LECTURES ON SHIMURA VARIETIES

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Abstract. The main goal of these lectures is to explain the representability of the moduli space of abelian varieties with polarizations, endomorphisms and level structures, due to Mumford [GIT], and the description of the set of its points over a finite field, due to Kottwitz [JAMS]. We also try to motivate the general definition of Shimura varieties and their canonical models as in the article of Deligne [Corvallis]. We will leave aside important topics like compactifications, bad reduction and the p-adic uniformization of Shimura varieties.

This is the notes of the lectures on Shimura varieties delivered by one of us in the Asia-French summer school organized at IHES in July 2006. It is basically based on the notes of a course delivered by the two of us in Université Paris-Nord in 2002.

Contents

1. Quotients of Siegel's upper half space

1.1. **Review on complex tori and abelian varieties.** Let V denote a complex vector space of dimension n and U a lattice in V which is by definition a discrete subgroup of V of rank $2n$. The quotient $X = V/U$ of V by U acting on V by translation, is naturally equipped with a structure of compact complex manifold and a structure of abelian group.

Lemma 1.1.1. *We have canonical isomorphisms from* $\mathrm{H}^r(X,\mathbb{Z})$ *to the group of alternating* r*-forms* $\bigwedge^r U \to \mathbb{Z}$.

Proof. Since $X = V/U$ with V contractible, $\mathrm{H}^1(X,\mathbb{Z}) = \mathrm{Hom}(U,\mathbb{Z})$. The cup-product defines a homomorphism

$$\bigwedge^r \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^r(X,\mathbb{Z})$$

which is an isomorphism since X is isomorphic to $(S_1)^{2n}$ as real manifolds (where $S_1 = \mathbb{R}/\mathbb{Z}$ is the unit circle). □

Let L be a holomorphic line bundle over the compact complex variety X. Its Chern class $c_1(L) \in \mathrm{H}^2(X,\mathbb{Z})$ is an alternating 2-form on U which can be made explicit as follows. By pulling back L to V by the quotient morphism $\pi : V \to X$, we get a trivial line bundle since every holomorphic line bundle over a complex vector space is trivial. We choose an isomorphism $\pi^*L \to \mathcal{O}_V$. For every $u \in U$, the canonical isomorphism $u^*\pi^*L \simeq \pi^*L$ gives rise to an automorphism of $\mathcal{O}_V$ which is given by an invertible holomorphic function

$$e_u \in \Gamma(V, \mathcal{O}_V^\times).$$

The collection of these invertible holomorphic functions for all $u \in U$, satisfies the cocycle equation

$$e_{u+u'}(z) = e_u(z+u')e_{u'}(z).$$

If we write $e_u(z) = e^{2\pi i f_u(z)}$ where $f_u(z)$ are holomorphic function well defined up to a constant in $\mathbb{Z}$, the above cocycle equation is equivalent to

$$F(u_1,u_2) = f_{u_2}(z+u_1) + f_{u_1}(z) - f_{u_1+u_2}(z) \in \mathbb{Z}.$$

The Chern class

$$c_1 : \mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X,\mathbb{Z})$$

sends the class of L in $\mathrm{H}^1(X,\mathcal{O}_X^\times)$ on the element $c_1(L) \in \mathrm{H}^2(X,\mathbb{Z})$ whose the corresponding 2-form $E : \bigwedge^2 U \to \mathbb{Z}$ is given by

$$(u_1,u_2) \mapsto E(u_1,u_2) := F(u_1,u_2) - F(u_2,u_1).$$

Lemma 1.1.2. *The Neron-Severi group* $\mathrm{NS}(X)$*, defined as the image of* $c_1 : \mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z})$ *consists of the alternating* 2*-form* $E : \bigwedge^2 U \to \mathbb{Z}$ *satisfying the equation*

$$E(iu_1, iu_2) = E(u_1, u_2)\,,$$

where E *still denotes the alternating* 2*-form extended to* $U \otimes_{\mathbb{Z}} \mathbb{R} = V$ *by* $\mathbb{R}$*-linearity.*

Proof. The short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X^\times \to \mathcal{O}_X \to 0$$

induces a long exact sequence which contains

$$\mathrm{H}^1(X, \mathcal{O}_X^\times) \to \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X).$$

It follows that the Neron-Severi group is the kernel of the map

$$\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X)\,.$$

This map is the composition of the obvious maps

$$\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X, \mathcal{O}_X).$$

The Hodge decomposition

$$\mathrm{H}^m(X, \mathbb{C}) = \bigoplus_{p+q=m} \mathrm{H}^p(X, \Omega_X^q)$$

where Ω_X^q is the sheaf of holomorphic q-forms on X, can be made explicit [13, page 4]. For $m = 1$, we have

$$\mathrm{H}^1(X, \mathbb{C}) = V_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}^* \oplus \overline{V}_{\mathbb{C}}^*\,,$$

where $V_{\mathbb{C}}^*$ is the space of $\mathbb{C}$-linear maps $V \to \mathbb{C}$, $V_{\mathbb{C}}^*$ is the space of conjugate $\mathbb{C}$-linear maps and $V_{\mathbb{R}}^*$ is the space of $\mathbb{R}$-linear maps $V \to \mathbb{R}$. There is a canonical isomorphism $\mathrm{H}^0(X, \Omega_X^1) = V_{\mathbb{C}}^*$ defined by evaluating a holomorphic 1-form on X on the tangent space V of X at the origine. There is also a canonical isomorphism $\mathrm{H}^1(X, \mathcal{O}_X) = \overline{V}_{\mathbb{C}}^*$.

By taking $\bigwedge^2$ on both sides, the Hodge decomposition of $\mathrm{H}^2(X, \mathbb{C})$ can also be made explicit. We have $\mathrm{H}^2(X, \mathcal{O}_X) = \bigwedge^2 \overline{V}_{\mathbb{C}}^*$, $\mathrm{H}^1(X, \Omega_X^1) = V_{\mathbb{C}}^* \otimes \overline{V}_{\mathbb{C}}^*$ and $\mathrm{H}^0(X, \Omega_X^2) = \bigwedge^2 V_{\mathbb{C}}^*$. It follows that the map $\mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(X, \mathcal{O}_X)$ is the obvious map $\bigwedge^2 U_{\mathbb{Z}}^* \to \bigwedge^2 V_{\mathbb{C}}^*$. Its kernel is precisely the set of integral 2-forms E on U which satisfy the relation $E(iu_1, iu_2) = E(u_1, u_2)$ (when they are extended to V by $\mathbb{R}$-linearity). □

Let $E : \bigwedge^2 U \to \mathbb{Z}$ be an integral alternating 2-form on U satisfying $E(iu_1, iu_2) = E(u_1, u_2)$ after extension to V by $\mathbb{R}$-linearity. The real 2-form E on V defines a Hermitian form λ on the $\mathbb{C}$-vector space V by

$$\lambda(x, y) = E(ix, y) + iE(x, y)$$

which in turn determines E by the relation $E = \mathrm{Im}(\lambda)$. The Neron-Severi group $\mathrm{NS}(X)$ can be described in yet another way as the group of the Hermitian forms λ on the $\mathbb{C}$-vector space V having an imaginary part which takes integral values on U.

Theorem 1.1.3 (Appell-Humbert)**.** *Isomorphy classes of holomorphic line bundles on $X = V/U$ correspond bijectively to pairs (λ, α), where*

- *$\lambda \in NS(X)$ is an Hermitian form on V such that its imaginary part takes integral values on U*
- *$\alpha : U \to S_1$ is a map from U to the unit circle S_1 satisfying the equation*

$$\alpha(u_1 + u_2) = e^{i\pi \mathrm{Im}(\lambda)(u_1,u_2)}\alpha(u_1)\alpha(u_2).$$

For every (λ, α) as above, the line bundle $L(\lambda, \alpha)$ is given by the Appell-Humbert cocycle

$$e_u(z) = \alpha(u)e^{\pi\lambda(z,u)+\frac{1}{2}\pi\lambda(u,u)}.$$

Let $\mathrm{Pic}(X)$ be the abelian group consisting of the isomorphy classes of line bundles on X and $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the kernel of the Chern class. We have an exact sequence :

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

Let us also write: $\hat{X} = \mathrm{Pic}^0(X)$; it is the group consisting of characters $\alpha : U \to S_1$ from U to the unit circle S_1. Let $V^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. There is a homomorphism $V^*_{\mathbb{R}} \to \hat{X}$ sending $v^* \in V^*_{\mathbb{R}}$ to the line bundle $L(0, \alpha)$, where $\alpha : U \to S_1$ is the character

$$\alpha(u) = \exp(2i\pi\langle u, v^*\rangle).$$

This induces an isomorphism $V^*_{\mathbb{R}}/U^* \to \hat{X}$, where

$$U^* = \{u^* \in V^*_{\mathbb{R}} \text{ such that } \forall u \in U,\ \langle u, u^*\rangle \in \mathbb{Z}\}.$$

Let us denote by the semi-linear dual, consisting of $\mathbb{C}$-semi-linear maps $V \to \mathbb{C}$. We can identify $\overline{V}^*_{\mathbb{C}}$ with the $\mathbb{R}$-dual $V^*_{\mathbb{R}}$ by the $\mathbb{R}$-linear bijection sending a semi-linear f to its imaginary part g (f can be recovered from g: use the formula $f(v) = -g(iv) + ig(v)$). This gives $\hat{X} = \overline{V}^*_{\mathbb{C}}/U^*$ a structure of complex torus; it is called *the dual complex torus of X*. With respect to this complex structure, the universal line bundle over $X \times \hat{X}$ given by Appell-Humbert formula is a holomorphic line bundle.

A Hermitian form on V induces a $\mathbb{C}$-linear map $V \to \overline{V}^*_{\mathbb{C}}$. If moreover its imaginary part takes integral values in U, the linear map $V \to \overline{V}^*_{\mathbb{C}}$ takes U into U^* and therefore induces a homomorphism $\lambda : X \to \hat{X}$ which is symmetric (i.e. such that $\hat{\lambda} = \lambda$ with respect to the obvious identification $X \simeq \hat{\hat{X}}$). In this way, we identify the Neron-Severi group $\mathrm{NS}(X)$ with the group of symmetric homomorphisms from X to $\hat{X}$.

Let (λ, α) be as in the theorem and let $\theta \in \mathrm{H}^0(X, L(\lambda, \alpha))$ be a global section of $L(\lambda, \alpha)$. Pulled back to V, θ becomes a holomorphic function on V which satisfies the equation

$$\theta(z+u) = e_u(z)\theta(z) = \alpha(u)e^{\pi\lambda(z,u)+\frac{1}{2}\pi\lambda(u,u)}\theta(z).$$

Such a function is called a theta-function with respect to the hermitian form λ and the multiplicator α. The Hermitian form λ needs to be positive definite for $L(\lambda, \alpha)$ to have a lot of sections, see [13, §3].

Theorem 1.1.4. *The line bundle $L(\lambda, \alpha)$ is ample if and only if the Hermitian form H is positive definite. In that case,*

$$\dim \mathrm{H}^0(X, L(\lambda, \alpha)) = \sqrt{\det(E)}.$$

Consider the case where H is degenerate. Let W be the kernel of H or of E, i.e.

$$W = \{x \in V | E(x, y) = 0, \forall y \in V\}.$$

Since E is integral on $U \times U$, $W \cap U$ is a lattice of W. In particular, $W/W \cap U$ is compact. For any $x \in X$, $u \in W \cap U$, we have

$$|\theta(x+u)| = |\theta(x)|$$

for all $d \in \mathbb{N}$, $\theta \in \mathrm{H}^0(X, L(\lambda, \alpha)^{\otimes d})$. By the maximum principle, it follows that θ is constant on the cosets of X modulo W and therefore $L(\lambda, \alpha)$ is not ample. Similar argument shows that if H is not positive definite, $L(H, \alpha)$ can not be ample, see [13, p.26].

If the Hermitian form H is positive definite, then the equality

$$\dim \mathrm{H}^0(X, L(\lambda, \alpha)) = \sqrt{\det(E)}$$

holds. In [13, p.27], Mumford shows how to construct a basis, well-defined up to a scalar, of the vector space $\mathrm{H}^0(X, L(\lambda, \alpha))$ after choosing a sublattice $U' \subset U$ of rank n which is Lagrangian with respect to the symplectic form E and such that $U' = U \cap \mathbb{R}U'$. Based on the equality $\dim \mathrm{H}^0(X, L(\lambda, \alpha)^{\otimes d}) = d^n\sqrt{\det(E)}$, one can prove $L(\lambda, \alpha)^{\otimes 3}$ gives rise to a projective embedding of X for any positive definite Hermitian form λ. See Theorem 2.2.3 for a more complete statement. □

Definition 1.1.5. (1) *An abelian variety is a complex torus that can be embedded into a projective space.*
(2) *A polarization of an abelian variety $X = V/U$ is an alternating form $\lambda : \bigwedge^2 U \to \mathbb{Z}$ which is the Chern class of an ample line bundle.*

With a suitable choice of a basis of U, λ can be represented by a matrix

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is a diagonal matrix $D = (d_1, \dots, d_n)$ for some non-negative integers $d_1, \dots, d_n$ such that $d_1|d_2|\dots|d_n$. The form E is non-degenerate if these integers are nonzero. We call $D = (d_1, \dots, d_n)$ the *type of the polarization* λ. A polarization is called *principal* if its type is $(1, \dots, 1)$.

Corollary 1.1.6 (Riemann)**.** *A complex torus $X = V/U$ can be embedded as a closed complex submanifold into a projective space if and only if there exists a positive definite hermitian form λ on V such that the restriction of its imaginary part to U is a (symplectic) 2-form with integral values.*

Let us rewrite Riemann's theorem in term of matrices. We choose a $\mathbb{C}$-basis $e_1, \dots, e_n$ for V and a $\mathbb{Z}$-basis $u_1, \dots, u_{2n}$ of U. Let Π be the $n \times 2n$-matrix $\Pi = (\lambda_{ji})$ with $u_i = \sum_{j=1}^{n} \lambda_{ji} e_j$ for all $i = 1, \dots, 2n$. Π is called *the period matrix.* Since $\lambda_1, \dots, \lambda_{2n}$ form an $\mathbb{R}$-basis of V, the $2n \times 2n$-matrix $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is invertible. The alternating form $E : \bigwedge^2 U \to \mathbb{Z}$ is represented by an alternating matrix, also denoted by E, with respect to the $\mathbb{Z}$-basis $u_1, \dots, u_{2n}$. The form $\lambda : V \times V \to \mathbb{C}$ given by $\lambda(x, y) = E(ix, y) + iE(x, y)$ is hermitian if and only if $\Pi E^{-1}\,{}^t\Pi = 0$. When this condition is satisfied, the Hermitian form λ is positive definite if and only if the symmetric matrix $i\Pi E^{-1}\,{}^t\overline{\Pi}$ is positive definite.

Corollary 1.1.7. *A complex torus $X = V/U$ defined by a period matrix Π is an abelian variety if and only if there is a nondegenerate alternating integral $2n \times 2n$ matrix E such that*

(1) $\Pi E^{-1}\,{}^t\Pi = 0$,
(2) $i\Pi E^{-1}\,{}^t\overline{\Pi} > 0$.

1.2. **Quotients of the Siegel upper half space.** Let X be an abelian variety of dimension n over $\mathbb{C}$ and let E be a polarization of X of type $D = (d_1, \dots, d_n)$. There exists a basis $u_1, \dots, u_n, v_1, \dots, v_n$ of $\mathrm{H}_1(X, \mathbb{Z})$ with respect to which the matrix of E is of the form

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

A datum $(X, E, (u_\bullet, v_\bullet))$ is called a *polarized abelian variety of type D with symplectic basis.* We are going to describe the moduli of polarized abelian varieties of type D with symplectic basis.

The Lie algebra V of X is an n-dimensional $\mathbb{C}$-vector space equipped with a lattice $U = \mathrm{H}_1(X, \mathbb{Z})$. Choose a $\mathbb{C}$-basis $e_1, \dots, e_n$ of V. The vectors $e_1, \dots, e_n, ie_1, \dots, ie_n$ form an $\mathbb{R}$-basis of V. The isomorphism $\Pi_{\mathbb{R}} : U \otimes \mathbb{R} \to V$ is given by an invertible real $2n \times 2n$-matrix

$$\Pi_{\mathbb{R}} = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}.$$

The complex $n \times 2n$-matrix $\Pi = (\Pi_1, \Pi_2)$ is related to $\Pi_{\mathbb{R}}$ by the relations $\Pi_1 = \Pi_{11} + i\Pi_{21}$ and $\Pi_2 = \Pi_{12} + i\Pi_{22}$.

Lemma 1.2.1. *The set of polarized abelian varieties of type D with symplectic basis is canonically in bijection with the set of $GL_{\mathbb{C}}(V)$ orbits of isomorphisms of real vector spaces $\Pi_{\mathbb{R}} : U \otimes \mathbb{R} \to V$ such that for* *all $x, y \in V$, we have $E(\Pi_{\mathbb{R}}^{-1} ix, \Pi_{\mathbb{R}}^{-1} iy) = E(\Pi_{\mathbb{R}}^{-1} x, \Pi_{\mathbb{R}}^{-1} y)$ and that the symmetric form $E(\Pi_{\mathbb{R}}^{-1} ix, \Pi_{\mathbb{R}}^{-1} y)$ is positive definite.*

There are at least two methods to describe this quotient. The first one is more concrete but the second one is more suitable for generalization.

In each $\mathrm{GL}_{\mathbb{C}}(V)$ orbit, there exists a unique $\Pi_{\mathbb{R}}$ such that $\Pi_{\mathbb{R}}^{-1} e_i = \frac{1}{d_i} v_i$ for $i = 1, \ldots, n$. Thus, the matrix $\Pi_{\mathbb{R}}$ has the form

$$\Pi_{\mathbb{R}} = \begin{pmatrix} \Pi_{11} & D \\ \Pi_{21} & 0 \end{pmatrix}$$

and Π has the form $\Pi = (Z, D)$, with

$$Z = \Pi_{11} + i\Pi_{21} \in M_n(\mathbb{C})$$

satisfying ${}^t Z = Z$ and $\mathrm{im}(Z) > 0$.

Proposition 1.2.2. *There is a canonical bijection from the set of polarized abelian varieties of type D with symplectic basis to the Siegel upper half-space*

$$\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}) | \, {}^t Z = Z, \mathrm{im}(Z) > 0\}.$$

On the other hand, an isomorphism $\Pi_{\mathbb{R}} : U \otimes \mathbb{R} \to V$ defines a cocharacter $h : \mathbb{C}^\times \to \mathrm{GL}(U \otimes \mathbb{R})$ by transporting the complex structure of V to $U \otimes \mathbb{R}$. It follows from the relation $E(\Pi_{\mathbb{R}}^{-1} ix, \Pi_{\mathbb{R}}^{-1} iy) = E(\Pi_{\mathbb{R}}^{-1} x, \Pi_{\mathbb{R}}^{-1} y)$ that the restriction of h to the unit circle S_1 defines a homomorphism $h_1 : S_1 \to \mathrm{Sp}_{\mathbb{R}}(U, E)$. Moreover, the $\mathrm{GL}_{\mathbb{C}}(V)$-orbit of $\Pi_{\mathbb{R}} : U \otimes \mathbb{R} \to V$ is well determined by the induced homomorphism $h_1 : S_1 \to \mathrm{Sp}_{\mathbb{R}}(U, E)$.

Proposition 1.2.3. *There is a canonical bijection from the set of polarized abelian varieties of type D with symplectic basis to the set of homomorphisms of real algebraic groups $h_1 : S_1 \to \mathrm{Sp}_{\mathbb{R}}(U, E)$ such that the following conditions are satisfied:*

1. *the complexification $h_{1,\mathbb{C}} : \mathbb{G}_m \to \mathrm{Sp}(U \otimes \mathbb{C})$ gives rise to a decomposition into direct sum of n-dimensional vector subspaces*
$$U \otimes \mathbb{C} = (U \otimes \mathbb{C})_+ \oplus (U \otimes \mathbb{C})_-$$
of weights $+1$ and -1;
2. *the symmetric form $E(h_1(i)x, y)$ is positive definite.*

This set is a homogenous space under the action of $\mathrm{Sp}(U \otimes \mathbb{R})$ acting by inner automorphisms.

