes \ldots \otimes a_n \tag{5.8}$$

for $i \le n$

Proposition 5.1.3. *The above formulas make $A^{\#}$ a simplicial graded space.*

5.2. **Cyclic objects.** A cyclic object of a category C is a simplicial object X together with morphisms $t : X_n \to X_n$ for all $n \ge 0$ such that

$$d_0 t = t d_1;\ d_1 t = t d_2;\ \ldots;\ d_{n-1} t = t d_n;\ d_n t = d_0 \tag{5.9}$$

as morphisms $X_n \to X_{n-1}$;

$$s_0 t = t s_1;\ s_1 t = t s_2;\ \ldots;\ s_{n-1} t = t s_n;\ s_n t = t^2 s_0 \tag{5.10}$$

as morphisms $X_n \to X_{n+1}$;

$$t^{n+1} = \mathrm{id} \tag{5.11}$$

on X_n. For a graded algebra A put

$$t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|a_0| \sum_{i\geq 1} a_i}(a_1 \otimes a_2 \otimes \ldots \otimes a_0)$$

Proposition 5.2.1. *The above formula for t, together with* (5.6), (5.7), (5.8)*, makes $A^{\#}$ a cyclic graded vector space.*

For a cyclic vector space X define

$$b : X_n \to X_{n-1};\ b = \sum_{i=0}^{n} (-1)^i d_i;\ \tau = (-1)^{n+1} t : X_n \to X_n; \tag{5.12}$$

$$N = \sum_{i=0}^{n} \tau^i;\ B : X_n \to X_{n+1};\ B = (1-\tau)sN \tag{5.13}$$

where

$$s = t^{-1} s_n : X_n \to X_{n+1}$$

Proposition 5.2.2.

$$b^2 = bB + Bb = B^2 = 0$$

Definition 5.2.3. *For a cyclic vector space put*

$$C_\bullet(X) = (X_\bullet, b)$$
$$CC_\bullet^-(X) = (X_\bullet[[u]], b + uB)$$
$$CC_\bullet(X) = (X_\bullet[u^{-1}, u]]/uX_\bullet[[u]], b + uB)$$
$$CC_\bullet^{\mathrm{per}}(X) = (X_\bullet[u^{-1}, u]], b + uB)$$

where u is a formal parameter of degree -2.

Let C be a small category and $X : C^{op} \to$ k-mod a functor. Let $k^{\#} : C \to$ k-mod be the constant functor whose value on all objects is k and on all morphisms id_k. One defines

$$X \otimes_C k^{\#} = \oplus_{i\in Ob(C)} X(i)/ < a - X(f)a | a \in X(i)\ ;\ f \in hom_C(j,i) >$$

One can show that $X \mapsto X \otimes_C k^{\#}$ is a right exact functor. Denote its left derived functors by $\mathrm{Tor}_\bullet^C(X, k^{\#})$.

Theorem 5.2.4. *(Connes) One has, for an associative algebra A,*

$$HH_\bullet(A) = \mathrm{Tor}_\bullet^{\Delta}(A^{\#}, k^{\#})$$

$$HC_\bullet(A) = \mathrm{Tor}_\bullet^{\Lambda}(A^{\#}, k^{\#})$$

6. Examples

6.1. **Smooth functions.** For a smooth manifold M one can compute the Hochschild and cyclic homology of the algebra $C^\infty(M)$ where the tensor product in the definition if the Hochschild complex is one of the following three:

$$C^\infty(M)^{\otimes n} = C^\infty(M^n); \tag{6.1}$$

$$C^\infty(M)^{\otimes n} = \mathrm{germs}_\Delta\, C^\infty(M^n); \tag{6.2}$$

$$C^\infty(M)^{\otimes n} = \mathrm{jets}_\Delta\, C^\infty(M^n) \tag{6.3}$$

where Δ is the diagonal.

Theorem 6.1.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes

$$C_\bullet(C^\infty(M)) \to (\Omega^\bullet(M), 0)$$

and a $\mathbb{C}[[u]]$-linear, (u)-adically continuous quasi-isomorphism

$$CC_\bullet^-(C^\infty(M)) \to (\Omega^\bullet(M)[[u]], ud)$$

Localizing with respect to u, we also get quasi-isomorphisms

$$CC_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]]/u\Omega^\bullet(M)[[u]], ud)$$

$$CC_\bullet^{\mathrm{per}}(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]], ud)$$

This theorem, for the tensor products (6.2, 6.3), is due essentially to Hochschild, Kostant and Rosenberg (the Hochschild case) and to Connes (the cyclic cases). For the tensor product (6.1), see [Te].

