On the Black-Scholes Implied Volatility at Extreme Strikes

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Abstract

We survey recent results on the behavior of the Black-Scholes implied volatility at extreme strikes. There are simple and universal formulae that give quantitative links between tail behavior and moment explosions of the underlying on one hand, and growth of the famous volatility smile on the other hand. Some original results are included as well.

1 Introduction

Let S be a nonnegative $\mathbb{P}$-martingale, and let $S_0 > 0$. Think of S as a forward price and $\mathbb{P}$ as forward risk-neutral measure. Write $\mathbb{E}$ for expectation with respect to $\mathbb{P}$.

For a fixed maturity T, let $C(k) := \mathbb{E}(S_T - S_0 e^k)^+$ be the forward price of a call as a function of moneyness k, the log of the strike-to-S_0 ratio.

Let $c(k) := C(k)/S_0$ be the S_0-normalized forward call price.

With

$$d_{1,2}(k,\sigma) := -k/\sigma \pm \sigma/2,$$

let

$$c_{BS}(k,\sigma) := \Phi(d_1) - e^k \Phi(d_2),$$

be the S_0-normalized forward Black-Scholes formula as a function of moneyness k and unannualized volatility σ.

For each k define the unannualized implied volatility $V(k)$ uniquely by

$$c(k) = c_{BS}(k, V(k)).$$

Our project is to study the $k \to \infty$ behavior of $V(k)$ and $V(-k)$. Two examples of applications are the choice of a functional form for the extrapolation of an implied volatility skew into the tails, and the inference of parameters of underlying dynamics, given observations of tail slopes of the volatility skew.

Unless otherwise stated, each limit, lim sup, lim inf, and asymptotic relation is taken as $k \to \infty$. In particular, $g(k) \sim h(k)$ means that $g(k)/h(k) \to 1$ as $k \to \infty$.

2 The Moment Formula

The *moment formula* [30] explicitly relates the $k \to \infty$ behavior of $V(k)$ to the right-hand critical exponent

$$\tilde{p} := \sup\{p : \mathbb{E}S_T^{1+p} < \infty\},$$

via the strictly decreasing function $\psi : [0, \infty] \to [0, 2]$ defined by

$$\psi(x) := 2 - 4\big(\sqrt{x^2 + x} - x\big).$$

Theorem 1 (Right-hand moment formula).

$$\limsup \frac{V^2(k)}{k} = \psi(\tilde{p}).$$

Some consequences are as follows:

The implied volatility tail cannot grow *faster* than $\sqrt{k}$, by which we mean that for k large enough, $V(k) \leq \sqrt{\beta k}$. The moment formula makes precise how small the constant coefficient β in that bound can be chosen.

Moreover, unless S_T has finite moments of all orders, the implied volatility tail cannot grow *slower* than $\sqrt{k}$, by which we mean that $V(k)$ cannot be $o(\sqrt{k})$.

These conclusions are fully model-independent, requiring *no distributional assumptions* on S_T.

2.1 Intuition of proof

Define $f_1 := f_-$ and $f_2 := f_+$ where

$$f_{\pm}(y) := \left(\frac{1}{\sqrt{y}} \pm \frac{\sqrt{y}}{2}\right)^2 = \frac{1}{y} \pm 1 + \frac{y}{4}. \tag{2.1}$$

Note that ψ is the inverse of $\frac{1}{2}f_1$ and that

$$f_j(\sigma^2/k) = d_j^2(k, \sigma)/k, \qquad j = 1, 2. \tag{2.2}$$

Using the normal CDF asymptotics

$$\Phi(-z) \sim \frac{e^{-z^2/2}}{\sqrt{2\pi}z}, \qquad z \to \infty, \tag{2.3}$$

we have, for constant $\beta \geq 0$,

$$c_{BS}(k, \sqrt{\beta k}) = \Phi(-\sqrt{f_1(\beta)k}) - e^k\Phi(-\sqrt{f_2(\beta)k}) \tag{2.4}$$

$$\sim \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-f_1(\beta)k/2}}{\sqrt{f_1(\beta)k}} - \frac{e^k e^{-f_2(\beta)k/2}}{\sqrt{f_2(\beta)k}}\right) = \frac{e^{-f_1(\beta)k/2}}{B\sqrt{k}}, \tag{2.5}$$

where B depends only on β, not k.

On the other hand, $c(k) = O(e^{-kp})$ holds for all p with $\mathbb{E}S_T^{1+p} < \infty$. So

$$c(k) \approx e^{-k\tilde{p}}, \tag{2.6}$$

where we will not define "$\approx$", as our purpose here is just to give intuition.

The decay rates in (2.5) and (2.6) match only if $\tilde{p} = f_1(\beta)/2$ or equivalently,

$$\beta = \psi(\tilde{p}). \tag{2.7}$$

Because βk was the square of the volatility argument in (2.4), this makes plausible the moment formula

$$\limsup V^2(k)/k = \psi(\tilde{p}). \tag{2.8}$$

Of course, we have not *proved* the moment formula here; see Lee [30] for the rigorous proof.

2.2 Left-hand moment formula

The left-hand moment formula explicitly relates the $k \to \infty$ behavior of $V(-k)$ to the left-hand moment index

$$\tilde{q} := \sup\{q : \mathbb{E}S_T^{-q} < \infty\}.$$

Theorem 2 (Left-hand moment formula).

$$\limsup \frac{V^2(-k)}{k} = \psi(\tilde{q}).$$

This follows from measure-change of the right-hand moment formula, as shown in [30]. Start by writing $V(-k) \equiv V(-k; S, \mathbb{P})$ to emphasize the dependence on the underlier and the measure. Then verify the symmetries

$$V(-k; S, \mathbb{P}) = V(k; 1/S, \mathbb{Q}) \tag{2.9}$$

$$\mathbb{E}S_T^{-p} = S_0 \mathbb{E}^{\mathbb{Q}} \left[(1/S_T)^{1+p} \right] \tag{2.10}$$

where the "foreign" risk-neutral measure $\mathbb{Q}$ is defined by $d\mathbb{Q}/d\mathbb{P} = S_T/S_0$ (provided that $\mathbb{P}(S_T > 0) = 1$, else a separate case is needed). Now apply Theorem 1 to obtain

$$\limsup \frac{V^2(-k)}{k} = \limsup \frac{V^2(k; 1/S, \mathbb{Q})}{k} = \psi(\sup\{p : \mathbb{E}^{\mathbb{Q}}(1/S_T)^{1+p} < \infty\}) = \psi(\tilde{q})$$

as claimed.

2.3 Conjectures

It is natural to conjecture the following two extensions of the moment formula.