Let Sp_D be the $\mathbb{Z}$-algebraic group of automorphisms of the symplectic form E of type D. The discrete group $\mathrm{Sp}_D(\mathbb{Z})$ acts simply transitively on the set of symplectic bases of U.

Proposition 1.2.4. *There is a canonical bijection between the set of isomorphy classes of polarized abelian varieties of type D and the quotient $\mathrm{Sp}_D(\mathbb{Z}) \backslash \mathfrak{H}_n$.*

According to H. Cartan, there is a way to give an analytical structure to this quotient and then to prove that this quotient "is" indeed a quasi-projective normal variety over $\mathbb{C}$ (more precisely, that it can be endowed with a canonical embedding into a complex projective space with an image whose closure is a projective normal variety).

1.3. **Torsion points and level structures.** Let $X = V/U$ be an abelian variety of dimension n. For every integer N, the group of N-torsion points $X[N] = \{x \in X | Nx = 0\}$ can be identified with the finite group $N^{-1}U/U$ that is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{2n}$. Let E be a polarization of X of type $D = (d_1, \ldots, d_n)$ with $(d_n, N) = 1$. The alternating form $E : \bigwedge^2 U \to \mathbb{Z}$ can be extended to a non-degenerate symplectic form on $U \otimes \mathbb{Q}$. The Weil pairing

$$(\alpha, \beta) \mapsto \exp(2i\pi N E(\alpha, \beta))$$

defines a symplectic non-degenerate form

$$e_N : X[N] \times X[N] \to \mu_N ,$$

where μ_N is the group of N-th roots of unity, provided that N is relatively prime to d_n. Let us *choose a primitive N-th root of unity*, so that the Weil pairing takes values in $\mathbb{Z}/N\mathbb{Z}$.

Definition 1.3.1. *Let N be an integer relatively prime to d_n. A principal N-level structure of an abelian variety X with a polarization E is an isomorphism from the symplectic module $X[N]$ to the standard symplectic module $(\mathbb{Z}/N\mathbb{Z})^{2n}$ given by the matrix*

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} ,$$

where I_n is the identity $n \times n$-matrix.

Let $\Gamma(N)$ be the subgroup of $\mathrm{Sp}_D(\mathbb{Z})$ consisting of the automorphisms of (U, E) which induce the trivial action on U/NU.

Proposition 1.3.2. *There is a natural bijection between the set of isomorphy classes of polarized abelian varieties of type D equipped with principal N-level structures and the quotient $\mathcal{A}^0_{n,N} = \Gamma_A(N)\backslash\mathfrak{H}_n$.*

For $N \geq 3$, the group $\Gamma(N)$ does not contain torsion elements and acts freely on the Siegel half-space $\mathfrak{H}_n$. The quotient $\mathcal{A}^0_{n,N}$ is therefore a smooth complex analytic space.

2. The moduli space of polarized abelian schemes

2.1. Polarizations of abelian schemes.

Definition 2.1.1. *An abelian scheme over a scheme S is a smooth proper group scheme with connected geometric fibers. Being a group scheme, X is equipped with the following structures:*

(1) *a unit section $e_X : S \to X$;*

(2) *a multiplication morphism* $X \times_S X \to X$*;*
(3) *an inverse morphism* $X \to X$*,*

such that the usual axioms for abstract groups hold.

Recall the following classical rigidity lemma.

Lemma 2.1.2. *Let* X *and* X' *be two abelian schemes over* S *and* $\alpha : X \to X'$ *be a morphism that sends the unit section of* X *to the unit section of* X'*. Then* α *is a homomorphism.*

Proof. We summarize the proof when S is a point. Consider the map $\beta : X \times X \to X'$ given by

$$\beta(x_1, x_2) = \alpha(x_1 x_2)\alpha(x_1)^{-1}\alpha(x_2)^{-1}.$$

We have $\beta(e_X, x) = e_{X'}$ for all $x \in X$. For any affine neighborhood U' of $e_{X'}$ in X', there exists an affine neighborhood U of e_X such that $\beta(U \times X) \subset U'$. For every $u \in U$, β maps the proper scheme $u \times X$ into the affine U'. It follows that the the restriction of β to $u \times X$ is constant. Since $\beta(u, e_X) = e_{X'}$, $\beta(u, x) = e_{X'}$ for any $x \in X$. It follows that $\beta(u, x) = e_{X'}$ for any $u, x \in X$ since X is irreducible. □

Let us mention two useful consequences of the rigidity lemma. Firstly, the abelian scheme is necessarily commutative since the inverse morphism $X \to X$ is a homomorphism. Secondly, given the unit section, a smooth proper scheme can have *at most one* abelian scheme structure. It suffices to apply the rigidity lemma for the identity of X.

An *isogeny* $\alpha : X \to X'$ is a surjective homomorphism whose kernel $\ker(\alpha)$ is a finite group scheme over S. Let d be a positive integer. Let S be a scheme such that all its residue characteristics are relatively prime to d. Let $\alpha : X \to X'$ be an isogeny of degree d and let $K(\alpha)$ be the kernel of α. For every geometric point $\overline{s} \in S$, $K(\alpha)_{\overline{s}}$ is a discrete group isomorphic to $\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$ with $d_1 | \cdots | d_n$ and $d_1 \ldots d_n = d$. The function defined on the underlying topological space $|S|$ of S which maps a point $s \in |S|$ to the type of $K(\alpha)_{\overline{s}}$ for any geometric point $\overline{s}$ over s is a locally constant function. So it makes sense to talk about the type of an isogeny of degree prime to all residue characteristics.

Let X be an abelian scheme over S. Consider the functor $\mathrm{Pic}_{X/S}$ from the category of S-schemes to the category of abelian groups which assigns to every S-scheme T the group of isomorphy classes of (L, ι), where L is an invertible sheaf on $X \times_S T$ and ι is a trivialization $e_X^* L \simeq \mathcal{O}_T$ along the unit section. See [2, p.234] for the following theorem.

Theorem 2.1.3. *Let* X *be a projective abelian scheme over* S*. Then the functor* $\mathrm{Pic}_{X/S}$ *is representable by a smooth separated* S*-scheme which is locally of finite presentation over* S*.*

The smooth scheme $\mathrm{Pic}_{X/S}$ equipped with the unit section corresponding to the trivial line bundle $\mathcal{O}_X$ admits a neutral component $\mathrm{Pic}^0_{X/S}$ which is an abelian scheme over S.

Definition 2.1.4. *Let X/S be a projective abelian scheme. The* dual abelian scheme *$\hat{X}/S$ is the neutral component* $\mathrm{Pic}^0(X/S)$ *of the Picard functor* $\mathrm{Pic}_{X/S}$. *The* Poincaré sheaf *P is the restriction of the universal invertible sheaf on $X \times_S \mathrm{Pic}_{X/S}$ to $X \times_S \hat{X}$.*

For every abelian scheme X/S, its bidual abelian scheme (i.e. the dual abelian scheme of $\hat{X}/S$) is canonically identified to X/S. For every homomorphism $\alpha : X \to X'$, we have a homomorphism $\hat{\alpha} : \hat{X}' \to \hat{X}$. If α is an isogeny, the same is true for $\hat{\alpha}$. A homomorphism $\alpha : X \to \hat{X}$ is called *symmetric* if the equality $\alpha = \hat{\alpha}$ holds.

Lemma 2.1.5. *Let $\alpha : X \to Y$ be an isogeny and let $\hat{\alpha} : \hat{Y} \to \hat{X}$ be the dual isogeny. There is a canonical perfect pairing*

$$\ker(\alpha) \times \ker(\hat{\alpha}) \to \mathbb{G}_m.$$

Proof. Let $\hat{y} \in \ker(\hat{\alpha})$ and let $L_{\hat{y}}$ be the corresponding line bundle on Y with a trivialization along the unit section. Pulling it back to X, we get the trivial line bundle equipped with yet another trivialization on $\ker(\alpha)$. The difference between the two trivializations gives rise to a homomorphism $\ker(\alpha) \to \mathbb{G}_m$ which defines the desired pairing. It is not difficult to check that this pairing is perfect, see [13, p.143]. □

Let $L \in \mathrm{Pic}_{X/S}$ be an invertible sheaf over X with trivialized neutral fibre $L_e = 1$. For any point $x \in X$ over $s \in S$, let $T_x : X_s \to X_s$ be the translation by x. The invertible sheaf $T_x^* L \otimes L^{-1} \otimes L_x^{-1}$ has trivialized neutral fibre

$$(T_x^* L \otimes L^{-1} \otimes L_x^{-1})_e = L_x \otimes L_e^{-1} \otimes L_x^{-1} = 1\,,$$

so, L defines a morphism $\lambda_L : X \to \mathrm{Pic}_{X/S}$. Since the fibres of X are connected, λ_L factors through the dual abelian scheme $\hat{X}$ and gives rise to a morphism

$$\lambda_L : X \to \hat{X}.$$

Since λ_L sends the unit section of X on the unit section of $\hat{X}$, the morphism of schemes λ_L is necessarily a homomorphism of abelian schemes. Let us denote by $K(L)$ the kernel of λ_L.

Lemma 2.1.6. *For every line bundle L on X with a trivialization along the unit section, the homomorphism $\lambda_L : X \to \hat{X}$ is symmetric. If moreover, $L = \hat{x}^* P$ for some section $\hat{x} : S \to \hat{X}$, then $\lambda_L = 0$.*

Proof. By construction, the homomorphism $\lambda_L : X \to \hat{X}$ represents the line bundle $m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ on $X \times X$, where m is the multiplication and p_1, p_2 are projections, equipped with the obvious trivialization along the unit section. As this line bundle is symmetric, the homomorphism λ_L is symmetric.

If $L = \mathcal{O}_X$ with the obvious trivialization along the unit section, it is immediate that $\lambda_L = 0$. Now for any $L = \hat{x}^*P$, L can be deformed continuously to the trivial line bundle and it follows that $\lambda_L = 0$. In order to make the argument rigorous, one can consider the universal family over $\hat{X}$ and apply the rigidity lemma. □

Definition 2.1.7. *A line bundle L over an abelian scheme X equipped with a trivialization along the unit section is called non-degenerate if $\lambda_L : X \to \hat{X}$ is an isogeny.*

In the case where the base S is $\mathrm{Spec}(\mathbb{C})$ and $X = V/U$, L is non-degenerate if and only if the associated Hermitian form on V is non-degenerate.

Let L be a non-degenerate line bundle on X with a trivialization along the unit section. The canonical pairing $K(L) \times K(L) \to \mathbb{G}_{m,S}$ is then symplectic. Assume S is connected with residue characteristics prime to the degree of λ_L. So, there exists $d_1 | \dots | d_s$ such that for every geometric point $\overline{s} \in S$ the abelian group $K(L)_{\overline{s}}$ is isomorphic to $(\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z})^2$. We call $D = (d_1, \dots, d_n)$ the type of the polarization λ

Definition 2.1.8. *Let X/S be an abelian scheme. A* polarization *of X/S is a symmetric isogeny $\lambda : X \to \hat{X}$ which locally for the étale topology of S, is of the form λ_L for some ample line bundle L of X/S.*

In order to make this definition workable, we will need to recall basic facts about cohomology of line bundles on abelian varieties. See corollary 2.2.4 in the next paragraph.

2.2. **Cohomology of line bundles on abelian varieties.** We are going to recollect some known facts about the cohomology of line bundles on abelian varieties. For the proofs, see [13, p.150]. Let X be an abelian variety over a field k. Let

$$\chi(L) = \sum_{i \in \mathbb{Z}} \dim_k \mathrm{H}^i(X, L)$$

be the Euler characteristic of L.

Theorem 2.2.1 (Riemann-Roch theorem)**.** *For any line bundle L on X, if $L = \mathcal{O}_X(D)$ for a divisor D, we have*

$$\chi(L) = \frac{(D^g)}{g!},$$

where (D^g) is the g-fold self-intersection number of D.

Theorem 2.2.2 (Mumford's vanishing theorem). *Let L be a line bundle on X such that $K(L)$ is finite. There exists a unique integer $i = i(L)$ with $0 \leq i \leq n = \dim(X)$ such that $\mathrm{H}^j(X, L) = 0$ for $j \neq i$ and $\mathrm{H}^i(X, L) \neq 0$. Moreover, L is ample if and only if $i(L) = 0$. For every $m \geq 1$, $i(L^{\otimes m}) = i(L)$.*

Assume $S = \mathrm{Spec}(\mathbb{C})$, $X = V/U$ with $V = \mathrm{Lie}(X)$ and U a lattice in V. Then the Chern class of L corresponds to a Hermitian form H and the integer $i(L)$ is the number of negative eigenvalues of H.

Theorem 2.2.3. *For any ample line bundle L on an abelian variety X, the line bundle $L^{\otimes m}$ is base-point free if $m \geq 2$ and it is very ample if $m \geq 3$.*

Since L is ample, $i(L) = 0$ and consequently $\dim_k \mathrm{H}^0(X, L) = \chi(L) > 0$. There exists an effective divisor D such that $L \simeq \mathcal{O}_X(D)$. Since $\lambda_L : X \to \hat{X}$ is a homomorphism, the divisor $T_x^*(D) + T_{-x}^*(D)$ is linearly equivalent to $2D$ and $T_x^*(D) + T_y^*(D) + T_{-x-y}^*(D)$ is linearly equivalent to $3D$. By moving $x, y \in X$ we get a lot of divisors linearly equivalent to $2D$ and to $3D$. The proof is based on this fact and on the formula for the dimension of $\mathrm{H}^0(X, L^{\otimes m})$. For a detailed proof, see [13, p.163]. □

Corollary 2.2.4. *Let $X \to S$ be an abelian scheme over a connected base and let L be an invertible sheaf on X such that $K(L)$ is a finite group scheme over S. If there exists a point $s \in S$ such that L_s is ample on X_s, then L is relatively ample for X/S.*

Proof. For t varying in S, the function $t \mapsto \dim \mathrm{H}^i(X_t, L_t)$ is upper semi-continuous. Hence $U_i := \{t \in S \mid \mathrm{H}^j(X, L) = 0 \text{for all } j \neq i\}$ is open. By Mumford's vanishing theorem, the collection of $U_i's$ is a disjoint open partition of S. Since L_s is ample, $\mathrm{H}^0(X_s, L_s) \neq 0$ thus $s \in U_0$. As $U_0 \neq \emptyset$ and S is connected, we have $U_0 = S$. If L_t is ample (which is the case for any $t \in S$ as we have just seen), L is relatively ample on X over a neighborhood of t in S. □

2.3. **An application of G.I.T.** Let us fix two positive integers $n \geq 1$, N and a type $D = (d_1, \ldots, d_n)$ with $d_1 | \ldots | d_n$, where d_n is prime to N. Let $\mathcal{A}$ be the functor which assigns to a $\mathbb{Z}[(Nd_n)^{-1}]$-scheme S the set of isomorphy classes of polarized S-abelian schemes of type D: for any such S, $\mathcal{A}(S)$ is the set of isomorphy classes of triples (X, λ, η), where

(1) X is an abelian scheme over S ;
(2) $\lambda : X \to \hat{X}$ is a polarization of type D ;
(3) η is a symplectic similitude $(\mathbb{Z}/N\mathbb{Z})^{2n} \simeq X[N]$.

In the third condition, $(\mathbb{Z}/N\mathbb{Z})^{2n}$ and $X[N]$ are respectively endowed with the symplectic pairing (1.3.1) and with the *Weil pairing*, which is the symplectic pairing $X[N] \times_S X[N] \to \mu_{N,S}$ obtained from the pairing $X[N] \times_S \hat{X}[N] \to \mu_{N,S}$ of lemma 2.1.5 (applied to the special case where α is the multiplication by N) by composing it with the morphism $X[N] \to \hat{X}[N]$ induced by λ.

Theorem 2.3.1. *If N is large enough (with respect to D; in the special case of principal polarizations, where $D = (1, \dots 1)$, any $N \geq 3$ is large enough) the functor $\mathcal{A}$ defined above is representable by a smooth quasi-projective $\mathbb{Z}[(Nd_n)^{-1}]$-scheme.*

Proof. Let X be an abelian scheme over S and $\hat{X}$ its dual abelian scheme. Let P be the Poincaré line bundle over $X \times_S \hat{X}$ equipped with a trivialization over the neutral section $e_X \times_S \mathrm{id}_{\hat{X}} : \hat{X} \to X \times_S \hat{X}$ of X. Let $L^{\Delta}(\lambda)$ be the line bundle over X obtained by pulling back the Poincaré line bundle P

$$L^{\Delta}(\lambda) = (\mathrm{id}_X, \lambda)^* P$$

by the composite homomorphism

$$(\mathrm{id}_X, \lambda) = (\mathrm{id}_X \times \lambda) \circ \Delta : X \to X \times_S X \to X \times_S \hat{X}\,,$$

where $\Delta : X \to X \times_S X$ is the diagonal. The line bundle $L^{\Delta}(\lambda)$ gives rise to a symmetric homomorphism $\lambda_{L^{\Delta}(\lambda)} : X \to \hat{X}$.

Lemma 2.3.2. *The equality $\lambda_{L^{\Delta}(\lambda)} = 2\lambda$ holds.*

Proof. Locally for étale topology, we can assume $\lambda = \lambda_L$ for some line bundle over X which is relatively ample. Then

$$L^{\Delta}(\lambda) = \Delta^*(\mathrm{id}_X \times \lambda)^* P = \Delta^*(\mu^* L \otimes \mathrm{pr}_1 L^{-1} \otimes \mathrm{pr}_2 L^{-1}).$$

It follows that

$$L^{\Delta}(\lambda) = (2)^* L \otimes L^{-2}$$

where $(2) : X \to X$ is the multiplication by 2. As for every $N \in \mathbb{N}$, $\lambda_{(N)^* L} = N^2 \lambda_L$, and in particular $\lambda_{(2)^* L} = 4\lambda_L$, we obtain the desired equality $\lambda_{L^{\Delta}(\lambda)} = 2\lambda$. $\square$

Since locally over S, $\lambda = \lambda_L$ for a relatively ample line bundle L, the line bundle $L^{\Delta}(\lambda)$ is a relatively ample line bundle, and $L^{\Delta}(\lambda)^{\otimes 3}$ is very ample. It follows that its higher direct images by $\pi : X \to S$ vanish

$$\mathrm{R}^i \pi_* L^{\Delta}(\lambda)^{\otimes 3} = 0 \text{ for all } i \geq 1$$

and that $M = \pi_* L^{\Delta}(\lambda)$ is a vector bundle of rank

$$m + 1 := 6^n d$$

over S, where $d = d_1 \cdots d_n$.

Definition 2.3.3. *A linear rigidification of a polarized abelian scheme (X, λ) is an isomorphism*

$$\alpha : \mathbb{P}^m_S \to \mathbb{P}_S(M)\,,$$

where $M = \pi_* L(\lambda)$. *In other words, a linear rigidification of a polarized abelian scheme* (X, λ) *is a trivialization of the* $\mathrm{PGL}(m+1)$*-torsor associated to the vector bundle* M *of rank* $m+1$.

Let $\mathcal{H}$ be the functor that assigns to every scheme S the set of isomorphy classes of quadruples $(X, \lambda, \eta, \alpha)$, where (X, λ, η) is a polarized abelian scheme over S of type D with level structure η and α is a linear rigidification. Forgetting α, we get a functorial morphism

$$\mathcal{H} \to \mathcal{A}$$

which is a $\mathrm{PGL}(m+1)$-torsor.