6.1.1. *Holomorphic functions.* Let M be a complex manifold with the structure sheaf $\mathcal{O}_\mathcal{M}$ and the sheaf of holomorphic forms $\Omega^\bullet_M$. If one uses one of the following definitions of the tensor product, then $C_\bullet(\mathcal{O}_M)$, etc. are complexes of sheaves:

$$\mathcal{O}_M^{\otimes n} = \mathrm{germs}_\Delta\, \mathcal{O}_{M^n}; \tag{6.4}$$

$$\mathcal{O}_M^{\otimes n} = \mathrm{jets}_\Delta\, \mathcal{O}_{M^n} \tag{6.5}$$

where Δ is the diagonal.

Theorem 6.1.2. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes of sheaves

$$C_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M, 0)$$

and a $\mathbb{C}[[u]]$*-linear,* (u)*-adically quasi-isomorphism of complexes of sheaves*

$$CC^-_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M[[u]], ud)$$

Similarly for the complexes $CC_\bullet$ and CC^{per}.

6.2. **Commutative algebras.**

6.2.1. *Smooth Noetherian algebras.* By definition, a commutative Noetherian k-algebra A is smooth if, for any k-algebra C and any ideal I of C such that $I^2 = 0$, the map $\mathrm{Hom}(A, C) \to \mathrm{Hom}(A, C/I)$ is surjective. The class of smooth algebras includes the class of coordinate rings of nonsingular affine varieties over k. Let $\Omega^1_{A/k}$ be the module of Kähler differentials of A and $\Omega^n_{A/k} = \wedge^n_A \Omega^1_{A/k}$

Theorem 6.2.1. *The map*

$$\mu : a_0 \otimes \ldots a_n \mapsto \frac{1}{n!} a_0 da_1 \ldots da_n$$

defines quasi-isomorphisms of complexes

$$C_\bullet(A) \to (\Omega^\bullet_{A/k}, 0)$$

$$CC^-_\bullet(A) \to (\Omega^\bullet_{A/k}[[u]], ud)$$

$$CC_\bullet(A) \to (\Omega^\bullet_{A/k}[u^{-1}, u]]/u\Omega^\bullet_{A/k}[[u]], ud)$$

$$CC^{\mathrm{per}}_\bullet(A) \to (\Omega^\bullet_{A/k}[u^{-1}, u]], ud)$$

Cf. [L], [LQ]

6.2.2. *Finitely generated commutative algebras.* For a finitely generated commutative algebra A over k, choose a surjective homomorphism $f : P \to A$ where P is a ring of polynomials. Let I be the kernel of f. Consider the complexes

$$0 \to P/I^{n+1} \xrightarrow{d} \Omega^1_{P/k}/I^n \xrightarrow{d} \ldots \Omega^n_{P/k}/I \xrightarrow{d} 0 \qquad (6.6)$$

for all $n \geq 0$. Denote their cohomologies by $H^*_{\mathrm{cris}}(A/k; n)$ Denote by $H^\bullet_{\mathrm{cris}}(P/k)$ the cohomology of the complex $(\widehat{\Omega}^\bullet_{P/k}, d)$, the I-adic completion of the De Rham complex of P. It is well known that the cohomologies above are independent on a choice of P and f.

Theorem 6.2.2. *1. There exists the canonical morphism*

$$\mu : HC_n(A) \to \bigoplus_{i\geq 0} H_{\mathrm{cris}}^{n-2i}(A/k; n-i);$$

if A is smooth then μ is induced by the map from Theorem 6.2.1.

2. If A *is a locally complete intersection then μ is an isomorphism.*

3. There exists the canonical isomorphism

$$\mu : HC_\bullet^{\mathrm{per}}(A) \to H_{\mathrm{cris}}^\bullet(A/k)$$

Cf. [FT1]

6.3. **Rings of differential operators.** For a C^∞ manifold X of dimension n let $D(X)$ be the ring of differential operators on X. We use the tensor products defined analogously to (6.2), (6.3).