First, can we replace the $\limsup$ with a limit? In other words,

Conjecture 1.

$$V^2(k)/k \to \psi(\tilde{p}). \tag{2.11}$$

Second, consider the complementary cumulative distribution function (CCDF) $\bar{F}$ of the log return:

$$\bar{F}(k) := 1 - F(k) \tag{2.12}$$

$$F(k) := \mathbb{P}(\log(S_T/S_0) \leq k) \tag{2.13}$$

where $\log 0 := -\infty$.

In the special case that $\bar{F} \sim e^{-ak}$, one could hope to argue that

$$\tilde{p} = a - 1 \sim -\log \bar{F}(k)/k - 1 \tag{2.14}$$

implies that (2.11) can be rewritten with $-\log \bar{F}/k - 1$ in place of $\tilde{p}$. One could conjecture more generally:

Conjecture 2. *For arbitrary $\bar{F}$,*

$$V^2(k)/k \sim \psi(-\log \bar{F}(k)/k - 1) \tag{2.15}$$

We construct an example of an S_T distribution for which neither conjectural generalization holds. Actually, instead of directly specifying a distribution, we can specify the distribution's call prices, as function h of strike, provided that h satisfies condition (b) in:

Proposition 3. *Let $h : [0, \infty) \to [0, \infty)$. The following are equivalent:*

(a) There exists (on some probability space) a nonnegative integrable random variable S_T such that $(S_T - K)^+$ has expectation $h(K)$ for all K.

(b) The function $H(K) := h(K)\mathbb{I}_{K\geq 0} + (h(0) - K)\mathbb{I}_{K<0}$ is convex on $\mathbb{R}$, and $\lim_{K\to\infty} h(K) = 0$.

To show that $(b) \Rightarrow (a)$, let S_T have distribution H'', which exists as a measure; we omit the details.

Proceeding with our example, let $S_0 = 1$ and choose $\beta \in (\psi(2), \psi(1))$. We will construct h such that

$$\mathbb{E}S_T^2 = \infty, \qquad \text{hence } \limsup V^2(k)/k \geq \psi(1) > \beta, \tag{2.16}$$

but such that there exists $k_n \equiv \log K_n \to \infty$ with

$$h(K_n) = c_{BS}(k_n, \sqrt{\beta k_n}), \qquad \text{hence } \liminf V^2(k)/k \leq \beta, \tag{2.17}$$

and with

$$-h'(K_n+) \leq K_n^{-3}, \text{ so } \psi(-\log \bar{F}(k_n)/k_n - 1) \leq \psi(2) < \beta = \frac{V^2(k_n)}{k_n}. \tag{2.18}$$

Indeed, let $K_0 := 0$, and $h(K_0) := S_0$. Given K_n and $h(K_n)$, define

$$\begin{aligned} K_{n+1} &:= K_n + \max(1/h(K_n), K_n^3 h(K_n)) \\ h(K_{n+1}) &:= c_{BS}(k_{n+1}, \sqrt{\beta k_{n+1}}). \end{aligned}$$

For all $K \neq K_n$, define $h(K)$ by linear interpolation.

The (b) condition holds, so h induces a legitimate distribution. Conjecture 1 fails by (2.16) and (2.17). Conjecture 2 fails by (2.18).

Therefore, without additional assumptions, the moment formula *cannot be sharpened* in the sense of (2.11) or (2.15). The next section will impose the additional assumption of *regular variation* to obtain results of the form (2.11) or (2.15).

3 Regular Variation and the Tail-Wing Formula

The example of section 2.3 shows that if the distribution of S_T is allowed to concentrate its mass arbitrarily, then it disconnects the asymptotics of $\bar{F}$ from the asymptotics of c (and hence of V), and moreover it allows implied volatility to oscillate, separating $\limsup$ from $\liminf$.

So in order to extend the moment formula in the sense of (2.11) or (2.15), we need to impose some additional regularity assumption on $\bar{F}$. A natural condition is that of *regular variation.*

Definition 4. *A positive*[1] *measurable function g satisfying*

$$g(\lambda x)/g(x) \to 1, \qquad x \to \infty$$

for all $\lambda > 0$ is said to be slowly varying.

In the following examples, let $p \in \mathbb{R}$ be a constant.

The following functions are slowly varying: any positive constant; the logarithm function; sums, products, and pth powers of slowly varying functions; any function asymptotically equivalent to a slowly varying function.

The following functions are not slowly varying: $2 + \sin x$ and x^p for $p \neq 0$.

Definition 5. *If $g(x) = x^\alpha g_0(x)$ where g_0 is slowly varying and $\alpha \in \mathbb{R}$, then we say that g is* regularly varying with index α *and we write $g \in R_\alpha$.*

With regular variation and a mild moment condition, Conjecture 2's conclusion holds, as shown by Benaim-Friz [11]:

Theorem 6 (Right-tail-wing formula). *Assume that $\mathbb{E}S_T^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$. Let φ denote either the CCDF $\bar{F}$ in (2.12) or, if it exists, the density f of $\log(S_T/S_0)$.*

If $-\log\varphi \in R_\alpha$ for some $\alpha > 0$, then

$$V^2(k)/k \sim \psi(-\log c(k)/k) \sim \psi(-\log\varphi(k)/k - 1). \tag{3.1}$$

[1]Positivity is a standard convention for regularly varying functions.

The tail-wing formula links tail-asymptotics of φ *on a logarithmic scale*[2] and the implied volatility at extreme strikes.

3.1 Outline of proof

Using Bingham's Lemma ([14], Thm 4.2.10), it can be shown that, in the case $\varphi = f$,

$$-\log f(k) \sim -\log \bar{F}(k) \in R_\alpha, \tag{3.2}$$

and that hence it suffices to consider the case $\varphi = \bar{F}$.

The $\mathbb{E}S_T^{1+\varepsilon} < \infty$ assumption implies $-d_{1,2} \to \infty$. So

$$\begin{aligned} c(k) = \Phi(d_1) - e^k \Phi(d_2) &\sim -\frac{1}{\sqrt{2\pi} d_1} e^{-d_1^2/2} + \frac{1}{\sqrt{2\pi} d_2} e^k e^{-d_2^2/2} \\ &\sim \frac{1}{\sqrt{2\pi}} \Big(-\frac{1}{d_1} + \frac{1}{d_2} \Big) e^{-d_1^2/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \log c(k) \ \sim \ \log(1/d_2 - 1/d_1) - d_1^2/2 &= \log(V/(k^2/V^2 - V^2/4)) - d_1^2/2 \\ &\sim -d_1^2/2 \end{aligned} \tag{3.3}$$

where the last step is justified by $-\log \bar{F} \in R_\alpha$, or the weaker condition that $-\log c \in R_\alpha$, or the still weaker condition that $\liminf \log V / \log k > -\infty$. Now divide by $-k$ and apply ψ, to obtain the first relation in (3.1):

$$V^2(k)/k \sim \psi(-\log c(k)/k).$$

For the second relation in (3.1), write

$$c(k) = \mathbb{E}(S_T/S_0 - e^k)^+ = \int_{e^k}^\infty \bar{F}(\log y)\mathrm{d}y = \int_k^\infty e^{x + \log \bar{F}(x)} \mathrm{d}x. \tag{3.4}$$

Bingham's Lemma states that, if $g \in R_\alpha$ with $\alpha > 0$, then

$$\log \int_k^\infty e^{-g(x)} \mathrm{d}x \sim -g(k). \tag{3.5}$$

So verify that $-x - \log \bar{F}(x) \in R_\alpha$ and apply (3.5) to obtain

$$\log c(k) \sim \log \bar{F}(k) + k. \tag{3.6}$$

Divide by $-k$ and apply ψ to conclude. For a complete proof see [11].