The line bundle $L^{\Delta}(\lambda)^{\otimes 3}$ provides a projective embedding

$$X \hookrightarrow \mathbb{P}_S(M).$$

Using the linear rigidification α, we can embed X into the standard projective space

$$X \hookrightarrow \mathbb{P}_S^m.$$

For every $r \in \mathbb{N}$, the higher direct images vanish

$$\mathrm{R}^i \pi_* L^{\Delta}(\lambda)^{\otimes 3r} = 0 \text{ for all } i > 0$$

and $\pi_* L^{\Delta}(\lambda)^{\otimes 3r}$ is a vector bundle of rank $6^n d r^n$ so that we have a functorial morphism

$$f : \mathcal{H}_n \to \mathrm{Hilb}^{Q(t),1}(\mathbb{P}_m)$$

(where $\mathrm{Hilb}^{Q(t),1}(\mathbb{P}_m)$ is the Hilbert scheme of 1-pointed subschemes of $\mathbb{P}^m$ with Hilbert polynomial $Q(t) = 6^n d t^n$) sending (X, λ, α) to the image of X in $\mathbb{P}^m$ pointed by the unit of X.

Proposition 2.3.4. *The morphism* f *identifies* $\mathcal{H}$ *with an open subfunctor of* $\mathrm{Hilb}^{Q(t),1}(\mathbb{P}^m)$ *which consist of pointed smooth subschemes of* $\mathbb{P}^m$.

Proof. Since a smooth projective pointed variety X has at most one abelian variety structure, the morphism f is injective. Following theorem 2.4.1 of the next paragraph, any smooth projective morphisme $f : X \to S$ over a geometrically connected base S with a section $e : S \to X$ has an abelian scheme structure if and only if one geometric fiber X_s does. □

Since a polarized abelian varieties with principal N-level structure has no trivial automorphisms (see [15]), $\mathrm{PGL}(m+1)$ acts freely on $\mathcal{H}$. We take $\mathcal{A}$ as the quotient of $\mathcal{H}$ by the free action of $\mathrm{PGL}(m+1)$. The construction of this quotient as a scheme requires nevertheless a quite technical analysis of stability. If N is large enough, then $X[N] \subset X \subset \mathbb{P}^m$ is not contained in any hyperplane; furthermore, no more than $N^{2n}/m+1$ points from these N-torsion points can lie in the same hyperplane of $\mathbb{P}^m$. In that case, $(A, \lambda, \eta, \alpha)$ is a stable point. In the general case, we can increase the level structures and then perform a quotient by a finite group. See [14, p.138] for a complete discussion. □

2.4. **Spreading the abelian scheme structure.** Let us now quote a theorem of Grothendieck [14, theorem 6.14].

Theorem 2.4.1. *Let S be a connected noetherian scheme. Let $X \to S$ be a smooth projective morphism equipped with a section $e : S \to X$. Assume for one geometric point $s = \mathrm{Spec}(\kappa(s))$, X_s is an abelian variety over $\kappa(s)$ with neutral point $\epsilon(s)$. Then X is an abelian scheme over S with neutral section ϵ.*

Let us consider first the infinitesimal version of this assertion.

Proposition 2.4.2. *Let $S = \mathrm{Spec}(A)$, where A is an Artin local ring. Let $\mathfrak{m}$ be the maximal ideal of A and let I be an ideal of A such that $\mathfrak{m}I = 0$. Let $S_0 = Spec(A/I)$. Let $f : X \to S$ be a proper smooth scheme with a section $e : S \to X$. Assume that $X_0 = X \times_S S_0$ is an abelian scheme with neutral section $e_0 = e|_{S_0}$. Then X is an abelian scheme with neutral section e.*

Proof. Let $k = A/\mathfrak{m}$ and $\overline{X} = X \otimes_A k$. Let $\mu_0 : X_0 \times_{S_0} X_0 \to X_0$ be the morphism $\mu_0(x, y) = x - y$ and let $\overline{\mu} : \overline{X} \times_k \overline{X} \to \overline{X}$ be the restriction of μ_0. The obstruction to extending μ_0 to a morphism $X \times_S X \to X$ is an element

$$\beta \in \mathrm{H}^1(\overline{X} \times \overline{X}, \overline{\mu}^* T_{\overline{X}} \otimes_k I)$$

where $T_{\overline{X}}$ is the tangent bundle of $\overline{X}$ which is a trivial vector bundle of fibre $\mathrm{Lie}(\overline{X})$. Thus, by Kunneth formula

$$\mathrm{H}^1(\overline{X} \times \overline{X}, \overline{\mu}^* T_{\overline{X}} \otimes_k I) = (\mathrm{Lie}(\overline{X}) \otimes_k \mathrm{H}^1(\overline{X})) \oplus (\mathrm{H}^1(\overline{X}) \otimes_k \mathrm{Lie}(\overline{X})) \otimes_k I.$$

Consider $g_1, g_2 : X_0 \to X_0 \times_{S_0} X_0$ with $g_1(x) = (x, e)$ and $g_2(x) = (x, x)$. The endomorphisms of X_0, $\mu_0 \circ g_1 = \mathrm{id}_{X_0}$ and $\mu_0 \circ g_2 = (e \circ f)$ extend in an obvious way to X so that the obstruction classes $\beta_1 = g_1^* \beta$ and $\beta_2 = g_2^* \beta$ must vanish. Since one can express β in terms of β_1 and β_2 by the Kunneth formula, β vanishes too.

The set of all extensions μ of μ_0 is a principal homogenous space under

$$\mathrm{H}^0(\overline{X} \times_k \overline{X}, \overline{\mu}^* T_{\overline{X}} \otimes_k I).$$

Among these extensions, there exists a unique μ such that $\mu(e, e) = e$ which provides a group scheme structure on X/S. □

We can extend the abelian scheme structure to an infinitesimal neighborhood of s. This structure can be algebrized and then descend to a Zariski neightborhood since the abelian scheme structure is unique if it exists. It remains to prove the following lemma due to Koizumi.

Lemma 2.4.3. *Let $S = \mathrm{Spec}(R)$, where R is a discrete valuation ring with generic point η. Let $f : X \to S$ be a proper and smooth morphism with a section $e : S \to X$. Assume that X_η is an abelian variety with neutral point $e(\eta)$. Then X is an abelian scheme with neutral section e.*

Proof. Suppose R is henselian. Since $X \to S$ is proper and smooth, the inertia group I acts trivially on $\mathrm{H}^i(X_{\overline{\eta}}, \mathbb{Q}_\ell)$. By Grothendieck-Ogg-Shafarevich's criterion, there exists an abelian scheme A over S with $A_\eta = X_\eta$ and A is the Néron model of A_η. By the universal property of Néron's model there exists a morphism $\pi : X \to A$ extending the isomorphism $X_\eta \simeq A_\eta$. Let $\mathcal{L}$ be a relatively ample invertible sheaf on X/S. Choose a trivialization on the unit point of $X_\eta = A_\eta$. Then $\mathcal{L}_\eta$ with the trivialization on the unit section extends uniquely on A to a line bundle $\mathcal{L}'$ since $\mathrm{Pic}(A/S)$ satisfies the valuative criterion for properness. It follows that, over the closed point s of S, $\pi^*\mathcal{L}'_s$ and $\mathcal{L}_s$ have the same Chern class. If π has a fiber of positive dimension then the restriction to that fiber of $\pi^*\mathcal{L}'_s$ is trivial. On the contrary, the restriction of $\mathcal{L}_s$ to that fiber is still ample. This contradiction implies that all fibers of π have dimension zero. The finite birational morphism $\pi : X \to A$ is necessarily an isomorphism according to Zariski's main theorem. □

2.5. **Smoothness.** In order to prove that $\mathcal{A}$ is smooth, we will need to review Grothendieck-Messing's theory of deformations of abelian schemes.

Let $S = \mathrm{Spec}(R)$ be a thickening of $\overline{S} = \mathrm{Spec}(R/I)$ with $I^2 = 0$, or more generally, locally nilpotent and equipped with a structure of divided power. According to Grothendieck and Messing, we can attach to an abelian scheme $\overline{A}$ of dimension n over $\overline{S}$ a locally free $\mathcal{O}_S$-module of rank $2n$

$$\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$$

such that

$$\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}} = \mathrm{H}^1_{\mathrm{dR}}(\overline{A}/\overline{S}).$$

We can associate with every abelian scheme A/S such that $A \times_S \overline{S} = \overline{A}$ a sub-$\mathcal{O}_S$-module

$$\omega_{A/S} \subset \mathrm{H}^1_{\mathrm{dR}}(A/S) = \mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$$

which is locally a direct factor of rank n and which satisfies

$$\omega_{A/S} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}} = \omega_{\overline{A}/\overline{S}}.$$

Theorem 2.5.1 (Grothendieck-Messing)**.** *The functor defined as above, from the category of abelian schemes A/S with $A \times_S \overline{S} = \overline{A}$ to the category sub-$\mathcal{O}_S$-modules $\omega \subset \mathrm{H}^1(\overline{A}/\overline{S})_S$ which are locally a direct factor such that*

$$\omega \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}} = \omega_{\overline{A}/\overline{S}}$$

is an equivalence of categories.

See [10, p.151] for the proof of this theorem.

Let $S = \mathrm{Spec}(R)$ be a thickening of $\overline{S} = \mathrm{Spec}(R/I)$ with $I^2 = 0$. Let $\overline{A}$ be an abelian scheme over S and $\overline{\lambda}$ be a polarization of $\overline{A}$ of type $(d_1, \ldots, d_s)$ with integers d_i relatively prime to the residual characteristics of $\overline{S}$. The polarization $\overline{\lambda}$ induces an isogeny

$$\psi_{\overline{\lambda}} : \overline{A} \to \overline{A}^{\vee},$$

where $\overline{A}^{\vee}$ is the dual abelian scheme of $\overline{A}/\overline{S}$. Since the degree of the isogeny is relatively prime to the residual characteristics, it induces an isomorphism

$$\mathrm{H}^1_{\mathrm{cris}}(\overline{A}^{\vee}/\overline{S})_S \to \mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$$

or a bilinear form $\psi_{\overline{\lambda}}$ on $\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$ which is a symplectic form. The module of relative differentials $\omega_{\overline{A}/\overline{S}}$ is locally a direct factor of $\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_{\overline{S}}$ which is isotropic with respect to the symplectic form $\psi_{\overline{\lambda}}$. It is known that the Lagrangian grassmannian is smooth so that one can lift $\omega_{\overline{A}/\overline{S}}$ to a locally direct factor of $\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$ which is isotropic. According to Grothendieck-Messing's theorem, we get a lifting of $\overline{A}$ to an abelian scheme A/S with a polarization λ that lifts $\overline{\lambda}$. □

2.6. **Adelic description and Hecke correspondences.** Let X and X' be abelian varieties over a base S. A homomorphism $\alpha : X \to X'$ is an isogeny if one of the following conditions is satisfied

- α is surjective and $\ker(\alpha)$ is a finite group scheme over S ;
- there exists $\alpha' : X' \to X$ such that $\alpha' \circ \alpha$ is the multiplication by N in X and $\alpha \circ \alpha'$ is the multiplication by N in X' for some positive integer N.

A *quasi-isogeny* is an equivalence class of pairs (α, N) formed by an isogeny $\alpha : X \to X'$ and a positive integer N, where $(\alpha, N) \sim (\alpha', N')$ if and only if $N'\alpha = N\alpha'$. Obviously, we think of the equivalence class (α, N) as $N^{-1}\alpha$.

Fix n, N, D as in 2.3. There is another description of the category $\mathcal{A}$ which is less intuitive but more convenient when we have to deal with level structures.

Let U be a free $\mathbb{Z}$-module of rank $2n$ and let E be an alternating form $U \times U \to M_U$ with value in some rank one free $\mathbb{Z}$-module M_U. Assume that the type of E is D. Let G be the group of symplectic similitudes of (U, M_U) which associates to any ring R the group $G(R)$ of pairs $(g, c) \in \mathrm{GL}(U \otimes R) \times R^{\times}$ such that

$$E(gx, gy) = cE(x, y)$$

for every $x, y \in U \otimes R$. Thus G is a group scheme defined over $\mathbb{Z}$ which is reductive over $\mathbb{Z}[1/d]$. For every prime $\ell \neq p$, let $K_{N,\ell}$ be the compact open subgroup of $G(\mathbb{Q}_\ell)$ defined as follows:

- if $\ell \not| N$, then $K_\ell = G(\mathbb{Z}_\ell)$;
- if $\ell | N$, then K_ℓ is the kernel of the homomorphism $G(\mathbb{Z}_\ell) \to G(\mathbb{Z}_\ell / N\mathbb{Z}_\ell)$.

Fix a prime p not dividing N nor D. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ obtained by inverting all the primes ℓ different from p.

For every scheme S whose residual characteristics are 0 or p, we consider the groupoid $\mathcal{A}'(S)$ defined as follows:

(1) objets of $\mathcal{A}'$ are triples $(X, \lambda, \tilde{\eta})$, where
  - X is an abelian scheme over S;
  - $\lambda : X \to \hat{X}$ is a $\mathbb{Z}_{(p)}$ multiple of a polarization of degree prime to p, such that for every prime ℓ and for every $s \in S$, the symplectic form induced by λ on $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ is similar to $U \otimes \mathbb{Q}_\ell$;
  - for every prime $\ell \neq p$, $\tilde{\eta}_\ell$ is a $K_{N,\ell}$-orbit of symplectic similitudes from $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ to $U \otimes \mathbb{Q}_\ell$ which is invariant under $\pi_1(S, s)$. We assume that for almost all prime ℓ, this $K_{N,\ell}$-orbit corresponds to the auto-dual lattice $\mathrm{H}_1(X_s, \mathbb{Z}_\ell)$.

(2) a homomorphism $\alpha \in \mathrm{Hom}_{\mathcal{A}'}((X, \lambda, \eta), (X', \lambda', \eta'))$ is a quasi-isogeny $\alpha : X \to X'$ of degree prime to p such that $\alpha^*(\lambda')$ and λ differ by a scalar in $\mathbb{Z}_{(p)}^\times$ and $\alpha^*(\eta') = \eta$.

Consider the functor $\mathcal{A} \to \mathcal{A}'$ which assigns to $(X, \lambda, \eta) \in \mathcal{A}(S)$ the triple $(X, \lambda, \tilde{\eta}) \in \mathcal{A}'(S)$, where the $\tilde{\eta}_\ell$ are defined as follows. Let s be a geometric point of S. Let ℓ be a prime not dividing N and D. Giving a symplectic similitude from $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ to $U \otimes \mathbb{Q}_\ell$ up to the action of $K_{N,\ell}$ is equivalent to giving an auto-dual lattice of $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$. The $K_{N,\ell}$-orbit is stable under $\pi_1(S, s)$ if and only the auto-dual lattice is invariant under $\pi_1(S, s)$. We pick the obvious choice $\mathrm{H}_1(X_s, \mathbb{Z}_\ell)$ as the auto-dual lattice of $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ which is invariant under $\pi_1(S, s)$. If ℓ divides D, we want a $\pi_1(S, s)$-invariant lattice such that the restriction of the Weil symplectic pairing is of type D. Again, $\mathrm{H}_1(X_s, \mathbb{Z}_\ell)$ fulfills this property. If ℓ divides N, giving a symplectic similitude from $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ to $U \otimes \mathbb{Q}_\ell$ up to the action of $K_{N,\ell}$ is equivalent to giving an auto-dual lattice of $\mathrm{H}_1(X_s, \mathbb{Q}_\ell)$ and a rigidification of the pro-ℓ-part of N torsion points of X_s. But this is provided by the level structure η_ℓ in the moduli problem $\mathcal{A}$.

Proposition 2.6.1. *The above functor is an equivalence of categories.*

Proof. As defined, it is obviously faithful. It is full because a quasi-isogeny $\alpha : X \to X'$ which induces an isomorphism $\alpha^* : \mathrm{H}_1(X', \mathbb{Z}_\ell) \to \mathrm{H}_1(X, \mathbb{Z}_\ell)$ is necessarily an isomorphism of abelian schemes. By assumption α carries λ to a rational multiple of λ'. But both λ and λ' are polarizations of the same type, so α must carry λ to λ'. This proves that the functor is fully faithful.

The essential surjectivity is derived from the fact that we can modify an abelian scheme X equipped with a level structure $\tilde{\eta}$ by a quasi-isogeny $\alpha : X \to X'$ so that the composed isomorphism

$$U \otimes \mathbb{Q}_p \simeq \mathrm{H}_1(X, \mathbb{Q}_p) \simeq \mathrm{H}_1(X', \mathbb{Q}_p)$$

identifies $U \otimes \mathbb{Z}_p$ with $\mathrm{H}_1(X', \mathbb{Z}_p)$. There is a unique way to choose a rigidification η of $X'[N]$ compatible with $\tilde{\eta}_p$ for $p|N$. Since the symplectic form E on U is of type (D, D), the polarization λ on X' is also of this type. □

Let us now describe the points of $\mathcal{A}'$ with values in $\mathbb{C}$. Consider an objet of $(X, \lambda, \tilde{\eta}) \in \mathcal{A}'(\mathbb{C})$ equipped with a symplectic basis of $\mathrm{H}_1(X, \mathbb{Z})$. In this case, since λ is a $\mathbb{Z}_{(p)}$-multiple of a polarization of X, it is given by an element of

$$\mathfrak{h}_n^{\pm} = \{Z \in M_n(\mathbb{C}) |\, {}^tZ = Z, \pm\mathrm{im}(Z) > 0\}\,.$$

For all $\ell \neq p$, $\tilde{\eta}_\ell$ defines an element of $G(\mathbb{Q}_p)/K_p$. At p, the integral Tate module $\mathrm{H}_1(X, \mathbb{Z}_p)$ defines an element of $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. It follows that

$$\mathcal{A}'(\mathbb{C}) = G(\mathbb{Q})\backslash[\mathfrak{h}_n^{\pm} \times G(\mathbb{A}_f)/K_N]\,,$$

where $K_N = \prod_{\ell \neq p} K_{N,\ell} \times G(\mathbb{Z}_p)$.

One of the advantages of the prime description of the moduli problem is that we can replace the principal compact open subgroups K_N by any compact open subgroup $K = \prod K_\ell \subset G(\mathbb{A}_f)$ such that $K_p = G(\mathbb{Z}_p)$, thus obtaining a $\mathbb{Z}_{(p)}$-scheme $\mathcal{A}_K$. In the general case, the proof of the representability is reduced to the principal case.

Using this description, it is also easy to define the Hecke operators, as follows. Let $K = K^{(p)} \times G(\mathbb{Z}_p)$, where $K^{(p)}$ is a compact open subgroup of $G(\mathbb{A}_f^{(p)})$, and let $g \in G(\mathbb{A}_f^{(p)})$. We have a morphism

$$\begin{array}{ccc} \mathcal{A}_K & \longrightarrow & \mathcal{A}_{g^{-1}Kg} \\ (X, \lambda, \tilde{\eta}K) & \mapsto & (X, \lambda, \tilde{\eta}K \circ g = \tilde{\eta} \circ g(g^{-1}Kg) \end{array}$$

(here we use the notation $\tilde{\eta}K$ for the K-orbit $\tilde{\eta}$ to emphasize the fact that it is an orbit). We then get a *Hecke correspondence*

$$\mathcal{A}_K \leftarrow \mathcal{A}_{K \cap gKg^{-1}} \to \mathcal{A}_{gKg^{-1}} \to \mathcal{A}_K\,,$$

where the right arrow is induced by g as above and the other ones are obvious morphisms (when $K' \subset K$, the orbit $\tilde{\eta}K'$ defines an orbit $\tilde{\eta}K$).

3. Shimura varieties of PEL type

3.1. **Endomorphisms of abelian varieties.** Let X be an abelian variety of dimension n over an algebraically closed field k. Let $\mathrm{End}(X)$ be the ring of endomorphisms of X and let $\mathrm{End}_{\mathbb{Q}}(X) = \mathrm{End}(X) \otimes \mathbb{Q}$. If $k = \mathbb{C}$ and $X = V/U$, then we have two faithful representations

$$\rho_a : \mathrm{End}(X) \to \mathrm{End}_{\mathbb{C}}(V) \text{ and } \rho_r : \mathrm{End}(X) \to \mathrm{End}_{\mathbb{Z}}(U).$$

It follows that $\mathrm{End}(X)$ is a torsion free abelian group of finite type. Over arbitrary field k, we need to recall the notion of Tate module. Let ℓ be a prime different from the characteristic of k then for every m, the kernel $X[\ell^m]$ of the multiplication by ℓ^m in X is isomorphic to $(\mathbb{Z}/\ell^m\mathbb{Z})^{2n}$.