Theorem 6.3.1. *(*[BG], [Wo1]*). There is a quasi-isomorphism*

$$C_\bullet(D(X)) \to (\Omega^{2n-\bullet}(T^*X), d)$$

which extends to a $\mathbb{C}[[u]]$-linear, (u)-adically continuous quasi-isomorphism

$$CC_\bullet^-(D(X)) \to (\Omega^{2n-\bullet}(T^*X)[[u]], d)$$

As in 6.1, one also has analogous statements for the cyclic and periodic cyclic complexes.

6.3.1. *Holomorphic differential operators.* Let X be a complex manifold of complex dimension n. For the sheaf D_X of holomorphic differential operators, define the Hochschild, cyclic, etc. complexes of sheaves using tensor products analogous to those in 6.1.1. Let $\pi : T^*X \to X$ be the projection.

Theorem 6.3.2. [Bry] *There exists an isomorphism*

$$\pi^{-1}C_\bullet(D_X) \to (\Omega_{T^*X}^\bullet[2n], d)$$

in the derived category of the category of sheaves on T^*X, which extends to a $\mathbb{C}[[u]]$-linear, (u)-adically continuous isomorphism in the derived category

$$\pi^{-1}CC_\bullet^-(D_X) \to (\Omega_{T^*X}^\bullet[2n][[u]], d)$$

As in 6.1.1, similar isomorphisms exist for the cyclic and periodic cyclic complexes.

6.4. **Rings of complete symbols.** For a compact smooth manifold X, let $CL(X)$ be the algebra of classical pseudo-differential operators. By $L_\infty(X)$ denote the algebra of smoothing operators (i.e. integral operators with smooth kernel), and put

$$CS(X) = CL(X)/L_\infty(X).$$

We use the projective tensor products.

For any manifold M, denote by $\widehat{\Omega}^*(M \times S^1)$ the space of power series

$$\sum_{\epsilon=0,\ 1;\ n=-\infty}^{N} \alpha_{i,\ \epsilon} z^i dz^\epsilon$$

where α_i are forms on M. Denote by S^*X the cosphere bundle of X.

Theorem 6.4.1. [Wo2] *There exists a quasi-isomorphism*

$$C_\bullet(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1), d)$$

which extends to a $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$CC_\bullet^-(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1)[[u]], d)$$

Similarly for $CC_\bullet$, $CC_\bullet^{\mathrm{per}}$. In particular:

Corollary 6.4.2.

$$HH_p(CS(X)) = H^{2n-p}(S^*X \times S^1)$$

$$HC_p(CS(X)) = \bigoplus_{i \geq 0} H^{2n-p+2i}(S^*X \times S^1)$$

Combined with the pairing with the fundamental class of $S^*X \times S^1$, the first of the above isomorphisms gives an isomorphism

$$HH_0(CS(X)) = CS(X)/[CS(X), CS(X)] \to \mathbb{C} \qquad (6.7)$$

This isomorphism is given by the Wodzicki noncommutative residue [Wo]. The above theorem has also a holomorphic version where the ring of complete symbols is replaced by the sheaf of micro-differential operators [BNT1].

6.5. **Rings of pseudo-differential operators.**

Theorem 6.5.1.

$$HH_0(L_\infty(X)) \simeq \mathbb{C}; \quad HH_p(L_\infty(X)) = 0,\ p > 0;$$

$$HC_{2p}(L_\infty(X)) \simeq \mathbb{C}; \quad HC_{2p+1}(L_\infty(X)) \to 0$$

Theorem 6.5.2. [Wo2]

$$HH_p(CL(X)) \simeq HH_p(CS(X)) \text{ for } p \neq 1;$$

there is an exact sequence

$$0 \to HH_1(CL(X)) \to HH_1(CS(X)) \to \mathbb{C} \to 0$$

Theorem 6.5.3. *For all $p \geq 0$*

$$HC_{2p}(CL(X)) \simeq HC_{2p}(CS(X))$$

and there is an exact sequence

$$0 \to HC_{2p+1}(CL(X)) \to HC_{2p+1}(CS(X)) \to HC_{2p}(\mathbb{C}) \to 0$$

Theorems 6.5.2 and 6.5.3 follow from Theorem 6.5.1 and from Wodzicki excision theorems [Wo3].