[2].. similar to the logarithmic scale of large deviations ...

3.2 Tail-wing formula for the left wing

To formulate the small-strike counterpart of Theorem 6, denote the S_0-normalized forward put price by $p(k) := \mathbb{E}(e^k - S_T/S_0)^+$.

Theorem 7 (Left-tail-wing formula). *Assume* $\mathbb{E}S_T^{-\varepsilon} < \infty$ *for some* $\varepsilon > 0$*. Let* ϕ *denote either the* CDF F *in* (2.13) *or, if it exists, the density* f *of* $\log(S_T/S_0)$*.*

If $-\log\phi(-k) \in R_\alpha$ *for some* $\alpha > 0$*, then*

$$V^2(-k)/k \sim \psi(-1 - \log p(-k)/k) \sim \psi(-\log\phi(-k)/k).$$

This follows from Theorem 6 and the measure-change argument of Section 2.2, using the symmetries: (2.9-2.10), and $\log c(k; 1/S, \mathbb{Q}) = \log p(-k; S; \mathbb{P}) + k$, and $\log f(k; 1/S, \mathbb{Q}) = \log f(-k; S, \mathbb{P}) - k$, and $\log \bar{F}(k; 1/S, \mathbb{Q}) \sim \log F(-k; S, \mathbb{P}) - k$. The extra arguments on c, p, f, F, $\bar{F}$ emphasize dependence on the underlier (S or $1/S$) and the measure ($\mathbb{P}$ or $\mathbb{Q}$).

4 Related Results

This section extends the results of the previous two sections.

Let us write, as earlier, φ for the CCDF or the density of $X := \log(S_T/S_0)$. The tail-wing formula has the following consequences.

First, if $-\log\varphi(k)/k$ has limit $L \in [1, \infty)$, then the moment formula's lim sup is a genuine limit; this gives a sufficient condition for Conjecture 1's conclusion that

$$V^2(k)/k \to \psi(\tilde{p}) = \psi(L - 1). \tag{4.1}$$

(For $L > 1$ the proof is by direct application of the tail-wing formula; for $L = 1$, the $\mathbb{E}S_t^{1+\varepsilon} < \infty$ assumption may fail, but the conclusion holds by dominating the tails of distributions having tail CCDF e^{-pk} for $p > 1$, for which the tail-wing formula *does* hold.) In turn, the question arises, of how to guarantee the convergence of $-\log\varphi(k)/k$. We answer this in Section 4.1 by finding sufficient conditions on the moment generating function of X.

Second, if $-\log\varphi(k)/k \to \infty$ and Theorem 6's assumptions hold, then the $x \to \infty$ relation $\psi(x) \sim 1/(2x)$ implies

$$V^2(k)/k \sim 1/(-2\log\varphi(k)/k), \tag{4.2}$$

which gives more precise information than the moment formula's conclusion that $V^2(k)/k \to 0$. Those assumptions entail that the log-return distribution decays faster than exponentially, but not *so* quickly that the R_α assumption fails. This excludes, for example, the case of exponential decay of the *underlying* S_T (hence *iterated* exponential decay of log-return), which requires a separate analysis, in Section 4.2.

We may state left-hand versions of these results in terms of ϕ, the CDF or density of X. If $-\log\phi(-k)/k$ has limit $L \in [0, \infty)$ then $V^2(-k)/k \to \psi(L)$;

if instead $-\log \phi(-k)/k \to \infty$, then $V^2(-k)/k \sim 1/(-2\log\phi(-k)/k)$, provided that Theorem 7's assumptions hold. With the change-of-measure argument seen earlier, we can and henceforth will restrict our discussion to the right tail.

4.1 MGFs and the Moment Formula

We first note that $\mathbb{E}S_T^r$ is a constant multiple of $M(r) := \mathbb{E}\left(e^{rX_T}\right)$, the moment generating function of X. For many models, such a MGF is available in closed form so that option pricing (and calibration in particular) can be based on fast Fourier-methods [15, 31]. For our purposes, explicit knowledge of M allows one to read off the critical exponent

$$\tilde{r} := \sup\{r : M(r) < \infty\}$$

simply by spotting the first singularity in M for positive r. Assuming that $\tilde{r} \in (1, \infty)$, we see that $\tilde{p} = \tilde{r} - 1$ is exactly what is needed for the (right-hand) moment formula and so

$$\limsup_{k\to\infty} \frac{V^2(k)}{k} = \psi(\tilde{p}). \tag{4.3}$$

We are now looking for practical conditions which will guarantee that

$$\lim_{k\to\infty} \frac{V^2(k)}{k} = \psi(\tilde{p}). \tag{4.4}$$

If $-\log\varphi(k)/k$ converges to a limit in $(1, \infty)$, then $\varphi \in R_1$ and $\mathbb{E}S_T^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$. All conditions of the tail-wing formula are satisfied, so (4.4) indeed follows. The problem with this criterion is that it requires knowledge of the tail asymptotics of φ which may be unknown. The good news is that the required tail asymptotics, at least on the logarithmic scale of interest to us, can be obtained from the MGF via Tauberian theory, in which conditions on a distribution's transform imply properties of the distribution. The following regularity criteria on the MGF will cover most, if not all, examples with known MGF of log-price, provided $\tilde{r} \in (1, \infty)$. In essence, Criterion I below says that M, or one of its derivatives, blows up in a regularly varying way as the argument approaches the critical value $\tilde{r}$. Criterion II deals with exponential blow up near $\tilde{r}$.

Criterion I. *For some integer $n \geq 0$ and some real $\rho > 0$,*

$$M^{(n)}(\tilde{r} - 1/s) \in R_\rho, \qquad s \to \infty.$$

Criterion II. *For some real $\rho > 0$*

$$\log M(\tilde{r} - 1/s) \in R_\rho, \qquad s \to \infty.$$

Theorem 8. *Let X be a real-valued random variable with moment generating function $M(r) := \mathbb{E}\left(e^{rX_T}\right)$, critical exponent $\tilde{r} := \sup\{r : M(r) < \infty\} \in (0, \infty)$ and CCDF $\bar{F}(k) := \mathbb{P}(X > k)$. If M satisfies Criterion I or II then*

$$\lim_{k\to\infty} \frac{\log \bar{F}(k)}{k} = -\tilde{r}. \tag{4.5}$$

The main idea of the proof is an Esscher-type change of measure which reduces the problem to the application of a more standard Tauberian theorem (e.g. Karamata's Tauberian theorem for Criterion I). Theorem 8 really belongs to Tauberian theory and we refer to [12] for the proof, see also [9] for full asymptotics under further assumptions.