Definition 3.1.1. *The Tate module $T_\ell(X)$ is the limit*

$$T_\ell(X) = \varprojlim X[\ell^n]$$

of the inverse system given by the multiplication by $\ell : X[\ell^{n+1}] \to X[\ell^n]$. As an $\mathbb{Z}_\ell$-module, $T_\ell(X) \simeq \mathbb{Z}_\ell^{2n}$ (non canonically). The rational Tate module is $V_\ell = T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

We can identify the Tate module $T_\ell(X)$ with the first étale homology $\mathrm{H}_1(X, \mathbb{Z}_\ell)$ which by definition is the dual of $\mathrm{H}^1(X, \mathbb{Z}_\ell)$. Similarly, $V_\ell(X) = \mathrm{H}_1(X, \mathbb{Q}_\ell)$.

Theorem 3.1.2. *For any abelian varieties X, Y over k,* $\mathrm{Hom}(X, Y)$ *is a finitely generated abelian group, and the natural map*

$$\mathrm{Hom}(X, Y) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), T_\ell(Y))$$

is injective.

See [13, p.176] for the proof.

Definition 3.1.3. *An abelian variety is called simple if it does not admit any strict abelian subvariety.*

Proposition 3.1.4. *If X is a simple abelian variety,* $\mathrm{End}_{\mathbb{Q}}(X)$ *is a division algebra.*

Proof. Let $f : X \to X$ be a non-zero endomorphism of X. The identity component of its kernel is a strict abelian subvariety of X which must be zero. Thus the whole kernel of f is a finite group and the image of f is all X for dimensional reason. It follows that f is an isogeny and therefore invertible in $\mathrm{End}_{\mathbb{Q}}(X)$. Thus $\mathrm{End}_{\mathbb{Q}}(X)$ is a division algebra. □

Theorem 3.1.5 (Poincaré)**.** *Every abelian variety X is isogenous to a product of simple abelian varieties.*

Proof. Let Y be an abelian subvariety of X. We want to prove the existence of a quasi-supplement of Y in X that is a subabelian variety Z of X such that the homomorphism $Y \times Z \to X$ is an isogeny. Let $\hat{X}$ be the dual abelian variety and let $\hat{\pi} : \hat{X} \to \hat{Y}$ be the dual homomorphism to the inclusion $Y \subset X$. Let L be an ample line bundle over X and $\lambda_L : X \to \hat{X}$ the isogeny attached to L. By restriction to Y, we get a homomorphism $\hat{\pi} \circ \lambda_L | Y : Y \to \hat{Y}$ which is surjective since $L|_Y$ is still an ample line bundle. Therefore the kernel Z of the homomorphism $\hat{\pi} \circ \lambda_L : X \to \hat{Y}$ is a quasi-complement of Y in X. □

Corollary 3.1.6. $\mathrm{End}_{\mathbb{Q}}(X)$ *is a semi-simple algebra of finite dimension over* $\mathbb{Q}$.

Proof. If X is isogenous to $\prod_i X_i^{m_i}$, where the X_i are mutually non-isogenous abelian varieties and $m_i \in \mathbb{N}$. Then $\mathrm{End}_{\mathbb{Q}}(X) = \prod_i M_{m_i}(D_i)$, where $M_{m_i}(D_i)$ is the algebra of $m_i \times m_i$-matrices over the skew-field $D_i = \mathrm{End}_{\mathbb{Q}}(X_i)$. □

We have a function

$$\deg : \mathrm{End}(X) \to \mathbb{N}$$

defined by the following rule : $\deg(f)$ is the degree of the isogeny f if f is an isogeny and $\deg(f) = 0$ if f is not an isogeny. Using the formula $\deg(mf) = m^{2n}\deg(f)$ for all $f \in \mathrm{End}(X)$, $m \in \mathbb{Z}$ and $n = \dim(X)$, we can extend this function to $\mathrm{End}_{\mathbb{Q}}(X)$

$$\deg : \mathrm{End}_{\mathbb{Q}}(X) \to \mathbb{Q}_+.$$

For every prime $\ell \neq \mathrm{char}(k)$, we have a representation of the endomorphism algebra

$$\rho_\ell : \mathrm{End}_{\mathbb{Q}}(X) \to \mathrm{End}(V_\ell).$$

These representations for different ℓ are related by the degree function.

Theorem 3.1.7. *For every* $f \in \mathrm{End}_{\mathbb{Q}}(X)$*, we have*

$$\deg(f) = \det \rho_\ell(f) \text{ and } \deg(n.1_X - f) = P(n)\,,$$

where $P(t) = \det(t - \rho_\ell(f))$ *is the characteristic polynomial of* $\rho_\ell(f)$*. In particular,* $\mathrm{tr}(\rho_\ell(f))$ *is a rational number which is independent of* ℓ.

Let $\lambda : X \to \hat{X}$ be a polarization of X. One attaches to λ an involution on the semi-simple $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(X)$. [1]

Definition 3.1.8. *The Rosati involution on* $\mathrm{End}_{\mathbb{Q}}(X)$ *associated with* λ *is the involution defined by the following formula*

$$f \mapsto f^* = \lambda^{-1}\hat{f}\lambda$$

for every $f \in \mathrm{End}_{\mathbb{Q}}(X)$.

The polarization $\lambda : X \to \hat{X}$ induces an alternating form $X[\ell^m] \times X[\ell^m] \to \mu_{\ell^m}$ for every m. By passing to the limit on m, we get a symplectic form

$$E : V_\ell(X) \times V_\ell(X) \to \mathbb{Q}_\ell(1).$$

By definition f^* is the adjoint of f for this symplectic form

$$E(fx, y) = E(x, f^*y).$$

Theorem 3.1.9. *The Rosati involution is positive. That is, for every* $f \in \mathrm{End}_{\mathbb{Q}}(X)$*,* $\mathrm{tr}(\rho_\lambda(ff^*))$ *is a positive rational number.*

[1]Our convention is that an involution of a non-commutative ring satisfies the relation $(xy)^* = y^*x^*$.

Proof. Let $\lambda = \lambda_L$ for some ample line bundle L. One can prove the formula

$$\mathrm{tr}\rho_\ell(ff^*) = \frac{2n(L^{n-1}.f^*(L))}{(L^n)}.$$

Since L is ample, the cup-product $(L^{n-1}.f^*(L))$ (resp. L^n) is the number of intersection of an effective divisor $f^*(L)$ (resp. L) with $n-1$ generic hyperplanes of $|L|$. Since L is ample, these intersection numbers are positive integers. □

Let X be an abelian variety over $\mathbb{C}$ equipped with a polarization λ. The semi-simple $\mathbb{Q}$-algebra $B = \mathrm{End}_{\mathbb{Q}}(X)$ is equipped with

(1) a complex representation ρ_a and a rational representation ρ_r satisfying $\rho_r \otimes_{\mathbb{Q}} \mathbb{C} = \rho_a \oplus \overline{\rho}_a$, and
(2) an involution $b \mapsto b^*$ such that for all $b \in B - \{0\}$, we have $\mathrm{tr}\rho_r(bb^*) > 0$.

Suppose that B is a simple algebra of center F. Then F is a number field equipped with a positive involution $b \mapsto b^*$ restricted from B. There are three possibilities:

(1) The involution is trivial on F. Then F is a totally real number field (*involution of first kind*). In this case, $B \otimes_{\mathbb{Q}} \mathbb{R}$ is a product of $M_n(\mathbb{R})$ or a product of $M_n(\mathbb{H})$, where $\mathbb{H}$ is the algebra of Hamiltonian quaternions, equipped with their respective positive involutions (case C and D).
(2) The involution is non-trivial on F. Then its fixed points form a totally real number field F_0 and F is a totally imaginary quadratic extension of F_0 (*involution of second kind*). In this case, $B \otimes_{\mathbb{Q}} \mathbb{R}$ is a product of $M_n(\mathbb{C})$ equipped with its positive involution (case A).

3.2. **Positive definite Hermitian forms.** Let B be a semisimple algebra of finite dimension over $\mathbb{R}$ with an involution. A Hermitian form on a B-module V is a symmetric form $V \times V \to \mathbb{R}$ such that $(bv, w) = (v, b^*w)$. It is positively definite if $(v, v) > 0$ for all $v \in V$.

Lemma 3.2.1. *The following assertions are equivalent*

(1) *There exists a faithful B-module V such that $\mathrm{tr}(xx^*, V) > 0$ for all $x \in B - \{0\}$.*
(2) *The above is true for every faithful B-module V.*
(3) $\mathrm{tr}_{B/\mathbb{R}}(xx^*) > 0$ *for all nonzero $x \in B$.*

3.3. **Skew-Hermitian modules.** Summing up what has been said in the last two sections, the tensor product with $\mathbb{Q}$ of the algebra of endomorphisms of a polarized abelian variety is a finite-dimensional semi-simple $\mathbb{Q}$-algebra equipped with a positive involution. For every prime $\ell \neq \mathrm{char}(k)$, this algebra has a representation on the Tate module

$V_\ell(X)$ which is equipped with a symplectic form. We are now going to look at this structure in a more axiomatic way.

Let k be a field. Let B be a finite-dimensional semisimple k-algebra equipped with an involution $*$. Let $\beta_1, \dots, \beta_r$ be a basis of B as a k-vector space. For any finite-dimensional B-module V we can define a polynomial $\det_V \in k[x_1, \dots, x_r]$ by the formula

$$\det_V = \det(x_1\beta_1 + \cdots + x_r\beta_r, V \otimes_k k[x_1, \dots, x_r])$$

Lemma 3.3.1. *Two finite-dimensional B-modules V and U are isomorphic if and only if $\det_V = \det_U$.*

Proof. If k is an algebraically closed field, B is a product of matrix algebras over k. The lemma follows from the classification of modules over a matrix algebra. Now let k be an arbitrary field and $\overline{k}$ its algebraic closure. The group of automorphisms of a B-module is itself the multiplicative group of a semi-simple k-algebra, so it has trivial Galois cohomology. This allows us to descend from the algebraic closure $\overline{k}$ to k. $\square$

Definition 3.3.2. *A skew-Hermitian B-module is a B-module U endowed with a symplectic form*

$$U \times U \to M_U$$

with values in a 1-dimensional k-vector space M_U such that $(bx, y) = (x, b^*y)$ for any $x, y \in V$ and $b \in B$.

An automorphism of a skew-Hermitian B-module U is pair (g, c), where $g \in \mathrm{GL}_B(U)$ and $c \in \mathbb{G}_{m,k}$ such that $(gx, gy) = c(x, y)$ for any $x, y \in U$. The group of automorphisms of the skew-Hermitian B-module U is denoted by $G(U)$.

If k is an algebraically closed field, two skew-Hermitian modules V and U are isomorphic if and only if $\det_V = \det_U$. In general, the set of skew-Hermitian modules V with $\det_V = \det_U$ is classified by $\mathrm{H}^1(k, G(U))$.

Let $k = \mathbb{R}$, let B be a finite-dimensional semi-simple algebra over $\mathbb{R}$ equipped with an involution and let U be a skew-Hermitian B-module. Let $h : \mathbb{C} \to \mathrm{End}_B(U_\mathbb{R})$ be such that $(h(z)v, w) = (v, h(\overline{z})w)$ and such that the symmetric bilinear form $(v, h(i)w)$ is positive definite.

Lemma 3.3.3. *Let $h, h' : \mathbb{C} \to \mathrm{End}_B(U_\mathbb{R})$ be two such homomorphisms. Suppose that the two $B \otimes_\mathbb{R} \mathbb{C}$-modules U induced by h and h' are isomorphic. Then h and h' are conjugate by an element of $G(\mathbb{R})$.*

Let B be a finite-dimensional simple $\mathbb{Q}$-algebra equipped with an involution and let $U_\mathbb{Q}$ be a skew-Hermitian module $U_\mathbb{Q} \times U_\mathbb{Q} \to M_{U_\mathbb{Q}}$. An integral structure is an order $\mathcal{O}_B$ of B and a free abelian group U equipped with multiplication by $\mathcal{O}_B$ and an alternating form $U \times U \to M_U$ of which the generic fibre is the skew Hermitian module $U_\mathbb{Q}$.

3.4. **Shimura varieties of type PEL.** Let us fix a prime p. We will describe the PEL moduli problem over a discrete valuation ring with residual characteristic p under the assumption that the PEL datum is unramified at p.

Definition 3.4.1. *A rational PE-structure (polarization and endomorphism) is a collection of data as follows:*

(1) *B is a finite-dimensional simple $\mathbb{Q}$-algebra such that that $B_{\mathbb{Q}_p}$ is a product of matrix algebra over unramified extensions of $\mathbb{Q}_p$;*
(2) *$*$ is a positive involution of B;*
(3) *$U_{\mathbb{Q}}$ is a skew-Hermitian B-module;*
(4) *$h : \mathbb{C} \to \mathrm{End}_B(U_{\mathbb{R}})$ is a homomorphism such that $(h(z)v, w) = (v, h(\overline{z})w)$ and such that the symmetric bilinear form $(v, h(i)w)$ is positive definite.*

The homomorphism h induces a decomposition $U_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = U_1 \oplus U_2$, where $h(z)$ acts on U_1 by z and on U_2 by $\overline{z}$. Let us choose a basis $\beta_1, \ldots, \beta_r$ of the $\mathbb{Q}$-vector space B. Let $x_1, \ldots, x_r$ be indeterminates. The determinant polynomial

$$\det_{U_1} = \det(x_1\beta_1 + \cdots + x_r\beta_r, U_1 \otimes \mathbb{C}[x_1, \ldots, x_r])$$

is a homogenous polynomial of degree $\dim_{\mathbb{C}} U_1$. The subfield of $\mathbb{C}$ generated by the coefficients of the polynomial $\det_{U_1}$ is a number field which is independent of the choice of the basis $\beta_1, \ldots, \beta_t$. The above number field E is called the *reflex field* of the PE-structure. Equivalently, E is the definition field of the *isomorphy class* of the $B_{\mathbb{C}}$-module U_1.

Definition 3.4.2. *An integral PE structure consists of a rational PE structure equipped with the following extra data:*

(5) *$\mathcal{O}_B$ is an order of B which is stable under $*$ and maximal at p;*
(6) *U is an $\mathcal{O}_B$-integral structure of the skew-Hermitian module $U_{\mathbb{Q}}$.*

Assuming that the basis $\beta_1, \ldots, \beta_r$ chosen above is a $\mathbb{Z}$-basis of $\mathcal{O}_B$, we see that the coefficients of the determinant polynomial $\det_{U_1}$ lie in $O = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Fix an integer $N \geq 3$. Consider the moduli problem $\mathcal{B}$ of abelian schemes with a PE-structure and with principal N-level structures. The functor $\mathcal{B}$ assigns to any O-scheme S the category $\mathcal{B}(S)$ whose objects are

$$(A, \lambda, \iota, \eta)\,,$$

where

(1) A is an abelian scheme over S;
(2) $\lambda : A \to \hat{A}$ is a polarization;
(3) $\iota : \mathcal{O}_B \to \mathrm{End}(A)$ is a homomorphism such that the Rosati involution induced by λ restricts to the involution $*$ of $\mathcal{O}_B$,

$$\det(\beta_1 X_1 + \cdots \beta_r x_r, \mathrm{Lie}(A)) = \det_{U_1}$$

and for every prime $\ell \neq p$ and every geometric point s of S, the Tate module $T_\ell(A_s)$, equipped with the action of $\mathcal{O}_B$ and with the alternating form induced by λ, is similar to $U \otimes \mathbb{Z}_\ell$;

(4) η is a similitude from $A[N]$ equipped with the symplectic form and the action of $\mathcal{O}_B$ to U/NU that can be lifted to an isomorphism $H_1(A_s, \mathbb{A}_f^p)$ with $U \otimes_{\mathbb{Z}} \mathbb{A}_f^p$, for every geometric point s of S.

Theorem 3.4.3. *The functor which assigns to each E-scheme S the set of isomorphy classes $\mathcal{B}(S)$ is smooth and representable by a quasi-projective scheme over $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$.*

Proof. For $\ell \neq p$, the isomorphy class of the skew-Hermitian module $T_\lambda(A_s)$ is locally constant with respect to s so that we can forget the condition on this isomorphy class in the representability problem.

By forgetting endomorphisms, we have a morphism $\mathcal{B} \to \mathcal{A}$. To have ι is equivalent to having actions of $\beta_1, \ldots, \beta_r$ satisfying certain conditions. Therefore, to prove that $\mathcal{B} \to \mathcal{A}$ is representable by a projective morphism it is enough to prove the following lemma.

Lemma 3.4.4. *Let A be a projective abelian scheme over a locally noetherian scheme S. Then the functor that assigns to every S-scheme T the set* $\mathrm{End}(A_T)$ *is representable by a disjoint union of projective schemes over S.*

Proof. The graph of an endomorphism b of A is a closed subscheme of $A \times_S A$ so that the functor of endomorphisms is a subfunctor of some Hilbert scheme. Let's check that this subfunctor is representable by a locally closed subscheme of the Hilbert scheme.

Let $Z \subset A \times_S A$ be a closed subscheme flat over a connected base S. Let's check that the condition $s \in S$ such that Z_s is a graph is an open condition. Suppose that $p_1 : Z_s \to A_s$ is an isomorphism over a point $s \in S$. By flatness, the relative dimension of Z over S is equal to that of A. For every $s \in S$ and every $a \in A$, the intersection $Z_s \cap \{a\} \times A_s$ is either of dimension bigger than 0 or consists of exactly one point since the intersection number is constant under deformation. This implies that the morphism $p_1 : Z \to A$ is a birationnal projective morphism. There is an open subset U of A over which $p_1 : Z \to A$ is an isomorphism. Since $\pi_A : A \to S$ is proper, $\pi_A(A - U)$ is closed. Its complement $S - \pi_A(A - U)$, which is open, is the set of $s \in S$ over which $p_1 : Z_s \to A_s$ is an isomorphism.

Let $Z \subset A \times_S A$ be the graph of a morphism $f : A \to A$. The morphism f is a homomorphism of abelian schemes if and only if f sends the unit to the unit, hence being a homomorphism of abelian schemes is a closed condition. So the functor which assigns to each S-scheme T the set $\mathrm{End}_T(A_T)$ is representable by a locally closed subscheme of a Hilbert scheme.

In order to prove that this subfunctor is represented by a closed subscheme of the Hilbert scheme, it is enough to check the valuative criterion of properness.

Let $S = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation ring with generic point η. Let A be an S-abelian scheme and let $f_\eta : A_\eta \to A_\eta$ be an endomorphism. Then f_η can be extended in a unique way to an endomorphism $f : A \to A$ by the following extension lemma, due to Weil. □

Theorem 3.4.5 (Weil)**.** *Let G be a smooth group scheme over S. Let X be smooth scheme over S and let $U \subset X$ be an open subscheme whose complement $Y = X - U$ has codimension ≥ 2. Then a morphism $f : U \to G$ can be extended to X. In particular, if G is an abelian scheme, using its properness we can always extend even without the condition on the codimension of $X - U$.*

3.5. **Adelic description.** Let G be the $\mathbb{Q}$-reductive group defined as the automorphism group of the skew-Hermitian module $U_{\mathbb{Q}}$. For every $\mathbb{Q}$-algebra R, let

$$G(R) = \{(g, c) \in \mathrm{GL}_B(U)(R) \times R^\times | (gx, gy) = c(x, y)\}.$$

For all $\ell \neq p$ we have a compact open subgroup $K_\ell \subset G(\mathbb{Q}_\ell)$ which consists of $g \in G(\mathbb{Q}_\ell)$ such that $g(U \otimes \mathbb{Z}_\ell) = (U \otimes \mathbb{Z}_\ell)$ and which, when $\ell | N$, satisfy the extra condition that the action induced by g on $(U \otimes \mathbb{Z}_\ell)/N(U \otimes \mathbb{Z}_\ell)$ is trivial.