The results of the two previous subsections were extended to more general rings of symbols and of pseudo-differential operators by Melrose-Nistor and Benameur-Nistor ([MN], [BN]). A survey of these results can be found in [BBN].

6.6. **Non-commutative tori.** Let q be a complex number, $|q| = 1$. By A_q denote the algebra over $\mathbb{C}$ with two generators $\mathbf{u}$ and $\mathbf{v}$ subject to one relation

$$\mathbf{vu} = q\mathbf{uv} \tag{6.8}$$

Theorem 6.6.1. [C]

$$H_\bullet(A_q) \simeq H_\bullet(\mathbb{T}^2)$$
$$HC_\bullet(A_q) \simeq H_\bullet(\mathbb{T})^2[[u, u^{-1}]/uH_\bullet(\mathbb{T})^2[[u, u^{-1}]$$
$$HC_\bullet^-(A_q) \simeq H_\bullet(\mathbb{T}^2)[[u]]$$
$$HC_\bullet^{\mathrm{per}}(A_q) \simeq H_\bullet(\mathbb{T}^2)[[u, u^{-1}]$$

where u is a formal variable of degree -2.

6.7. **Deformation quantization.** Let M be a smooth manifold. By a deformation quantization of M we mean a formal product

$$f * g = fg + \sum_{k=1}^{\infty} t^k P_k(f, g) \tag{6.9}$$

where P_k are bidifferential expressions, $*$ is associative, and $1 * f = f * 1 = f$ for all f. Given such a product (which is called a star product), we define

$$\mathbb{A}^t(M) = (C^\infty(M)[[t]], *) \tag{6.10}$$

This is an associative algebra over $\mathbb{C}[[t]]$. By $\mathbb{A}_c^t(M)$ we denote the ideal $C_c^\infty(M)[[t]]$ of this algebra. An isomorphism of two deformations is by definition a power series $T(f) = f + \sum_{k=1}^{\infty} T_k(f)$ where all T_k are differential operators and which is an isomorphism of algebras.

Given a star product on M, for $f,\ g \in C^\infty(M)$ let

$$\{f,g\} = P_1(f,g) - P_1(g,f) = \frac{1}{t}[f,g]|t=0 \tag{6.11}$$

This is a Poisson bracket corresponding to some Poisson structure on M. If this Poisson structure is defined by a symplectic form ω, we say that $\mathbb{A}^t(M)$ is a deformation of the symplectic manifold $(M,\ \omega)$. In defining the Hochschild and cyclic complexes, we use $k = \mathbb{C}[[t]]$ as the ring of scalars, and put

$$\mathbb{A}^t(M)^{\otimes n} = \mathrm{jets}_\Delta\, C^\infty(M^n)[[t]] \tag{6.12}$$

Let $\mathbb{A}^t(M)$ be a deformation of a symplectic manifold (M,ω).

Theorem 6.7.1. *There exists a quasi-isomorphism*

$$C_\bullet(\mathbb{A}^t(M), \mathbb{A}^t(M))[t^{-1}] \to (\Omega^{2n-\bullet}(M)[[t,t^{-1}], td)$$

which extends to a $\mathbb{C}[[t,u]]$-linear, (t,u)-adically continuous quasi-isomorphism

$$CC_\bullet^-(\mathbb{A}^t(M))[t^{-1}] \to (\Omega^{2n-\bullet}(M)[[u,t]][t^{-1}], td)$$

An analogous theorem holds for $\mathbb{A}_c^t(M)$ if we replace $\Omega^\bullet$ by $\Omega_c^\bullet$.

6.7.1. *The canonical trace.* Combining the first map from Theorem 6.7.1 in the compactly supported case with integrating over M and dividing by $\frac{1}{n!}$, one gets the canonical trace of Fedosov

$$Tr : \mathbb{A}_c^t(M) \to \mathbb{C}[[t,t^{-1}]$$

6.7.2. *Deformation quantization of complex manifolds.* For a complex manifold M with an open cover $\mathfrak{U}$, define a deformation of the structure sheaf of algebras $\mathcal{O}_M$ to be the following:

a) a collection of holomorphic star products $*_U$ on every $U \in \mathfrak{U}$

b) a collection of holomorphic isomorphisms G_{UV} between $*_U$ and $*_V$ for every $U,\ V \in \mathfrak{U}$, such that

$$G_{UV}G_{VW} = G_{UW} \tag{6.13}$$

for all $U,\ V,\ W$.