It must be emphasized that conclusion (4.5) fails if one omits the regularity criteria on M. (For a counterexample, consider the distribution specified by $\mathbb{P}(X \leq k) = 1 - \exp\{-e^{[\log k]}\}$, where $[\cdot]$ denotes the integer part of a real number). What remains true without regularity assumptions is a $\limsup$ statement: by Chebyshev we have $\mathbb{P}(X > k) \leq e^{-rk}\mathbb{E}\left(e^{rX_T}\right)$ so that $\log \bar{F}(k) \leq -rk + \log M(r) \sim -rk$ for all $r \in (0, \tilde{r})$; it easily follows that

$$\limsup_{k\to\infty} \frac{\log \bar{F}(k)}{k} \leq -r$$

and, using the very definition of $\tilde{r}$, one sees that equality holds with $\tilde{r}$. A formal insertion in the tail-wing formula would bring us back to the moment formula in its $\limsup$ form. More interestingly, we see that (4.5) leads in full rigor to (4.4). Omitting a similar "left-hand" formula we summarize our findings in

Theorem 9 (Right-hand moment formula, for regular MGFs). *Assume that $X = \log(S_T/S_0)$ has MGF M with critical exponent $\tilde{r} \equiv \tilde{p} + 1 \in (1, \infty)$ such that M satisfies Criterion I or II. Then*

$$V^2(k)/k \to \psi(\tilde{p}) \text{ as } k \to \infty.$$

4.2 Tail-wing formula for exponential decay

In Section 3 we discussed the limiting behavior of the implied volatility when the log of the distribution function of the returns is regularly varying. This condition is sometimes violated, for example if the log of the distribution function of the underlying is regularly varying, and therefore the distribution function of the returns will decay more quickly; we will discuss such an example below. Nonetheless, if one views the tail-wing formula as a meta-theorem in which the regular variation condition is replaced by the (undefined) "reasonable tail-behavior" one can hope that the tail-wing formula still gives the correct result. We will now prove this for another type of "reasonable tail behavior."

We will make the following assumption on the distribution of the underlying:

Assumption 1. *The log of the distribution function of the underlying is regularly varying with positive exponent. That is:*

$$-\log \bar{F}_{S_T} \in R_\alpha \text{ for } \alpha > 0.$$

To compare with the assumption we made originally, note that this implies that the distribution function of the returns satisfies:

$$\log\{-\log \bar{F}_{\log S_T/S_0}(x)\} \sim \alpha x. \tag{4.6}$$

If we assume this, a straightforward application of Bingham's Lemma allows us to obtain an expression for the call price in the large strike limit:

Lemma 10. *Under Assumption 1, the call price $C(K)$ as a function of strike satisfies:*

$$\log C(K) \sim \log \bar{F}_{S_T}(K) \ \textit{as} \ K \to \infty$$

Proof. Using Fubini,

$$C(K) = \mathbb{E}(S_T - K)^+ = \mathbb{E}\int_K^\infty \mathbb{I}(S_T > u)\mathrm{d}u = \int_K^\infty \bar{F}_{S_T}(u)\mathrm{d}u.$$

The result follows from Bingham's Lemma. ∎

To use this to analyse the implied volatility, we need to approximate the Black-Scholes formula in the range of interest to us. The calculations in the proof of the tail-wing formula do not apply directly to our case because the assumptions behind (3.3) are violated here. We note that the implied volatility for a model satisfying Assumption 1 is bounded as the strike goes to infinity because the density must ultimately be dominated by any Gaussian density. We can therefore use the following approximation.

Lemma 11. *Given any $\varepsilon > 0$ and $\bar{\sigma} > 0$, there exists a real number k_1 such that for all $k > k_1$ and for all $0 < \sigma < \bar{\sigma}$,*

$$-(1+\varepsilon)\frac{k^2}{2\sigma^2} \le \log c_{BS}(k,\sigma) \le -(1-\varepsilon)\frac{k^2}{2\sigma^2}.$$

Proof. The Black-Scholes formula satisfies

$$\begin{aligned} c_{BS}(k,\sigma) &> c_{BS}(k(1+\varepsilon),\sigma) \\ &= \Phi\Big(-\frac{k(1+\varepsilon)}{\sigma} + \frac{\sigma}{2}\Big) - e^{k(1+\varepsilon)}\Phi\Big(-\frac{k(1+\varepsilon)}{\sigma} - \frac{\sigma}{2}\Big), \end{aligned}$$

and the normal distribution function Φ has the following well-known bounds, obtainable by integration by parts (or other methods):

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}x}\Big(1 - \frac{1}{x^2}\Big) \le \Phi(-x) \le \frac{e^{-x^2/2}}{\sqrt{2\pi}x}, \qquad x > 0. \tag{4.7}$$

Therefore, for $k > \bar{\sigma}^2$ and $0 < \sigma < \bar{\sigma}$,

$$c_{BS}(k,\sigma) > \frac{e^{-(k(1+\varepsilon)/\sigma - \sigma/2)^2/2}}{\sqrt{2\pi}} Err(k,\sigma),$$

where

$$\begin{aligned} Err(k,\sigma) &:= \frac{1}{k(1+\varepsilon)/\sigma - \sigma/2} - \frac{1}{k(1+\varepsilon)/\sigma + \sigma/2} - \frac{1}{(k(1+\varepsilon)/\sigma - \sigma/2)^3} \\ &= \frac{\sigma^3}{k^2(1+\varepsilon)^2 - \sigma^4/4} - \frac{\sigma^3}{(k(1+\varepsilon) - \sigma^2/2)^3} \end{aligned}$$

satisfies

$$|Err(k,\sigma)| \le \frac{\bar\sigma^3}{k^2(1+\varepsilon)^2-\bar\sigma^4/4} + \frac{\bar\sigma^3}{(k(1+\varepsilon)-\bar\sigma^2/2)^3} = O\Big(\frac{1}{k^2}\Big),$$

as $k\to\infty$, uniformly in σ. Taking logs, we have

$$\begin{aligned}\log c_{BS}(k,\sigma) &> -\frac{1}{2}\Big(\frac{k(1+\varepsilon)}{\sigma}-\frac{\sigma}{2}\Big)^2 + O(\log k)\\ &> -\frac{k^2(1+\varepsilon)^2}{2\sigma^2} + \frac{k}{2} - \frac{\bar\sigma^2}{8} + O(\log k)\\ &= -\frac{k^2(1+\varepsilon)^2}{2\sigma^2} + O(k)\end{aligned}$$

as required.