Lemma 3.5.1. *There exists a unique smooth group scheme $\mathcal{G}_{K_\ell}$ over $\mathbb{Z}_\ell$ such that $\mathcal{G}_{K_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = G \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and $\mathcal{G}_{K_\ell}(\mathbb{Z}_\ell) = K_\ell$.*

Consider the functor $\mathcal{B}'$ which assigns to any E-scheme S the category $\mathcal{B}'(S)$ defined as follows. An object of this category is a quintuple

$$(A, \lambda, \iota, \tilde{\eta}),$$

where

1. A is a S-abelian schemes over S,
2. $\lambda : A \to \hat{A}$ is a $\mathbb{Z}_{(p)}$-multiple of a polarization,
3. $\iota : \mathcal{O}_B \to \mathrm{End}(A)$ is a homomorphism such that the Rosati involution induced by λ restricts to the involution $*$ of $\mathcal{O}_B$ and such that
$$\det(\beta_1 X_1 + \cdots \beta_t X_t, \mathrm{Lie}(A)) = \det{}_{U_1}$$
4. fixing a geometric point s of S, for every prime $\ell \neq p$, $\tilde{\eta}_\ell$ is a K_ℓ-orbit of isomorphisms from $V_\ell(A_s)$ to $U \otimes \mathbb{Q}_\ell$ compatible with symplectic forms and action of $\mathcal{O}_B$ and stable under the action of $\pi_1(S, s)$.

A morphism from $(A, \lambda, \iota, \tilde{\eta})$ to $(A', \lambda, \iota, \tilde{\eta})$ is a quasi-isogeny $\alpha : A \to A'$ of degree prime to p carrying λ to a scalar (in $\mathbb{Q}^\times$) multiple of λ' and carrying $\tilde{\eta}$ to $\tilde{\eta}'$.

Proposition 3.5.2. *The obvious functor $\mathcal{B} \to \mathcal{B}'$ is an equivalence of categories.*

The proof is the same as in the Siegel case. □

3.6. **Complex points.** The isomorphy class of an object $(A, \lambda, \iota, \tilde{\eta}) \in \mathcal{B}'(\mathbb{C})$ gives rises to

(1) a skew-Hermitian B-module $\mathrm{H}_1(A, \mathbb{Q})$ and
(2) for every prime ℓ, a $\mathbb{Q}_\ell$-similitude $\mathrm{H}_1(A, \mathbb{Q}_\ell) \simeq U \otimes \mathbb{Q}_\ell$ as skew-Hermitian $B \otimes \mathbb{Q}_\ell$-modules, defined up to the action of K_ℓ.

For $\ell \neq p$, this is required in the moduli problem. For the prime p, for every $b \in B$, we have $\mathrm{tr}(b, \mathrm{H}_1(A, \mathbb{Q}_p)) = \mathrm{tr}(b, U \otimes \mathbb{Q})$ because both are equal with $\mathrm{tr}(b, \mathrm{H}_1(A, \mathbb{Q}_\ell))$ for any $\ell \neq p$. It follows that the skew-Hermitian modules $\mathrm{tr}(b, \mathrm{H}_1(A, \mathbb{Q}_p))$ and $\mathrm{tr}(b, \Lambda \otimes \mathbb{Q}_p)$ are isomorphic after base change to a finite extension of $\mathbb{Q}_p$ and therefore the isomorphy class of the skew-Hermitian module defines an element $\xi_p \in \mathrm{H}^1(\mathbb{Q}_p, G)$. Now, in the groupoïd $\mathcal{B}'(\mathbb{C})$ the arrows are given by prime to p isogenies, so $\mathrm{H}_1(A, \mathbb{Z}_p)$ is a well-defined self-dual lattice stable by multiplication by $\mathcal{O}_B$. It follows that the class $\xi_p \in \mathrm{H}^1(\mathbb{Q}_p, G)$ mentioned above comes from a class in $\mathrm{H}^1(\mathbb{Z}_p, G_{\mathbb{Z}_p})$, where $G_{\mathbb{Z}_p}$ is the reductive group scheme over $\mathbb{Z}_p$ which extends $G_{\mathbb{Q}_p}$. In the case where $G_{\mathbb{Z}_p}$ has connected fibres, this implies the vanishing of ξ_p. Kottwitz gave a further argument in the case where G is not connected.

The first datum gives rise to a class

$$\xi \in \mathrm{H}^1(\mathbb{Q}, G)$$

and the second datum implies that the images of ξ in each $\mathrm{H}^1(\mathbb{Q}_\ell, G)$ vanishes. We have

$$\xi \in \ker^1(\mathbb{Q}, G) = \ker(\mathrm{H}^1(\mathbb{Q}, G) \to \prod_\ell \mathrm{H}^1(\mathbb{Q}_\ell, G)).$$

According to Borel and Serre, $\ker^1(\mathbb{Q}, G)$ is a finite set. For every $\xi \in \ker^1(\mathbb{Q}, G)$, fix a skew-Hermitian B-module $V^{(\xi)}$ whose class in $\ker^1(\mathbb{Q}, G)$ is ξ and fix a $\mathbb{Q}_\ell$-similitude of $V^{(\xi)} \otimes \mathbb{Q}_\ell$ with $U \otimes \mathbb{Q}_\ell$ as skew-Hermitian $B \otimes \mathbb{Q}_\ell$-module and also a similitude over $\mathbb{R}$.

Let $\mathcal{B}^{(\xi)}(\mathbb{C})$ be the subset of $\mathcal{B}^{(\xi)}(\mathbb{C})$ consisting of those $(A, \lambda, \iota, \tilde{\eta})$ such that $\mathrm{H}_1(A, \mathbb{Q})$ is isomorphic to $V^{(\xi)}$. Let $(A, \lambda, \iota, \tilde{\eta}) \in \mathcal{B}^{(\xi)}(\mathbb{C})$ and let β be an isomorphism of skew-Hermitian B-modules from $\mathrm{H}_1(A, \mathbb{Q})$ to $V^{(\xi)}$. The set of quintuple $(A, \lambda, \iota, \tilde{\eta}, \beta)$ can be described as follows.

(1) $\tilde{\eta}$ defines an element $\tilde{\eta} \in G(\mathbb{A}_f)/K$.
(2) The complex structure on $\mathrm{Lie}(A) = V \otimes_{\mathbb{Q}} \mathbb{R}$ defines a homomorphism $h : \mathbb{C} \to \mathrm{End}_B(V_{\mathbb{R}})$ such that $h(\overline{z})$ is the adjoint operator of $h(z)$ for the symplectic form on $V_{\mathbb{R}}$. Since $\pm\lambda$ is a polarization, $(v, h(i)w)$ is positive or negative definite. Moreover the isomorphisy class of the $B \otimes \mathbb{C}$-module V is specified by the

determinant condition on the tangent space. It follows that h lies in a $G(\mathbb{R})$-conjugacy class X_∞.

Therefore the set of quintuples is $X_\infty \times G(\mathbb{A}_f)/K$. Two different trivializations β and β' differs by an automorphism of the skew-Hermitian B-module $V^{(\xi)}$. This group is the inner form $G^{(\xi)}$ of G obtained by the image of $\xi \in \mathrm{H}^1(\mathbb{Q}, G)$ in $\mathrm{H}^1(\mathbb{Q}, G^{\mathrm{ad}})$. In conclusion we get

$$\mathcal{B}^{(\xi)}(\mathbb{C}) = G^{(\xi)}(\mathbb{Q})\backslash[X_\infty \times G(\mathbb{A}_f)/K]$$

and

$$\mathcal{B}(\mathbb{C}) = \bigsqcup_{\xi \in \ker^1(\mathbb{Q},G)} \mathcal{B}^{(\xi)}(\mathbb{C}).$$

4. Shimura varieties

4.1. **Review on Hodge structures.** See [Deligne, Travaux de Griffiths]. Let Q be a subring of $\mathbb{R}$; we think specifically about the cases $Q = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. A Q-Hodge structure will be called respectively an integral, rational or real Hodge structure.

Definition 4.1.1. *A Q-Hodge structure is a projective Q-module V equipped with a bi-grading of $V_\mathbb{C} = V \otimes_Q \mathbb{C}$*

$$V_\mathbb{C} = \bigoplus_{p,q} H^{p,q}$$

such that $H^{p,q}$ and $H^{q,p}$ are complex conjugate, i.e. the semi-linear automorphism σ of $V_\mathbb{C} = V \otimes_Q \mathbb{C}$ given by $v \otimes z \mapsto v \otimes \overline{z}$ satisfies the relation $\sigma(H^{p,q}) = H^{q,p}$ for every $p, q \in \mathbb{Z}$.

The integers $h^{p,q} = \dim_\mathbb{C}(H^{p,q})$ are called Hodge numbers. We have $h^{p,q} = h^{q,p}$. If there exists an integer n such that $H^{p,q} = 0$ unless $p + q = n$ then the Hodge structure is said to be pure of weight n. When the Hodge structure is pure of weight n, the Hodge filtration $F^pV = \bigoplus_{r\geq p} V^{rs}$ determines the Hodge structure by the relation $V^{pq} = F^pV \cap \overline{F^qV}$.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the real algebraic torus defined as the Weil restriction from $\mathbb{C}$ to $\mathbb{R}$ of $\mathbb{G}_{m,\mathbb{C}}$. We have a norm homomorphism $\mathbb{C}^\times \to \mathbb{R}^\times$ whose kernel is the unit circle S^1. Similarly, we have an exact sequence of real tori

$$1 \to \mathbb{S}^1 \to \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}} \to 1.$$

We have an inclusion $\mathbb{R}^\times \subset \mathbb{C}^\times$ whose cokernel can be represented by the homomorphism $\mathbb{C}^\times \to S^1$ given by $z \mapsto z/\overline{z}$. We have the corresponding exact sequence of real tori

$$1 \to \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S} \to \mathbb{S}^1 \to 1.$$

The inclusion $w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ is called the weight homomorphism.

Lemma 4.1.2. *Let $G = GL(V)$ be the linear group defined over Q. A Hodge structure on V is equivalent to a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}} = G \otimes_Q \mathbb{R}$. The Hodge structure is pure of weight n if the restriction of h to $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$ factors through the center $\mathbb{G}_{m,\mathbb{R}} = Z(G_{\mathbb{R}})$ and the homomorphism $\mathbb{G}_{m,\mathbb{R}} \to Z(G_{\mathbb{R}})$ is given by $t \mapsto t^n$.*

Proof. A bi-grading $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ is the same as a homomorphism $h_{\mathbb{C}} : \mathbb{G}^2_{m,\mathbb{C}} \to G_{\mathbb{C}}$. The complex conjugation of $V_{\mathbb{C}}$ exchanges the factors $V^{p,q}$ and $V^{q,p}$ if and only if $h_{\mathbb{C}}$ descends to a homomorphism of real algebraic groups $h : \mathbb{S} \to G_{\mathbb{R}}$. □

Definition 4.1.3. *A polarization of a Hodge structure $(V_{\mathbb{Q}}, V^{p,q})$ of weight n is a bilinear form Ψ_Q on V_Q such that the induced form Ψ on $V_{\mathbb{R}}$ is invariant under $h(S^1)$ and such that the form $\Psi(x, h(i)y)$ is symmetric and positive definite.*

It follows from the identity $h(i)^2 = (-1)^n$ that the bilinear form $\Psi(x, y)$ is symmetric if n is even and alternating if n is odd :

$$\Psi(x, y) = (-1)^n \Psi(x, h(i)^2 y) = (-1)^n \Psi(h(i)y, h(i)x) = (-1)^n \Psi(y, x).$$

Example. An abelian variety induces a typical Hodge structure. Let $X = V/U$ be an abelian variety. Let G be $GL(U \otimes \mathbb{Q})$ as an algebraic group defined over $\mathbb{Q}$. The complex structure V on the real vector space $U \otimes \mathbb{R} = V$ induces a homomorphism of real algebraic groups

$$\phi : \mathbb{S} \to G_{\mathbb{R}}$$

so that U is equipped with an integral Hodge structure of weight -1. A polarization of X is a symplectic form E on V, taking integral values on U such that $E(ix, iy) = E(x, y)$ and such that $E(x, iy)$ is a positive definite symmetric form.

Let V be a projective Q-module of finite rank. A Hodge structure on V induces Hodge structures on tensor products $V^{\otimes m} \otimes (V^*)^{\otimes n}$. Fix a finite set of tensors (s_i)

$$s_i \in V^{\otimes m_i} \otimes (V^*)^{\otimes n_i}.$$

Let $G \subset \mathrm{GL}(V)$ be the stabilizer of these tensors.

Lemma 4.1.4. *There is a bijection between the set of Hodge structures on V for which the tensors s_i are of type $(0, 0)$ and the set of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$.*

Proof. A homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ factors through $G_{\mathbb{R}}$ if and only if the image $h(\mathbb{S})$ fixes all tensors s_i. This is equivalent to saying that these tensors are of type $(0, 0)$ for the induced Hodge structures. □

There is a related notion of Mumford-Tate group.

Definition 4.1.5. *Let G be an algebraic group over $\mathbb{Q}$. Let $\phi : S^1 \to G_{\mathbb{R}}$ be a homomorphism of real algebraic groups. The Mumford-Tate group of (G, ϕ) is the smallest algebraic subgroup $H = \mathrm{Hg}(\phi)$ of G defined over $\mathbb{Q}$ such that ϕ factors through $H_{\mathbb{R}}$.* [2]

Let $\mathbb{Q}[G]$ be the ring of algebraic functions over G and $\mathbb{R}[G] = \mathbb{Q}[G] \otimes_{\mathbb{Q}} \mathbb{R}$. The group S^1 acts on $\mathbb{R}[G]$ through the homomorphism ϕ. Let $\mathbb{R}[G]^{\phi=1}$ be the subring of functions fixed by $\phi(S^1)$ and consider the subring

$$\mathbb{Q}[G] \cap \mathbb{R}[G]^{\phi=1}$$

of $\mathbb{Q}[G]$. For every $v \in \mathbb{Q}[G] \cap \mathbb{R}[G]^{\phi=1}$, let G_v be the stabilizer subgroup of G at v. Since G_v is defined over $\mathbb{Q}$ and ϕ factors through $G_{v,\mathbb{R}}$, we have the inclusion $H \subset G_v$. In particular, $v \in \mathbb{Q}[G]^H$. It follows that

$$\mathbb{Q}[G] \cap \mathbb{R}[G]^{\phi=1} = \mathbb{Q}[G]^H.$$

This property does not however characterize H. In general, for any subgroup H of G, we have an obvious inclusion

$$H \subset H' = \bigcap_{v \in \mathbb{Q}[G]^H} G_v.$$

which is strict in general. If the equality $H = H'$ occurs, we say that H is an *observable* subgroup of G. To prove that this is indeed the case for the Mumford-Tate group of an abelian variety, we will need the following general lemma.

Lemma 4.1.6. *Let H be a reductive subgroup of a reductive group G. Then H is observable.*

Proof. Assume the base field $k = \mathbb{C}$. According to Chevalley, see [Borel], for every subgroup H of G, there exists a representation $\rho : G \to \mathrm{GL}(V)$ and a vector $v \in V$ such that H is the stabilizer of the line kv. Since H is reductive, there exists a H-stable complement U of kv in V. Let $kv^* \subset V^*$ be the line orthogonal to U with some generator v^*. Then H is the stabilizer of the vector $v \otimes v^* \in V \otimes V^*$.

Let $G = \mathrm{GL}(U_{\mathbb{Q}})$ and let $\phi : S^1 \to G_{\mathbb{R}}$ be a homomorphism such that $\phi(i)$ induces a Cartan involution on $G_{\mathbb{R}}$. Let $\mathcal{C}$ be the smallest tensor subcategory stable by subquotient of the category of Hodge structures that contains $(U_{\mathbb{Q}}, \phi)$. There is the forgetful functor $\mathrm{Fib}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$.

Lemma 4.1.7. *H is the automorphism group of the functor $\mathrm{Fib}_{\mathcal{C}}$.*

Proof. Let V be a representation of G defined over $\mathbb{Q}$ equipped with the Hodge structure defined by ϕ. Let U be a subvector space of V compatible with the Hodge structure. Then H must stabilize U. It follows that H acts naturally on $\mathrm{Fib}_{\mathcal{C}}$, i.e. we have a natural homomorphism $H \to \mathrm{Aut}^{\otimes}(\mathrm{Fib}_{\mathcal{C}})$. □

[2] Originally, Mumford called $\mathrm{Hg}(\phi)$ the Hodge group.

Proposition 4.1.8. *The Mumford-Tate group of a polarizable Hodge structure is a reductive group.*

Proof. Since we are working over fields of characteristic zero, H is reductive if and only if the category of representations of H is semi-simple. Using the Cartan involution, we can exhibit a positive definite bilinear form on V. This implies that every subquotient of $V^{\otimes m} \otimes (V^*)^{\otimes n}$ is a subobject. $\square$

4.2. **Variations of Hodge structures.** Let S be a complex analytic variety. The letter Q denotes a ring contained in $\mathbb{R}$ which could be $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$.

Definition 4.2.1. *A variation of Hodge structures (VHS) on S of weight n consists of the following data:*

1. *a local system of projective Q-modules V;*
2. *a decreasing filtration $F^p\mathcal{V}$ on the vector bundle $\mathcal{V} = V \otimes_Q \mathcal{O}_S$ such that the Griffiths transversality is satisfied, i.e. for every integer p*
$$\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_S$$
where $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$ is the connection $v \otimes f \mapsto v \otimes df$ for which $V \otimes_Q \mathbb{C}$ is the local system of horizontal sections;
3. *for every $s \in S$, the filtration induces on V_s a pure Hodge structure of weight n.*

There are obvious notions of the dual VHS and tensor product of VHS. The Leibnitz formula $\nabla(v \otimes v') = \nabla(v) \otimes v' + v \otimes \nabla(v')$ assures that the Griffiths transversality is satisfied for the tensor product.

Typical examples of polarized VHS are provided by cohomology of smooth projective morphisms. Let $f : X \to S$ be a smooth projective morphism over a complex analytic variety S. Then $H^n = \mathrm{R}^n f_* \mathbb{Q}$ is a local system of $\mathbb{Q}$-vector spaces. Since $\mathrm{H}^n \otimes_{\mathbb{Q}} \mathcal{O}_S$ is equal to the de Rham cohomology $\mathrm{H}^n_{dR} = \mathrm{R}^n f_* \Omega^\bullet_{X/S}$, where $\Omega^\bullet_{X/S}$ is the relative de Rham complex, and the Hodge spectral sequence degenerates on E^2, the abutments H^n_{dR} are equipped with a decreasing filtration by subvector bundles $F^p(\mathrm{H}^n_{dR})$ with

$$(F^p/F^{p+1})\mathrm{H}^n_{dR} = \mathrm{R}^q f_* \Omega^p_{X/S}$$

with $p + q = n$. The connexion ∇ satisfies the Griffiths transversality. By Hodge's decomposition, we have instead a direct sum

$$\mathrm{H}^n_{dR}(X_s) = \bigoplus_{pq} H^{p,q}$$

with $H^{p,q} = \mathrm{H}^q(X_s, \Omega^p_{X_s})$ and $\overline{H^{p,q}} = H^{q,p}$ so that all the axioms of VHS are satisfied.

If we choose a projective embedding $X \to \mathbb{P}^d_S$, the line bundle $\mathcal{O}_{\mathbb{P}^d}(1)$ defines a class

$$c \in \mathrm{H}^0(S, \mathrm{R}^2 f_* \mathbb{Q}).$$

By the hard Lefschetz theorem, the cup product by c^{d-n} induces an isomorphism

$$\mathrm{R}^n f_* \mathbb{Q} \to \mathrm{R}^{2d-n} f_* \mathbb{Q} \text{ defined by } \alpha \mapsto c^{d-n} \wedge \alpha$$

so that by Poincaré duality we get a polarization on $\mathrm{R}^n f_* \mathbb{Q}$.

4.3. **Reductive Shimura-Deligne data.** The torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ plays a particular role in the formalism of Shimura varieties shaped by Deligne in [4], [5].