Such a data defines a sheaf of algebras $\mathbb{A}_M^t$.

(The above definition is enough for the purposes of this article; there is, however, a better definition in which (6.13) is replaced by $G_{UV}G_{VW} = \mathrm{Ad}(c_{UVW})G_{UW}$ for some invertible elements c_{UVW} subject to a natural cocycle condition on intersections of any four open subsets). Using the definition of tensor products analogous to 6.1, we define complexes of sheafs $C_\bullet(\mathbb{A}_M^t)$, etc.

Theorem 6.7.2. *Let $\mathbb{A}^t_M$ be a deformation of a complex manifold M with a holomorphic symplectic structure ω. There exists an isomorphism in the derived category of sheaves*

$$\mathbb{C}_\bullet(\mathbb{A}^t_M)[t^{-1}] \to (\Omega^{2n-\bullet}_M[[t, t^{-1}], td) \tag{6.14}$$

which extends to a $\mathbb{C}[[t, u]]$-linear, (t, u)-adically continuous isomorphism in the derived category

$$CC^-_\bullet(\mathbb{A}^t_M)[t^{-1}] \to (\Omega^{2n-\bullet}_M[[t, u]][t^{-1}], td) \tag{6.15}$$

Similarly for $CC_\bullet$, $CC^{\mathrm{per}}_\bullet$.

If one wants to avoid the language of derived categories, one can say that there may be no morphism of sheaves (6.15) but there is, canonical up to homotopy, morphism of Cech complexes of corresponding sheaves. One can also use sheaf resolutions other than the Cech resolution, for example the complex of Dolbeault forms with values in Hochschild, cyclic, De Rham, etc. complexes of the algebra of jets [BNT].

6.8. **The Brylinski spectral sequence.** Consider a deformation quantization $\mathbb{A}^t(M)$ of a smooth manifold M. As we saw above, such a deformation gives rise to a Poisson bracket on $C^\infty(M)$. Denote by π_0 the corresponding Poisson bivector: $\pi_0 \in \Gamma(M, \wedge^2 TM)$ such that $\{f,\ g\} =< \pi_0, df \wedge dg >$. Denote by i_{π_0} the operator of contracting a form by π_0, and put

$$L_{\pi_0} = di_{\pi_0} - i_{\pi_0} d \tag{6.16}$$

The Hochschild and cyclic complexes of the deformed algebra $\mathbb{A}^t(M)$ are complete with respect to the filtration $F_p C_\bullet = t^p C_\bullet$. Computing the spectral sequences corresponding to this filtration, one obtains the following

Theorem 6.8.1. *1. There exists a spectral sequence converging to*

$$HH_\bullet(\mathbb{A}^t(M))$$

with the E^1 term equal to the homology of the complex

$$(\Omega^\bullet(M)[[t]], tL_{\pi_0}) \tag{6.17}$$

2. There exists a spectral sequence converging to

$$HC^-_\bullet(\mathbb{A}^t(M))$$

with the E^1 term equal to the homology of the complex

$$(\Omega^\bullet(M)[[t, u]], tL_{\pi_0} + ud) \tag{6.18}$$

and similarly for the cyclic and periodic cyclic homologies.

If π_0 is symplectic then the complex 6.17, resp. 6.18, is quasi-isomorphic to the complex $(\Omega^{2n-\bullet}(M)[[t]], td)$, resp. $(\Omega^{2n-\bullet}(M)[[t,u]], td)$.