For the upper bound, observe that for $k>\bar\sigma^2$ and $0<\sigma<\bar\sigma$,

$$c_{BS}(k,\sigma) < \Phi\Big(-\frac{k}{\sigma}+\frac{\sigma}{2}\Big) \le \frac{1}{\sqrt{2\pi}} e^{-(k/\sigma-\sigma/2)^2/2}\frac{\sigma}{k-\sigma^2/2},$$

where the last inequality follows again from (4.7). So

$$\frac{\sigma}{k-\sigma^2/2} \le \frac{\bar\sigma}{k-\bar\sigma^2/2} = O\Big(\frac{1}{k}\Big),$$

as $k\to\infty$. Taking logs, we have

$$\begin{aligned}\log c_{BS}(k,\sigma) &< -\frac{1}{2}\Big(\frac{k}{\sigma}-\frac{\sigma}{2}\Big)^2 + O(\log k)\\ &= -\frac{k^2}{2\sigma^2} + O(k),\end{aligned}$$

as required. ∎

Combining the lemmas yields

Theorem 12 ([10]). *Under Assumption 1,*

$$V^2(k) \sim \frac{k^2}{-2\log \bar F_{S_T}(e^k)} = \frac{k^2}{-2\log \bar F_{\log S_T}(k)} \text{ as } k\to\infty.$$

Proof. Because V is bounded for k large enough, Lemma 11 implies that, for any $\varepsilon>0$, for k large enough,

$$-(1-\varepsilon)\frac{k^2}{2\log c_{BS}(k,V(k))} \le V^2(k) \le -(1+\varepsilon)\frac{k^2}{2\log c_{BS}(k,V(k))},$$

hence

$$V^2(k) \sim \frac{k^2}{-2\log c_{BS}(k,V(k))} \sim \frac{k^2}{-2\log \bar F_{S_T}(e^k)}$$

by Lemma 10. ∎

5 Applications

5.1 Exponential Lévy Models

We first note that the recent Lévy-tail estimates from Albin-Bengtsson [1, 2], in conjunction with the tail-wing formula, allow to compute the implied volatility asymptotics of virtually *all* exponential Lévy models, regardless wether the underlying has finite moments of all orders or not. Only in the latter case do moment formulae (in the regular-MGF form or not) give quantitative information. A nice example where all methods discussed work is *Barndorff-Nielsen's Normal Inverse Gaussian model* in which $X = \log(S_T/S_0) \sim NIG(\alpha, \beta, \mu T, \delta T)$. The moment generating function is given by

$$M(r) = \exp\left[T\left(\delta\left\{\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+r)^2}\right\} + \mu r\right)\right]$$

When r approaches $\tilde{r} = \alpha - \beta$ the argument of the second square-root approaches the branching point singularity 0. It then follows from the moment formula that

$$\limsup_{k\to\infty} V^2(k)/k = \psi(\tilde{p}) \text{ with } \tilde{p} = \tilde{r} - 1.$$

Observe that $M(r)$ does not blow up as $r \uparrow \tilde{r}$ but its first derivative does. Indeed, $M'(\tilde{r}-1/s) \sim 2\delta\alpha\sqrt{2\alpha}s^{1/2}M(\tilde{r})$ as $s \to \infty$ and we see that Criterion I holds with $n = 1$ and the regular-MGF moment formula implies $\lim_{k\to\infty} V^2(k)/k = \psi(\tilde{p})$. Alternatively, one can take a direct route as follows. It is known, from [8] and the references therein, that X has density f with asymptotics

$$f(k) \sim C\,|k|^{-3/2}\,e^{-\alpha|k|+\beta k} \text{ as } k \to \pm\infty.$$

These are more than enough to see that $-\log f$ is regularly varying (with index 1) and $-\log f(k)/k \to \alpha - \beta$ as $k \to +\infty$. The Right-tail-wing formula now leads to $V^2(k)/k \to \psi(\alpha - \beta - 1)$ which is, of course, in agreement with our findings above.

Among the Lévy examples to which one can apply the regular-MGF moment formula (with Criterion I, $n = 0$), we mention *Carr-Madan's Variance Gamma model*. An example to which Criterion II applies is given by *Kou's Double Exponential model*. See [12] for details.

As remarked earlier, in models in which the underlying has finite moments of all orders moment formulae only give sublinear behavior of implied variance, namely $V^2(k)/k \to 0$; whereas the tail-wing formula still provides a complete asymptotic answer. Among the exponential Lévy examples with sublinear behavior of implied variance, we mention the *Black-Scholes model* as a sanity check example, *Merton's jump diffusion* as a borderline example in which the sublinear behavior comes from a subtle logarithmic correction term, and *Carr-Wu's Finite Moment Logstable model* for which tail asymptotics can be derived by Kasahara's Tauberian theorem. All these examples are discussed in detail in [11].

5.2 Time-Changed Lévy Models

Consider a Lévy process $L = L(t)$ described through its cumulant generating function (CGF) at time 1, denoted by K_L where

$$K_L(v) = \log \mathbb{E}\left[\exp\left(vL_1\right)\right],$$

and an independent random clock $\tau = \tau(\omega) \geq 0$ with CGF K_τ.

It follows that the MGF of $X \equiv L \circ \tau$ is given by

$$M(v) = \mathbb{E}\left[\mathbb{E}\left(e^{vL_\tau}|\tau\right)\right] = \mathbb{E}\left[e^{K_L(v)\tau}\right] = \exp\left[K_\tau(K_L(v))\right].$$

Frequently used random clocks [36, 12] are the *Gamma-Ornstein-Uhlenbeck* clock and the *Cox-Ingersoll-Ross* clock. More information on time-changed Lévy processes can be found in the text books [35, 16].

What matters for our purposes is that the MGF is explicitly known (provided K_τ, K_L are explicitly known) so that one can hope to apply the moment formula (for regular MGFs) in order to understand the implied volatility at extreme strikes for such models. The following result translates the regularity conditions (i.e. Criterion I and II) on M into "manageable" conditions in terms of K_τ, K_L. (The algebraic expression for M may be explicit but can be complicated!)

Define $M_\tau \equiv \exp\left(K_\tau\right)$ and $M_L \equiv \exp\left(K_L\right)$ and set

$$\tilde{r}_L = \sup\left\{r : M_L(r) < \infty\right\}, \qquad \tilde{r}_\tau = \sup\left\{r : M_\tau(r) < \infty\right\}.$$

We then have the following result (a similar "left-hand" result is omitted).