Definition 4.3.1. *A Shimura-Deligne datum is a pair (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following properties:*

(SD1) *For $h \in X$, only the characters $z/\overline{z}, 1, \overline{z}/z$ occur in the representation of $\mathbb{S}$ on* $\mathrm{Lie}(G)$*;*

(SD2) $\mathrm{ad}h(i)$ *is a Cartan involution of G^{ad}, i.e. the real Lie group $\{g \in G^{\mathrm{ad}}(\mathbb{C}) \mid \mathrm{ad}(h(i))\sigma(g) = g\}$ is compact (where σ denotes the complex conjugation and $G^{ad} = G/Z(G)$ is the adjoint group of G).*

The action $\mathbb{S}$, restricted to $\mathbb{G}_{m,\mathbb{R}}$ is trivial on $\mathrm{Lie}(G)$ so that $h : \mathbb{S} \to G_{\mathbb{R}}$ sends $\mathbb{G}_{m,\mathbb{R}}$ into the center $Z_{\mathbb{R}}$ of $G_{\mathbb{R}}$. The induced homomorphism $w = h|_{\mathbb{G}_{m,\mathbb{R}}} : \mathbb{G}_{m,\mathbb{R}} \to Z_{\mathbb{R}}$ is independent of the choice of $h \in X$. We call w the *weight homomorphism*.

After base change to $\mathbb{C}$, we have $\mathbb{S} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{G}_m \times \mathbb{G}_m$, where the factors are ordered in the way that $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C})$ is the map $z \mapsto (z, \overline{z})$. Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ the homomorphism defined by $z \to (z, 1)$. If $h : \mathbb{S} \to \mathrm{GL}(V)$ is a Hodge structure, then $\mu_h = h_{\mathbb{C}} \circ \mu : \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}})$ determines its Hodge filtration.

Siegel case. An abelian variety $A = V/U$ is equipped with a polarization E which is a non-degenerate symplectic form on $U_{\mathbb{Q}} = U \otimes \mathbb{Q}$. Let GSp be the group of symplectic similitudes

$$\mathrm{GSp}(U_{\mathbb{Q}}, E) = \{(g, c) \in \mathrm{GL}(U_{\mathbb{Q}}) \times \mathbb{G}_{m,\mathbb{Q}} | E(gx, gy) = cE(x, y)\}.$$

The scalar c is called the similitude factor. Base changed to $\mathbb{R}$, we get the group of symplectic similitudes of the real symplectic space $(U_{\mathbb{R}}, E)$. The complex vector space structure on $V = U_{\mathbb{R}}$ induces a homomorphism

$$h : \mathbb{S} \to \mathrm{GSp}(U_{\mathbb{R}}, E).$$

In this case, X is the set of complex structures on $U_{\mathbb{R}}$ such that $E(h(i)x, h(i)y) = E(x, y)$ and $E(x, h(i)y)$ is a positive definite symmetric form.

PEL case. Suppose B is a simple $\mathbb{Q}$-algebra of center F equipped with a positive involution $*$. Let $F_0 \subset F$ be the fixed field by $*$. Let G be the group of symplectic similitudes of a skew-Hermitian B-module V

$$G = \{(g, c) \in \mathrm{GL}_B(V) \times \mathbb{G}_{m,\mathbb{Q}} | (gx, gy) = c(x, y)\}.$$

So, the h of definition 3.4.1 induces a morphism $\mathbb{S} \to G_{\mathbb{R}}$.

Let G_1 be the subgroup of G defined by

$$G_1(R) = \{g \in \mathrm{GL}_B(V) \times \mathbb{G}_{m,\mathbb{Q}} | (gx, gy) = (x, y)\}$$

for any $\mathbb{Q}$-algebra R. We have an exact sequence

$$1 \to G_1 \to G \to \mathbb{G}_m \to 1.$$

The group G_1 is a scalar restriction of a group G_0 defined over F_0.

Since a simple $\mathbb{R}$-algebra with positive involution must be $M_n(\mathbb{C})$, $M_n(\mathbb{R})$ or $M_n(\mathbb{H})$ with their standard involutions, there will be three cases to be considered.

(1) Case (A) : If $[F : F_0] = 2$, then F_0 is a totally real field and F is a totally imaginary extension. Over $\mathbb{R}$, $B \otimes_{\mathbb{Q}} \mathbb{R}$ is product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{C})$. $G_1 = \mathrm{Res}_{F_0/\mathbb{Q}} G_0$,where G_0 is an inner form of the quasi-split unitary group attached to the quadratic extension F/F_0.

(2) Case (C) : If $F = F_0$, then F is a totally real field. and $B \otimes \mathbb{R}$ is isomorphic to a product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{R})$ equipped with their positive involution. In this case, $G_1 = \mathrm{Res}_{F_0/\mathbb{Q}} G_0$, where G_0 is an inner form of a quasi-split symplectic group over F_0.

(3) Case (D) : $B \otimes \mathbb{R}$ is isomorphic to a product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{H})$ equipped with positive involutions. The simplest case is $B = \mathbb{H}$, V is a skew-Hermitian quaternionic vector space. In this case, $G_1 = \mathrm{Res}_{F_0/\mathbb{Q}} G_0$, where G_0 is an even orthogonal group.

Tori case. In the case where $G = T$ is a torus over $\mathbb{Q}$, both conditions (SD1) and (SD2) are obvious since the adjoint representation is trivial. The conjugacy class of $h : \mathbb{S} \to T_{\mathbb{R}}$ contains just one element since T is commutative.

Deligne proved the following statement in [5, prop. 1.1.14] which provides a justification for the not so natural notion of Shimura-Deligne datum.

Proposition 4.3.2. *Let (G, X) be a Shimura-Deligne datum. Then X has a unique structure of a complex manifold such that for every representation $\rho : G \to \mathrm{GL}(V)$, $(V, \rho \circ h)_{h \in X}$ is a variation of Hodge structures which is polarizable.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation of G. Since $w_h \subset Z_G$, the weight filtration of V_h is independent of h. Since the weight filtration is fixed, the Hodge structure is determined by the Hodge filtration. It follows that the morphism to the Grassmannian

$$\omega : X \to \mathrm{Gr}(V_{\mathbb{C}})$$

which sends h to the Hodge filtration attached to h is injective. We need to prove that this morphism identifies X with a complex subvariety of $\mathrm{Gr}(V_{\mathbb{C}})$. It suffices to prove that

$$d\omega : T_h X \to T_{\omega(h)}\mathrm{Gr}(V_{\mathbb{C}})$$

identifies $T_h X$ with a complex vector subspace of $\mathrm{Gr}(V_{\mathbb{C}})$.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation. Let G_h be the centralizer of h, and $\mathfrak{g}_h$ its Lie algebra. We have $\mathfrak{g}_h = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{g}^{0,0}$ for the Hodge structure on $\mathfrak{g}$ induced by h. It follows that the tangent space to the real analytic variety X at h is

$$T_h X = \mathfrak{g}_{\mathbb{R}}/\mathfrak{g}_{\mathbb{C}}^{0,0} \cap \mathfrak{g}_{\mathbb{R}}.$$

Let W be a pure Hodge structure of weight 0. Consider the $\mathbb{R}$-linear morphism

$$W_{\mathbb{R}}/W_{\mathbb{R}} \cap W_{\mathbb{C}}^{0,0} \to W_{\mathbb{C}}/F^0 W_{\mathbb{C}}$$

which is injective. Since both vector spaces have the same dimension over $\mathbb{R}$, it is also surjective. It follows that $W_{\mathbb{R}}/W_{\mathbb{R}} \cap W_{\mathbb{C}}^{0,0}$ admits a canonical complex structure.

Since the above isomorphism is functorial on the category of pure Hodge structures of weight 0, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{R}}/\mathfrak{g}_{\mathbb{C}}^{0,0} \cap \mathfrak{g}_{\mathbb{R}} & \longrightarrow & \mathrm{End}(V_{\mathbb{R}})/\mathrm{End}(V_{\mathbb{R}}) \cap \mathrm{End}(V_{\mathbb{C}})^{0,0} \\ \downarrow & & \downarrow \\ \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} & \longrightarrow & \mathrm{End}(V_{\mathbb{C}})/F^0\mathrm{End}(V_{\mathbb{C}}) \end{array}$$

which proves that the image of $T_h X = \mathfrak{g}_{\mathbb{R}}/\mathfrak{g}_{\mathbb{C}}^{0,0}$ in $T_{\omega(h)}\mathrm{Gr}(V_{\mathbb{C}}) = \mathrm{End}(V_{\mathbb{C}})/F^0\mathrm{End}(V_{\mathbb{C}})$ is a complex vector subspace.

The Griffiths transversality of $V \otimes \mathcal{O}_X$ follows from the same diagram. There is a commutative triangle of vector bundles

$$\begin{array}{ccc} TX & \longrightarrow & \mathrm{End}(V \otimes \mathcal{O}_X) \\ & \searrow & \downarrow \\ & & \mathrm{End}(V \otimes \mathcal{O}_X)/F^0\mathrm{End}(V \otimes \mathcal{O}_X) \end{array}$$

where the horizontal arrow is the derivation in $V \otimes \mathcal{O}_X$. The Griffiths transversality of $V \otimes \mathcal{O}_X$ is satisfied if and only if the image of the derivation is contained in $F^{-1}\mathrm{End}(V \otimes \mathcal{O}_X)$. But this follows from the fact that

$$\mathfrak{g}_{\mathbb{C}} = F^{-1}\mathfrak{g}_{\mathbb{C}}$$

and the map $TX \to \mathrm{End}(V \otimes \mathcal{O}_X)/F^0\mathrm{End}(V \otimes \mathcal{O}_X)$ factors through $(\mathfrak{g}_C \otimes \mathcal{O}_X)/F^0(\mathfrak{g}_C \otimes \mathcal{O}_X)$. $\square$

4.4. **The Dynkin classification.** Let (G, X) be a Shimura-Deligne datum. Over $\mathbb{C}$, we have a conjugacy class of cocharacter

$$\mu^{ad} : \mathbb{G}_{m,\mathbb{C}} \to G^{ad}_{\mathbb{C}}.$$

The complex adjoint semi-simple group G^{ad} is isomorphic to a product of complex adjoint simple groups $G^{ad} = \prod_i G_i$. The simple complex adjoint groups are classified by their Dynkin diagrams. The axiom $SD1$ implies that μ^{ad} induces an action of $\mathbb{G}_{m,\mathbb{C}}$ on $\mathfrak{g}_i$ of which the set of weights is $\{-1, 0, 1\}$. Such cocharacters are called *minuscules.* Minuscule coweights are some of the fundamental coweights and therefore can be specified by special nodes in the Dynkin diagram. Every Dynkin diagram has at least one special node except the three diagrams F4, G2, E8. We can classify Shimura-Deligne data over the complex numbers with the help of Dynkin diagrams.

4.5. **Semi-simple Shimura-Deligne data.**

Definition 4.5.1. *A semi-simple Shimura-Deligne datum is a pair* (G, X^+) *consisting of a semi-simple algebraic group* G *over* $\mathbb{Q}$ *and a* $G(\mathbb{R})^+$*-conjugacy class of homomorphisms* $h^1 : \mathbb{S}^1 \to G_{\mathbb{R}}$ *satisfying the axioms* (SD1) *and* (SD2). *Here* $G(\mathbb{R})^+$ *denotes the neutral component of* $G(\mathbb{R})$ *for the real topology.*

Let (G, X) be a reductive Shimura-Deligne datum. Let G^{ad} be the adjoint group of G. Every $h \in X$ induces a homomorphism $h^1 : \mathbb{S}^1 \to G^{\mathrm{ad}}$. The $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of h^1 is isomorphic to the connected component of h in X.

The spaces X^+ are exactly the so-called Hermitian symmetric domains with the symmetry group $G(\mathbb{R})^+$.

Theorem 4.5.2 (Baily-Borel)**.** *Let* Γ *be a torsion free arithmetic subgroup of* $G(\mathbb{R})^+$. *The quotient* $\Gamma\backslash X^+$ *has a canonical realization as a Zariski open subset of a complex projective algebraic variety. In particular, it has a canonical structure of a complex algebraic variety.*

These quotients $\Gamma\backslash X^+$, considered as complex algebraic varieties, are called *connected Shimura varieties.* The terminology is a little bit confusing, because they are not those Shimura varieties which are connected but connected components of Shimura varieties.

4.6. **Shimura varieties.** Let (G, X) be a Shimura-Deligne datum. For a compact open subgroup K of $G(\mathbb{A}_f)$, consider the double coset space

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)/K]$$

in which $G(\mathbb{Q})$ acts on X and $G(\mathbb{A}_f)$ on the left and K acts on $G(\mathbb{A}_f)$ on the right.

Lemma 4.6.1. *Let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$, where $G(\mathbb{R})_+$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{R})$ under the morphism $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$. Let X_+ be a connected component of X. Then there is a homeomorphism*

$$G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)/K] = \bigsqcup_{\xi \in \Xi} \Gamma_\xi \backslash X_+ \,,$$

where ξ runs over a finite set Ξ of representatives of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ and $\Gamma_\xi = \xi K \xi^{-1} \cap G(\mathbb{Q})$.

Proof. The map

$$\bigsqcup_{\xi \in \Xi} \Gamma_\xi \backslash X_+ \to G(\mathbb{Q})_+\backslash[X_+ \times G(\mathbb{A}_f)/K]$$

sending the class of $x \in X_+$ to the class of $(x, \xi) \in X_+ \times G(\mathbb{A}_f)$ is bijective by the very definition of the finite set Ξ and of the discrete groups Γ_ξ.

It follows from the theorem of real approximation that the map

$$G(\mathbb{Q})_+\backslash[X_+ \times G(\mathbb{A}_f)/K] \to G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)/K]$$

is a bijection.

Lemma 4.6.2 (Real approximation)**.** *For any connected group G over $\mathbb{Q}$, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$.*

See [17, p.415]. □

Remarks.

(1) The group $G(\mathbb{A}_f)$ acts on the inverse limit

$$\varprojlim_K G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f]/K \,.$$

On Shimura varieties of finite level, there is an action of Hecke algebras by correspondences.

(2) In order to be arithmeticcally significant, Shimura varieties must have models over a number field. According to the theory of canonical model, there exists a number field called the reflex field E depending only on the Shimura-Deligne datum over which the Shimura variety has a model which can be characterized by certain properties.

(3) The connected components of Shimura varieties have canonical models over abelian extensions of the reflex field E; these extensions depend not only on the Shimura-Deligne datum but also on the level structure.

(4) Strictly speaking, the moduli of abelian varieties with PEL structures is not a Shimura variety but a disjoint union of Shimura varieties. The union is taken over the set $\ker^1(\mathbb{Q}, G)$. For each class $\xi \in \ker^1(\mathbb{Q}, G)$, we have a $\mathbb{Q}$-group $G^{(\xi)}$ which is isomorphic to G over $\mathbb{Q}_p$ and over $\mathbb{R}$ but which might not be isomorphic to G over $\mathbb{Q}$.

(5) The Langlands correspondence has been proved in many particular cases by studing the commuting action of Hecke operators and of Galois groups of the reflex field on the cohomology of Shimura varieties.

5. CM tori and canonical models

5.1. **The PEL moduli attached to a CM torus.** Let F be a totally imaginary quadratic extension of a totally real number field F_0 of degree f_0 over $\mathbb{Q}$. We have $[F:\mathbb{Q}] = 2f_0$. Such a field F is called a CM field. Let τ_F denote the non-trivial element of $\mathrm{Gal}(F/F_0)$. This involution acts on the set $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ of cardinal $2f_0$.

Definition 5.1.1. *A CM-type of F is a subset $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ of cardinal f_0 such that*

$$\Phi \cap \tau(\Phi) = \emptyset \text{ and } \Phi \cup \tau(\Phi) = \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}).$$

A CM type is a pair (F, Φ) constisting of a CM field F and a CM type Φ of F.

Let (F, Φ) be a CM type. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$. Let E be the fixed field of the open subgroup

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) | \sigma(\Phi) = \Phi\}.$$

For every $b \in F$,

$$\sum_{\phi \in \Phi} \phi(b) \in E$$

and conversely E can be characterized as the subfield of $\overline{\mathbb{Q}}$ generated by the sums $\sum_{\phi\in\Phi} \phi(b)$ for $b \in F$.

For any number field K, we denote by $\mathcal{O}_K$ the maximal order of K. Let Δ be the finite set of primes where $\mathcal{O}_F$ is ramified over $\mathbb{Z}$. By construction, the scheme $Z_F = \mathrm{Spec}(\mathcal{O}_F[\ell^{-1}]_{\ell\in\Delta})$ is finite étale over $\mathrm{Spec}(\mathbb{Z}) - \Delta$. By construction the reflex field E is also unramified away from Δ and let $Z_E = \mathrm{Spec}(\mathcal{O}_E[\ell^{-1}]_{\ell\in\Delta})$. Then we have a canonical isomorphism

$$Z_F \times Z_E = (Z_{F_0} \times Z_E)_\Phi \sqcup (Z_{F_0} \times Z_E)_{\tau(\Phi)},$$

where $(Z_{F_0} \times Z_E)_\Phi$ and $(Z_{F_0} \times Z_E)_{\tau(\Phi)}$ are two copies of $(Z_{F_0} \times Z_E)$ with $Z_{F_0} = \mathrm{Spec}(\mathcal{O}_{F_0}[p^{-1}]_{p\in\Delta})$.

To complete the *PE-structure*, we take U to be the $\mathbb{Q}$-vector space F. The Hermitian form on U is given by

$$(b_1, b_2) = \mathrm{tr}_{F/\mathbb{Q}}(cb_1\tau(b_2))$$

for some element $c \in F$ such that $\tau(c) = -c$. The reductive group G associated to this PE-structure is a $\mathbb{Q}$-torus T equipped with a morphism $h : \mathbb{S} \to T$ which can be made explicit as follows.

Let $\widetilde{T} = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. The CM-type Φ induces an isomorphism of $\mathbb{R}$-algebras and of tori

$$F \otimes_{\mathbb{Q}} \mathbb{C} = \prod_{\phi\in\Phi} \mathbb{C} \text{ and } \widetilde{T}(\mathbb{R}) = \prod_{\phi\in\Phi} \mathbb{C}^\times.$$

According to this identification, $\tilde{h} : \mathbb{S} \to \widetilde{T}_{\mathbb{R}}$ is the diagonal homomorphism

$$\mathbb{C}^\times \to \prod_{\phi\in\Phi} \mathbb{C}^\times.$$

The complex conjugation τ induces an involution τ on $\widetilde{T}$. The norm N_{F/F_0} given by $x \mapsto x\tau(x)$ induces a homomorphism $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{F_0/\mathbb{Q}}\mathbb{G}_m$.

The torus T is defined as the pullback of the diagonal subtorus $\mathbb{G}_m \subset \mathrm{Res}_{F_0/\mathbb{Q}}\mathbb{G}_m$. In particular

$$T(\mathbb{Q}) = \{x \in F^\times | x\tau(x) \in \mathbb{Q}^\times\}.$$

The morphism $\tilde{h} : \mathbb{S} \to \widetilde{T}_{\mathbb{R}}$ factors through T and defines a morphism $h : \mathbb{S} \to T$. As usual h defines a cocharacter

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$$

defined on points by

$$\mathbb{C}^\times \to \prod_{\phi\in\mathrm{Hom}(F,\overline{\mathbb{Q}})} \mathbb{C}^\times,$$

where the projection on the component $\phi \in \Phi$ is the identity and the projection on the component $\phi \in \tau(\Phi)$ is trivial. The reflex field E is the field of definition of μ.

Let $p \notin \Delta$ be an unramified prime of $\mathcal{O}_F$. Choose an open compact subgroup $K^p \subset T(\mathbb{A}_f^p)$ and take $K_p = T(\mathbb{Z}_p)$.

We consider the functor $\mathrm{Sh}(T, h_\Phi)$ which associates to a Z_E-scheme S the set of isomorphy classes of

$$(A, \lambda, \iota, \eta),$$

where

- A is an abelian scheme of relative dimension f_0 over S ;
- $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ is an action of F on A such that for every $b \in F$ and every geometric point s of S, we have
$$\mathrm{tr}(b, \mathrm{Lie}(A_s)) = \sum_{\phi\in\Phi} \phi(a);$$
- λ is a polarization of A whose Rosati involution induces on F the complex conjugation τ ;
- η is a level structure.