6.9. **Deformation quantization: general case.** Recall first Kontsevich's classification of deformation quantizations. We call a series

$$\pi = \sum_{i \geq 0} t^i \pi_i,$$

$\pi_i \in \Gamma(M, \wedge^2 T_M)$, *a formal Poisson structure* if $[\pi, \pi] = 0$ where $[\ ,\ ]$ is the Schouten bracket; in other words, π is a formal Poisson structure if

$$\{f, g\}_\pi = < \pi, df \wedge dg >$$

defines a Lie algebra structure on $C^\infty(M)[[t]]$. Two formal Poisson structures are called *equivalent* if there exists a series

$$X = \sum_{i \geq 0} t^i X_i,$$

$X_i \in \Gamma(M, T_M)$, such that

$$\pi' = \exp \operatorname{ad} L_X(P)$$

Theorem 6.9.1. *(Kontsevich) There is a bijection between the set of isomorphism classes of deformation quantizations of $C^\infty(M)$ and the set of equivalence classes of formal Poisson structures on M.*

For a formal Poisson structure π the above theorem defines up to isomorphism a deformed associative algebra $\mathbb{A}^t_\pi(M)$.

Theorem 6.9.2. *There are quasi-isomorphisms of complexes*

$$C_\bullet(\mathbb{A}^t_\pi(M)) \to (\Omega^\bullet(M)[[t]], tL_\pi)$$

$$CC^-_\bullet(\mathbb{A}^t_\pi(M)) \to (\Omega^\bullet(M)[[t,u]], tL_\pi + ud)$$

In particular, if π is an ordinary Poisson structure, then the Brylinski spectral sequence degenerates.

6.10. **Group rings.** Let G be a discrete group.

Theorem 6.10.1.

$$HH_\bullet(k[G]) \simeq \bigoplus_{<x>} H_\bullet(G_x, k)$$

where the sum is taken over all conjugacy classes of G and G_x is the centralizer of an element x of G.

The Hochschild homology can also be decomposed into components corresponding to conjugacy classes; the construction of these components depends on whether a class is elliptic (i.e. of the form $<x>$ where x is of finite order) or not.

Theorem 6.10.2. [Bu]

$$HC_n(k[G]) \simeq \bigoplus_{<x>\mid \operatorname{ord} x<\infty} \bigoplus_i H_{n-2i}(G_x,k) \oplus \bigoplus_{<x>\mid \operatorname{ord} x=\infty} H_{n-2i}(G_x/(x),k)$$

Remark 6.10.3. The exact sequence of the triangle (2.5) splits into a direct sum over the set of conjugacy classes of G. For the summands corresponding to elliptic classes, the component of the differential $HH_\bullet \to HC_\bullet$ is injective. In the non-elliptic case, the corresponding component of the exact sequence is isomorphic to the exact sequence of the fibration

$$B\mathbb{Z} \to BG_x \to B(G_x/(x))$$

Simirarly, the components of the spectral sequence (2.6) which correspond to elliptic classes degenerate at E^1. The component corresponding to a non-elliptic class $<x>$ is isomorphic to the spectral sequence of the fibration

$$BG_x \to B(G_x/(x)) \to K(\mathbb{Z},2)$$

7. Index theorems

7.1. **Index theorem for deformations of symplectic structures.** Let M be a smooth symplectic manifold. Let $\mathbb{A}^t_M$ be a deformation of the sheaf of functions on M. Recall that there exists canonical up to homotopy equivalence quasi-isomorphism (6.15)

$$\mu^t:\ CC^-_\bullet(\mathbb{A}^t(M))[t^{-1}] \to (\Omega^{2n-\bullet}(M)[[t,u]][t^{-1}], td) \tag{7.1}$$

Localizing in u, we obtain a quasi-isomorphism

$$\mu^t:\ CC^{\mathrm{per}}_\bullet(\mathbb{A}^t(M))[t^{-1}] \to (\Omega^{2n-\bullet}(M)[[t,u,t^{-1},u^{-1}], td) \tag{7.2}$$

Definition 7.1.1. *The above morphisms are called the trace density morphisms.*

The index theorem compares the trace density morphism to the principal symbol morphism. To define the latter, consider the cyclic complex of the deformed algebra where the scalar ring is $\mathbb{C}$ instead of $\mathbb{C}[[t]]$. Consider the composition

$$CC^-_\bullet(\mathbb{A}^t(M)) \to CC^-_\bullet(C^\infty(M)) \to$$
$$\to (\Omega^\bullet(M)[[u]], ud) \to (\Omega^\bullet(M)[[u]][[t]], ud)$$

where the first morphism is reduction modulo t, the second one is μ from Theorem 6.1.1, and the third one is induced by the embedding $\mathbb{C} \to \mathbb{C}[[t]]$. We will denote this composition, followed by localization in t, by

$$\mu : CC_\bullet^{\mathrm{per}}(\mathbb{A}^t(M)) \to (\Omega^\bullet(M)[[u, t, u^{-1}, t^{-1}], ud) \tag{7.3}$$