Theorem 13. *Let $\bar{F}$ denote the CCDF of $X = L \circ \tau$. Assume $\tilde{r}_L, \tilde{r}_\tau \in (0, \infty)$. If both M_τ, M_L satisfy Criterion I or II, then M does.*

Moreover, if $K_L(r) = \tilde{r}_\tau$ for some $r \in [0, \tilde{r}_L]$ then $r = \tilde{r}$, the critical exponent of M. Otherwise, if $K_L(r) < \tilde{r}_\tau$ for all $r \in [0, \tilde{r}_L]$, then $\tilde{r}_L = \tilde{r}$. Either way, we have

$$V^2(k)/k \sim \psi\left(\tilde{p}\right) \quad \textit{with } \tilde{p} = \tilde{r} - 1.$$

The proof is little more than a careful analysis of $\exp\left[K_\tau(K_L(v))\right]$ with regard to our MGF regularity criteria, and is found in [12]. In the same paper, as illustration and straightforward application of the last theorem, the *Variance Gamma with Gamma-OU time change* and *Normal Inverse Gaussian with CIR time* models are discussed. Applied with parameters obtained from real-world market calibrations, the asymptotic regime for the implied volatility becomes visible at remarkably low level of moneyness k and several plots are given in [12].

5.3 Heston Model

Heston's stochastic volatility model seems to require no introduction these days! It suffices to say that its MGF M is known, see [25] for instance, and a direct analysis shows that the moment formula (for regular MGFs) is applicable. (For zero correlation, the Heston model becomes a Brownian motion run with an independent CIR clock, which falls into the previous section of time-changed Lévy models, simplifying the discussion a bit.) The critical exponent of M is computed by Andersen-Piterbarg [3]; the authors then apply the original moment formula with $\limsup$ statements. Tail asymptotics for the Heston models are also known [20].

6 CEV and SABR

We now discuss the Constant Elasticity of Variance (CEV) model, followed by its extension to stochastic volatility, the SABR model [27]. Both are of interest to practitioners, and both have an interesting behavior of implied volatility at extreme strikes.

6.1 CEV Model

This model generates a skew via the stochastic differential equation

$$\mathrm{d}S_t = \sigma S_t^{1-\beta}\mathrm{d}W_t$$

where σ and $\beta \in (0,1)$ are constants. (When $\beta > 1/2$, boundary conditions at zero have to be specified.) The density in this model can be written explicitly in terms of the modified Bessel function [18], but the following heuristic argument using large deviation theory (which has in common with the tail-wing formula the same crude, logarithmic scale) may be more enlightening. Large deviations for stochastic differential equations (also known as Freidlin-Wentzell theory [19, 38]) describe the family of solutions when $\mathrm{d}W_t$ above is replaced by $\epsilon\ \mathrm{d}W_t$ (or equivalently, the Brownian motion is run at speed $\epsilon^2 t$). Closely related are the Varadhan asymptotics for diffusions which have been used in the context of the implied volatility smile in [5, 6, 13].

In general, asymptotic probabilities as $\epsilon \to 0$ are unrelated to the behavior of spacial asymptotic probabilities of the form $\{S_T > K\}$ with $K \to \infty$. In the CEV model, however, one can switch from the $K \to \infty$ regime to the $\epsilon \to 0$ regime by a scaling property. To wit,

$$\mathrm{d}\tilde{S} \equiv \mathrm{d}(S/K) = \sigma(S/K)^{1-\beta}\epsilon\ \mathrm{d}W = \sigma\tilde{S}^{1-\beta}\epsilon\ \mathrm{d}W,$$

with $\epsilon = 1/K^\beta \to 0$ for K large. From Freidlin-Wentzell's estimate[3], now writing S^ϵ for $\tilde{S}$,

$$\epsilon^2 \log \mathbb{P}(S_T^\epsilon > 1) \sim -\frac{1}{2T}d^2(S_0^\epsilon, 1) \sim -\frac{1}{2T}d^2(0,1)$$

as $S_0^\epsilon \to 0$, where $d(0,1)$ is the Riemannian distance from 0 to 1 given by

$$d(0,1) = \int_0^1 \frac{1}{\sigma x^{1-\beta}}\mathrm{d}x = \frac{1}{\beta\sigma}.$$

[3]The geodesic connecting S_0^ϵ and 1 stays away from the boundary at zero. Hence, we don't expect boundary conditions at zero to play a role for the right tail.

Unwrapping the definition leads to

$$\log \mathbb{P}(S_T > K) \sim -\frac{K^{2\beta}}{2\beta^2\sigma^2 T}, \tag{6.1}$$

which can alternatively be derived, rigorously, from the explicitly known density of S_T. As a consistency check, note that (6.1) recovers normal "Bachelier" asymptotics in the case $\beta = 1$.

If we express this in terms of moneyness $k = \log(K/S_0)$ then $\{S_T > K\} = \{\log(S_T/S_0) > k\}$ and the probability of this event decays exponentially fast as $k \to \infty$, so that the CCDF is *not* of regular variation. This is one of the rare examples we are aware of where an application of the tail-wing formula as stated in Section 3 is not justified.

What *does* apply here is Theorem 12, which completes rigorously the proof of

$$V^2(k)/k \sim k(S_0 e^k)^{-2\beta}\sigma^2\beta^2 T \text{ as } k \to \infty, \tag{6.2}$$

a relationship reported in [22].

6.2 SABR Model

Combining the CEV model with stochastic volatility, the SABR model is defined by the dynamics

$$\mathrm{d}S_t = \sigma_t S_t^c \mathrm{d}W_t^1 \tag{6.3}$$

$$\mathrm{d}\sigma_t = \varepsilon\sigma_t \mathrm{d}W_t^2 \tag{6.4}$$

where ε and $c \leq 1$ are constants, and W^1 and W^2 are Brownian motions with correlation ρ. This model is popular partly because it has an explicit solution for call prices in the limit as time to maturity tends to 0, and this can be used to produce an expansion in time to maturity for the implied volatility that is accurate as long as log strike and time are not too large.

The expansion for annualized implied volatility in the case[4] $c = 1$ is as follows (see [27]):

$$\frac{V(k)}{\sqrt{T}} \approx \sigma_{SABR}(k,T) := \frac{\sigma_0 z}{x(z)}\Big(1 + \Big(\frac{1}{4}\rho\varepsilon\sigma_0 + \frac{2-3\rho^2}{24}\varepsilon^2\Big)T\Big),$$

$$x(z) := \log\Big(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\Big),$$

where $z := -\varepsilon k/\sigma_0$. After some simplification it becomes clear that, as $|k| \to \infty$,

$$\sigma_{SABR}(k,T) \sim \frac{\varepsilon|k|}{\sigma_0 \log|k|}\Big(1 + \Big(\frac{1}{4}\rho\varepsilon\sigma_0 + \frac{2-3\rho^2}{24}\varepsilon^2\Big)T\Big),$$

[4]The expression for $c < 1$ is similar, but uses a power series expansion in k, which diverges in the limits we are looking at, so we will not discuss it here.

which is incompatible with Theorems 1 and 2.2, and so the approximation $\sigma_{SABR}(k,T)$ cannot be accurate in the large $|k|$ regime. Interestingly, it was proved in [13] that $\sigma_{SABR}(k,0)$ equals the $T \to 0$ limit of implied volatility, which demonstrates that the limits $T \to 0$ and $k \to \pm\infty$ are not interchangeable in general.