Proposition 5.1.2. *$\mathrm{Sh}(T, h_\Phi)$ is a finite étale scheme over Z_E.*

Proof. Since $\mathrm{Sh}(T, h_\Phi)$ is quasi-projective over Z_E, it suffices to check the valuative criterion for properness and the unique lifting property of étale morphism.

Let $S = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation ring with generic point $\mathrm{Spec}(K)$ and with closed point $\mathrm{Spec}(k)$. Pick a point $x_K \in \mathrm{Sh}(T, h_\Phi)(K)$ with

$$x_K = (A_K, \iota_K, \lambda_K, \eta_K).$$

The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on the $F \otimes \overline{\mathbb{Q}}_\ell$-module $\mathrm{H}^1(A \otimes_K \overline{K}, \overline{\mathbb{Q}}_\ell)$. It follows that $\mathrm{Gal}(\overline{K}/K)$ acts semisimply, and its inertia subgroup acts through a finite quotient. Using the Ogg-Shafarevich criterion for good reduction, we see that after replacing K by a finite extension K', and R by its normalization R' in K', A_K acquires a good reduction, i.e. there exists an abelian scheme over R' whose generic fiber is $A_{K'}$. The endomorphisms extend by Weil's extension theorem. The polarization needs a little more care. The symmetric homomorphism $\lambda_K : A_K \to \hat{A}_K$ extends to a symmetric homomorphism $\lambda : A \to \hat{A}$. After finite étale base change of S, there exists an invertible sheaf L on A such that $\lambda = \lambda_L$. By assumption L_K is an ample invertible sheaf over A_K. λ is an isogeny, and L is non degenerate on the generic and on the special fibre. Mumford's vanishing theorem implies that $\mathrm{H}^0(X_K, L_K) \neq 0$. By the upper semi-continuity property, $\mathrm{H}^0(X_s, L_s) \neq 0$. But since L_s is non-degenerate, Mumford'vanishing theorem says that L_s is ample. This proves that $\mathrm{Sh}(T, h_\Phi)$ is proper.

Let $S = \mathrm{Spec}(R)$, where R is a local artinian $\mathcal{O}_E$-algebra with residue field $\overline{k}$, and let $\overline{S} = \mathrm{Spec}(\overline{R})$ with $\overline{R} = R/I$, $I^2 = 0$. Let $s = \mathrm{Spec}(\overline{k})$ be the closed point of S and $\overline{S}$. Let $\overline{x} \in \mathrm{Sh}(T, h_\Phi)(\overline{S})$ with $\overline{x} = (\overline{A}, \overline{\iota}, \overline{\lambda}, \overline{\eta})$. We have the exact sequence

$$0 \to \omega_{\overline{A}} \to \mathrm{H}^1_{dR}(\overline{A}) \to \mathrm{Lie}(\widehat{\overline{A}}) \to 0$$

with a compatible action of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_E$. As $\mathcal{O}_{Z_F \times Z_E}$-module, ω_{A_s} is supported by $(Z_{F_0} \times Z_E)_\Phi)$ and $\mathrm{Lie}(\widehat{\overline{A}})$ is supported by $(Z_{F_0} \times Z_E)_{\tau(\Phi)}$ so that the above exact sequence splits. This extends to a canonical splitting of the cristalline cohomology $\mathrm{H}^1_{\mathrm{cris}}(\overline{A}/\overline{S})_S$. According to Grothendieck-Messing, this splitting induces a lift of the abelian scheme $\overline{A}/\overline{S}$ to an abelian scheme A/S. We are also able to lift the additional structures $\overline{\lambda}, \overline{\iota}, \overline{\eta}$ by the functoriality of Grothendieck-Messing's theory. □

5.2. **Description of its special fibre.** We will keep the notations of the previous paragraph. Pick a place v of the reflex field E which does not lie over the finite set Δ of primes where $\mathcal{O}_F$ is ramified. $\mathcal{O}_E$ is unramified ovec $\mathbb{Z}$ at the place v. We want to describe the set $\mathrm{Sh}_K(T, h_\Phi)(\overline{\mathbb{F}}_p)$ equipped with the operator of Frobenius Frob_v.

Theorem 5.2.1. *There is a natural bijection*

$$\mathrm{Sh}_K(T, h_\Phi)(\overline{\mathbb{F}}_p) = \bigsqcup_\alpha T(\mathbb{Q})\backslash Y^p \times Y_p \,,$$

where

(1) *α runs over the set of (compatible with the action of $\mathcal{O}_F$)-isogeny classes ;*
(2) *$Y^p = T(\mathbb{A}_f^p)/K^p$;*
(3) *$Y_p = T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$;*
(4) *for every $\lambda \in T(\mathbb{A}_f^p)$ we have $\lambda(x^p, x_p) = (\lambda x^p, x_p)$;*
(5) *the Frobenius Frob_v acts by the formula*

$$(x^p, x_p) \mapsto (x^p, \mathrm{N}_{E_v/\mathbb{Q}_p}(\mu(p^{-1}))x_p)\,.$$

Proof. Let $x_0 = (A_0, \lambda_0, \iota_0, \eta_0) \in \mathrm{Sh}_K(T, h_\Phi)(\overline{\mathbb{F}}_p)$. Let X be the set of pairs (x, ρ), where $x = (A, \lambda, \iota, \eta) \in \mathrm{Sh}_K(T, h_\Phi)(\overline{\mathbb{F}}_p)$ and

$$\rho : A_0 \to A$$

is a quasi-isogeny which is compatible with the action of $\mathcal{O}_F$ and transforms λ_0 into a rational multiple of λ.

We will need to prove the following two assertions:

(1) $X = Y^p \times Y_p$ with the prescribed action of Hecke operators and of Frobenius ;
(2) the group of quasi-isogenies of A_0 compatible with ι_0 and transforming λ_0 into a rational multiple is $T(\mathbb{Q})$.

Quasi-isogenies of degrees relatively prime to p. Let Y^p be the subset of X where the degree of the quasi-isogeny is relatively prime to p. Consider the prime description of the moduli problem. A point $(A, \lambda, \iota, \tilde{\eta})$ consists of

- an abelian variety A up to isogeny,
- a rational multiple λ of a polarization,
- the multiplication ι by $\mathcal{O}_F$ on A, and
- a class $\tilde{\eta}_\ell$ modulo the action of an open compact subgroup K_ℓ of isomorphisms from $\mathrm{H}_1(A, \mathbb{Q}_\ell)$ to U_ℓ which are compatible with ι and transform λ into a rational multiple of the symplectic form on U_ℓ.

By this description, an isogeny of degree prime to p compatible with ι and preserving the $\mathbb{Q}$-line of the polarization, is given by an element $g \in T(\mathbb{A}_f^p)$. The isogeny corresponding to g defines an isomorphism in the category $\mathcal{B}'$ if and only if $g\tilde{\eta} = \tilde{\eta}'$. Thus

$$Y^p = T(\mathbb{A}_f^p)/K^p$$

with the obvious action of Hecke operators and trivial action of Frob_v.

Quasi-isogenies whose degree is a power of p. Let Y_p the subset of X where the degree of the quasi-isogeny is a power of p. We will use the covariant Dieudonné theory to describe the set Y_p equipped with action of Frobenius operator.

Let $W(\overline{\mathbb{F}}_p)$ be the ring of Witt vectors with coefficients in $\overline{\mathbb{F}}_p$. Let L be the field of fractions of $W(\overline{\mathbb{F}}_p)$; we will write $\mathcal{O}_L$ instead of $W(\overline{\mathbb{F}}_p)$. The Frobenius automorphism $\sigma : x \mapsto x^p$ of $\overline{\mathbb{F}}_p$ induces by functoriality an automorphism σ on the Witt vectors. For every abelian variety A over $\overline{\mathbb{F}}_p$, $\mathrm{H}^1_{cris}(A/\mathcal{O}_L)$ is a free $\mathcal{O}_L$-module of rank $2n$ equipped with an operator φ which is σ-linear. Let $D(A) = \mathrm{H}_1^{cris}(A/\mathcal{O}_L)$ be the dual $\mathcal{O}_L$-module of $\mathrm{H}^1_{cris}(A/\mathcal{O}_L)$, where φ acts in σ^{-1}-linearly. Furthermore, there is a canonical isomorphism

$$\mathrm{Lie}(A) = D(A)/\varphi D(A).$$

A quasi-isogeny $\rho : A_0 \to A$ induces an isomorphism $D(A_0) \otimes_{\overline{\mathcal{O}}_L} L \simeq D(A) \otimes_{\overline{\mathcal{O}}_L} L$ compatible with the multiplication by $\mathcal{O}_F$ and preserving the $\mathbb{Q}$-line of the polarizations. The following proposition is an immediate consequence of the Dieudonné theory.

Proposition 5.2.2. *Let $H = D(A_0) \otimes_{\overline{\mathcal{O}}_L} L$. The above construction defines a bijection between Y_p and the set of lattices $D \subset H$ such that*

(1) *$pD \subset \varphi D \subset D$,*
(2) *D is stable under the action of $\mathcal{O}_B$ and*

$$\mathrm{tr}(b, D/\varphi D) = \sum_{\phi \in \Phi} \phi(b)$$

for every $b \in \mathcal{O}_B$,

(3) *D is autodual up to a scalar in $\mathbb{Q}_p^\times$.*

Moreover, the Frobenius operator on Y_p that transforms the quasi-isogeny $\rho : A_0 \to A$ into the quasi-isogeny $\varphi \circ \rho : A_0 \to A \to \sigma^* A$ acts on the above set of lattices by sending D to $\varphi^{-1} D$.

Since $\mathrm{Sh}(T, h_\Phi)$ is étale, there exists a unique lifting

$$\tilde{x} \in \mathrm{Sh}(T, h_\Phi)(\mathcal{O}_L)$$

of $x_0 = (A_0, \lambda_0, \iota_0, \eta_0) \in \mathrm{Sh}(T, h_\Phi)(\overline{\mathbb{F}}_p)$. By assumption,

$$D(A_0) = \mathrm{H}_1^{dR}(\tilde{A})$$

is a free $\mathcal{O}_F \otimes \mathcal{O}_L$-module of rank 1 equipped with a pairing given by an element $c \in (\mathcal{O}_{F_p}^\times)^{\tau = -1}$. The σ^{-1}-linear operator φ on $H = D(A_0) \otimes_{\overline{\mathcal{O}}_L} L$ is of the form

$$\varphi = t(1 \otimes \sigma^{-1})$$

for an element $t \in T(L)$.

Lemma 5.2.3. *The element t lies in the coset $\mu(p)T(\mathcal{O}_L)$.*

Proof. H is a free $\mathcal{O}_F \otimes L$-module of rank 1 and

$$\mathcal{O}_F \otimes L = \prod_{\psi \in \mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p)} L$$

is a product of $2f_0$ copies of L. By ignoring the autoduality condition, t can be represented by an element

$$t = (t_\psi) \in \prod_{\psi \in \mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p)} L^\times$$

It follows from the assumption

$$pD_0 \subset \varphi D_0 \subset D_0$$

that for all $\psi \in \mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p)$ we have

$$0 \leq \mathrm{val}_p(t_\psi) \leq 1.$$

Remember the decomposition

$$Z_F \times Z_E = (Z_{F_0} \times Z_E)_\Phi \sqcup (Z_{F_0} \times Z_E)_{\tau(\Phi)} ,$$

induced by the CM type Φ. So, the embedding of the residue field of v in $\overline{\mathbb{F}}_p$ induces a decomposition $\mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p) = \Psi \sqcup \tau(\Psi)$ such that

$$\mathrm{tr}(b, D_0/\varphi D_0) = \sum_{\psi \in \Psi} \psi(b)$$

for all $b \in \mathcal{O}_F$. It follows that

$$\mathrm{val}_p(t_\psi) = \begin{cases} 0 & \text{if} \quad \psi \notin \Psi \\ 1 & \text{if} \quad \psi \in \Psi \end{cases}$$

By the definition of μ, it follows that $t \in \mu(p)T(\mathcal{O}_L)$. □

Description of Y_p *continued.* A lattice D stable under the action of $\mathcal{O}_F$ and autodual up to a scalar can be uniquely written in the form

$$D = mD_0$$

for $m \in T(L)/T(\mathcal{O}_L)$. The condition $pD \subset \varphi D \subset D$ and the trace condition on the tangent space are equivalent to $m^{-1}t\sigma(m) \in \mu(p)T(\mathcal{O}_L)$ and thus m lies in the group of σ-fixed points in $T(L)/T(\mathcal{O})_L$, that is,

$$m \in [T(L)/T(\mathcal{O}_L)]^{\langle\sigma\rangle} .$$

Now there is a bijection between the cosets $m \in T(L)/T(\mathcal{O}_L)$ fixed by σ and the cosets $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ by considering the exact sequence

$$1 \to T(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \to [T(L)/T(\mathcal{O}_L]^{\langle\sigma\rangle} \to \mathrm{H}^1(\langle\sigma\rangle, T(\mathcal{O}_L)) ,$$

where the last cohomology group vanishes by Lang's theorem. It follows that

$$Y_p = T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$$

and φ acts on it as $\mu(p)$.

On H, $\mathrm{Frob}_v(1 \otimes \sigma^r)$ acts as φ^{-r} so that

$$\begin{aligned} \mathrm{Frob}_v(1 \otimes \sigma^r) &= (\mu(p)(1 \otimes \sigma^{-1}))^{-r} \\ &= \mu(p^{-1})\sigma(\mu(p^{-1}))\ldots\sigma^{r-1}(\mu(p^{-1}))(1 \otimes \sigma^r). \end{aligned}$$

Thus the Frobenius Frob_v acts on $Y^p \times Y_p$ by the formula

$$(x^p, x_p) \mapsto (x^p, \mathrm{N}_{E_v/\mathbb{Q}_p}(\mu(p^{-1}))x_p).$$

Auto-isogenies. For every prime $\ell \neq p$, $\mathrm{H}^1(A_0, \mathbb{Q}_\ell)$ is a free $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module of rank one. So

$$\mathrm{End}_{\mathbb{Q}}(A_0, \iota_0) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

It follows that $\mathrm{End}_{\mathbb{Q}}(A_0, \iota_0) = F$. The auto-isogenies of A_0 form the group $F^\times$ and those who transport the polarization λ_0 to rational multiples of λ_0 form by definition the subgroup $T(\mathbb{Q}) \subset F^\times$. □

5.3. **Shimura-Taniyama formula.** Let (F, Φ) be a CM-type. Let $\mathcal{O}_F$ be an order of F which is maximal almost everywhere. Let p be a prime where $\mathcal{O}_F$ is unramified. We can either consider the moduli space of polarized abelian schemes edowed with a CM-multiplication of CM-type Φ as in the previous paragraphs or consider the moduli space of abelian schemes endowed wit a with CM-multiplication of CM-type Φ. Everything works in the same way for properness, étaleness, and the description of points, but we lose the obvious projective morphism to the Siegel moduli space. But since we know a posteriori that there is only a finite number of points, this loss is not serious.

Let (A, ι) be an abelian scheme over a number field K which is unramified at p equipped with a sufficiently large level structure. K must contains the reflex field E but might be bigger. Let $\mathfrak{q}$ be a place of K over p, and $\mathcal{O}_{K,\mathfrak{q}}$ be the localization of $\mathcal{O}_K$ at $\mathfrak{q}$, and let q be the cardinal of the residue field of $\mathfrak{q}$. By étaleness of the moduli space, A can be extended to an abelian scheme over $\mathrm{Spec}(\mathcal{O}_{K,\mathfrak{q}})$ equipped with a multiplication ι_v by $\mathcal{O}_F$. As we have already seen, the CM type Φ and the choice of the place $\mathfrak{q}$ of K define a decomposition $\mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p) = \Psi \sqcup \tau(\Psi)$ such that

$$\mathrm{tr}(b, \mathrm{Lie}\,(A_v)) = \sum_{\psi \in \Psi} \psi(b)$$

for all $b \in \mathcal{O}_F$.

Let $\pi_{\mathfrak{q}}$ be the relative Frobenius of A_v. Since $\mathrm{End}_{\mathbb{Q}}(A_v, \iota_v) = F$, $\pi_{\mathfrak{q}}$ defines an element of F.

Theorem 5.3.1 (Shimura-Taniyama formula)**.** *For all prime v of F, we have*

$$\frac{\mathrm{val}_v(\pi_{\mathfrak{q}})}{\mathrm{val}_v(q)} = \frac{|\Psi \cap H_v|}{|H_v|},$$

where $H_v \subset \mathrm{Hom}(\mathcal{O}_F, \overline{\mathbb{F}}_p)$ is the subset formed by the morphisms $\mathcal{O}_f \to \overline{\mathbb{F}}_p$ which factor through v.

Proof. As in the description of the Frobenius operator in Y_p, we have

$$\pi_{\mathfrak{q}} = \varphi^{-r},$$

where $q = p^r$. It is an elementary exercice to relate the Shimura-Taniyama formula to the group theoretical description of φ. □

5.4. **Shimura varieties of tori.** Let T be a torus defined over $\mathbb{Q}$ and $h : \mathbb{S} \to T_{\mathbb{R}}$ a homomorphism. Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}\mathbb{C}$ be the associated cocharater. Let E be the number field of definition of μ. Choose an open compact subgroup $K \subset T(\mathbb{A}_f)$. The Shimura variety attached to these data is

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$$

since the conjugacy class X of h has just one element. This finite set is the set of $\mathbb{C}$-point of a finite étale scheme over $\mathrm{Spec}(E)$. We need to define how the absolute Galois group $\mathrm{Gal}(E)$ acts on this set.

The Galois group $\mathrm{Gal}(E)$ will act through its maximal abelian quotient $\mathrm{Gal}^{ab}(E)$. For almost all prime v of E, we will define how the Frobenius π_v at v acts.

A prime p is said unramified if T can be extended to a torus T over $\mathbb{Z}_p$ and of $K_p = T(\mathbb{Z}_p)$. Let v be a place of E over an unramified prime p. Then p is an uniformizing element of $\mathcal{O}_{E,v}$. The cocharacter $\mu : \mathbb{G}_m \to T$ is defined over $\mathcal{O}_{E,v}$ so that $\mu(p^{-1})$ is well-defined element of $T(E_v)$. We require that the π_v acts on $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ as the element

$$\mathrm{N}_{E_v/\mathbb{Q}_p}(\mu(p^{-1})) \in T(\mathbb{Q}_p).$$

By class field theory, this rule defines an action of $\mathrm{Gal}^{ab}(E)$ on the finite set $T(F)\backslash T(\mathbb{A}_f)/K$.

5.5. **Canonical models.** Let (G, h) be a Shimura-Deligne datum. Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the attached cocharater. Let E be the field of definition of the conjugacy class of μ and which is called the *reflex field* of (G, h).

Let (G_1, h_1) and (G_2, h_2) be two Shimura-Deligne data and let $\rho : G_1 \to G_2$ be an injective homomorphism of reductive $\mathbb{Q}$-group which sends the conjugacy class h_1 to the conjugacy class h_2. Let E_1 and E_2 be the reflex fields of (G_1, h_1) and (G_2, h_2). Since the conjugacy class of $\mu_2 = \rho \circ \mu_1$ is defined over E_1, we have the inclusion $E_2 \subset E_1$.

Definition 5.5.1. *A canonical model of* $\mathrm{Sh}(G, h)$ *is an algebraic variety defined over E such that for all Shimura-Deligne datum (G_1, h_1), where G_1 is a torus, and any injective homomorphism $(G_1, h_1) \to (G, h)$, the morphism*

$$\mathrm{Sh}(G_1, h_1) \to \mathrm{Sh}(G, h)$$

is defined over E_1, where E_1 is the reflex field of (G_1, h_1) and the E_1-structure of $\mathrm{Sh}(G_1, h_1)$ *was defined in the last paragraph.*

Theorem 5.5.2 (Deligne)**.** *There exists at most one canonical model up to unique isomorphism.*

Theorem 5.2.1 proves more or less that the moduli space gives rise to a canonical model for symplectic group. It follows that PEL moduli space also gives rise to canonical model. The same for Shimura varieties of Hodge type and abelian type. Some other crucial cases were obtained by Shih afterward. The general case, the existence of the canonical model is proved by Borovoi and Milne.