To compare μ and μ^t, let us identify the right hand sides by the isomorphism

$$(\Omega^{2n-\bullet}(M)[[u, t, u^{-1}, t^{-1}], td) \to (\Omega^\bullet(M)[[u, t, u^{-1}, t^{-1}], ud)$$

which is equal to $(\frac{t}{u})^{n-k}$ on $\Omega^k(M)[[u, t, u^{-1}, t^{-1}]$. After this identification, we obtain two morphisms

$$\mu,\ \mu^t : CC_\bullet^{\mathrm{per}}(\mathbb{A}^t(M)) \to (\Omega^\bullet(M)[[u, t, u^{-1}, t^{-1}], ud)$$

where the right hand side is defined as the periodic cyclic complex with respect to the ground ring $\mathbb{C}$.

Theorem 7.1.2. *At the level of cohomology,*

$$\mu^t = \sum_{p=0}^{\infty} u^p (\widehat{A}(M) e^\theta)_{2p} \cdot \mu$$

where $\widehat{A}(M)$ is the $\widehat{A}$ class of the tangent bundle of M viewed as a complex bundle (with an almost complex structure compatible to the symplectic form), and $\theta \in \frac{1}{t}[\omega] + H^2(M, \mathbb{C}[[t]])$ is the characteristic class of the deformation.

Note that the canonical trace $\mathrm{Tr}_{\mathrm{can}}$ is the composition of μ^t with the integration $\Omega^{2n}[[t, t^{-1}] \to \mathbb{C}[[t, t^{-1}]$. Let P and Q be $N \times N$ matrices over $\mathbb{A}^t(M)$ such that $P^2 = P$, $Q^2 = Q$, and $P - Q$ is compactly supported. Let P_0, Q_0 be reductions of P, Q modulo t. They are idempotent matrix-valued functions; their images $P_0\mathbb{C}^N$, $Q_0\mathbb{C}^N$ are vector bundles on M. Applying μ^t to the the difference of Chern characters of P and Q, we obtain the following index theorem of Fedosov (cf. also [NT4]).

Theorem 7.1.3.

$$\mathrm{Tr}_{\mathrm{can}}(P - Q) = \int_M (\mathrm{ch}(P_0\mathbb{C}^N) - \mathrm{ch}(Q_0\mathbb{C}^N)) \widehat{A}(M) e^\theta$$

7.2. **Index theorem for holomorphic symplectic deformations.** Let M be a complex manifold with a holomorphic symplectic form ω. Let $\mathbb{A}_M^t$ be a deformation of the sheaf of holomorphic functions on M. One defines two morphisms in the derived category of sheaves

$$\mu,\ \mu^t : CC_\bullet^{\mathrm{per}}(\mathbb{A}_M^t) \to (\Omega_M^\bullet[[u, t, u^{-1}, t^{-1}], ud)$$

exactly as in the smooth case.

Theorem 7.2.1.

$$\mu^t = \sum_{p=0}^{\infty} u^p (\sqrt{\mathrm{Td}(M)} e^{\theta})_{2p} \cdot \mu$$

in the derived category of sheaves, where $\mathrm{Td}(M)$ *is the Todd class of the holomorphic tangent bundle of* M*.*

7.3. **General index theorem for deformations.** Let M be a C^∞ manifold. Let π be a formal Poisson structure on M and $\mathbb{A}^t_\pi(M)$ the deformation corresponding to π under the Kontsevich isomorphism from Theorem 6.9.1. We construct the quasi-isomorphisms

$$\mu,\ \mu^t : CC^{\mathrm{per}}_\bullet(\mathbb{A}^t_\pi(M)) \to (\Omega^\bullet(M)[[u, t, u^{-1}, t^{-1}], ud)$$

as follows. The principal symbol map μ is constructed exactly as in (7.3). The trace density morphism μ^t is defined as follows. Start with the quasi-isomorphism

$$CC^-_\bullet(\mathbb{A}^t_\pi(M)) \to (\Omega^\bullet(M)[[t, u]], tL_\pi + ud)$$

from Theorem 6.9.2. Then localize it with respect to t and u, and then follow it by the isomorphism

$$\exp(-\frac{t}{u} i_\pi) : (\Omega^\bullet(M)[[t, u, t^{-1}, u^{-1}], tL_\pi + ud) \to (\Omega^\bullet(M)[[t, u, t^{-1}, u^{-1}], ud$$

.