For large $|k|$, therefore, we turn to the techniques presented in this article. However, because neither the moment generating function nor the distribution function is known for SABR, we need to approximate one of these to apply our results. Let us consider the MGF.

The tail behavior in this model depends on the value of c. When $c = 1$, we can determine which moments of S_T are finite, and use the moment formula. Solving (6.3) and (6.4), we have

$$\begin{aligned} S_t &= S_0 \exp\Big(\int_0^t \sigma_s \mathrm{d}W_s^1 - \frac{1}{2}\int_0^t \sigma_s^2 \mathrm{d}s\Big) \\ \sigma_t &= \sigma_0 \exp(\varepsilon W_t^2 - \varepsilon^2 t/2). \end{aligned}$$

Taking the expectation of S_T^p conditional on W^2, we have

$$\begin{aligned} \mathbb{E}[S_T^p] &= \mathbb{E}[\mathbb{E}[S_T^p \mid W^2]] \\ &= S_0^p\ \mathbb{E}\exp\left(p\rho\int_0^T \sigma_s \mathrm{d}W_s^2 + \frac{p^2(1-\rho^2)-p}{2}\int_0^T \sigma_s^2 \mathrm{d}s\right) \\ &= S_0^p\ \mathbb{E}\exp\left(\frac{p\rho(\sigma_T - \sigma_0)}{\varepsilon} + \frac{p^2(1-\rho^2)-p}{2}\int_0^T \sigma_s^2 \mathrm{d}s\right). \end{aligned} \tag{6.5}$$

Because $\mathbb{E}\exp(a\int_0^T \sigma_s^2 \mathrm{d}s) = \infty$ for any positive a and T (see for example [3]), we would expect (6.5) to be infinite whenever the coefficient of $\int_0^T \sigma_s^2 \mathrm{d}s$ (which we would expect to dominate the σ_T term) in (6.5) is positive. This would imply that, for[5] $\rho \le 0$, we have

$$\mathbb{E}[S_T^p] = \infty \text{ iff } p > 1/(1-\rho^2) \text{ or } p < 0. \tag{6.6}$$

For $p > 1, \rho < 0$, this is indeed proved in [32, Theorem 2.3.]; the case $p > 1, \rho = 0$ follows directly from (6.5) as does the case $p < 0, \rho \le 0$ since in that case the coefficients of σ_T and $\int_0^T \sigma_s^2 \mathrm{d}s$ appearing in (6.5) are ≥ 0 resp. > 0. In summary, the moment formula then implies

$$\limsup_{k\to\infty} V^2(k)/k = \psi\Big(\frac{1}{1-\rho^2} - 1\Big)$$

and

$$\limsup_{k\to\infty} V^2(-k)/k = \psi(0) = 2$$

[5] By [29, 37], if $\rho > 0$ and $c = 1$, then S would not be a martingale.

which concludes the analysis of the case $c = 1$.

Now consider the case $c < 1$. Because S can reach 0 with positive probability, we assume, as usual, an absorbing[6] boundary condition at 0, which leads to [7]

$$\lim_{k\to\infty} V^2(-k)/k = 2.$$

To calculate the right wing, we need a sufficiently good approximation for the distribution function of S_T, which we will obtain below (via Kasahara's Tauberian theorem) from a sufficiently good approximation of the moment generating function of $\log S_T$ (equivalently, the moments of S_T).

Andersen and Piterbarg [3, Prop 5.2] calculated the upper bounds

$$\mathbb{E}[S_t^p] \le \left[S_0^{2(1-c)} + (1-c)(p-1) \int_0^t \mathbb{E}\big(\sigma_s^{p/(1-c)}\big)^{2(1-c)/p} \mathrm{d}s \right]^{\frac{p}{2(1-c)}}.$$

Therefore we have

$$\begin{aligned}
\limsup_{p\to\infty} \frac{\log \mathbb{E}[S_T^p]}{p^2} &\le \limsup_{p\to\infty} \frac{1}{2p(1-c)} \log \int_0^T \mathbb{E}\big(\sigma_s^{p/(1-c)}\big)^{2(1-c)/p} \mathrm{d}s \\
&= \limsup_{p\to\infty} \frac{1}{2p(1-c)} \log \int_0^T e^{(p^2/(1-c)^2 - p/(1-c))\varepsilon^2 s(1-c)/p} \mathrm{d}s \\
&\le \limsup_{p\to\infty} \frac{1}{2p(1-c)} \left(\frac{p}{1-c} - 1 \right) \varepsilon^2 T = \frac{\varepsilon^2 T}{2(1-c)^2}. \qquad (6.7)
\end{aligned}$$

To apply the tail wing formula, we need to show that this bound is sharp enough. We can do so, at least when the correlation ρ is 0, which we shall assume from here on. We do emphasize, however, that $\rho = 0$ still allows for skew in the implied volatility.

Proposition 14. *Assume $\rho = 0$. For $n \in \mathbb{N}$ define $p_n = 1 + 2(1-c)n$. Then we have the lower bound*

$$\mathbb{E}[S_T^{p_n}] \ge S_0 \sigma_0^{2n} \varepsilon^{-2n} e^{(4n^2-2n)\varepsilon^2 T/2} (1 + O(e^{-n})) \prod_{i=1}^{n} \frac{p_i(p_i - 1)}{2n^2 - n - 2(i-1)^2 + (i-1)}$$

Proof. By Itô's formula,

$$S_t^p - S_0^p = \int_0^t p S_s^{p-1} S_s^c \sigma_s \mathrm{d}W_s^1 + \int_0^t p \frac{p-1}{2} S_s^{p-2} S_s^{2c} \sigma_s^2 \mathrm{d}s.$$

[6] Note that for $c \in (1/2, 1)$ this is the only possible boundary condition, and for $c \le 1/2$ this is the only possibility that ensures that S_t is a martingale.

[7] The limsup statement is clear. To see that one has a genuine lim it suffices to compare put prices (in the small strike limit) with those obtained from a model where returns decay like $exp(-p|k|)$ as $k \to \infty$. By monotonicity of the Black-Scholes prices in volatility, and the tail-wing formula applied to the "comparison-model", this leads to

$$\liminf_{k\to\infty} V^2(-k)/k \ge \psi(p),$$

for all $p > 0$. As p tends to zero, the right-hand-side approaches 2 which finishes the argument.