Theorem 5.5.3 (Borovoi, Milne)**.** *The canonical model exists.*

5.6. **Integral models.** An integral model comes naturally with every PEL moduli problem. More generally, in the case of Shimura varieties of Hodge type, Vasiu proves the existence of a "canonical" integral model. In this case, the integral model is nothing but the closure of the canonical model (over the reflex field) in the Siegel moduli space (base-changed to the maximal order of the reflex field). Vasiu proved that this closure has good properties in particular the smoothness. A good place to begin with integral models is the article [12] by B. Moonen.

6. Points of Siegel varieties over a finite field

6.1. **Abelian varieties over a finite field up to isogeny.** Let $k = \mathbb{F}_q$ be a finite field of characteristic p with $q = p^s$ elements. Let A be a simple abelian variety defined over k and $\pi_A \in \mathrm{End}_k(A)$ its geometric Frobenius.

Theorem 6.1.1 (Weil)**.** *The subalgebra $\mathbb{Q}(\pi_A) \subset \mathrm{End}_k(A)_{\mathbb{Q}}$ is a finite extension of $\mathbb{Q}$ such that for every inclusion $\phi : \mathbb{Q}(\pi_A) \hookrightarrow \mathbb{C}$, we have $|\phi(\pi_A)| = q^{1/2}$.*

Proof. Choose a polarization and let τ be the associated Rosati involution. We have

$$(\pi_A x, \pi_A y) = q(x, y)$$

so that $\tau(\pi_A)\pi_A = q$. For every complex embedding $\phi : \mathrm{End}(A) \to \mathbb{C}$, τ corresponds to the complex conjugation. It follows that $|\phi(\tau_A)| = q^{1/2}$. □

Definition 6.1.2. *An algebraic number satisfying the conclusion of the above theorem is called a Weil q-number.*

Theorem 6.1.3 (Tate)**.** *The homomorphism*

$$\mathrm{End}_k(A) \to \mathrm{End}_{\pi_A}(V_\ell(A))$$

is an isomorphism.

The fact that there is a finite number of abelian varieties over a finite field with a given polarization type plays a crucial role in the proof of this theorem.

Theorem 6.1.4 (Honda-Tate)**.**
(1) *The category $M(k)$ of abelian varieties over k with $\mathrm{Hom}_{M(k)}(A, B) = \mathrm{Hom}(A, B) \otimes \mathbb{Q}$ is a semi-simple category.*

(2) *The application $A \mapsto \pi_A$ defines a bijection between the set of isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and the set of Galois conjugacy classes of Weil q-numbers.*

Corollary 6.1.5. *Let A, B be abelian varieties over $\mathbb{F}_q$ of dimension n. They are isogenous if and only if the characteristic polynomials of π_A on $\mathrm{H}_1(\overline{A}, \mathbb{Q}_\ell)$ and π_B on $\mathrm{H}_1(\overline{B}, \mathbb{Q}_\ell)$ are the same.*

6.2. **Conjugacy classes in reductive groups.** Let k be a field and G be a reductive group over k. Let T be a maximal torus of G. The finite group $W = N(T)/T$ acts on T. Let

$$T/W := \mathrm{Spec}([k[T]^W])$$

where $k[T]$ is the ring of regular functions on T, i.e. $T = \mathrm{Spec}(k[T])$ and $k[T]^W$ is the ring of W-invariants regular functions on T. The following theorem is from [18].

Theorem 6.2.1 (Steinberg)**.** *There exists a G-invariant morphism*

$$\chi : G \to T/W\,,$$

which induces a bijection between the set of semi-simple conjugacy classes of $G(k)$ and $(T/W)(k)$ if k is an algebraically closed field.

If $G = \mathrm{GL}(n)$, the map

$$[\chi](k) : \{\text{ semisimple conjugacy class of } G(k)\} \to (T/W)(k)$$

is still a bijection for any field of characteristic zero. For arbitrary reductive group, this map is neither injective nor surjective.

For $a \in (T/W)(k)$, the obstruction to the existence of a (semi-simple) k-point in $\chi^{-1}(a)$ lies in some Galois cohomology group H^2. In some important cases this group vanishes.

Proposition 6.2.2 (Kottwitz)**.** *If G is a quasi-split group with G^{der} simply connected, then $[\chi](k)$ is surjective.*

For now, we will assume that G is quasi-split and G^{der} is simply connected. In this case, the elements $a \in (T/W)(k)$ are called *stable conjugacy classes*. For every stable conjugacy class $a \in (T/W)(k)$, there might exist several semi-simple conjugacy classes of $G(k)$ contained in $\chi^{-1}(a)$.

Examples. If $G = \mathrm{GL}(n)$, $(T_n/W_n)(k)$ is the set of monic polynomials of degree n

$$a = t^n + a_1 t^{n-1} + \cdots + a_0$$

with $a_0 \in k^\times$. If $G = \mathrm{GSp}(2n)$, $(T/W)(k)$ is the set of pairs (P, c), where P is a monic polynomial of degree $2n$ and $c \in k^\times$, satisfying

$$a(t) = c^{-n} t^{2n} a(c/t).$$

In particular, if $a = t^{2n} + a_1 t^{2n-1} + \cdots + a_{2n}$ then $a_{2n} = c^n$. The homomorphism $\mathrm{GSp}(2n) \to \mathrm{GL}(2n) \times \mathbb{G}_m$ induces a closed immersion

$$T/W \hookrightarrow (T_{2n}/W_{2n}) \times \mathbb{G}_m.$$

Semi-simple elements of $\mathrm{GSp}(2n)$ are stably conjugate if and only if they have the same characteristic polynomials and the same similitude factors.

Let $\gamma_0, \gamma \in G(k)$ be semisimple elements such that $\chi(\gamma_0) = \chi(\gamma) = a$. Since γ_0, γ are conjugate in $G(\overline{k})$, there exists $g \in G(\overline{k})$ such that $g\gamma_0 g^{-1} = \gamma$. It follows that for every $\varsigma \in \mathrm{Gal}(\overline{k}/k)$, $\varsigma(g)\gamma_0\varsigma(g)^{-1} = \gamma$ and thus

$$g^{-1}\varsigma(g) \in G_{\gamma_0}(\overline{k}).$$

The cocycle $\varsigma \mapsto g^{-1}\varsigma(g)$ defines a class

$$\mathrm{inv}(\gamma_0, \gamma) \in \mathrm{H}^1(k, G_{\gamma_0})$$

with trivial image in $\mathrm{H}^1(k, G)$. For $\gamma_0 \in \chi^{-1}(a)$ the set of semi-simple conjugacy classes stably conjugate to γ_0 is in bijection with

$$\ker(\mathrm{H}^1(k, G_{\gamma_0}) \to \mathrm{H}^1(k, G)).$$

It happens often that instead of an element $\gamma \in G(k)$ stably conjugate to γ_0, we have a G-torsor $\mathcal{E}$ over k with an automorphism γ such that $\chi(\gamma) = a$. We can attach to the pair $(\mathcal{E}, \gamma)$ a class in $\mathrm{H}^1(k, G_{\gamma_0})$ whose image in $\mathrm{H}^1(k, G)$ is the class of $\mathcal{E}$.

Consider the simplest case where γ_0 is semisimple and strongly regular. For $G = \mathrm{GSp}$, (g, c) is semisimple and strongly regular if and only if the characteristic polynomial of g is a separable polynomial. In this case, $T = G_{\gamma_0}$ is a maximal torus of G. Let $\hat{T}$ be the complex dual torus equipped with a finite action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$

Lemma 6.2.3 (Tate-Nakayama)**.** *If k is a non-archimedian local field, then $\mathrm{H}^1(k, T)$ is the group of characters $\hat{T}^\Gamma \to \mathbb{C}^\times$ which have finite order.*

6.3. **Kottwitz triples** $(\gamma_0, \gamma, \delta)$**.** Let $\mathcal{A}$ be the moduli space of abelian schemes of dimension n with polarizations of type D and principal N-level structure. Let $U = \mathbb{Z}^{2n}$ be equipped with an alternating form of type D

$$U \times U \to M_U\,,$$

where M_U is a rank one free $\mathbb{Z}$-module. Let $G = \mathrm{GSp}(2n)$ be the group of automorphisms of the symplectic module U.

Let $k = \mathbb{F}_q$ be a finite field with $q = p^r$ elements. Let $(A, \lambda, \tilde{\eta}) \in \mathcal{A}'(\mathbb{F}_q)$. Let $\overline{A} = A \otimes_{\mathbb{F}_q} \overline{k}$ and $\pi_A \in \mathrm{End}(\overline{A})$ its relative Frobenius endomorphism. Let a be the characteristic polynomial of π_A on $\mathrm{H}_1(\overline{A}, \mathbb{Q}_\ell)$. This polynomial has rational coefficients and satisfies

$$a(t) = q^{-n} t^{2n} a(q/t)$$

so that (a, q) determines a stable conjugacy class a of $\mathrm{GSp}(\mathbb{Q})$. Weil's theorem implies that this is an elliptic class in $G(\mathbb{R})$. Since GSp is quasi-split and its derived group Sp is simply connected, there exists $\gamma_0 \in G(\mathbb{Q})$ lying in the stable conjugacy class a.

The partition of $\mathcal{A}'(\mathbb{F}_q)$ with respect to the stable conjugacy classe a in $G(\mathbb{Q})$ is the same as the partition by the isogeny classes of the underlying abelian variety A (forgetting the polarization). This follows from the fact that stable conjugacy classes in GSp are intersection of conjugacy classes of GL with GSp. In the general PEL case, we need a more involved description.

Now, let us partition such an isogeny class into classes of isogenies *respecting the polarization up to a multiple* and let us pick such a class. As in section 5, we choose a base point A_0, λ_0 in this class and define the set Y whose elements are quadruples $(A, \lambda, \tilde{\eta}, f : A \to A_0)$, where $(A, \lambda, \tilde{\eta}) \in \mathcal{A}'(\mathbb{F}_q)$ and f is a quasi-isogeny transforming λ to a $\mathbb{Q}^\times$-multiple of λ_0. The isogeny class is recovered as $I(\mathbb{Q})\backslash Y$, where $I(\mathbb{Q})$ is the group of quasi-isogenies of A_0 to itself sending λ_0 to a $\mathbb{Q}^\times$-multiple of itself.

Again, as in section 5, we can write $Y = Y^p \times Y_p$, where Y^p is the subset of Y consisting of the quadruples for which the degree of the quasi-isogey f is prime to p (i.e., is an element of $\mathbb{Z}_{(p)}^\times$) and Y_p is the subset consisting of the quadruples for which the degree of f is a power of p. We will successively describe Y^p and Y^p.

Description of Y^p. For any prime $\ell \neq p$, $\rho_\ell(\pi_A)$ is an automorphism of the adelic Tate module $\mathrm{H}_1(\overline{A}, \mathbb{A}_f^p)$ preserving the symplectic form up to a similitude factor q

$$(\rho_\ell(\pi_A)x, \rho_\ell(\pi_A)y) = q(x, y).$$

The rational Tate module $\mathrm{H}_1(\overline{A}, \mathbb{A}_f^p)$ with the Weil pairing is similar to $U \otimes \mathbb{A}_f^p$ so that π_A defines a $G(\mathbb{A}_f^p)$-conjugacy class in $G(\mathbb{A}_f^p)$. We have

$$Y^p = \{\tilde{\eta} \in G(\mathbb{A}_f^p)/K^p | \tilde{\eta}^{-1}\gamma\tilde{\eta} \in K^p\}.$$

Note that for every prime $\ell \neq p$, γ_0 and γ_ℓ are stably conjugate. In the case where γ_0 is strongly regular semisimple, we have an invariant

$$\alpha_\ell : \hat{T}^{\Gamma_\ell} \to \mathbb{C}^\times$$

which is a character of finite order.

Description of Y_p. Recall that $\pi_A : \overline{A} \to \overline{A}$ is the composite of an isomorphism $u : \sigma^r(\overline{A}) \to \overline{A}$ and the r-th power of the Frobenius $\varphi^r : \overline{A} \to \sigma^r(\overline{A})$

$$\pi_A = u \circ \varphi^r.$$

On the covariant Dieudonné module $D = \mathrm{H}_1^{cris}(\overline{A}/\mathcal{O}_L)$, the operator φ acts σ^{-1}-linearly and u acts σ^r-linearly. We can extend these actions to $H = D \otimes_{\mathcal{O}_L} L$. Let $G(H)$ be the group of auto-similitudes of H and we form the semi-direct product $G(H) \rtimes \langle\sigma\rangle$. The elements u, φ and π_A can be seen as commuting elements of this semi-direct product.

Since $u : \sigma^r(A) \to A$ is an isomorphism, u fixes the lattice $u(D) = D$. This implies that

$$H_r = \{x \in H \mid u(x) = x\}$$

is an L_r-vector space of dimension $2n$ over the field of fractions L_r of $W(\mathbb{F}_{p^r})$ and equipped with a symplectic form. Autodual lattices in H fixed by u must come from autodual lattices in H_r.

Since $\varphi \circ u = u \circ \varphi$, φ stabilizes H_r and its restriction to H_r induces a σ^{-1}-linear operator of which the inverse will be denoted by $\delta \circ \sigma$ (with $\delta \in G(L)$). We have

$$Y_p = \{g \in G(L_r)/G(\mathcal{O}_{L_r}) \,|\, g^{-1}\delta\sigma(g) \in K_p\mu(p^{-1})K_p\}.$$

There exists an isomorphism H with $U \otimes L$ that transports π_A to γ_0 and carries φ to an element $b\sigma \in T(L) \rtimes \langle\sigma\rangle$. By the definition of δ, the element $N\delta = \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ is stably conjugate to γ_0). Following Kottwitz, the σ-conjugacy class of b in $T(L)$ determines a character

$$\alpha_p : \hat{T}^{\Gamma_p} \to \mathbb{C}^\times.$$

The set of σ-conjugacy classes in $G(L)$ for any reductive group G is described in [7].

Invariant at ∞. Over $\mathbb{R}$, T is an elliptic maximal torus. The conjugacy class of the cocharacter μ induces a well-defined character

$$\alpha_\infty : \hat{T}^{\Gamma_\infty} \to \mathbb{C}^\times.$$

Let us state Kottwitz theorem in a particular case which is more or less equivalent to theorem 5.2.1. The proof of the general case is much more involved.

Proposition 6.3.1. *Let $(\gamma_0, \gamma, \delta)$ be a triple with γ_0 semisimple strongly regular such that γ and $N\delta$ are stably conjugate to γ_0. Assume that the torus $T = G_{\gamma_0}$ is unramified at p. There exists a pair $(A, \lambda) \in \mathcal{A}(\mathbb{F}_q)$ for the triple $(\gamma_0, \gamma, \delta)$ if and only if*

$$\sum_v \alpha_v|_{\hat{T}^\Gamma} = 0.$$

In that case there are $\ker^1(\mathbb{Q}, T)$ isogeny classes of $(A, \lambda) \in \mathcal{A}(\mathbb{F}_q)$ which are mapped to the triple $(\gamma_0, \gamma, \delta)$.

Let γ_0 as in the statement and $a \in \mathbb{Q}[t]$ its characteristic polynomial which is a monic polynomial of degree $2n$ satisfying the equation

$$a(t) = q^{-n}t^{2n}a(q/t).$$

The algebra $F = \mathbb{Q}[t]/a$ is a product of CM-fields which are unramified at p. The moduli space of polarized abelian varieties with multiplication by $\mathcal{O}_F$ and with a given CM type is finite and étale at p. A point

$A \in \mathcal{A}(\mathbb{F}_q)$ mapped to $(\gamma_0, \gamma, \delta)$ belongs to one of these Shimura varieties of dimension 0 by letting t act as the Frobenius endomorphism Frob_q.

We can lift A to a point $\tilde{A}$ with coefficients in $W(\mathbb{F}_q)$ by the étaleness. By choosing a complex embedding of $W(\mathbb{F}_q)$, we obtain a symplectic $\mathbb{Q}$-vector space by taking the first Betti homology $\mathrm{H}_1(\tilde{A} \otimes_{W(\mathbb{F}_q)} \mathbb{C}, \mathbb{Q})$ which is equipped with a non-degenerate symplectic form and multiplication by $\mathcal{O}_F$. This defines a conjugacy class in $G(\mathbb{Q})$ within the stable conjugacy class defined by the polynomial a. For every prime $\ell \neq p$, the ℓ-adic homology $\mathrm{H}_1(A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)$ is a symplectic vector space equipped with an action of $t = \mathrm{Frob}_q$. This defines a conjugacy class γ_ℓ in $G(\mathbb{Q}_\ell)$. By the comparison theorem, we have a canonical isomorphism

$$\mathrm{H}_1(\tilde{A} \otimes_{W(\mathbb{F}_q)} \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \mathrm{H}_1(A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)$$

compatible with action of t so that the invariant $\alpha_\ell = 0$ for $\ell \neq p$.

The cancellation between α_p ans α_∞ is essentially the equality $\varphi = \mu(p)(1 \otimes \sigma^{-1})$ occuring in the proof of Theorem 5.2.1. □

Kottwitz stated and proved a more general statement for all γ_0 and for all PEL Shimura varieties of type (A) and (C). In particular, he derived a formula for the number of points on $\mathcal{A}$

$$\mathcal{A}(\mathbb{F}_q) = \sum_{(\gamma_0,\gamma,\delta)} n(\gamma_0, \gamma, \delta) T(\gamma_0, \gamma, \delta)\,,$$

where $n(\gamma_0, \gamma, \delta) = 0$ unless Kottwitz vanishing condition is satisfied. In that case

$$n(\gamma_0, \gamma, \delta) = \ker^1(\mathbb{Q}, I)$$

and

$$T(\gamma_0, \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q} \backslash I(\mathbb{A}_f)) O_\gamma(1_{K^p}) TO_\delta(1_{K_p \mu(p^{-1}) K_p})\,,$$

where I is an inner form of G_{γ_0}.

It is expected that this formula can be compared to Arthur-Selberg's trace formula, see [8].

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References

[1] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math 84(1966).

[2] S. Bosch, W. Lutkebohmert, M. Raynaud, *Neron models*, Ergeb. der Math. 21. Springer Verlag 1990.

[3] P. Deligne, *Travaux de Griffiths*, séminaire Bourbaki 376, in Springer Lecture Notes in Math. 180, Springer Verlag

[4] P. Deligne, *Travaux de Shimura*, séminaire Bourbaki 389, in Springer Lecture Notes in Math. 244, Springer Verlag

[5] P. Deligne, *Interprétation modulaire et techniques de construction de modèles canoniques*. Proc. Symp. Pure Math. 33 (2).

[6] R. Kottwitz, *Rational conjugacy classes in reductive groups*, Duke Math. J. **49** (1982), no. 4, 785-806.

[7] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. 56 (1985), no. 2, 201–220.

[8] R. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 161-209, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[9] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc., **5** (1992), 373-444.

[10] W. Messing, *The crystals associated to Barsotti-Tate groups : with applications to abelian schemes*, Springer Lecture Notes in Math. 264, Springer Verlag.

[11] J. Milne, *Introduction to Shimura varieties* in Harmonic analysis, the trace formula, and Shimura varieties, 265–378, Amer. Math. Soc. 2005.

[12] B. Moonen, *Models of Shimura varieties in mixed characteristics.* in Galois representations in arithmetic algebraic geometry, 267–350, Cambridge Univ. Press, 1998

[13] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Oxford University Press, London 1970

[14] D. Mumford, *Geometric invariant theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34 Springer-Verlag 1965.

[15] J-P Serre, *Rigidité du foncteur de Jacobi d'échelon $n \geq 3$*, appendice à l' exposé 17 du séminaire Cartan 1960-61.

[16] D. Mumford, *Families of abelian varieties*, in Algebraic Groups and Discontinuous Subgroups, PSPM 9, pp. 347–351 Amer. Math. Soc.

[17] Platonov, Rapinchuk *Algebraic groups and number theory,* Pure and Applied Mathematics, 139. Academic Press, Inc., Boston, MA, 1994

[18] R. Steinberg *Conjugacy classes in algebraic groups* Springer Lecture Notes in Math. 366 Springer Verlag 1974.