Theorem 7.3.1. *At the level of cohomology,*

$$\mu^t = \sum_{p=0}^{\infty} u^p (\sqrt{\mathrm{Td}(M)})_{2p} \cdot \mu$$

where $\mathrm{Td}(M)$ *is the Todd class of the tangent bundle of* M*.*

Theorem 7.1.2 can be deduced from the theorem above. Note the absence of the factor e^θ, explained by the presence of $\exp(-\frac{t}{u} i_\pi)$ in the definition of the trace density map μ^t.

Let us stress again that all the ingredients of the theorem above (the construction of the deformation $\mathbb{A}^t_\pi(M)$) from π and the construction of the trace density map) are not canonical but depend on a choice of a Drinfeld associator.

8. Riemann-Roch theorem for D-modules

Let us recall the definition and examples from [SS]. Let X be a complex manifold. An elliptic pair $(\mathcal{M}, \mathcal{F})$ on X is a coherent D_X-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf $\mathcal{F}$ on X such that

$$\operatorname{char}(\mathcal{M}) \bigcap SS(\mathcal{F}) \subset T^*_X X \tag{8.1}$$

More generally, $(\mathcal{M}, \mathcal{F})$, can be a pair of complexes, or rather of objects of corresponding derived categories.

Examples

(1) Let $\mathcal{M}$ be a coherent D_X-module. Then the pair $(\mathcal{M}, \mathbb{C}_X)$ is elliptic.
(2) Let U be an open subset of X. Then the pair $(\mathcal{M}, \mathbb{C}_U)$ is elliptic if and only if ∂U is noncharacteristic for $\mathcal{M}$.
(3) Let X be a complexification of a real analytic manifold X_0 and $\mathcal{M}$ a complexification of a real analytic D_{X_0}-module $\mathcal{M}_0$ on X_0. Then $(\mathcal{M}, \mathbb{C}_{X_0})$ is an elliptic pair if and only if $\mathcal{M}_0$ is elliptic in the classical sense.
(4) If $\mathcal{F}$ is $\mathbb{R}$-constructible on X then $(\mathcal{O}_X, \mathcal{F})$ is an elliptic pair.

We assume that $\mathcal{M}$ is endowed with a good filtration. Recall that the order of differential operators defines a filtration F_p on D_X; a filtration G_q on a module is good if $F_p D_X \cdot G_q \subset G_{p+q}$.

Consider the derived tensor product $\mathcal{M} \otimes^{\mathbb{L}}_{D_X} \mathcal{O}_X$. Schapira and Schneiders showed that $(\mathcal{M} \otimes^{\mathbb{L}}_{D_X} \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{F}$ has finite-dimensional cohomology. Let $\mu\,\mathrm{eu}(\mathcal{F})$ be Kashiwara's microlocal Euler class of $\mathcal{F}$. Let Λ be the characteristic variety of $\mathcal{M}$. The good filtration on $\mathcal{M}$ defines a coherent $\mathrm{gr}_F\, D_X$-module $\mathrm{gr}\,\mathcal{M}$; since $\mathrm{gr}_F\, D_X$ is the sheaf of functions on T^*X which are polynomial along the fibres, by extension of scalars we obtain a coherent $\mathcal{O}_{T^*X}$-module $\sigma_\Lambda(\mathcal{M})$ supported on Λ. One can define Chern character ch_Λ of such a module in $H^\bullet_\Lambda(T^*X)$.

The following was conjectured in [SS] and proven in [BNT] by reducing it to Theorem 7.2.1.

Theorem 8.0.2.

$$\chi(R\Gamma((\mathcal{M} \otimes^{\mathbb{L}}_{D_X} \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{F})) = \int \mu\,\mathrm{eu}(\mathcal{F}) \smile \mathrm{ch}_\Lambda(\sigma_\Lambda(\mathcal{M})) \smile \pi^* \operatorname{Td}(X)$$

where π is the projection from T^*X to X.

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