We now take expectations conditional on the second Brownian motion W^2 and see that

$$E[S_t^p \mid W^2] \geq \int_0^t p\frac{p-1}{2}\mathbb{E}[S_s^{p-2+2c} \mid W^2]\sigma_s^2 \mathrm{d}s.$$

Note that the first integral disappeared because it is a martingale[8] on the filtration of W^1 which is independent of W^2. Then

$$\mathbb{E}[S_t^{p_1} \mid W^2] \geq \int_0^t p_1\frac{p_1-1}{2}\mathbb{E}[S_s \mid W^2]\sigma_s^2 \mathrm{d}s = \int_0^t p_1\frac{p_1-1}{2}S_0\sigma_s^2 \mathrm{d}s$$

and the same reasoning yields the recurrence relation

$$\mathbb{E}[S_t^{p_n} \mid W^2] \geq \int_0^t p_n\frac{p_n-1}{2}\mathbb{E}[S_s^{p_{n-1}} \mid W^2]\sigma_s^2 \mathrm{d}s.$$

By iteration and taking the total expectation, we can therefore bound the p_n^{th} moments. Moreover, because σ_s/σ_u and σ_u are independent for $s > u$, it is relatively easy to do so. This leads us to

$$\mathbb{E}[S_T^{p_n}] \geq S_0\sigma_0^{2n}\varepsilon^{-2n}e^{(4n^2-2n)\varepsilon^2 T/2}(1+O(e^{-n}))\prod_{i=1}^{n}\frac{p_i(p_i-1)}{2n^2-n-2(i-1)^2+(i-1)}$$

where $O(e^{-n})$ depends tacitly on T. ■

Now, for each p, define $N(p)$ to be the integer such that

$$p \in [p_{N(p)}, p_{N(p)+1}) \equiv [1+2(1-c)N(p), 1+2(1-c)(N(p)+1)).$$

Then

$$\begin{aligned}\liminf_{p\to\infty}\frac{\log\mathbb{E}[S_T^p]}{p^2} &\geq \liminf_{p\to\infty}\frac{\log\mathbb{E}[S_T^{p_{N(p)}}]}{p^2}\\ &= \liminf_{p\to\infty}\frac{2[N(p)]^2\varepsilon^2 T}{p^2} = \frac{\varepsilon^2 T}{2(1-c)^2}.\end{aligned}$$

Combining this with (6.7), we obtain

$$\log\mathbb{E}\left[S_T^p\right] = \log\mathbb{E}\left[\exp\left(p\log S_T\right)\right] \sim \frac{\varepsilon^2 T}{(1-c)^2}\times\frac{p^2}{2},$$

which matches the growth of the MGF of a standard Gaussian with variance $\varepsilon^2 T/(1-c)^2$. One suspects that the CCDF of $\log S_T$, denoted by $\bar{F}_{\log S_T}$, has (at least at logarithmic scale) a matching Gaussian tail, i.e.

$$-\log\bar{F}_{\log S_T}(k) \sim \frac{(1-c)^2}{2\varepsilon^2 T}k^2.$$

We now make this rigorous via *Kasahara's exponential Tauberian theorem* [14, Theorem 4.12.7 p253].

[8]It is straightforward to show that it is a martingale and not just a local martingale using the fact that σ and S have finite moments of all orders

Theorem 15 (Kasahara). *Let μ be a measure on $(0, \infty)$ such that*

$$M(\lambda) := \int_0^\infty e^{\lambda x} \mathrm{d}\mu(x) < \infty$$

for all $\lambda > 0$. If $0 < \alpha < 1$, $\phi \in R_\alpha$, put $\theta(\lambda) := \lambda/\phi(\lambda) \in R_{1-\alpha}$.

Then, for $B > 0$,

$$-\log \mu(x, \infty) \sim B\phi^{\leftarrow}(x) \text{ as } x \to \infty$$

if and only if

$$\log M(\lambda) \sim (1-\alpha)(\alpha/B)^{\alpha/(1-\alpha)}\theta^{\leftarrow}(\lambda) \text{ as } \lambda \to \infty$$

where $f^{\leftarrow}$ denotes the generalised inverse of f.

If we let $\mu(x, y) = \mathbb{P}[S_T \in (x, y)]$, $\phi(x) = x^{1/2} = \theta(x)$, $B = (1-c)^2/(2\varepsilon^2 T)$, and $\alpha = 1/2$, then

$$\log M(p) \sim \log \mathbb{E}\left[\exp\left(p \log S_T\right)\right] \sim \frac{\varepsilon^2 T}{(1-c)^2} \times \frac{p^2}{2},$$

and so

$$-\log \bar{F}_{\log S_T}(k) \sim -\log \mu(k, \infty) \sim \frac{(1-c)^2}{2\varepsilon^2 T} k^2,$$

as expected.

The right-tail wing formula now leads immediately to the following result as conjectured by Piterbarg in [33].

Proposition 16 ([10]). *Let $V(k)$ denote the (unannualized) implied volatility for the SABR model with $\rho = 0$ and $c < 1$. Then*

$$\lim_{k \to \infty} V^2(k) = \frac{\varepsilon^2 T}{(1-c)^2}.$$

Remark 17. Hagan, Lesniewski and Woodward [28], see also [13, 4] find that the pdf of S_t is "approximately" Gaussian with respect to distance

$$d\left(S_0, S\right) = \frac{1}{\varepsilon} \log \frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1 - \rho},$$

$$\zeta = \frac{\varepsilon}{\sigma_0} \int_{S_0}^{S} \frac{1}{u^c} du \sim \frac{\varepsilon}{\sigma_0} \frac{S^{1-c}}{1-c}.$$

To compare with our result above, let $\rho = 0$. Then, as $S \to \infty$,

$$d(S_0, S) \sim \frac{1}{\varepsilon} \log \zeta \sim \frac{1-c}{\varepsilon} \log S$$

and

$$-\log \mathbb{P}(S_T \in \mathrm{d}S) \approx \frac{1}{2T} d\left(S_0, S\right)^2 \sim \frac{(1-c)^2}{2\varepsilon^2 T} \left(\log S\right)^2.$$

If f denotes the pdf of $\log S_t$, this easily implies that

$$-\log f(k) \sim \frac{(1-c)^2}{2\varepsilon^2 T} k^2,$$

and the tail-wing formula gives the same asymptotic implied volatility as Kasahara's Tauberian theorem above. Heat-kernel estimates may provide the key to make such heuristics rigorous and extend the discussion to arbitrary stochastic volatility models. For stochastic volatility models with specific structure, smile asymptotics have been discussed early on, see [24, 7]. We also note that heat-kernel bounds have been explored (e.g. [22]) towards large strike asymptotics of implied volatility in local volatility models.

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